ASSET PRICES, DEBT CONSTRAINTS AND INEFFICIENCY

GAETANO BLOISE AND PIETRO REICHLIN

Current version: March 18, 2009

ABSTRACT. We consider economies with (possibly endogenous) solvency constraints under uncertainty over an infinite horizon. Constrained inefficiency corresponds to a feasible redistribution yielding a welfare improvement beginning from every contingency reached by the economy. A sort of Cass Criterion (Cass [13]) characterizes constrained inefficiency. This criterion involves only observable prices and requires low interest rates in the long period, exactly as it happens for canonical inefficiency in economies of overlapping generations. In addition, when quantitative limits to private liabilities arise from participation constraints (depending on the value of debt repudiation for individuals), the existence of a feasible welfare improvement, subject to participation, coincides with the introduced notion of constrained inefficiency.

KEYWORDS. Private debt; solvency constraints; default; Cass Criterion; asset prices; incomplete markets; constrained inefficiency; transversality condition.

JEL CLASSIFICATION NUMBERS. D50, D52, D61, E44, G13.

CONTENTS

1. INTRODUCTION

Models with debt constraints have been used to explain the time series of output, asset prices and interest rates (Scheinkman and Weiss [33]), to understand and quantify the size of precautionary savings (Aiyagari [2]), to derive the optimal quantity of money (Bewley [9]), or public debt (Woodford [34]), and to prove the existence of asset bubbles (Scheinkman and Weiss [33], Kocherlakota [24], Santos and Woodford [32]). More recently, there has been a great deal of research on the endogenous determination of debt constraints, assuming limited enforcement and incentive constraints (among others, Kehoe and Levine [21], Kocherlakota [23] and Alvarez and Jermann [5]). These studies have reconsidered and quantified the same issues addressed in previous models with exogenous debt limits, as well as other issues, such as limited risk sharing (Kocherlakota [23], Krueger and Perri [25]), macroeconomic implications for asset pricing (Alvarez and Jerman [6]) and debt sustainability (Eaton and Gersovitz [18], Bulow and Rogoff [12], Hellwig and Lorenzoni [20]).

Debt limits prevent the economy from attaining a social optimum because individuals are unable to exploit all trading opportunities and disparities in subjective evaluations of risk persist at equilibrium. The interesting issue is whether competitive equilibria are constrained optimal, that is, whether there may be benefits from redistributions under the condition that debt constraints cannot be removed. This notion of optimality is particularly relevant when debt limits are endogenous, since, most likely, policy intervention fails in sidestepping the incentive constraints from which debt limits arise. The purpose of this work is to verify whether, at a competitive equilibrium, the mere observation of prices completely reveals constrained inefficiency.

We here consider an economy under uncertainty with sequentially complete asset markets and arbitrarily specified debt constraints, that is, quantitative limits to private liabilities. This formulation encompasses not-too-tight debt constraints of Alvarez and Jermann [5] and self-enforcing private debt of Hellwig and Lorenzoni [20], as well as the extreme cases of natural debt limits (Levine and Zame [27], Santos and Woodford [32]) and of no private liabilities or pure borrowing constraints (Bewley [9], Aiyagari [2]). In general, the more severe are debt constraints, the more severe is incomplete insurance, or market incompleteness. Thus, increasing severity of limits to private liabilities amplifies welfare losses. This notwithstanding, the understanding of constrained inefficiency admits a unified treatment, independently of the specific nature of debt constraints.

In a recent literature, endogenous solvency constraints serve to sustain debt obligations under limited commitment. Thus, they are generated by reservation utilities (the value of debt repudiation for individuals) and allow for natural notions of constrained inefficiency, along the lines of Kehoe and Levine [21]. Alvarez and Jermann [5] postulate that debt repudiation induces a permanent exclusion from financial markets, though individuals maintain labor incomes; thus, their notion

of not-too-tight debt constraints corresponds to participation constraints at reservation utilities ensuring that individuals would not benefit, at every contingency, from reverting permanently to autarchy (Kehoe and Levine [21] and Kocherlakota [23]). Hellwig and Lorenzoni [20], instead, assume that unhonored debt deprives from issuing further debt obligations, though individuals might still participate into financial markets for lending (Bulow and Rogoff [12]); their notion of self-enforcing debt coincides with participation constraints at reservation utilities guaranteeing that individuals would not profit from repudiating their debt and participating into financial markets subject to no borrowing constraints. Furthermore, participation constraints naturally emerge under a variety of assumptions, such as that bankruptcy induces an exclusion from asset markets for a limited number of periods only, or the imposition of collateral requirements (see Phelan [31], Kiyotaki and Moore [22], Lustig [28, 29], Krueger and Uhlig [26] for different formulations of outside values for borrowers). In all these instances, a natural notion of *constrained inefficiency* is given by a feasible welfare improvement subject to participation constraints at reservation utilities sustaining endogenous debt limits. An hypothetical planner, thus, is restricted by constraints analogous to those inducing market incompleteness. It is worth remarking that this notion of constrained optimum is *conditional* on reservation utilities varying from autarchy (the most severe punishment) to welfare evaluated at planned consumptions (the most lenient punishment).

Along analogous lines, we introduce the concept of *unconditional constrained inefficiency*. This corresponds to the occurrence of a feasible welfare improvement beginning from any contingency along the infinite horizon of the economy. More precisely, given consumption plans of individuals, a feasible reallocation guarantees (weakly) higher utilities to all individuals conditional on the realization of uncertainty at every period of trade. Thus, unconditional constrained inefficiency admits sequential benefits from a feasible redistribution with respect to planned consumptions.

The interest for unconditional constrained inefficiency is motivated by three observations. First, even though debt constraints arise because of limited commitment, it is realistic, or prudential, to assume that the planner might not know the exact specification of reservation utilities across individuals. These might not be completely or unambiguously identified by legislation or by some public signals, as individuals might have different opportunities of renegotiating debt contracts or obtaining loans from other financial intermediaries after default. Unconditional constrained inefficiency does not rely on a precise knowledge of reservation utilities of individuals: a welfare improvement occurs under the most lenient punishment for debt repudiation and, hence, under the most severe participation constraints for the planner. Second, as unconditional constrained inefficiency is independent of reservation utilities, it maintains an autonomous role even in economies with *ad hoc* debt constraints, where no natural notion of constrained inefficiency exists. Third, the concept of unconditional constrained inefficiency might interpreted as defining a failure of transversality. In this perspective, we show that constrained inefficiency in economies with endogenous solvency constraints (hence, well-defined reservation utilities) is equivalent to unconditional constrained inefficiency, that is, to the existence of a feasible redistribution yielding a welfare improvement at every contingency across periods of trade. This is basically the only source of inefficiency conditionally on the fact that those market imperfections cannot be removed.

Inspired by Cass [13], we adopt the view that a failure of optimality is to be revealed by observable economic variables alone, not relying on any direct knowledge of preferences or of subjective evaluations of risk. We show that (conditional as well as unconditional) constrained inefficiency occurs if and only if *low implied interest rates* prevail at equilibrium, that is, when interest rates (net of growth) are recurrently and sufficiently negative conditionally on some non-negligible event. The intuition is straightforward at steady states: in every period of trade, a planner might reduce current consumption of an unconstrained individual for an equal compensation in the following period of trade; when the rate of interest is strictly negative, as the marginal rate of substitution coincides with gross interest rate, this reallocation yields an increase in the welfare of this individual; and, as this balanced redistribution can be continued indefinitely over the infinite horizon of the economy, it proves a failure of constrained optimality. Under uncertainty and when interest rate fluctuates over time, a (Modified) Cass Criterion serves to precisely identify the domain of low implied interest rates. In its more transparent formulation, the (Modified) Cass Criterion requires the existence of a sequence of bounded positive transfers of commodities, $\{v_t\}$, satisfying, for some ρ in $(0, 1)$,

$$\rho \mathbb{E}_t m_{t,t+1} v_{t+1} \geq v_t,$$

where $\{m_{t,t+1}\}$ is the sequence of stochastic discount factors commonly used for asset pricing in macroeconomics. The value of the parameter ρ might be interpreted as an upper bound on the average safe (gross) interest rate prevailing in the long period conditionally on some non-negligible event, whereas the identified transfers yield a sequential welfare improvement when redistributed across unconstrained individuals. Thus, when constrained efficiency fails, prices retain all the information about relevant welfare improving feasible redistributions. Instead, in order to produce a welfare improvement at a constrained optimum, the reallocation of risk need necessarily depend on unobservable marginal evaluations of individuals.

Several other studies have used the Cass Criterion for identifying a failure of optimality in the allocation of resources. The original criterion was introduced by Cass [13] for the overaccumulation of capital and it was exploited by Balasko and Shell [8] and by Okuno and Zilcha [30] for a complete characterization of inefficiency in overlapping generations economies. Its domain was extended to uncertainty by Chattopadhyay and Gottardi [14], whereas its modification was presented by Demange and Laroque [17] and by Bloise and Calciano [11]. Being grounded only on observable economic magnitudes, alternative formulations of the Cass Criterion were used for empirical studies (among others, Abel, Mankiw, Summers and Zeckhauser [1] and Barbie, Hagedorn and Kaul [7]).

It is worth remarking that, as a matter of fact, the application of the Cass Criterion to constrained inefficiency is absent in the established literature. To the best of our knowledge, only Alvarez and Jermann [5] present an analysis with similar motivations and analogous purposes. They provide a characterization of constrained optimality when default produces permanent exclusion from financial markets. They claim that *high implied interest rates* (that is, a finite present value of intertemporal aggregate endowment) are sufficient and, to some extent, necessary for constrained optimality. Clearly, the (Modified) Cass Criterion cannot be satisfied at high implied interest rates. However, our findings show that high implied interest rates are not necessary for constrained optimality. In fact, under non-stationary endowments, we provide an example of a non-autarchic constrained

efficient allocation, according to the notion adopted by these authors, violating high implied interest rates.

The proposed characterization of constrained inefficiency further clarifies the analogy between economies with debt constraints and economies of overlapping generations. Economies of overlapping generations might exhibit locally indeterminate competitive equilibria and might sustain a positive value of outside money, or speculative bubbles, at equilibrium (*e.g.*, Geanakoplos and Polemarchakis [19]). Similar phenomena emerge in economies with debt constraints (*e.g.*, Santos and Woodford [32]). Indeed, debt limits produce a fragmentation of the intertemporal budget constraint, so that impatient individuals do in fact act over a sequence of limited horizons (Bewley [10]), as in overlapping generations economies. The analogy also extends to welfare properties of equilibria. Inefficiency (or, possibly, constrained inefficiency) of competitive equilibria can be understood, in both cases, as a failure of the transversality condition and is characterized by low implied interest rates. Whereas in overlapping generations economies transversality fails because no individual holds a positive fraction of the aggregate endowment, in economies with borrowing constraints transversality is violated because, at equilibrium, no individual is never credit constrained at all date events. We clarify that dynamic efficiency is not restricted to the case of high implied interest rates (*i.e.*, a finitely valued aggregate endowment), but it is also verified when implied interest rates are neither high nor low (a sort of *golden rule*, in the terminology borne out by the literature on overlapping generations economies).

The essay is organized as follows. In section 2, we present the basic assumptions on fundamentals. In sections 3 and 4, we discuss the notions of conditional and unconditional constrained inefficiency. In section 5, we introduce the notions of equilibrium and of price support. In section 6, we provide the characterization of unconditional constrained inefficiency in terms of equilibrium prices. In section 7, we extend the characterization to constrained inefficiency, conditionally on reservation utilities consistent with debt constraints, and discuss the relation between our analysis and the analysis of Alvarez and Jermann [5]. Finally, in order to simplify the presentation throughout the body of the manuscript, we provide some appendixes on lateral issues. In particular, we compare the Modified Cass Criterion with its canonical formulation in appendix A; we provide an example in appendix B; we present a complement to Kehoe and Levine's [21] Second Welfare Theorem in appendix C. All proofs are collected in appendix D.

2. Fundamentals

2.1. **Time and uncertainty.** Time and uncertainty are represented by an event-tree $\mathcal{S}$, a countably infinite set, endowed with ordering $\succeq$. For a date-event σ in $\mathcal{S}$, $t(\sigma)$ in $\mathcal{T} = \{0, 1, 2, \ldots, t, \ldots\}$ denotes its date and

$$\sigma_+ = \{\tau \in \mathcal{S}(\sigma) : t(\tau) = t(\sigma) + 1\}$$

is the non-empty finite set of all immediate direct successors, where

$$\mathcal{S}(\sigma) = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}.$$

The initial date-event is ϕ in $\mathcal{S}$, with $t(\phi) = 0$, that is, $\sigma \succeq \phi$ for every σ in $\mathcal{S}$. This construction is canonical (Debreu [16, Chapter 7]).

2.2. **Vector space notation and terminology.** As far as notation and terminology for vector spaces are concerned, we basically follow Aliprantis and Border [4, Chapters 5-8]. Consider the vector space of all real maps on $\mathcal{S}$, $\mathbb{R}^{\mathcal{S}}$, endowed with the canonical (product) ordering. An element v of $\mathbb{R}^{\mathcal{S}}$ is positive (respectively, strictly positive) if $v_\sigma \geq 0$ for every σ in $\mathcal{S}$ (respectively, $v_\sigma > 0$ for every σ in $\mathcal{S}$). For a positive element v of $\mathbb{R}^{\mathcal{S}}$, we simply write $v \geq 0$ and, when v in $\mathbb{R}^{\mathcal{S}}$ is non-null, $v > 0$. For an element v of $\mathbb{R}^{\mathcal{S}}$, v^+ in $\mathbb{R}^{\mathcal{S}}$ and v^- in $\mathbb{R}^{\mathcal{S}}$ are, respectively, its positive part and its negative part, so that $v = v^+ - v^-$ in $\mathbb{R}^{\mathcal{S}}$ and $|v| = v^+ + v^-$ in $\mathbb{R}^{\mathcal{S}}$. Also, for an arbitrary collection $\left\{v^j\right\}_{j \in \mathcal{J}}$ of elements of $\mathbb{R}^{\mathcal{S}}$, its supremum and its infimum in $\mathbb{R}^{\mathcal{S}}$, if they exist, are denoted, respectively, by

$$\bigvee_{j \in \mathcal{J}} v^j \text{ and } \bigwedge_{j \in \mathcal{J}} v^j.$$

Finally, the positive cone of any (Riesz) vector subspace F of $\mathbb{R}^{\mathcal{S}}$ is $\{v \in F : v \geq 0\}$.

2.3. **Commodity space.** There exists a single commodity that is traded and consumed at every date-event. The *(reduced) commodity space* is L, the (Riesz) vector space of all *bounded* real maps on $\mathcal{S}$. The vector space L is endowed with the supremum norm given by

$$\|v\| = \inf\left\{\lambda > 0 : |v| \leq \lambda u\right\},$$

where here u denotes the unit of L. Notice that, as far as the aggregate endowment is bounded, the restriction to a reduced commodity space only serves to simplify our presentation. Furthermore, growth could be straightforwardly encompassed in our analysis by strengthening the hypotheses on preferences.

A linear functional f on L is strictly positive if, for every non-null positive element v of L, $f \cdot v > 0$. It is order-continuous if, for every element v of L,

$$f \cdot v = \sum_{\sigma \in \mathcal{S}} f \cdot v_\sigma,$$

where we use the decomposition $\mathbb{R}^{\mathcal{S}} = \oplus_{\sigma \in \mathcal{S}} \mathbb{R}_\sigma$. Order-continuity expresses the fact that the linear functional admits a sequential representation.

2.4. **Preferences.** There is a finite set $\mathcal{J}$ of individuals. For every individual i in $\mathcal{J}$, the *consumption space* X^i is the positive cone of L. A consumption plan x^i in X^i is interior if there exists $\lambda > 0$ such that $x^i \geq \lambda u$, where here u denotes the unit of L. Though more general preferences can be encompassed in our analysis, it simplifies to assume time additively separable utilities. Preferences of individual i in $\mathcal{J}$ on X^i are represented by

$$U^i\left(x^i\right) = \sum_{\sigma \in \mathcal{S}} \pi^i_\sigma u^i\left(x^i_\sigma\right),$$

where π^i is a strictly positive order-continuous linear functional on L and $u^i : \mathbb{R}_+ \to \mathbb{R}$ is a bounded, smooth, smoothly strictly increasing and smoothly strictly concave per-period utility function. For every date-event σ in $\mathcal{S}$, let

$$U^i_\sigma\left(x^i\right) = \frac{1}{\pi^i_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} \pi^i_\tau u^i\left(x^i_\tau\right),$$

so that

$$U_\sigma^i\left(x^i\right) = u^i\left(x_\sigma^i\right) + \frac{1}{\pi_\sigma^i} \sum_{\tau \in \sigma_+} \pi_\tau^i U_\tau^i\left(x^i\right).$$

Finally, we assume that there exists a sufficiently small $1 > \eta > 0$ satisfying, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

(UI) $$\pi_\sigma^i \geq \eta \sum_{\sigma \in \mathcal{S}(\sigma)} \pi_\tau^i.$$

This hypothesis imposes *uniform impatience* across individuals at interior consumption plans (see, for instance, Levine and Zame [27, Assumption 5] or Santos and Woodford [32, Assumption 2]).

2.5. Allocation. An *allocation* x is an element of $X = \prod_{i \in \mathcal{J}} X^i$. An allocation x in X is interior if, for every individual i in $\mathcal{J}$, the consumption plan x^i in X^i is interior. The hypothesis of interiority is stronger than necessary and is maintained only to simplify presentation.

2.6. Subjective prices. For an individual i in $\mathcal{J}$, at an interior consumption plan x^i in X^i, the *subjective price* p^i is an element of P^i, the set of all strictly positive order-continuous linear functionals on L, satisfying, at every consumption plan z^i in X^i,

(SP) $$U^i\left(z^i\right) - U^i\left(x^i\right) \leq \sum_{\sigma \in \mathcal{S}} p_\sigma^i\left(z_\sigma^i - x_\sigma^i\right).$$

Subjective prices exist under the maintained hypotheses on preferences at interior consumption plans. Indeed, for every individual i in $\mathcal{J}$,

$$\left(p_\sigma^i\right)_{\sigma \in \mathcal{S}} = \left(\pi_\sigma \partial u^i\left(x_\sigma^i\right)\right)_{\sigma \in \mathcal{S}}.$$

2.7. Stationarity. In part of the analysis, we limit attention to stationarity, rendering this restriction explicit whenever it occurs. An economy is stationary if uncertainty can be represented as a Markov process over a finite state space and preferences are measurable with respect to this state space. Formally, for some *finite* state space, S,

$$\mathcal{S} = \bigcup_{t \in \mathcal{T}} S^t,$$

where S^t denotes the set of histories of length t in $\mathcal{T}$. (The initial history ϕ in $\mathcal{S}$ is given by some state s_0 in S, that is, by convention, $S^0 = \{s_0\}$.) This induces an obvious finite partition $(\mathcal{S}^s)_{s \in S}$ of $\mathcal{S}$, given by the identification of every $\sigma = (s_0, s_1, \ldots, s_t)$ in $\mathcal{S}$ with the last state s_t in S appearing in the given history. Stationarity of the economy requires that, for every individual i in $\mathcal{J}$, the map

$$\sigma \mapsto \left(\frac{\pi_\tau^i}{\pi_\sigma^i}\right)_{\tau \in \sigma_+}$$

be measurable with respect to the finite partition $(\mathcal{S}^s)_{s \in S}$ of $\mathcal{S}$. Finally, in a stationary economy, an allocation x in X is stationary if it is measurable with respect to the finite partition $(\mathcal{S}^s)_{s \in S}$ of $\mathcal{S}$.

3. Inefficiency

An allocation x in X is *Pareto dominated* by an alternative allocation z in X if, for every individual i in $\mathcal{J}$,

$$U^i\left(z^i\right) - U^i\left(x^i\right) \geq 0$$

and, for some individual i in $\mathcal{J}$,

$$U^i\left(z^i\right) - U^i\left(x^i\right) > 0.$$

To introduce a general notion of constrained inefficiency, we allow for participation constraints at arbitrarily given reservation utilities. By varying reservation utilities, this serves to capture different hypotheses on sustainable reallocations.

Given reservation utilities ν in V, the vector space $\mathbb{R}^{\mathcal{S} \times \mathcal{J}}$, we define the set $X^{\mathrm{PC}}\left(\nu\right)$ of all allocations z in X satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

(PC) $$U^i_\sigma\left(z^i\right) - \nu^i_\sigma \geq 0.$$

This is the set of allocations z in X fulfilling a sort of participation constraint when reservation utilities are given at values ν in V. By progressive specification, given an allocation e in X, we define the set $X^{\mathrm{PC}}\left(e\right)$ of all allocations z in X satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U^i_\sigma\left(z^i\right) - U^i_\sigma\left(e^i\right) \geq 0.$$

This is the set of allocations z in X fulfilling a sort of participation constraint when reservation utilities are induced by allocation e in X, that is, allocations producing (weakly) higher utility for all individuals beginning from any date-event with respect to the reference allocation.

An allocation x in X is *constrained inefficient conditionally on reservation utilities ν in V* if it is Pareto dominated by an allocation z in $X^{\mathrm{PC}}\left(\nu\right)$ satisfying

(CF) $$\sum_{i \in \mathcal{J}} z^i \leq \sum_{i \in \mathcal{J}} x^i.$$

By progressive specification, an allocation x in X is *constrained inefficient conditionally on allocation e in X* if it is constrained inefficient conditionally on reservation utilities induced by allocation e in X. Finally, an allocation x is X is simply *(unconditionally) constrained inefficient* if it is constrained inefficient conditionally on allocation x in X.

The introduced notion of constrained inefficiency is strengthened in part of the analysis. Strong inefficiency occurs when, along some subtree of the economy, a welfare improving redistribution, satisfying participation constraints, is feasible even though a constant (however small) share of the aggregate endowment is destroyed. This redistribution, in addition, leaves consumptions unaltered in the remaining part of the economy. Formally, an allocation x in X is *strongly constrained inefficient conditionally on reservation utilities ν in V* if, for some non-empty subset $\mathcal{F}$ of $\mathcal{S}$ such that

(T) $$\sigma \notin \mathcal{F} \text{ if and only if } \mathcal{F} \cap \mathcal{S}\left(\sigma\right) = \emptyset,$$

it is Pareto dominated by an allocation z in $X^{\mathrm{PC}}\left(\nu\right)$ satisfying, for some $\epsilon > 0$,

(SF-1) $$\epsilon u_{\mathcal{F}} + \sum_{i \in \mathcal{J}} z^i \leq \sum_{i \in \mathcal{J}} x^i$$

and

(SF-2)
$$\epsilon \sum_{i \in \mathcal{J}} \left| z^i - x^i \right| \leq u_{\mathcal{F}},$$

where $u_{\mathcal{F}}$ denotes the unit of the vector space

$$L_{\mathcal{F}} = \{ v \in L : v_{\sigma} = 0 \text{ for every } \sigma \in (\mathcal{S}/\mathcal{F}) \}.$$

Again, by progressive specification, an allocation x in X is *strongly constrained inefficient conditionally on allocation e in X* if it is constrained inefficient conditionally on reservation utilities induced by allocation e in X. Finally, an allocation x is X is simply *strongly (unconditionally) constrained inefficient* if it is strongly constrained inefficient conditionally on allocation x in X.

Strong inefficiency is meant to capture robust welfare losses occurring at equilibrium. In a tradition of general equilibrium (Debreu [15]), a measure of inefficiency is given by the *coefficient of resource utilization*, that is, by the largest share of the aggregate endowment whose destruction is consistent with a feasible welfare improving redistribution. Strong inefficiency occurs when this measure of inefficiency is positive. Over an infinite horizon, however, inefficiency might persist even though the mentioned measure of inefficiency vanishes. Though we do not explore this matter in depth, a non-strong inefficiency corresponds to the circumstance of benefits from the redistribution vanishing over time and, typically, of allocation approaching a constrained optimum at infinity. Finally, it is worth remarking that strong and canonical constrained inefficiency coincide if attention is limited to stationary allocations (§4).

Our main interest is for (unconditional) constrained inefficiency, or constrained inefficiency conditional on reservation utilities evaluated at planned consumptions. Constrained inefficiency conditional on initial endowments is introduced for a comparison with Kehoe and Levine [21] and Alvarez and Jermann [5]. Constrained inefficiency conditional on particular reservation utilities serves to encompass the formulation of Hellwig and Lorenzoni [20]. The first is suitable for a general characterization at any specification of debt constraints. The other two require a consistent specification of debt constraints. As far as our characterization is concerned, we clarify the exact differences and analogies in depth later on (§7).

4. STATIONARITY

We here show that, under the hypothesis of stationarity, every constrained inefficient allocation, among stationary allocations, is also strongly constrained inefficient. This might be regarded as a digression to ascertain the loss in generality induced by the strong form rather than the canonical form of inefficiency. Stationarity, indeed, ensures the existence of uniformly positive benefits from the welfare improving redistribution at constrained inefficient allocations.

Proposition 1 (Strongly *versus* canonical constrained inefficiency)**.** *In a stationary economy, a stationary interior allocation x in X is Pareto dominated by a stationary allocation z in $X^{\mathrm{PC}}(x)$, satisfying $\sum_{i \in \mathcal{J}} z^i \leq \sum_{i \in \mathcal{J}} x^i$, only if it is strongly constrained inefficient.*

The underlying logic can be illustrated as follows. At stationary allocations, inefficiency entails the comparison of *finitely many variations* in utility across date-events. In addition, by strict convexity of preferences, at no loss of generality, if

the redistribution leaves utility unaltered at some date-event, then it also leaves consumptions unaltered at all succeeding date-events. Hence, a slight contraction of consumptions, at all date-events at which the redistribution occurs, preserves the strict increase in utility. This leaves uniformly positive quantities of resources undistributed at all date-events along some subtree of the economy.

5. Equilibrium

Individuals participate into financial markets, represented as a complete spectrum of elementary Arrow securities. However, their holdings of securities are restricted by quantitative limits. The nature of such debt, or solvency, constraints is irrelevant for the purpose of our analysis, insofar as consumption and financial plans of individuals do not bear any direct effect on debt constraints. In particular, the construction is consistent with that of Alvarez and Jermann [5] and of Hellwig and Lorenzoni [20], as well as with those of Aiyagari [2] and Bewley [9] (except for the fact that those formulations assume a single risk-less security, instead of a full set of elementary Arrow securities, available at every date-event).

At every date-event, simple Arrow securities are traded subject to debt constraints. A price p is an element of

$$P = \left\{ p \in \mathbb{R}^{\mathcal{S}} : p_\sigma > 0 \text{ for every } \sigma \in \mathcal{S} \right\}.$$

Prices are expressed in present values and are comparable with Arrow-Debreu prices, or contingent prices, or state prices. Relevantly, prices need not assign finite values to (bounded) consumption plans over the entire infinite horizon. Thus, the duality between price and commodity spaces might fail, as in economies of overlapping generations.

Debt constraints are quantitative limits to liabilities held by individuals at non-initial date-events. For an individual i in $\mathcal{J}$, debt constraints f^i are an element of

$$F^i = \left\{ f^i \in \mathbb{R}^{\mathcal{S}} : f^i_\sigma \geq 0 \text{ for every non-initial } \sigma \in \mathcal{S} \right\}.$$

Across individuals, debt constraints f are elements of F. Notice that, as debt constraints are positive at non-initial date-events, saving is unrestricted, though borrowing might be inhibited by debt limits. In addition, to the only purpose of simplifying notation, the initial value of debt constraints serves to represent initial claims, or liabilities, held by individuals.

At price p in P, for an initial endowment of commodities e^i in X^i and debt constraints f^i in F^i, the budget set of individual i in $\mathcal{J}$ is given by

$$B^i_p\left(e^i, f^i\right) = \left\{ x^i \in X^i : \sum_{\tau \in \sigma_+} p_\tau v^i_\tau + p_\sigma \left(x^i_\sigma - e^i_\sigma\right) \leq p_\sigma v^i_\sigma \text{ for some } v^i \in V^i\left(f^i\right) \right\},$$

where

$$V^i\left(f^i\right) = \left\{ v^i \in \mathbb{R}^{\mathcal{S}} : v^i + f^i \geq 0, \text{ with } v^i_\phi + f^i_\phi = 0 \right\}.$$

The set $V^i\left(f^i\right)$ represents allowed financial plans. These are restricted by limits to debt and by given initial claims, or liabilities, both captured by f^i in F^i.

Debt constraints reflect solvency requirements. Under perfect financial markets, solvency is guaranteed whenever debt constraints do not exceed the present value of future endowment. However, when debt might not be honored, debt constraints serve to prevent incentives to default. Alvarez and Jermann [5] assume that, when

default occurs, an individual is excluded from financial markets. Hellwig and Lorenzoni [20], instead, postulate that individuals are prohibited to borrow, though they might participate into financial markets for lending. Bewley [9] simply excludes borrowing and introduces positive outside money. Though the specific nature of debt constraints varies across all such instances, solvency requirements are specified as quantitative limits to the amount of liabilities held by individuals, so that they are all consistent with our representation of budget sets.

We are only concerned with prices that can be observed at equilibrium for *some* debt constraints and *some* initial endowments of commodities. Thus, the only relevant feature of equilibrium is optimality of consumption plans for individuals (that is, a sort of price support). A preliminary observation shows that it suffices to restrict attention to consumption plans that are optimal, for some debt constraints, when they are distributed to individuals as initial endowments. The logic is straightforward. If a consumption plan is optimal, it is sustained by some financial plan that satisfies some debt constraints. Thus, any *net* variation of this financial plan, consistent with given debt constraints, cannot yield higher utility. It follows that the consumption plan remains optimal when it corresponds to the initial endowment and debt constraints are given as the sum of initial debt constraints and the optimal financial plan. Clearly, in this transformation, saving and lending are to be interpreted as net positions, corresponding to variations with respect to the initial financial plan. For instance, if initial debt constraints prohibit borrowing (as in Bewley [9]), a negative net position, when the consumption plan is given as initial endowment, corresponds to a reduction of savings.

Proposition 2 (Price support). *Given a price p in P, for every $\left(e^i, g^i\right)$ in $X^i \times F^i$, a consumption plan x^i in X^i is U^i-optimal in the budget set $B_p^i\left(e^i, g^i\right)$ only if, for some debt constraints f^i in F^i, it is U^i-optimal in the budget set $B_p^i\left(x^i, f^i\right)$.*

An allocation x in X is supported by price p in P at debt-constraints f in F if, for every individual i in $\mathcal{J}$, the consumption plan x^i in X^i is U^i-optimal in the budget constrain $B_p\left(x^i, f^i\right)$. An allocation x in X is *supported* (respectively, *non-trivially supported*) by price p in P if it is supported by price p in P at some debt-constraints f in F (respectively, at some debt constraints f in F satisfying, at every non-initial date-event σ in $\mathcal{S}$, $\sum_{i \in \mathcal{J}} f_\sigma^i > 0$). Non-trivial support requires that, at every date-event, some individual is allowed to borrow (*i.e.*, to reduce savings), so ruling out fundamentally autarchic equilibria.

Price support admits a first-order characterization based on elementary arbitrage arguments, as in Alvarez and Jermann [5]. First, for every individual, the subjective evaluation of transfers at succeeding date-events cannot exceed their market evaluation (FOC-1). Second, whenever an individual is allowed to borrow against income at some succeeding date-event, subjective and market evaluations need coincide (FOC-2). These necessary conditions are also sufficient for optimality, provided that boundedness of debt constraints ensures transversality.

Proposition 3 (First-order characterization). *An interior allocation x in X is supported by price p in P at debt-constraints f in F only if, at every date-event σ in $\mathcal{S}$,*

(FOC-1)
$$\bigvee_{i \in \mathcal{J}} \left(\frac{p_\tau^i}{p_\sigma^i}\right)_{\tau \in \sigma_+} \leq \left(\frac{p_\tau}{p_\sigma}\right)_{\tau \in \sigma_+}$$

and

$$(\text{FOC-2}) \qquad \sum_{\tau \in \sigma_+} \left(\frac{p_\tau^i}{p_\sigma^i} \right) f_\tau^i = \sum_{\tau \in \sigma_+} \left(\frac{p_\tau}{p_\sigma} \right) f_\tau^i,$$

where, for every individual i in $\mathcal{J}$, p^i in P^i is the subjective price at interior consumption plan x^i in X^i. Furthermore, an interior allocation x in X is supported by price p in P at bounded *debt-constraints f in F if, at the initial date-event ϕ in $\mathcal{S}$, $f_\phi = 0$ and, at every date-event σ in $\mathcal{S}$, conditions* (FOC-1)-(FOC-2) *are satisfied.*

6. CHARACTERIZATION

We here provide an equivalent characterization of (unconditional) constrained inefficiency in terms of supporting prices. In particular, we show that prices reveal this sort of inefficiency independently of the nature of debt constraints. This characterization exploits a Modified Cass Criterion, exactly as in economies of overlapping generations. The Modified Cass Criterion is a variation of the original criterion proposed by Cass [13] for capital theory and was initially introduced by Demange and Laroque [17] and by Bloise and Calciano [11] for stochastic overlapping generations economies.

Modified Cass Criterion. *A price p in P fulfills the* Modified Cass Criterion *if there exists a non-null positive element v of L satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,*

$$\rho \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

Similarly, it fulfills the Weak Modified Cass Criterion *if there exists a non-null positive element v of L satisfying, at every σ in $\mathcal{S}$,*

$$\sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

Finally, it fulfills the Strong Modified Cass Criterion *if there exists a non-null positive element v of L satisfying, for some $1 > \rho > 0$ and for some $\epsilon > 0$, at every σ in $\mathcal{F}$,*

$$\rho \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma + \epsilon p_\sigma,$$

where the non-empty subset $\mathcal{F}$ of $\mathcal{S}$ is such that $\{\sigma \in \mathcal{S} : v_\sigma > 0\} \subset \mathcal{F}$ and

$$\sigma \notin \mathcal{F} \text{ if and only if } \mathcal{F} \cap \mathcal{S}(\sigma) = \emptyset.$$

The Modified Cass Criterion admits equivalent specifications in terms of weighted infinite sum of the reciprocals of prices and of dominant root (and, to some extent, spectral radius) of a suitably defined positive linear operator (see Proposition 1 and Remarks 1-2 in Bloise and Calciano [11] and appendix A), the latter being suitable for direct applications in empirical studies. The parameter $(\rho - 1)$ represents an appropriate estimation of (an upper bound on) the implicit average real rate of interest in the long period conditionally on some non-negligible event.

Prices fulfilling the Modified Cass Criterion might be regarded as involving *low interest rates*. Prices exhibit *high interest rates*, according to the terminology borne out by Alvarez and Jermann [5], when they are summable, that is,

$$\sum_{\sigma \in \mathcal{S}} p_\sigma \text{ is finite.}$$

Clearly, high interest rates are inconsistent with the Modified Cass Criterion. Finally, prices involve *neither high nor low interest rates* when they neither satisfy the Modified Cass Criterion (though they do satisfy the Weak Modified Cass Criterion) nor are summable. The latter circumstance reveals a null interest rate in the long period and corresponds to a *golden rule* in the terminology for overlapping generations economies. High interest rates, in turn, guarantee a finite pricing of all intertemporal consumption profiles, so preserving the duality between commodity and price spaces. As our characterization of inefficiency exploits low interest rates, prices are consistent with an efficient allocation of resources even when not involving high interest rates and, hence, an infinite value of the aggregate endowment.

In the formulation of Hellwig and Lorenzoni [20], when repudiating their debt, individuals are not excluded by participation into financial markets, though they are not allowed to hold liabilities anymore. Debt constraints are determined so as to prevent individuals from default and to sustain self-enforcing private debt. The characterization of Hellwig and Lorenzoni [20] shows that debt is self-enforcing if and only if debt constraints allow for exact roll-over, that is, in our notation, at every non-initial date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau f_\tau = p_\sigma f_\sigma.$$

When debt constraints are bounded, exact roll-over implies that prices never involve high interest rates. Our Modified Cass Criterion is of particular relevance in this situation.

We begin with proving sufficiency of the Modified Cass Criterion.

Proposition 4 (Sufficiency). *An interior allocation x in X, with non-trivially supporting price p in P, is constrained inefficient if price p in P satisfies the Modified Cass Criterion. Furthermore, it is strongly constrained inefficient if price p in P satisfies the Strong Modified Cass Criterion.*

The logic underlying welfare improvement is extremely simple. For an elementary illustration, suppose that there is no uncertainty (that is, $\mathcal{S}$ can be identified with $\mathcal{T}$). By hypothesis, all consumption plans are interior and, at every date-event, at least one individual is unconstrained (that is, has a subjective evaluation of transfers to the following period coinciding with the market evaluation). Thus, at every period t in $\mathcal{T}$, for some unconstrained individual i in $\mathcal{J}$, the modification of consumptions, described by

$$\left(x_t^i, x_{t+1}^i\right) \mapsto \left(x_t^i - v_t, x_{t+1}^i + v_{t+1}\right),$$

induces a first-order effect on welfare that can be estimated as

$$-p_t^i v_t + p_{t+1}^i v_{t+1} = \left(\frac{p_t^i}{p_t}\right)\left(-p_t v_t + p_{t+1} v_{t+1}\right) \geq \left(\frac{1-\rho}{\rho}\right) p_t^i v_t.$$

This estimate exploits the fact that, for an unconstrained individual, subjective and market evaluations coincide. By iterating this sort of transfers across periods of trade, the final redistribution yields a positive first-order effect on utilities beginning from every period. As second-order effects are uniformly bounded, smoothness suffices to prove a welfare improvement beginning from all date-events.

We now prove necessity of the Modified Cass Criterion. This requires a strengthening of inefficiency to capture non-vanishing benefits from the redistribution across periods of trade. As mentioned earlier, this sort of strong inefficiency corresponds

to inefficiency of positive measure according to Debreu's [15] coefficient of resource utilization.

Proposition 5 (Necessity). *An interior allocation x in X, with supporting price p in P, is constrained inefficient only if price p in P satisfies the Weak Modified Cass Criterion. Furthermore, it is strongly constrained inefficient only if price p in P satisfies the Strong Modified Cass Criterion.*

Necessity is also straightforwardly explained. For every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau^i v_\tau^i + p_\sigma^i \left(z_\sigma^i - x_\sigma^i \right) = p_\sigma^i v_\sigma^i.$$

Here, v^i in L represents the subjectively evaluated (first-order) benefit, in terms of current consumption, from the redistribution. This benefit needs be positive at all date-events. Thus, exploiting the fact that subjective evaluation cannot exceed market evaluation (FOC-1)-(FOC-2), at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau^i + \left(z_\sigma^i - x_\sigma^i \right) \geq v_\sigma^i.$$

Only market prices appear in this inequality. Aggregating across individuals,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau \sum_{i \in \mathcal{J}} v_\tau^i + \sum_{i \in \mathcal{J}} \left(z_\sigma^i - x_\sigma^i \right) \geq \sum_{i \in \mathcal{J}} v_\sigma^i.$$

Feasibility proves the claim, as the aggregate subjectively evaluated benefit $\sum_{i \in \mathcal{J}} v^i$ in L satisfies

$$\sum_{\tau \in \sigma_+} p_\tau \sum_{i \in \mathcal{J}} v_\tau^i \geq p_\sigma \sum_{i \in \mathcal{J}} v_\sigma^i.$$

Finally, the strong form of inefficiency allows for a small uniform contraction preserving positive subjectively-evaluated gains from the redistribution.

7. Consistent Debt Constraints

We here verify to which extent our characterization is preserved under the notion of constrained inefficiency conditional on given reservation utilities, rather than on welfare at planned consumptions. This allows for a direct comparison with the characterization provided by Alvarez and Jermann [5]. In addition, it provides insights into constrained inefficiency at equilibrium with self-enforcing debt as in Hellwig and Lorenzoni [20].

Sufficiency is obviously unaltered. If planned consumptions are individually rational at some given reservation utilities, any welfare improving reallocation guaranteeing sequential participation at planned consumptions is *a fortiori* a welfare improving reallocation guaranteeing sequential participation at the given reservation utilities.

Proposition 6 (Sufficiency with consistent debt constraints). *Given reservation utilities ν in V, an interior allocation x in $X^{\mathrm{PC}}(\nu)$, with non-trivially supporting price p in P, is constrained inefficient conditionally on reservation utilities ν in V if price p in P satisfies the Modified Cass Criterion. Furthermore, it is strongly constrained inefficient conditionally on reservation utilities ν in V if price p in P satisfies the Strong Modified Cass Criterion*

14

As far as necessity is concerned, we preliminarily add restrictions on debt constraints consistent with Alvarez and Jermann's [5] and Hellwig and Lorenzoni's [20] formulations. Given reservation utilities ν in V, an allocation x in $X^{\mathrm{PC}}(\nu)$ is supported by price p in P *at debt constraints consistent with reservation utilities ν in V* if it is supported by price p in P at debt constraints f in F satisfying, for every individual i in $\mathcal{J}$, at every non-initial date-event σ in $\mathcal{S}$,

(DC) $$U^i_\sigma \left(x^i \right) - \nu^i_\sigma > 0 \text{ only if } f^i_\sigma > 0.$$

The underlying logic of this notion is that, whenever subjective welfare exceeds reservation utility at some date-event, debt constrains allow for borrowing at that date-event, that is, for (locally) unrestricted participation into financial markets.

Proposition 7 (Necessity with consistent debt constraints). *Given reservation utilities ν in V, an interior allocation x in $X^{\mathrm{PC}}(\nu)$, with supporting price p in P at debt constraints consistent with reservation utilities ν in V, is constrained inefficient conditionally on reservation utilities ν in V only if price p in P satisfies the Weak Cass Criterion. Furthermore, it is strongly constrained inefficient conditionally on reservation utilities ν in V only if price p in P satisfies the Strong Modified Cass Criterion.*

The proof of this claim requires a minor amendment of the previous argument for necessity (proposition 5). For an individual i in $\mathcal{J}$, the subjectively evaluated benefit from the redistribution v^i in L need not be positive at all date-events, though it is positive at the initial date-event. (Indeed, at some non-initial date-event, subjective welfare might fall below utility at planned consumptions.) However, notice that, when an individual is constrained in transferring resources at a date-event, consistent debt constraints ensure that the individual will positively benefit, with respect to planned consumptions, from the redistribution at that date-event. Hence, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v^i_\tau + \left(z^i_\sigma - x^i_\sigma \right) \geq v^i_\sigma.$$

The argument then unfolds as in the proof of proposition 5, using only the non-null positive part of aggregate subjectively-evaluated benefit $\sum_{i \in \mathcal{J}} v^i$ in L.

Loosely interpreted, our complete characterization proves that constrained inefficiency at initial endowments (that is, constrained inefficiency as defined by Kehoe and Levine [21] and Alvarez and Jermann [5]) coincides with *low interest rates*. Alvarez and Jermann [5] show, on the one side, that every equilibrium allocation involving high interest rates is constrained efficient (Corollary 4.7) and, on the other side, that every non-autarchic constrained efficient allocation involves high interest rates (Proposition 4.10). Therefore, according to Alvarez and Jermann [5], *high interest rates* fully characterize non-autarchic constrained efficiency. What about *neither high nor low interest rates*, that is, a null interest rate over the long period?

In appendix B, we provide an example of a non-autarchic equilibrium with not-too-tight debt constraints that is constrained efficient, conditionally on reservation utilities evaluated at initial endowments, and involves a constant null interest rate. This example, though it requires non-stationary initial endowments, shows that a null interest rate over the long period can be sustained at non-autarchic equilibrium with not-too-tight debt constraints. In turn, non-stationary endowments might be

15

of interest for applications to the sustainability of sovereign debt, when some countries face a decline, or a deindustrialization, and some other countries an expansion, or an industrialization. Consistently, our characterization is tight.

In appendix C, we also complement Kehoe and Levine's [21] and Alvarez and Jermann's [5] Second Welfare Theorem in order to prove that, when both consumptions and endowments are stationary, a non-autarchic constrained optimum requires high interest rates. As a conclusion, limiting attention to non-autarchic *stationary* consumptions, constrained efficiency conditional on initial *stationary* endowments is fully characterized by *high interest rates*.

8. Conclusion

Our contribution in this manuscript is twofold. On the one side, we show that a failure of constrained optimality in the allocation of resources is completely revealed by (observable) prices, without requiring any precise knowledge of fundamentals (consumption plans, endowments, preferences). Inefficiency corresponds to low implied interest rates as captured by the (Modified) Cass Criterion. Furthermore, this criterion exploits properties of the stochastic discount factor, commonly used in macroeconomic theory, and is suitable for empirical studies. On the other side, we prove that, when a planner is restricted by participation constraints preventing debt repudiation, constrained inefficiency coincides with a feasible recursive welfare improvement independent of the particular nature of solvency constraints, that is, with a feasible redistribution yielding (weakly) higher utilities at all contingencies along the infinite horizon of the economy. This might be interpreted as a social failure of transversality.

References

[1] A.B. Abel, N.G. Mankiw, L.H. Summers and R.J. Zeckhauser. Assessing dynamic efficiency: Theory and evidence. *Review of Economic Studies*, 56, 120, 1989.

[2] S.R. Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *Quarterly Journal of Economics*, 109, 659-684, 1994.

[3] S.R. Aiyagari and D. Peled. Dominant root characterization of Pareto optimality and the existence of optimal monetary equilibria in stochastic OLG models. *Journal of Economic Theory*, 54, 69-83, 1991.

[4] C.D. Aliprantis and K.C. Border. *Infinite Dymensional Analysis: A Hitchhiker's Guide*. Springer-Verlag, 1999.

[5] F.A. Alvarez and U.J. Jermann. Efficiency, equilibrium, and asset pricing with risk of default. *Econometrica*, 68, 775-797, 2000.

[6] F.A. Alvarez and U.J. Jermann. Quantitative Asset Pricing Implications of Endogenous Solvency Constraints. *Review of Financial Studies*, 14, 1117-1151, 2001.

[7] M. Barbie, M. Hagedorn and A. Kaul. Assessing aggregate tests of efficiency for dynamic economies. *Topics in Macroeconomics*, 4, 2004.

[8] Y. Balasko and K. Shell. The overlapping generations model, I: The case of pure exchange without money. *Journal of Economic Theory*, 23, 281-306, 1980.

[9] T. Bewley. The optimum quantity of money. In J.H. Kareken and N. Wallace (eds.), *Models of Monetary Economies*, Minneapolis: Federal Reserve Bank, 1980.

[10] T. Bewley. Dynamic implications of the form of the budget constraint. In H.F. Sonnenschein (ed.), *Models of Economic Dynamics*, New York: Springer-Verlag, 1986.

[11] G. Bloise and F.L. Calciano. A characterization of inefficiency in stochastic overlapping generations economies. *Journal of Economic Theory*, 143, 442-468, 2008.

[12] J. Bulow and K. Rogoff. Sovereign debt: Is to forgive to forget? *American Economic Review*, 79, 43-50, 1989.

[13] D. Cass. On capital overaccumulation in the aggregative neoclassical model of economic growth: A complete characterization. *Journal of Economic Theory*, 4, 200-223, 1972.

[14] S. Chattopadhyay and P. Gottardi. Stochastic OLG models, market structure, and optimality. *Journal of Economic Theory*, 89, 21-67, 1999.

[15] G. Debreu. The coefficient of resource utilization. *Econometrica*, 19, 273-292, 1951.

[16] G. Debreu. *Theory of Value.* New York: Wiley, 1959.

[17] G. Demange and G. Laroque. Social Security, Optimality, and Equilibria in a Stochastic Overlapping Generations Economy. *Journal of Public Economic Theory*, 2, 1-23, 2000.

[18] J. Eaton and M. Gersovitz. Debt with potential repudiation. *Review of Economic Studies*, 48, 289-309, 1981.

[19] J.D. Geanakoplos and H.M. Polemarchakis. Overlapping generations. In W. Hildenbrand and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, vol. IV, New-York: North-Holland, 1891-1960, 1991.

[20] M. Hellwig and G. Lorenzoni. Bubbles and self-enforcing debt. Mimeograph, October 2006.

[21] T.J. Kehoe and D.K. Levine. Debt-constrained asset markets. *Review of Economic Studies*, 60, 865-888, 1993.

[22] K. Kiyotaki and J. Moore. Credit cycles. *Journal of Political Economy*, 105, 211-248, 1997.

[23] N.R. Kocherlakota. Implications of efficient risk sharing without commitment. *Review of Economic Studies*, 63, 595-609, 1996.

[24] N.R. Kocherlakota. Bubbles and constraints on debt accumulation. *Journal of Economic Theory*, 57, 245-256, 1992.

[25] D. Krueger and F. Perri. Does income inequality lead to consumption inequality? Evidence and theory. *Review of Economic Studies*, 73, 163-193, 2006.

[26] D. Krueger and H. Uhlig. Competitive risk sharing contracts with one-sided commitment. SFB 649 Discussion Paper 2005-003, 2005.

[27] D.K. Levine and W.R. Zame. Debt constraints and equilibrium in infinite horizon economies with incomplete markets. *Journal of Mathematical Economics*, 26, 103-131, 1996.

[28] H. Lustig. Secured lending and asset prices. Mimeograph, Stanford University, 2000.

[29] H. Lustig. The market price of aggregate risk and the wealth distribution. Mimeograph, Stanford University, 2001.

[30] M. Okuno and I. Zilcha. On the efficiency of a competitive equilibrium in infinite horizon monetary economies. *Review of Economic Studies*, 47, 797-807, 1980.

[31] C. Phelan. Repeated moral hazard and one sided commitment. *Journal of Economic Theory*, 66, 488-506, 1995.

[32] M. Santos and M. Woodford. Rational asset pricing bubbles. *Econometrica*, 64, 19-57, 1997.

[33] J. Scheinkman and L. Weiss. Borrowing constraints and aggregate econommic activity. *Econometrica*, 45, 23-45, 1986.

[34] M. Woodford. Public debt as private liquidity. *American Economic Review*, 80, 382-388, 1990.

Appendix A. Cass Criterion

In this appendix, we compare our Modified Cass Criterion with the Canonical Cass Criterion, as established by Chattopadhyay and Gottardi [14], and we explain difficulties in using the latter for a full characterization.

Canonical Cass Criterion (Chattopadyay and Gottardi [14])**.** *A price p in P fulfills the* Canonical Cass Criterion *if there exists a weight function λ in L such that, for every non-initial σ in $\mathcal{S}$,*

$$\sum_{\nu \in \mathcal{P}(\sigma)} \frac{\lambda_\nu}{p_\nu} \leq 1,$$

where $\mathcal{P}(\sigma) = \{\nu \in \mathcal{S} : \sigma \succ \nu\}$ is the set of all date-events ν in $\mathcal{S}$ strictly preceding date-event σ in $\mathcal{S}$ and a weight function is a non-null positive element λ of L such

that, at every σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} \lambda_\tau \geq \lambda_\sigma.$$

For a more transparent comparison between the Canonical and the Modified Cass Criterion, the former might be reformulated as follows.

Claim 1 (Canonical Cass Criterion)**.** *A price p in P fulfills the Canonical Cass Criterion if and only if there exists a non-null positive element v of L satisfying, at every σ in $\mathcal{S}$,*

$$\sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma + p_\sigma v_\sigma^2.$$

Proof of claim 1. Sufficiency directly follows from the proof of Theorem 1 in Chattopadhyay and Gottardi [14]. To verify necessity (see also the proof of Theorem 2 in Chattopadhyay and Gottardi [14]), define $v_\phi = 0$ and, at every non-initial σ in $\mathcal{S}$,

$$v_\sigma = \frac{\lambda_\sigma}{p_\sigma} \sum_{\nu \in \mathcal{P}(\sigma)} \frac{\lambda_\nu}{p_\nu}.$$

Clearly, by the Canonical Cass Criterion, v is a non-null positive element of L. For every non-initial σ in $\mathcal{S}$,

$$
\begin{aligned}
\sum_{\tau \in \sigma_+} p_\tau v_\tau &= \sum_{\tau \in \sigma_+} \lambda_\tau \sum_{\nu \in \mathcal{P}(\tau)} \frac{\lambda_\nu}{p_\nu} \\
&= \sum_{\tau \in \sigma_+} \lambda_\tau \sum_{\nu \in \mathcal{P}(\sigma)} \frac{\lambda_\nu}{p_\nu} + \sum_{\tau \in \sigma_+} \lambda_\tau \left(\frac{\lambda_\sigma}{p_\sigma} \right) \\
&\geq \lambda_\sigma \sum_{\nu \in \mathcal{P}(\sigma)} \frac{\lambda_\nu}{p_\nu} + \lambda_\sigma \left(\frac{\lambda_\sigma}{p_\sigma} \right) \\
&= p_\sigma \left(\frac{\lambda_\sigma}{p_\sigma} \sum_{\nu \in \mathcal{P}(\sigma)} \frac{\lambda_\nu}{p_\nu} \right) + p_\sigma \left(\frac{\lambda_\sigma}{p_\sigma} \right)^2 \\
&\geq p_\sigma \left(\frac{\lambda_\sigma}{p_\sigma} \sum_{\nu \in \mathcal{P}(\sigma)} \frac{\lambda_\nu}{p_\nu} \right) + p_\sigma \left(\frac{\lambda_\sigma}{p_\sigma} \sum_{\nu \in \mathcal{P}(\sigma)} \frac{\lambda_\nu}{p_\nu} \right)^2 \\
&= p_\sigma v_\sigma + p_\sigma v_\sigma^2.
\end{aligned}
$$

The first inequality uses the definition of a weight function; the last inequality exploits the Canonical Cass Criterion. □

Clearly, the Canonical Cass Criterion is implied by the Modified Cass Criterion, but the opposite is not necessarily true. Observe, however, that the Modified Cass Criterion is implied by the Canonical Cass Criterion wherever there exists a sufficiently small $\epsilon > 0$ such that, at every date-event σ in $\mathcal{S}$,

$$v_\sigma^2 \geq \epsilon v_\sigma$$

or, equivalently,

$$v_\sigma > 0 \text{ only if } v_\sigma \geq \epsilon.$$

Thus, a failure of coincidence between Canonical and Modified Cass Criterion only occurs if implicit welfare-improving transfers are not uniformly positive.

It can be showed that the Canonical Cass Criterion is sufficient for constrained inefficiency. In order to establish necessity of the Modified Cass Criterion (and, hence, of the Canonical Cass Criterion), however, we exploit a stronger form of inefficiency. The additional difficulty is caused by the fact that, in general, there is not a uniform upper bound on the length of the horizons over which an individual is not debt constrained. Similar difficulties would emerge in overlapping generations economies with possibly not uniformly bounded finite horizons of generations.

As explained in Bloise and Calciano [11], the Modified Cass Criterion is satisfied if and only if the positive bounded linear operator $T : L \to L$ admits an eigenvalue larger than the unity, where, at every date-event σ in $\mathcal{S}$,

$$T\left(v\right)_\sigma = \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau.$$

This is the analogous of the Dominant Root Characterization of Aiyagari and Peled [3] for stationary equilibria in stochastic overlapping generations economies. In stochastic overlapping generations economies with capital accumulation, a similar characterization is provided by Demange and Laroque [17] in terms of spectral radius, $r\left(T\right)$, of the positive linear operator $T : L \to L$, where

$$r\left(T\right) = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}}..$$

The Modified Cass Criterion is satisfied only if the spectral radius is larger than the unity, $r\left(T\right) > 1$. Exploiting compactness and continuity assumptions that are not necessarily fulfilled in our formulation, Demange and Laroque [17] prove that the Modified Cass Criterion is satisfied if the spectral radius is larger than the unity, $r\left(T\right) > 1$. Their analysis fails in completely solving the borderline case where $r\left(T\right) = 1$. Our analysis establishes that, in this case, the allocation is not *strongly* inefficient.

It is worth noticing that the Modified Cass Criterion is robust to perturbations, a property that might fail for the Canonical Cass Criterion. For heuristic purposes, consider an allocation x in X, with (non-trivially) supporting price p in P, and an alternative allocation $\hat{x}$ in X, with (non-trivially) supporting price $\hat{p}$ in P. Observe that, by first-order conditions for individual optimality, at every σ in $\mathcal{S}$,

$$
\begin{aligned}
\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau &= \sum_{\tau \in \sigma_+} \left(\bigvee_{i \in \mathcal{J}} \frac{p_\tau^i}{p_\sigma^i} \right) v_\tau \\
&= \sum_{\tau \in \sigma_+} \left[\left(\bigvee_{i \in \mathcal{J}} \frac{\hat{p}_\tau^i}{\hat{p}_\sigma^i} \right)^{-1} \left(\bigvee_{i \in \mathcal{J}} \frac{p_\tau^i}{p_\sigma^i} \right) \right] \left(\bigvee_{i \in \mathcal{J}} \frac{\hat{p}_\tau^i}{\hat{p}_\sigma^i} \right) v_\tau \\
&= \sum_{\tau \in \sigma_+} \left[\left(\bigvee_{i \in \mathcal{J}} \frac{\hat{p}_\tau^i}{\hat{p}_\sigma^i} \right)^{-1} \left(\bigvee_{i \in \mathcal{J}} \frac{p_\tau^i}{p_\sigma^i} \right) \right] \left(\frac{\hat{p}_\tau}{\hat{p}_\sigma} \right) v_\tau \\
&\leq \left(\frac{\hat{\rho}}{\rho} \right) \frac{1}{\hat{p}_\sigma} \sum_{\tau \in \sigma_+} \hat{p}_\tau v_\tau,
\end{aligned}
$$

where

$$\hat{\rho} = \rho \bigvee_{\sigma \in \mathcal{S}} \left(\bigvee_{i \in \mathcal{J}} \frac{\hat{p}_\tau^i}{\hat{p}_\sigma^i} \right)^{-1} \left(\bigvee_{i \in \mathcal{J}} \frac{p_\tau^i}{p_\sigma^i} \right).$$

Therefore, at every σ in $\mathcal{S}$,

$$\rho \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma \text{ only if } \hat{\rho} \sum_{\tau \in \sigma_+} \hat{p}_\tau v_\tau \geq \hat{p}_\sigma v_\sigma.$$

And, if interior allocation $\hat{x}$ in X is sufficiently close to interior allocation x in X in the supremum norm, $1 > \hat{\rho} > 0$. Hence, the Modified Cass Criterion is satisfied for any slightly perturbed allocation.

Appendix B. Example

In this appendix, we provide an example of a constrained efficient allocation, according to Alvarez and Jermann [5], violating the hypothesis of high interest rates. In particular, a null interest rate sustains a stationary allocation as non-autarchic equilibrium at not-too-tight debt constraints. Initial endowments are non-stationary and are constructed so as to approach the equilibrium stationary allocation in the long period. Non-stationarity of either endowments or consumptions is necessary for a non-autarchic constrained optimum not to involve high interest rates, as shown in appendix C.

Before presenting the example, we shall produce necessary conditions for constrained inefficiency. To simplify, we shall assume that there is no uncertainty, that is, $\mathcal{S}$ can be identified with $\mathcal{T}$; also, that there is a common discount factor, $1 > \delta > 0$, and that the common per-period utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is smooth on $\mathbb{R}_+$ (that is, to be precise, it can be extended as a twice continuously differentiable function on some open set containing $\mathbb{R}_+$); finally, that $u'(1) < \delta u'(0)$. Recall that, given an initial allocation e in X, an interior allocation x in $X^{\text{PC}}(e)$ is supported by price p in P at not-too-tight debt constraints with respect to initial allocation e in X if it is supported by price p in P such that, for every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

(FOC-1) $$\frac{p_{t+1}}{p_t} \geq \frac{p_{t+1}^i}{p_t^i}$$

and

(FOC-2) $$\frac{p_{t+1}}{p_t} = \frac{p_{t+1}^i}{p_t^i} \text{ if } U_{t+1}^i(x^i) - U_{t+1}^i(e^i) > 0,$$

where p^i in P^i is the subjective price at interior consumption plan x^i in X^i.

Claim 2 (Constrained inefficiency). *Given an initial allocation e in X, an interior allocation x in $X^{\text{PC}}(e)$, with supporting price p in P at not-too-tight debt constraints with respect to initial allocation e in X, is Pareto dominated by an allocation z in $X^{\text{PC}}(e)$, satisfying $\sum_{i \in \mathcal{J}} z^i \leq \sum_{i \in \mathcal{J}} x^i$, only if there exists a strictly positive element v of L satisfying, for some sufficiently small $\epsilon > 0$, at every t in $\mathcal{T}$,*

$$\frac{p_{t+1}}{p_t} v_{t+1} \geq v_t + \epsilon \sum_{i \in \mathcal{J}} \left(z_t^i - x_t^i \right)^2$$

and

$$\sum_{s \in \mathcal{T}} \delta^s \sum_{i \in \mathcal{J}} \left| z_{t+s}^i - x_{t+s}^i \right| \geq \epsilon v_t.$$

Proof of claim 2. Preliminarily observe that, for consumptions varying in a compact interval of $\mathbb{R}_+$, there exists a sufficiently small $\epsilon > 0$ satisfying

$$u(c') - u(c) \le u'(c)(c'-c) - \epsilon u'(c)(c'-c)^2.$$

This shows a sort of quadratic concavity of intertemporal utility.

For every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$, define

$$v_t^i = \frac{1}{p_t^i} \sum_{s \in \mathcal{T}} p_{t+s}^i \left(z_{t+s}^i - x_{t+s}^i\right) - \epsilon \frac{1}{p_t^i} \sum_{s \in \mathcal{T}} p_{t+s}^i \left(z_{t+s}^i - x_{t+s}^i\right)^2.$$

Notice that, for every individual i in $\mathcal{J}$, v^i is an element of L. Indeed, this follows from feasibility and uniform impatience (that is, restriction (UI)). Define $v = \sum_{i \in \mathcal{J}} v^i$, an element itself of L, and observe that, by Pareto dominance and quadratic concavity, $v_0 = \sum_{i \in \mathcal{J}} v_0^i > 0$. In addition, at every t in $\mathcal{T}$,

$$\frac{1}{\epsilon} \sum_{i \in \mathcal{J}} \sum_{s \in \mathcal{T}} \delta^s \left|z_{t+s}^i - x_{t+s}^i\right| \ge \sum_{i \in \mathcal{J}} \frac{1}{p_t^i} \sum_{s \in \mathcal{T}} p_{t+s}^i \left|z_{t+s}^i - x_{t+s}^i\right| \ge v_t,$$

where the first inequality, as $\epsilon > 0$ can be assumed to be arbitrarily small, follows from bounded derivatives of per-period utility $u : \mathbb{R}_+ \to \mathbb{R}$ over a compact interval of $\mathbb{R}_+$.

For every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

$$\frac{p_{t+1}^i}{p_t^i} v_{t+1}^i + \left(z_t^i - x_t^i\right) \ge v_t^i + \epsilon \left(z_t^i - x_t^i\right)^2.$$

As debt constraints are not-too-tight,

$$\frac{p_{t+1}}{p_t} > \frac{p_{t+1}^i}{p_t^i} \text{ only if } U_{t+1}^i\left(x^i\right) - U_{t+1}^i\left(e^i\right) = 0.$$

Hence, as $U_{t+1}^i\left(z^i\right) - U_{t+1}^i\left(x^i\right) \ge 0$, $v_{t+1}^i \ge 0$. We consistently conclude that, for every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

$$\frac{p_{t+1}}{p_t} v_{t+1}^i + \left(z_t^i - x_t^i\right) \ge v_t^i + \epsilon \left(z_t^i - x_t^i\right)^2.$$

Aggregating across individuals, by feasibility, this proves our claim. $\qquad\square$

For the example, it suffices to consider only two individuals, $\mathcal{J} = \{e, o\}$, associated with even, e, and odd, o, periods of trade. Let $x_e > 0$ and $x_o > 0$ satisfy $x_e + x_o = 1$ and

(SS) $$u'(x_e) = \delta u'(x_o).$$

Allocation x in X is given by

$$\begin{aligned} x^e &= (x_e, x_o, x_e, x_o, \ldots), \\ x^o &= (x_o, x_e, x_o, x_e, \ldots). \end{aligned}$$

At allocation x in X, the supporting price p in P is

$$(p_t)_{t \in \mathcal{T}} = (1, 1, 1, \ldots, 1, \ldots),$$

whereas the subjective price p^i in P^i of individual i in $\mathcal{J}$ is given by

$$\left(p_t^i\right)_{t \in \mathcal{T}} = \left(\delta^t u'\left(x_t^i\right)\right)_{t \in \mathcal{T}}.$$

We need to construct initial endowments e in X which are consistent with price support at not-too-tight debt constraints.

Claim 3 (Not-too-tight debt constraints). *There exists an initial allocation e in X, satisfying*

$$\sum_{i \in \mathcal{J}} x^i = \sum_{i \in \mathcal{J}} e^i$$

and

$$\sum_{i \in \mathcal{J}} U_t^i\left(x^i\right) - \sum_{j \in \mathcal{J}} U_t^i\left(e^i\right) > 0 \text{ at every } t \in \mathcal{T},$$

such that allocation x in $X^{\mathrm{PC}}\left(e\right)$ is supported by price p in P at not-too-tight debt constraints with respect to initial allocation e in X.

Proof of claim 3. Consider the (local) difference equation

(*) $\qquad h\left(\xi_t, \xi_{t+1}\right) = u\left(x_e\right) + \delta u\left(x_o\right) - u\left(x_e + \xi_t\right) - \delta u\left(x_o - \xi_{t+1}\right) = 0.$

It is easy to verify that this difference equation admits a strictly positive solution $\left(\xi_t\right)_{t \in \mathcal{T}}$ in L satisfying $\lim_{t \in \mathcal{T}} \xi_t = 0$. (Indeed, observe that $\xi > 0$ implies $h\left(\xi, \xi\right) > 0$ and $h\left(\xi, 0\right) < 0$, so that $h\left(\xi, \xi'\right) = 0$ for some $\xi > \xi' > 0$ by the Intermediate Value Theorem.) Endowments e in X are given by

$$\begin{aligned} e^e &= \left(x_e + \xi_0, x_o - \xi_1, x_e + \xi_2, x_o - \xi_3, \ldots\right), \\ e^o &= \left(x_o - \xi_0, x_e + \xi_1, x_o - \xi_2, x_e + \xi_3, \ldots\right). \end{aligned}$$

In addition, because of restriction (*), at every t in $\{0, 2, 4, \ldots\}$,

$$U_t^e\left(x^e\right) = U_t^e\left(e^e\right)$$

and

$$U_t^o\left(x^o\right) \geq u\left(x_o\right) + \delta U_{t+1}^o\left(x^o\right) > u\left(x_o - \xi_t\right) + \delta U_{t+1}^o\left(e^o\right) \geq U_t^o\left(e^o\right);$$

at every t in $\{1, 3, 5, \ldots\}$,

$$U_t^o\left(x^o\right) = U_t^o\left(e^o\right)$$

and

$$U_t^e\left(x^e\right) \geq u\left(x_o\right) + \delta U_{t+1}^e\left(x^e\right) > u\left(x_o - \xi_t\right) + \delta U_{t+1}^e\left(e^e\right) \geq U_t^e\left(e^e\right).$$

Because of restriction (ss), this suffices to prove the claim. $\qquad \square$

We now conclude that allocation x in X is a constrained optimum at initial allocation e in X.

Claim 4 (Constrained optimum). *Given the constructed initial allocation e in X, allocation x in $X^{\mathrm{PC}}\left(e\right)$ is not Pareto dominated by an alternative allocation z in $X^{\mathrm{PC}}\left(e\right)$ satisfying $\sum_{i \in \mathcal{J}} z^i \leq \sum_{i \in \mathcal{J}} x^i$.*

Proof of claim 4. Supposing not, because of claim 3, we can apply the characterization of claim 2. Exploiting the stationarity of supporting price p in P, this characterization imposes the existence of a strictly positive element v of L satisfying, for some sufficiently small $\epsilon > 0$, at every t in $\mathcal{T}$,

(*) $$v_{t+1} \geq v_t + \epsilon \sum_{i \in \mathcal{J}} \left(z_t^i - x_t^i\right)^2$$

and

(**) $$\sum_{s \in \mathcal{T}} \delta^s \sum_{i \in \mathcal{J}} \left|z_{t+s}^i - x_{t+s}^i\right| \geq \epsilon v_t.$$

Clearly, the sequence $(v_t)_{t \in \mathcal{T}}$ in L converges, so that condition (*) yields

$$\lim_{t \in \mathcal{T}} v_{t+1} \geq v_0 + \epsilon \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{J}} \left(z_t^i - x_t^i \right)^2.$$

Therefore,

$$\lim_{t \in \mathcal{T}} \sum_{i \in \mathcal{J}} \left| z_t^i - x_t^i \right| = 0.$$

This is inconsistent with condition (**) as the sequence $(v_t)_{t \in \mathcal{T}}$ in L is (weakly) increasing. $\qquad \square$

Summing up, we have provided an example of a constrained optimum, according to Alvarez and Jermann [5], which is not autarchic and does not involve high interest rates, as supporting prices exhibit a null interest rate. It is to be remarked that, *strictu sensu*, this is not a counter-example to Proposition 4.10 of Alvarez and Jermann [5], as they also assume stationary endowments, though, in the proof, stationarity of endowments seems not being exploited.

Appendix C. Second Welfare Theorem

We here provide a version of the Second Welfare Theorem as in Kehoe and Levine [21, Proposition 5]. The Second Welfare Theorem of Kehoe and Levine [21] is exploited by Alvarez and Jermann [5, Proposition 4.10] to prove necessity of high interest rates at non-autarchic constrained efficient allocations.

Given an initial allocation e in X, an allocation x in $X^{\mathrm{PC}}(e)$ is an *abstract equilibrium with transfers at initial allocation e in X* if there exists a positive linear functional φ on L such that, given any allocation z in $X^{\mathrm{PC}}(e)$, for every individual i in $\mathcal{J}$,

$$U^i \left(z^i \right) - U^i \left(x^i \right) > 0 \text{ implies } \varphi \cdot \left(z^i - x^i \right) > 0.$$

Claim 5 (Second Welfare Theorem under Stationarity)**.** *In a stationary economy, given a stationary allocation e in X, a stationary interior allocation x in $X^{\mathrm{PC}}(e)$, satisfying*

$$\sum_{i \in \mathcal{J}} x^i - \sum_{i \in \mathcal{J}} e^i = 0$$

and

(sw) $$\sum_{i \in \mathcal{J}} U_\sigma^i \left(x^i \right) - \sum_{i \in \mathcal{J}} U_\sigma^i \left(e^i \right) > 0 \text{ at every } \sigma \in \mathcal{S},$$

is not constrained inefficient at initial allocation e in X only if it is an abstract equilibrium with transfers at initial allocation e in X.

Proof of claim 5. By the Separating Hyperplane Theorem (see Kehoe and Levine [21]), there exists a non-null positive linear functional φ on L such that, for every allocation z in $X^{\mathrm{PC}}(e)$ that (weakly) Pareto dominates allocation x in X,

$$\sum_{i \in \mathcal{J}} \varphi \cdot \left(z^i - x^i \right) \geq 0.$$

Clearly, by positivity of the supporting linear functional, $\varphi \cdot u > 0$, where u is any interior positive element of L. We shall prove that the linear functional φ on L is strictly positive (that is, for every non-null positive element v of L, $\varphi \cdot v > 0$). By canonical arguments, this suffices to prove the claim.

Assuming not, then there exists $v > 0$ in L such that $\varphi \cdot v = 0$ and, for all but finitely many σ in $\mathcal{S}$, $v_\sigma = 0$. For any sufficiently small $1 > \lambda > 0$, consider the interior allocation z in X that is defined, for every individual i in $\mathcal{J}$, by

$$z^i = (1 - \lambda) x^i + \lambda e^i + v.$$

By strict monotonicity and continuity of preferences, allocation z in X strictly Pareto dominates allocation x in X, provided that $1 > \lambda > 0$ is sufficiently small. By strict monotonicity and strict convexity of preferences, allocation z lies in $X^{\mathrm{PC}}(e)$ and, in addition, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$(*) \qquad U_\sigma^i\left(z^i\right) - U_\sigma^i\left(e^i\right) = 0 \text{ implies } \left(z_\tau^i\right)_{\tau \in \mathcal{S}(\sigma)} = \left(e_\tau^i\right)_{\tau \in \mathcal{S}(\sigma)}.$$

Also, consider the collection $\left(\mathcal{F}^i\right)_{i \in \mathcal{J}}$ determined, for every individual i in $\mathcal{J}$, by $\mathcal{F}^i = \left\{\sigma \in \mathcal{S} : U_\sigma^i\left(z^i\right) - U_\sigma^i\left(e^i\right) > 0\right\}$. Notice that, by stationarity, provided that $1 > \lambda > 0$ is sufficiently small, it can be assumed that, for every individual i in $\mathcal{J}$, $\left\{\sigma \in \mathcal{S} : U_\sigma^i\left(x^i\right) - U_\sigma^i\left(e^i\right) > 0\right\} \subset \mathcal{F}^i$, so that, using condition (sw),

$$(**) \qquad \bigcup_{i \in \mathcal{J}} \mathcal{F}^i = \mathcal{S}.$$

Finally, observe that, as v in L vanishes at all but finitely many date-events σ in $\mathcal{S}$, for every individual i in $\mathcal{J}$, the map

$$\sigma \mapsto \left(z_\tau^i\right)_{\tau \in \mathcal{S}(\sigma)}$$

is measurable with respect to some finite partition of $\mathcal{S}$.

By the last observation and restriction (*), there exists $1 > \theta > 0$ such that the alternative interior allocation y in X, defined, for every individual i in $\mathcal{J}$, by

$$y^i = z^i - \theta \sum_{\sigma \in \mathcal{F}^i} x_\sigma^i,$$

lies in $X^{\mathrm{PC}}(e)$ and Pareto dominates allocation x in X. (Here, to simplify notation, we use the decomposition $\mathbb{R}^{\mathcal{S}} = \oplus_{\sigma \in \mathcal{S}} \mathbb{R}_\sigma$.) Hence, by separation,

$$(\#\mathcal{J}) \, \varphi \cdot v - \theta \varphi \cdot \sum_{i \in \mathcal{J}} \sum_{\sigma \in \mathcal{F}^i} x_\sigma^i \geq \varphi \cdot \left(\sum_{i \in \mathcal{J}} y^i - \sum_{i \in \mathcal{J}} x^i\right) \geq 0,$$

that is,

$$0 \geq \left(\frac{\#\mathcal{J}}{\theta}\right) \varphi \cdot v \geq \varphi \cdot \sum_{i \in \mathcal{J}} \sum_{\sigma \in \mathcal{F}^i} x_\sigma^i.$$

Observing that allocation x in X is interior and that condition (**) holds, this is a contradiction, as $\varphi \cdot u > 0$ for every interior positive element u of L. $\qquad \square$

APPENDIX D. PROOFS

Proof of proposition 1. The stationary allocation x in X is Pareto dominated by an alternative stationary allocation z in $X^{\mathrm{PC}}(x)$ satisfying

$$\sum_{i \in \mathcal{J}} z^i \leq \sum_{i \in \mathcal{J}} x^i.$$

At no loss of generality, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$U_\sigma^i\left(z^i\right) - U_\sigma^i\left(x^i\right) = 0 \text{ implies } \left(z_\tau^i\right)_{\tau \in \mathcal{S}(\sigma)} = \left(x_\tau^i\right)_{\tau \in \mathcal{S}(\sigma)}.$$

(Indeed, if not, by strict convexity of preferences, one could use, for some sufficiently large $1 > \lambda > 0$, the alternative stationary allocation $\lambda (z - x) + x$ in X.) For an individual i in $\mathcal{S}$, let $\mathcal{F}^i$ be the set consisting of all date-events σ in $\mathcal{S}$ such that

$$U^i_\sigma \left(z^i \right) - U^i_\sigma \left(x^i \right) > 0.$$

For some $1 > \lambda > 0$, define an alternative allocation y in X by setting, for every individual i in $\mathcal{J}$,

$$y^i = z^i - \lambda \sum_{\sigma \in \mathcal{F}^i} z^i_\sigma.$$

(For notational convenience, we use the decomposition $\mathbb{R}^{\mathcal{S}} = \oplus_{\sigma \in \mathcal{S}} \mathbb{R}_\sigma$.) By stationarity of preferences, there exists a sufficiently small $1 > \lambda > 0$ preserving welfare improvement. (This is so because stationarity requires to satisfy welfare improvement for finitely many continuous utility functions.) By interiority of allocation x in X, strong constrained inefficiency obtains at the subtree

$$\mathcal{F} = \bigcup_{i \in \mathcal{J}} \mathcal{F}^i.$$

This proves the claim. $\qquad\square$

Proof of proposition 2. As consumption plan x^i in X^i is optimal in the budget set $B^i_p \left(e^i, g^i \right)$, for some financial plan v^i in $V^i \left(g^i \right)$, at every date-event σ in $\mathcal{S}$,

$$(*) \qquad\qquad \sum_{\tau \in \sigma_+} p_\tau v^i_\tau + p_\sigma \left(x^i_\sigma - e^i_\sigma \right) = p_\sigma v^i_\sigma.$$

Consider debt constraints $f^i = v^i + g^i$ in F^i, which are positive as v^i is in $V^i \left(g^i \right)$. Suppose that consumption plan z^i in X^i lies in the budget set $B^i_p \left(x^i, f^i \right)$. It follows that, for some financial plan w^i in $V^i \left(f^i \right)$, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau w^i_\tau + p_\sigma \left(z^i_\sigma - x^i_\sigma \right) \le p_\sigma w^i_\sigma.$$

Hence, at every date-event σ in $\mathcal{S}$,

$$- \sum_{\tau \in \sigma_+} p_\tau v^i_\tau + \sum_{\tau \in \sigma_+} p_\tau \left(w^i_\tau + v^i_\tau \right) + p_\sigma \left(z^i_\sigma - x^i_\sigma \right) \le p_\sigma \left(w^i_\sigma + v^i_\sigma \right) - p_\sigma v^i_\sigma.$$

That is, using condition (*),

$$\sum_{\tau \in \sigma_+} p_\tau \left(w^i_\tau + v^i_\tau \right) + p_\sigma \left(z^i_\sigma - e^i_\sigma \right) \le p_\sigma \left(w^i_\sigma + v^i_\sigma \right).$$

In addition, as w^i lies in $V^i \left(f^i \right)$, financial plan $w^i + v^i$ is an element of $V^i \left(g^i \right)$. It follows that consumption plan z^i in X^i belongs to the budget set $B^i_p \left(e^i, g^i \right)$, so proving the claim. $\qquad\square$

Proof of proposition 3. Necessity of this first-order characterization is established by Alvarez and Jermann [5]. To prove sufficiency, for an individual i in $\mathcal{J}$, observe that consumption plan x^i lies in the budget set $B^i_p \left(x^i, f^i \right)$ and consider any consumption plan z^i in the budget set $B^i_p \left(x^i, f^i \right)$. It follows that, for some financial plan v^i in $V^i \left(f^i \right)$, at every date-event σ in $\mathcal{S}$,

$$-p^i_\sigma \sum_{\tau \in \sigma_+} \left(\frac{p_\tau}{p_\sigma} \right) f^i_\tau + p^i_\sigma \sum_{\tau \in \sigma_+} \left(\frac{p_\tau}{p_\sigma} \right) \left(v^i_\tau + f^i_\tau \right) + p^i_\sigma \left(z^i_\sigma - x^i_\sigma \right) \le p^i_\sigma v^i_\sigma,$$

25

where p^i in P^i is the subjective price at consumption plan x^i in X^i. Using condition (FOC-1), along with the fact that v^i lies in $V^i\left(f^i\right)$, this yields

$$-p_\sigma^i \sum_{\tau \in \sigma_+} \left(\frac{p_\tau}{p_\sigma}\right) f_\tau^i + \sum_{\tau \in \sigma_+} p_\tau^i \left(v_\tau^i + f_\tau^i\right) + p_\sigma^i \left(z_\sigma^i - x_\sigma^i\right) \le p_\sigma^i v_\sigma^i.$$

Using condition (FOC-2), this finally becomes

$$-\sum_{\tau \in \sigma_+} p_\tau^i f_\tau^i + \sum_{\tau \in \sigma_+} p_\tau^i \left(v_\tau^i + f_\tau^i\right) + p_\sigma^i \left(z_\sigma^i - x_\sigma^i\right) \le p_\sigma^i \left(v_\sigma^i + f_\sigma^i\right) - p_\sigma^i f_\sigma^i.$$

Adding up, one obtains

$$-\sum_{\sigma \in \mathcal{S}_t} \sum_{\tau \in \sigma_+} p_\tau^i f_\tau^i + \sum_{\sigma \in \mathcal{S}^t} p_\sigma^i \left(z_\sigma^i - x_\sigma^i\right) \le 0,$$

where, for every t in $\mathcal{T}$, $\mathcal{S}_t = \{\sigma \in \mathcal{S} : t(\sigma) = t\}$ and $\mathcal{S}^t = \{\sigma \in \mathcal{S} : t(\sigma) \le t\}$. Observing that debt-constrains f in F are bounded and subjective price p^i in P^i defines an order-continuous linear functional on L,

$$\sum_{\sigma \in \mathcal{S}} p_\sigma^i \left(z_\sigma^i - x_\sigma^i\right) \le 0.$$

This, because of (SP), suffices to prove the claim. $\qquad\square$

Proof of proposition 4. We first prove constrained inefficiency when the Modified Cass Criterion is satisfied. At no loss of generality, as x in X is an interior allocation, it can be assumed that

$$\bigwedge_{i \in \mathcal{J}} x^i \ge v.$$

Consider a partition $\left(\mathcal{P}^i\right)_{i \in \mathcal{J}}$ of the set of *non-initial* date-events in $\mathcal{S}$ such that, for every non-initial date-event σ in $\mathcal{S}$, σ belongs to $\mathcal{P}^i$ only if $f_\sigma^i > 0$. This construction is consistent as price support is non-trivial. Also, for every individual i in $\mathcal{J}$, let $\mathcal{N}^i = \left\{\sigma \in \mathcal{S} : \sigma_+ \cap \mathcal{P}^i \ne \emptyset\right\}$. Finally, for every date-event σ in $\mathcal{S}$, define $\mathcal{P}^i(\sigma) = \mathcal{P}^i \cap \mathcal{S}(\sigma)$ and $\mathcal{N}^i(\sigma) = \mathcal{N}^i \cap \mathcal{S}(\sigma)$.

For every individual i in $\mathcal{J}$, define

$$z^i = x^i + \sum_{\sigma \in \mathcal{P}^i} v_\sigma - \sum_{\sigma \in \mathcal{N}^i} \left(\frac{\sum_{\tau \in \sigma_+ \cap \mathcal{P}^i} p_\tau v_\tau}{\sum_{\tau \in \sigma_+} p_\tau v_\tau}\right) v_\sigma.$$

(For notational convenience, we use the decomposition $\mathbb{R}^{\mathcal{S}} = \oplus_{\sigma \in \mathcal{S}} \mathbb{R}_\sigma$.) For every individual i in $\mathcal{J}$, the underlying redistribution increases consumption at date-events in $\mathcal{P}^i$ and decreases consumption at date-events in $\mathcal{N}^i$. Clearly, z in X is a feasible allocation, that is, it satisfies (CF). Also, notice that, by construction, for

every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$
\begin{aligned}
\sum_{\nu \in \mathcal{N}^i(\sigma)} \left(\frac{\sum_{\tau \in \nu_+ \cap \mathcal{P}^i} p_\tau v_\tau}{\sum_{\tau \in \nu_+} p_\tau v_\tau} \right) p_\nu^i v_\nu \quad &\leq \quad \sum_{\nu \in \mathcal{N}^i(\sigma)} p_\nu^i \left(\frac{p_\nu v_\nu}{\sum_{\tau \in \nu_+} p_\tau v_\tau} \right) \frac{1}{p_\nu} \sum_{\tau \in \nu_+ \cap \mathcal{P}^i} p_\tau v_\tau \\
&\leq \quad \sum_{\nu \in \mathcal{N}^i(\sigma)} \left(\frac{p_\nu v_\nu}{\sum_{\tau \in \nu_+} p_\tau v_\tau} \right) \sum_{\tau \in \nu_+ \cap \mathcal{P}^i} p_\tau^i v_\tau \\
&\leq \quad \rho \sum_{\nu \in \mathcal{N}^i(\sigma)} \sum_{\tau \in \nu_+ \cap \mathcal{P}^i} p_\tau^i v_\tau \\
&\leq \quad \rho \sum_{\tau \in \mathcal{P}^i(\sigma)} p_\tau^i v_\tau.
\end{aligned}
$$

The first inequality is a simple manipulation; the second inequality uses the fact that subjective and market evaluations coincide; the third inequality is a consequence of the Modified Cass Criterion; the last inequality uses the construction of subsets $\mathcal{P}^i$ and $\mathcal{N}^i$ of $\mathcal{S}$. Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$
(*) \qquad \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left(z_\tau^i - x_\tau^i \right) \geq (1 - \rho) \sum_{\tau \in \mathcal{P}^i(\sigma)} p_\tau^i v_\tau \geq (1 - \rho) \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left(z_\tau^i - x_\tau^i \right)^+.
$$

Manipulating inequality (*), we obtain

$$
\sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left(z_\tau^i - x_\tau^i \right) \geq \left(\frac{1 - \rho}{\rho} \right) \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left(z_\tau^i - x_\tau^i \right)^- \geq (1 - \rho) \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left(z_\tau^i - x_\tau^i \right)^- ..
$$

Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$
(**) \qquad \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left(z_\tau^i - x_\tau^i \right) \geq \left(\frac{1 - \rho}{2} \right) \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left| z_\tau^i - x_\tau^i \right|.
$$

Condition (**) guarantees a first-order positive welfare effect beginning from every date-event σ in $\mathcal{S}$. To obtain a welfare improvement, we show that higher order effects are uniformly bounded. As allocation x in X is interior, for a sufficiently small $\epsilon > 0$, any allocation y in $B_\epsilon(x)$ is also interior, where

$$
B_\epsilon(x) = \left\{ y \in X : \sum_{i \in \mathcal{J}} \left\| y^i - x^i \right\| \leq \epsilon \right\}.
$$

Notice that per-period utility $u^i : \mathbb{R}_+ \to \mathbb{R}$ exhibits a bounded second-order term over any compact interval in $\mathbb{R}_{++}$. Thus, it can be assumed that there exists a sufficiently large $\mu > 0$ satisfying, given any allocation y in $B_\epsilon(x)$, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$
u^i \left(y_\sigma^i \right) - u^i \left(x_\sigma^i \right) \geq \partial u^i \left(x_\sigma^i \right) \left(y_\sigma^i - x_\sigma^i \right) - \left(\frac{\mu}{2} \right) \left| y_\sigma^i - x_\sigma^i \right| \partial u^i \left(x_\sigma^i \right) \left| y_\sigma^i - x_\sigma^i \right|.
$$

Also, possibly contracting v in L, at no loss of generality,

$$
\bigvee_{i \in \mathcal{J}} \left\| z^i - x^i \right\| \leq \| v \| \leq \epsilon \wedge \left(\frac{1 - \rho}{\mu} \right).
$$

Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$
u^i \left(z_\sigma^i \right) - u^i \left(x_\sigma^i \right) \geq \partial u^i \left(x_\sigma^i \right) \left(z_\sigma^i - x_\sigma^i \right) - \left(\frac{1 - \rho}{2} \right) \partial u^i \left(x_\sigma^i \right) \left| z_\sigma^i - x_\sigma^i \right|.
$$

This, because of condition (**), shows weak Pareto dominance. By strict convexity of preferences, this suffices to prove the first claim.

To prove strong constrained inefficiency, the argument requires only a minor amendment. At no loss of generality, as x in X is an interior allocation, it can be assumed that

$$\bigwedge_{i \in \mathcal{J}} x^i \geq v + \epsilon u,$$

where u is the unit of L. Also, to simplify, it can be assumed that $\mathcal{F}$ coincides with $\mathcal{S}$, for, otherwise, one could consider the reduction of the economy to $\mathcal{F}$. For every individual i in $\mathcal{J}$, define

$$z^i = x^i + \sum_{\sigma \in \mathcal{P}^i} v_\sigma - \sum_{\sigma \in \mathcal{N}^i} \left(\frac{\sum_{\tau \in \sigma_+ \cap \mathcal{P}^i} p_\tau v_\tau}{\sum_{\tau \in \sigma_+} p_\tau v_\tau} \right) (v_\sigma + \epsilon).$$

Clearly, z in X is a feasible allocation, even though a constant share of the aggregate endowment is destroyed, that is, it satisfies (SF-1)-(SF-2). Also, notice that, by construction, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$
\begin{aligned}
\sum_{\nu \in \mathcal{N}^i(\sigma)} \left(\frac{\sum_{\tau \in \nu_+ \cap \mathcal{P}^i} p_\tau v_\tau}{\sum_{\tau \in \nu_+} p_\tau v_\tau} \right) p_\nu^i (v_\nu + \epsilon) \quad & \leq \quad \sum_{\nu \in \mathcal{N}^i(\sigma)} p_\nu^i \left(\frac{p_\nu v_\nu + \epsilon p_\nu}{\sum_{\tau \in \nu_+} p_\tau v_\tau} \right) \frac{1}{p_\nu} \sum_{\tau \in \nu_+ \cap \mathcal{P}^i} p_\tau v_\tau \\
& \leq \quad \sum_{\nu \in \mathcal{N}^i(\sigma)} \left(\frac{p_\nu v_\nu + \epsilon p_\nu}{\sum_{\tau \in \nu_+} p_\tau v_\tau} \right) \sum_{\tau \in \nu_+ \cap \mathcal{P}^i} p_\tau^i v_\tau \\
& \leq \quad \rho \sum_{\nu \in \mathcal{N}^i(\sigma)} \sum_{\tau \in \nu_+ \cap \mathcal{P}^i} p_\tau^i v_\tau \\
& \leq \quad \rho \sum_{\tau \in \mathcal{P}^i(\sigma)} p_\tau^i v_\tau.
\end{aligned}
$$

Here, the third inequality is a consequence of the Strong Modified Cass Criterion. The proof then unfolds as previously explained. $\square$

Proof of proposition 5. As allocation z lies in $X^{\mathrm{PC}}(x)$, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$v_\sigma^i = \frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left(z_\tau^i - x_\tau^i \right) \geq 0.$$

In addition, $v = \sum_{i \in \mathcal{J}} v^i$ is a non-null positive element of $\mathbb{R}^{\mathcal{S}}$, as welfare is higher for at least one individual at some date-event. By feasibility and the bound on subjective prices (UI), as a matter of fact, for every individual i in $\mathcal{J}$, v^i is a positive element of L and, across individuals, v is a non-null positive element of L.

Observe that, by construction, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma^i} \sum_{\tau \in \sigma_+} p_\tau^i v_\tau^i + \left(z_\sigma^i - x_\sigma^i \right) = v_\sigma^i.$$

By first-order conditions (FOC-1)-(FOC-2) and the positivity of v^i in L,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau^i + \left(z_\sigma^i - x_\sigma^i \right) \geq v_\sigma^i.$$

Summing among individuals,

$$(*) \qquad \frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau + \sum_{i \in \mathcal{J}} \left(z_\sigma^i - x_\sigma^i \right) \geq v_\sigma,$$

where $v = \sum_{i \in \mathcal{J}} v^i$ in L. We here distinguish two cases.

Assuming constrained inefficiency, condition (*) delivers, at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq v_\sigma.$$

This proves the claim.

Assuming strong constrained inefficiency, observe that, at every σ in $\mathcal{S}$, $v_\sigma > 0$ only if σ belongs to $\mathcal{F}$. (Indeed, if σ is not in $\mathcal{F}$, then, for every individual i in $\mathcal{J}$,

$$\left(z_\tau^i \right)_{\tau \in \mathcal{S}(\sigma)} = \left(x_\tau^i \right)_{\tau \in \mathcal{S}(\sigma)}$$

and, hence, $v_\sigma^i = 0$.) Furthermore, as v is a bounded element in L, for some sufficiently large $1 > \rho > 0$, at every σ in $\mathcal{F}$,

$$\epsilon \geq \left(\frac{1-\rho}{\rho} \right) v_\sigma + \left(\frac{1-\rho}{\rho} \right) \epsilon.$$

Hence, condition (*) delivers, at every date-event σ in $\mathcal{F}$,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq v_\sigma + \epsilon \geq \left(\frac{1}{\rho} \right) v_\sigma + \left(\frac{1-\rho}{\rho} \right) \epsilon.$$

This proves the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of proposition 6. By proposition 4, allocation x in X is constrained inefficient conditionally on allocation x in X. As allocation x lies in $X^{\mathrm{PC}}(\nu)$, this simple observation suffices to prove the claim. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of proposition 7. For every individual i in $\mathcal{J}$, define, at every date-event σ in $\mathcal{S}$,

$$v_\sigma^i = \frac{1}{p_\sigma^i} \sum_{\tau \in \mathcal{S}(\sigma)} p_\tau^i \left(z_\tau^i - x_\tau^i \right).$$

By Pareto dominance, at the initial date-event ϕ in $\mathcal{S}$, $\sum_{i \in \mathcal{J}} v_\phi^i > 0$. In addition, as allocation x in X is interior, by uniform impatience (UI), for every individual i in $\mathcal{J}$, v^i is an element of L. In addition, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$(*) \qquad \frac{1}{p_\sigma^i} \sum_{\tau \in \sigma_+} p_\tau^i v_\tau^i + \left(z_\sigma^i - x_\sigma^i \right) = v_\sigma^i.$$

For every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$v_\sigma^i < 0 \text{ implies } U_\sigma^i \left(z^i \right) - U_\sigma^i \left(x^i \right) < 0.$$

Therefore, as allocation z lies in $X^{\mathrm{PC}}(\nu)$,

$$v_\sigma^i < 0 \text{ implies } U_\sigma^i \left(x^i \right) - \nu_\sigma^i > 0.$$

Using the consistency requirement (DC), this yields

$$v_\sigma^i < 0 \text{ implies } f_\sigma^i > 0.$$

Hence, by first-order conditions (FOC-1)-(FOC-2), condition (*) delivers, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau^i + \left(z_\sigma^i - x_\sigma^i\right) \geq v_\sigma^i.$$

Summing up across individuals,

(**) $$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau + \sum_{i \in \mathcal{J}} \left(z_\sigma^i - x_\sigma^i\right) \geq v_\sigma,$$

where $v = \sum_{i \in \mathcal{J}} v^i$ in L. We here distinguish two cases.

Assuming constrained inefficiency, condition (**) delivers, at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau^+ \geq v_\sigma^+,$$

where v^+ in L is the non-null positive part of v in L This proves the claim.

Assuming strong constrained inefficiency, observe that, at every σ in $\mathcal{S}$, $v_\sigma > 0$ only if σ belongs to $\mathcal{F}$. Furthermore, as v is a bounded element in L, for some sufficiently large $1 > \rho > 0$, at every σ in $\mathcal{F}$,

$$\epsilon \geq \left(\frac{1-\rho}{\rho}\right) v_\sigma + \left(\frac{1-\rho}{\rho}\right) \epsilon.$$

Define recursively a non-null positive element $\hat{v}$ of L by means of $\hat{v}_\phi = v_\phi^+ > 0$ and, at every σ in $\mathcal{S}$, $(\hat{v}_\tau)_{\tau \in \sigma_+} = (v_\tau^+)_{\tau \in \sigma_+}$, if $\hat{v}_\sigma > 0$, and $(\hat{v}_\tau)_{\tau \in \sigma_+} = 0$, if $\hat{v}_\sigma = 0$. Notice that, at every σ in $\mathcal{S}$, $\hat{v}_\sigma > 0$ only if $\hat{v}_\sigma = v_\sigma$. Hence, condition (**) delivers, at every date-event σ in $\hat{\mathcal{F}}$,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau \hat{v}_\tau \geq \hat{v}_\sigma + \epsilon \geq \left(\frac{1}{\rho}\right) \hat{v}_\sigma + \left(\frac{1-\rho}{\rho}\right) \epsilon,$$

where

$$\hat{\mathcal{F}} = \{\sigma \in \mathcal{F} : \hat{v}_\sigma > 0\}.$$

Observing that, by construction,

$$\sigma \notin \hat{\mathcal{F}} \text{ if and only if } \hat{\mathcal{F}} \cap \mathcal{S}(\sigma) = \emptyset,$$

this proves the claim. $\qquad \square$

Gaetano Bloise, Department of Economics, University of Rome III, Italy
E-mail address: gaetano.bloise@uniroma3.it

Pietro Reichlin, Department of Economics, LUISS 'Guido Carli', Rome, Italy; CEPR; EIEF
E-mail address: preichlin@luiss.it