The many routes to the ubiquitous Bradley-Terry model

Ian Hamilton

University of Warwick

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Quiz Format

- Six rounds : Five topic rounds + a quick-fire round.
- Three teams.
- In the five topic rounds, each team will be set a different question. A point will be awarded for a correct answer and a point for the working.
- In each topic round, there will be an extra bonus question open to all teams.
- The 'Just a Minute' rule applies - in the absence of a correct answer, a sufficiently amusing answer may also gain a point.
- Quizmaster accepts no responsibility for the quality of the questions or answers, and his judgement is final.

Round 1: Games and pastimes

1. Sudden death - Suppose we have two teams i and j involved in a 'sudden death' shoot-out. They play a game where in each round they succeed with independent probabilities p_i and p_j respectively, where $\pi_i = \frac{p_i}{1-p_i}$. The winner is the team who first has more successes than the other team. What is the probability that team i wins? (answer to be given in terms of π_i and π_j).
2. Poisson scoring - Suppose we have two teams i and j who score according to independent Poisson processes with rate parameters π_i and π_j respectively. The winner is the first team to score. What is the probability that team i wins?
3. Accumulated win ratio - suppose we have a sequence of matches between two players, i and j, where the probability that team i wins is proportional to the accumulated wins in previous matches. Suppose that the probability that i wins the first match is $\pi_i/(\pi_i + \pi_j)$. What is the probability that i wins the nth match?

Round 1: Games and pastimes - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 1: Games and pastimes - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$
2. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 1: Games and pastimes - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$
2. $\frac{\pi_i}{\pi_i+\pi_j}$
3. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 1: Games and pastimes - Answers Q1

$$
\begin{aligned}
p_{ij} &= \sum_{n=1}^{\infty} P(i \succ j)_n \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} p_i(1-p_j) \binom{n-1}{k} (p_i p_j)^k ((1-p_i)(1-p_j))^{n-k-1} \\
&= p_i(1-p_j) \sum_{m=0}^{\infty} (p_i p_j + (1-p_i)(1-p_j))^m \\
&= \frac{p_i(1-p_j)}{p_i + p_j - 2p_i p_j} \\
&= \frac{p_i(1-p_j)}{p_i(1-p_j) + p_j(1-p_i)} \\
&= \frac{\frac{p_i}{1-p_i}}{\frac{p_i}{1-p_i} + \frac{p_j}{1-p_j}} \\
&= \frac{\pi_i}{\pi_i + \pi_j}, \quad \text{where } \pi_i = \frac{p_i}{1-p_i}.
\end{aligned}
$$

Round 1: Games and pastimes - Answers Q2

$$
\begin{aligned}
p_{ij} &= P(X_i = 1 \mid X_i + X_j = 1) \\
&= \frac{P(X_i = 1)P(X_j = 0)}{P(X_i + X_j = 1)} \\
&= \frac{e^{-\pi_i}\pi_i e^{-\pi_j}}{e^{-(\pi_i+\pi_j)}(\pi_i + \pi_j)} \\
&= \frac{\pi_i}{\pi_i + \pi_j} \quad .
\end{aligned}
$$

Round 1: Games and pastimes - Answers Q3

$p_{ij} = \frac{\pi_i}{\pi_i + \pi_j}$

Proof by induction. Let $P[(i \succ j)_n]$ be the probability that i beats j in round n. It is given that it is true for $n = 1$. Suppose true for $n = k$. then

$$\begin{aligned} P[(i \succ j)_{k+1}] &= P[(i \succ j)_{k+1} \mid (i \succ j)_k] P[(i \succ j)_k] \\ &\quad + P[(i \succ j)_{k+1} \mid (j \succ i)_k] P[(j \succ i)_k] \\ &= \frac{\pi_i + 1}{\pi_i + 1 + \pi_j} \frac{\pi_i}{\pi_i + \pi_j} + \frac{\pi_i}{\pi_i + 1 + \pi_j} \frac{\pi_j}{\pi_i + \pi_j} \\ &= \frac{\pi_i(\pi_i + 1 + \pi_j)}{(\pi_i + 1 + \pi_j)(\pi_i + \pi_j)} \\ &= \frac{\pi_i}{\pi_i + \pi_j} \quad . \end{aligned}$$

Round 1: Games and pastimes Bonus question

Q: Suppose we have an alternative sudden death contest where in each round competitors i and j score with probability p_i and p_j respectively, but now the winner is the team that is the first to have r more successes than the opposition. Then the probability that i beats j is $p_{ij} = \frac{\pi_i}{\pi_i + \pi_j}$, but what is π_i in terms of p_i?

Round 1: Games and pastimes Bonus answer

A: $\pi_i = \left(\frac{p_i}{1-p_i}\right)^r$

Let A_i be the event that i wins and A^{r+k} be the event that a result, either i or j winning, occurs after the winning team has scored exactly $r+k$ times, then defining $q_i = p_i/(1-p_i)$,

$$\begin{aligned}
p_{ij} = P(A_i) &= \sum_{k=0}^{\infty} P(A_i|A^{r+k})P(A^{r+k}) \\
&= \sum_{k=0}^{\infty} \frac{q_i^{r+k} q_j^k}{q_i^{r+k} q_j^k + q_i^k q_j^{r+k}} P(A^{r+k}) \\
&= \frac{q_i^r}{q_i^r + q_j^r} \sum_{k=0}^{\infty} P(A^{r+k}) \\
&= \frac{q_i^r}{q_i^r + q_j^r} \\
&= \frac{\pi_i}{\pi_i + \pi_j}
\end{aligned}$$

Round 2: Discriminal Processes

Consider a scenario where the strength of two entities in a pairwise interaction is observed, with the winner being that with the greatest observed strength. The observed strength of competitor i may be considered to be a random variable with a distribution $F_i(x)$. This is the model of Thurstone's 'discriminal processes'. For your distribution below, what is the probability that i beats j?

1. Gumbel distribution

$$F_i(x) = \exp\left(-\pi_i \mathrm{e}^{-\alpha x}\right) \text{ for } x \in \mathbb{R} \text{ and } \alpha > 0,$$

2. Weibull distribution

$$F_i(x) = 1 - \exp\left(-x^{\alpha}/\pi_i\right) \text{ for } x \in \mathbb{R}^+ \text{ and } \alpha > 0.$$

3. Frechet distribution

$$F_i(x) = \exp\left(-\pi_i x^{-\alpha}\right) \text{ for } x \in \mathbb{R}^+ \text{ and } \alpha > 0.$$

Round 2: Discriminal Processes - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 2: Discriminal processes - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$
2. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 2: Discriminal Processes - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$
2. $\frac{\pi_i}{\pi_i+\pi_j}$
3. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 2: Discriminal Processes - Answers Q1

$$
\begin{aligned}
p_{ij} &= \int_{-\infty}^{\infty} F_j(x) F_i'(x) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \exp\left(-\pi_j \mathrm{e}^{-\alpha x}\right) \alpha \pi_i \exp\left(-\alpha x - \pi_i \mathrm{e}^{-\alpha x}\right) \mathrm{d}x \\
&= \frac{\pi_i}{\pi_i + \pi_j} \int_{-\infty}^{\infty} \alpha(\pi_i + \pi_j) \exp\left(-\alpha x - (\pi_i + \pi_j)\mathrm{e}^{-\alpha x}\right) \mathrm{d}x \\
&= \frac{\pi_i}{\pi_i + \pi_j} \quad .
\end{aligned}
$$

Round 2: Discriminal Processes - Answers Q2

$$
\begin{aligned}
p_{ij} &= \int_0^\infty F_j(x) F_i'(x) \mathrm{d}x \\
&= \int_0^\infty \left(1 - \exp\left(-\frac{x^\alpha}{\pi_j}\right)\right) \frac{\alpha x^{\alpha-1}}{\pi_i} \exp\left(-\frac{x^\alpha}{\pi_j}\right) \mathrm{d}x \\
&= 1 - \int_0^\infty \frac{\alpha x^{\alpha-1}}{\pi_i} \exp\left(-x^\alpha \left(\frac{1}{\pi_i} + \frac{1}{\pi_j}\right)\right) \mathrm{d}x \mathrm{d}x \\
&= \frac{\pi_i}{\pi_i + \pi_j} \quad .
\end{aligned}
$$

Round 2: Discriminal Processes - Answers Q3

$$
\begin{aligned}
p_{ij} &= \int_0^\infty F_j(x) F_i'(x) \mathrm{d}x \\
&= \int_0^\infty \exp\left(-\pi_j x^{-\alpha}\right) \frac{\pi_i \alpha}{x^{\alpha+1}} \exp\left(-\pi_i x^{-\alpha}\right) \mathrm{d}x \\
&= \frac{\pi_i}{\pi_i + \pi_j} \int_0^\infty \alpha \frac{\pi_i + \pi_j}{x^{\alpha+1}} \exp\left(-(\pi_i + \pi_j) x^{-\alpha}\right) \mathrm{d}x \\
&= \frac{\pi_i}{\pi_i + \pi_j} \quad .
\end{aligned}
$$

Round 2: Discriminal Processes Bonus question

Q: The Gumbel, Weibull, and Frechet are the three types of what family of distributions?

Round 2: Discriminal Processes Bonus answer

A: Extreme value distributions

We might also note that if $F_i(x) = G^{\pi_i}(x)$ where $G(x)$ is itself a distribution then

$$p_{ij} = \int_{x_i > x_j} dF^{\pi_i}(x_i) dF^{\pi_j}(x_j) = \frac{\pi_i}{\pi_i + \pi_j}$$

Round 3: Quasi-symmetry and popular ranking methods - Questions

Consider a ('nice') quasi-symmetric competition matrix. What well-known rating do we derive given the method described?.

1. We rank teams by their win-loss ratio, but realise this fails to account for strength of opposition, so we weight their wins by the win-loss ratio of the defeated team, but then we decide that isn't fair because it fails to account for the strength of the opposition's opposition and so we weight the teams' wins by the rating we got by the 'opposition's strength' rating, but then we decide this isn't fair because...you get the idea.
2. We have the same thought process as in question 1, but we decide we shouldn't completely throw away the rating derived from thinking about the opposition's strength, or the opposition's opposition's strength, or the...so instead we decide to make our rating the mean of all of these.
3. A glory-seeking fan chooses a team to support at random. He then chooses another team to support at random from the set of teams that has beaten the first team he chose. He then chooses another team to support at random from the set of teams that has beaten that team...and so on. The rating is devised by the proportion of time that the glory-seeking fan spends supporting each team.

Round 3: Quasi-symmetry and popular ranking methods - Answers

1. The Bradley-Terry model (Adjusted Wei-Kendall)

Round 3: Quasi-symmetry and popular ranking methods - Answers

1. The Bradley-Terry model (Adjusted Wei-Kendall)
2. The Bradley-Terry model (Adjusted Ratings Percentage Index (RPI))

Round 3: Quasi-symmetry and popular ranking methods - Answers

1. The Bradley-Terry model (Adjusted Wei-Kendall)
2. The Bradley-Terry model (Adjusted Ratings Percentage Index (RPI))
3. Undamped PageRank

Round 3: Quasi-symmetry and popular ranking methods - Answers Q1

Let $C = [c_{ij}]$ be the competition matrix, $D = [d_{ij}]$ be the diagonal matrix of column sums, where $d_{jj} = \sum_k c_{kj}$, and $\tilde{C} = D^{-1}C$. Then the rating described is

$$\boldsymbol{\pi} = \lim_{k\to\infty} \tilde{C}^k e,$$

which by Perron-Frobenius Theorem is equal to the leading eigenvector of $\tilde{C}$. A vector $\boldsymbol{\pi}$ is an eigenvector for $\tilde{C} = D^{-1}C$ with an eigenvalue of 1 if and only if

$$\sum_j c_{ij}\pi_j = d_{ii}\pi_i \quad \text{for all } i,$$

but if $C = AS$ is quasi-symmetric such that A is a diagonal matrix and S is symmetric then if we choose $\pi_i = a_{ii}$ then the left hand side is

$$\sum_j c_{ij}\pi_j = \sum_j a_{ii}s_{ij}a_{jj} = a_{ii}\sum_j s_{ji}a_{jj} = \pi_i \sum_j c_{ji} = d_{ii}\pi_i \quad \text{for all } i,$$

so that $\boldsymbol{\pi}$ is the diagonal component of a quasi-symmetric matrix, or equivalently the Bradley-Terry rating vector.

Round 3: Quasi-symmetry and popular ranking methods - Answers Q2

The rating described is

$$\boldsymbol{\pi} = \lim_{r \to \infty} \frac{1}{r} \sum_{k=1}^{r} \tilde{C}^k e.$$

which by Perron-Frobenius Theorem is equal to the leading eigenvector of $\tilde{C}$.

Round 3: Quasi-symmetry and popular ranking methods - Answers Q3

This is the definition of undamped PageRank.

Round 3: Quasi-symmetry and popular ranking methods - Bonus question

Q: In terms of D and the undamped PageRank rating vector $\boldsymbol{\alpha}_{PR}$, how may one express the Bradley-Terry rating vector $\boldsymbol{\pi}$?

Round 3: Quasi-symmetry and popular ranking methods - Bonus answer

A: $\boldsymbol{\pi} = D^{-1}\boldsymbol{\alpha}_{PR}$

$$\boldsymbol{\alpha}_{PR} = CD^{-1}\boldsymbol{\alpha}_{PR}$$

$$\boldsymbol{\pi} = D^{-1}\boldsymbol{\alpha}_{PR} = D^{-1}CD^{-1}\boldsymbol{\alpha}_{PR} = D^{-1}C\boldsymbol{\pi},$$

Round 4: Objective function maximisation

In each case you will be given an objective function, and a constraint or constraints. What is the unique rating you get when the objective function is maximised subject to the constraint(s). ($M = C + C^T$, assume $m_{ij} > 0$)

1. $S(\boldsymbol{p})$ is the Shannon entropy, $S(\boldsymbol{p}) = -\sum_{i \neq j} m_{ij} p_{ij} \log p_{ij}$, subject to $\sum_{j \neq i} c_{ij} = \sum_{j \neq i} m_{ij} p_{ij}$ for all teams i.
2. $S(\boldsymbol{p})$ is the log-likelihood, $S(\boldsymbol{p}) = -\sum_{i \neq j} c_{ij} \log p_{ij}$, subject to $\sum_{j \neq i} c_{ij} = \sum_{j \neq i} m_{ij} p_{ij}$ for all teams i.'
3. Assume $p_{ij} = f(\pi_i, \pi_j)$ then define objective function $S(p_{ij}) =$ $1/$(the number of arithmetic operations $(+, -, \times, \div)$ used to define p_{ij}) subject to $p_{ij} = \frac{1}{2}$ when $\pi_i = \pi_j$; $\lim_{\pi_i \to \infty, \pi_j \text{ fixed}} p_{ij} = 1$; $\lim_{\pi_i \to 0, \pi_j \text{ fixed}} p_{ij} = 0$; $\lim_{\pi_j \to \infty, \pi_i \text{ fixed}} p_{ij} = 0$; $\lim_{\pi_j \to 0, \pi_i \text{ fixed}} p_{ij} = 1$.

Round 4: Objective function maximisation - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 4: Objective function maximisation - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$
2. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 4: Objective function maximisation - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$
2. $\frac{\pi_i}{\pi_i+\pi_j}$
3. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 4: Objective function maximisation - Answers Q1

Taking the Lagrangian as

$$\mathcal{L}(p, \boldsymbol{\lambda}) = S(p) - \sum_{i=1}^{n} \lambda_i \Bigg(\sum_{j=1, j\neq i}^{n} (m_{ij} p_{ij} - c_{ij}) \Bigg),$$

and setting $\frac{\partial \mathcal{L}}{\partial p_{ij}} = 0$ for all p_{ij} in the normal way gives that

$$\frac{\partial S(p)}{\partial p_{ij}} = \frac{\partial}{\partial p_{ij}} \sum_{r=1}^{n} \lambda_r \Bigg(\sum_{r=1, r\neq s}^{n} (m_{rs} p_{rs} - c_{rs}) \Bigg) \quad \text{for all } i, j.$$

So that for all i, j such that $m_{ij} \neq 0$,

$$-\log p_{ij} + \log(1 - p_{ij}) = \lambda_i - \lambda_j,$$

or equivalently

$$p_{ij} = \frac{\pi_i}{\pi_i + \pi_j},$$

Round 4: Objective function maximisation - Answers Q2

$$\begin{aligned}\frac{\partial}{\partial \lambda_k} S(\boldsymbol{p}) &= \frac{\partial}{\partial \lambda_k}\left[\sum_{i,j \in N} c_{ij} \log(p_{ij})\right] \\ &= \sum_{j \in N} c_{kj} \frac{\partial}{\partial \lambda_k} \log(p_{kj}) + c_{jk} \frac{\partial}{\partial \lambda_k} \log(p_{jk}),\end{aligned}$$

and that

$$\sum_{j \in N} c_{kj} - m_{kj} p_{kj} = \sum_{j \in N} c_{kj}(1 - p_{kj}) - c_{jk}(1 - p_{jk}),$$

and so a solution may be defined by setting

$$\frac{\partial}{\partial \lambda_k} \log(p_{kj}) = (1 - p_{kj}) \text{ and } \frac{\partial}{\partial \lambda_k} \log(p_{jk}) = -(1 - p_{jk}),$$

which gives

$$\frac{\partial p_{kj}}{\partial \lambda_k} = p_{kj}(1 - p_{kj}) \text{ and } \frac{\partial p_{jk}}{\partial \lambda_k} = -p_{jk}(1 - p_{jk}).$$

so that

$$p_{ij} = \frac{1}{1 + e^{-(\lambda_i - \lambda_j)}} = \frac{\pi_i}{\pi_i + \pi_j} \text{ where } \pi_i = e^{\lambda_i}.$$

Round 4: Objective function maximisation - Answers Q3

The formula $p_{ij} = \pi_i \div (\pi_i + \pi_j)$, defining the Bradley-Terry model, includes two operators. There are a finite number of alternative functions with two or fewer operators. It may be shown through exhaustive consideration of these alternatives that no other function of the same or greater simplicity can be defined that meets these criteria. In fact, no other function of the same or greater simplicity can be defined that meets any more than three of these criteria.

Round 4: Objective function maximisation - Bonus question

Q: What is the link between the maximum entropy and maximum likelihood distributions within an exponential family of distributions?

Round 4: Objective function maximisation - Bonus answer

A: They are the same.

Round 5: Axioms

In each case give the rating that results from the assumption (applied to pairwise comparisons).

1. Transitivity - Consider four teams i, j, k, l. Suppose that the probability that j beats k is greater than the probability that j beats l, $p_{jk} > p_{jl}$, then it is somewhat intuitive to think that the probability that i beats k will be greater than the probability that i beats l, $p_{ik} > p_{il}$. A simple way to model this would be by insisting on the transitivity of odds $\frac{p_{ij}}{p_{ji}} \times \frac{p_{jk}}{p_{kj}} = \frac{p_{ik}}{p_{ki}}$.
2. $s_i = \sum_j c_{ij}$ are the points gained by team i. Assume that the score vector $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ is a sufficient statistic such that the likelihood $\prod_{i \neq j} {p_{ij}}^{c_{ij}}$ is dependent on C only through $\boldsymbol{s}$.
3. Suppose we are selecting from a set of items, and we do so by a two-stage process where initially we select a set S from the total set T, and then we select an item i from the set S. Assume that the probability of choosing i does not depend on the intermediate set S. (then apply the result to a pairwise comparison)

Round 5: Axioms - Answers

1. $\frac{\pi_i}{\pi_i + \pi_j}$

Round 5: Axioms - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$
2. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 5: Axioms - Answers

1. $\frac{\pi_i}{\pi_i+\pi_j}$
2. $\frac{\pi_i}{\pi_i+\pi_j}$
3. $\frac{\pi_i}{\pi_i+\pi_j}$

Round 5: Axioms - Answers Q1

Let $p_{ij}/p_{ji} = \exp(\tau(\theta_i, \theta_j))$, where θ_i can be thought of as a parameter summarising the strength of i, then

$$\tau(\theta_i, \theta_j) + \tau(\theta_j, \theta_k) = \tau(\theta_i, \theta_k).$$

Then setting $\theta_j = \theta_i$ it may be noted that $\tau(\theta_i, \theta_i) = 0$ for all i. By setting $\theta_k = \theta_i$ it may be noted that τ is an antisymmetric function. And by differentiating with respect to θ_i it may be noted that the partial derivative of $\tau(\theta_i, \theta_j)$ with respect to θ_i is independent of θ_j, so that $\tau(\theta_i, \theta_j)$ is a function of θ_i alone and θ_j alone, and since it is antisymmetric it must be of the form $t(\theta_i) - t(\theta_j)$. $t(\theta_i)$ may be taken as an increasing continuous function of θ_i, and $\lambda_i = t(\theta_i)$ can be used as a parameter for the strength of i also, so that we have

$$\frac{p_{ij}}{p_{ji}} = \exp(\lambda_i - \lambda_j) \quad \text{for all } i, j,$$

giving

$$p_{ij} = \frac{1}{1 + e^{-(\lambda_i - \lambda_j)}} = \frac{\pi_i}{\pi_i + \pi_j},$$

Round 5: Axioms - Answers Q2

Consider the competition matrix $C = [c_{ij}]$ with c_{kl}, c_{lm}, c_{mk} non-zero, for the triplet (k, l, m). Now consider an alternative C' with $c'_{kl} = c_{kl} - 1$,$c'_{lm} = c_{lm} - 1$, $c'_{mk} = c_{mk} - 1$ and $c'_{lk} = c_{lk} + 1$,$c'_{ml} = c_{ml} + 1$, $c'_{km} = c_{km} + 1$, and all else the same. Then the score vector is identical and so if score is a sufficient statistic then the log-likelihoods must be identical. And so

$$\log \frac{p_{kl}}{p_{lk}} + \log \frac{p_{lm}}{p_{ml}} + \log \frac{p_{mk}}{p_{km}} = 0,$$

giving the Bradley-Terry model as we saw in Question 1.

Round 5: Axioms - Answers Q3

The assumption states that $P_T(S)P_S(i)$ is not dependent on S, but if that is the case then we may take $S = T$ and note that this is equal to $P_T(i)$.
A complete system satisfies the Choice Axiom if and only if there exist a set of numbers $\pi_1, \pi_2, \dots \pi_n$ for every i and $S \subseteq T$ such that

$$p_S(i) = \frac{\pi_i}{\sum_{k \in S} \pi_k} \quad .$$

In order to see this, let

$$\pi_i = \kappa p_T(i), \quad \kappa > 0,$$

then

$$p_S(i) = \frac{p_T(i)}{\sum_{k \in S} p_T(k)} = \frac{\kappa p_T(i)}{\sum_{k \in S} \kappa p_T(k)} = \frac{\pi_i}{\sum_{k \in S} \pi_k}.$$

This is unique up to a multiplicative constant since suppose there is another π_i' satisfying this condition, then

$$\pi_i = \kappa p_T(i) = \frac{\kappa \pi_i'}{\sum_{k \in T} \pi_k'},$$

and setting $\kappa' = \kappa / \sum_{k \in T} \pi_k'$ then $\pi = \kappa' \pi_i'$.
By taking S to be the two member set $\{i, j\}$ we get the Bradley-Terry model.

Round 5: Axioms - Bonus question

Q: Kolmogorov's criterion and the product rule are alternative framings for which of the criteria we saw in the first round of questions - transitivity, score as a sufficient statistic, or the choice axiom?

Round 5: Axioms - Bonus answer

A: Transitivity

$$\frac{p_{ij}}{p_{ji}} \times \frac{p_{jk}}{p_{kj}} = \frac{p_{ik}}{p_{ki}} \text{ for all triplets } (i, j, k),$$

if and only if

$$p_{ij} p_{jk} p_{ki} = p_{ik} p_{kj} p_{ji} \text{ for all triplets } (i, j, k),$$

Round 5: Axioms - Bonus question

Q: Kolmogorov's criterion and the product rule are alternative framings for which of the criteria we saw in the first round of questions - transitivity, score as a sufficient statistic, or the choice axiom?

Final Round: Quickfire