A Simple and Unifying Approach to the Existence of Nash Equilibrium in Discontinuous Games

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Abstract

We introduce a new condition, weak better-reply security, and show that every compact, quasiconcave and weakly better-reply secure game has a Nash equilibrium. This result is established using simple generalizations of classical ideas. Furthermore, we show that, when players' action spaces are metric, it implies the existence results of Reny (1999) and Carmona (2009), as well as a recent result of Barelli and Soza (2009).

1 Introduction

The classical approach to the existence of Nash equilibrium in a normal form game consists in applying a fixed point theorem to its best-reply correspondence (e.g., Nash (1950)). When discontinuities on players' payoff functions prevent this approach to be used, the classical solution consists in appropriately approximating the original game by a sequence of sufficiently well-behaved games in the following sense: a fixed point theorem can be applied to the best-reply correspondence of each of the approximating games, and limit points the resulting Nash equilibria are themselves a Nash

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equilibrium of the original game (e.g., Dasgupta and Maskin (1986), Simon (1987) and Simon and Zame (1990)). Clearly, an equally suitable approach is to use a fixed point theorem to guarantee the existence of ε-equilibrium in the approximating games, and to show that limit points of the resulting ε-equilibria, with ε converging to zero, are themselves a Nash equilibrium of the original game (see Fudenberg and Levine (1986)).

In this note, we combine and generalize these ideas in order to obtain a new existence theorem that, when players' action spaces are metric, implies the main results of Reny (1999) and Carmona (2009), and the generalization of Reny's Theorem obtained by Barelli and Soza (2009). In this way, these results are unified under an existence result established in a relatively simple way using the classical ideas mentioned above.

The use of these classical ideas requires a generalization of the notion of an approximate equilibrium. This is done by noting that a Nash equilibrium is a strategy yielding, to each player, a payoff at least as high as the value of his value function evaluated at the equilibrium strategies of the other players. Then, our notion of approximate equilibrium is obtained by replacing, in this definition of Nash equilibrium, each player's value function by a function strictly below it and defined also on the action space of the other players.

The next step in our approach is to define a condition, weak better-reply security, which guarantees that the original game is appropriately approximated with a well-behaved game in the following sense: First, players' value functions in the approximating game are lower semicontinuous and can therefore be "closely" approximated from below by continuous functions; second, a fixed point theorem can be applied to a subcorrespondence of the approximate best-reply correspondence (in the approximating game) defined using those continuous functions; and, third, every limit point of the resulting approximate equilibria of the approximating game is a Nash equilibrium of the original game.

Clearly, our approach corresponds closely to the classical ideas discussed above. As it name suggest, our condition generalizes the notion of better-reply security of

Reny (1999) and, in fact, it generalizes the notion of generalized better-reply security of Barelli and Soza (2009). Furthermore, weak better-reply security also implies the (combined) notion of weak payoff security and weak upper semicontinuity considered in Carmona (2009). Thus, in the case where players' action spaces are metric, Reny (1999, Theorem 3.1), Barelli and Soza (2009, Corollary 6.8) and Carmona (2009, Corollary 2) can be established using our general existence result, and more importantly, be seen as a consequence of the classical approach on which our result is based upon.

We note that our approach is as a synthesis of several recent developments in the literature. The idea of approximating the original game by a better behaved game is based on Reny (1999) and the approximation of players' value functions is achieved using Reny's approximation lemma. Furthermore, our sufficient condition for the existence of equilibrium is inspired by Reny's better-reply security. The idea of approximating players' value functions in the approximating game and of applying a fixed point to a subcorrespondence of the resulting approximate best-reply correspondence is based on Prokopovych (2009). The use of a fixed point theorem to a subcorrespondence of the approximate best-reply correspondence requires, first, allowing players to use a well-behaved correspondence defined locally to secure payoff and, second, using a partition of the unity to obtain a well-behaved correspondence defined globally from the well-behaved correspondences defined locally; both of these ideas are based on Barelli and Soza (2009). Finally, the idea that limit points of approximate equilibria of the approximating game are themselves Nash equilibria of the original game is based on Bagh (2009), Carmona (2009) and Prokopovych (2009). The strength of our approach is that these ideas are organized in such a way that it allows for a general existence result based on (simple generalizations of) classic ideas, which is strong enough to imply several known existence theorems. This, in turn, allows for a clear understanding of both the new and the previous existence results.

We also note that our approach leaves out the existence results based on the existence of a maximal element of binary relations, such as the ones of Shafer and

Sonnenschein (1975) and Baye, Tian, and Zhou (1993).[1] A unifying approach to both these and the existence results we have focused on can be found in Barelli and Soza (2009). In contrast with their work, our goal is not to obtain a general existence theorem that can generalize all such results, but rather to show that several of them can be obtained by an appropriate generalization of classical ideas.

2 Notation and Definitions

A *normal form game* $G = (X_i, u_i)_{i\in N}$ consists of a finite set of players $N = \{1, \ldots, n\}$ and, for all players $i \in N$, a pure strategy space X_i and a payoff function $u_i : X \to \mathbb{R}$, where $X = \prod_{i\in N} X_i$. We assume that, for all $i \in N$, X_i is a subset of a metric vector space.

We say that a game G is *compact* if, for all $i \in N$, X_i is compact and u_i is bounded. Given a player $i \in N$, the symbol $-i$ denotes "all players but i". Also, $X_{-i} = \prod_{j\neq i} X_j$. Furthermore, we say that G is *quasiconcave* if, for all $i \in N$, X_i is convex and $u_i(\cdot, x_{-i})$ is quasiconcave for all $x_{-i} \in X_{-i}$.

Let $G = (X_i, u_i)_{i\in N}$ be a game and let, for all $i \in N$, $v_i : X_{-i} \to \mathbb{R}$ be defined by $v_i(x_{-i}) = \sup_{x_i\in X_i} u_i(x_i, x_{-i})$; the function v_i is player i's value function. A strategy $x^* \in X$ is *Nash equilibrium* if $u_i(x^*) \geq v_i(x^*_{-i})$ for all $i \in N$.

We define the notion of an approximate equilibrium by replacing v_i with a function strictly below it in the inequality above. Let $F(G)$ denote the set of all functions $f = (f_1, \ldots, f_n)$ such that f_i is a real-valued function on X_{-i} and $f_i(x_{-i}) < v_i(x_{-i})$ for all $x_{-i} \in X_{-i}$ and $i \in N$. For all $f \in F(G)$, we say a strategy $x^* \in X$ is an *f-equilibrium* if $u_i(x^*) \geq f_i(x^*_{-i})$ for all $i \in N$. Note that this notion generalizes the concept of ε-equilibrium since, for all $\varepsilon > 0$, $x^* \in X$ is an ε-equilibrium if it is an f-equilibrium for $f = (v_1 - \varepsilon, \ldots, v_n - \varepsilon)$.

[1] The existence theorem in Baye, Tian, and Zhou (1993) can be easily proven by applying Theorem 2 in Tian (1993) on the existence of maximal elements to the binary relation on the joint action space defined by $x \succ y$ if and only if $U(x, y) > U(y, y)$, where $U(x, y) = \sum_{i\in N} u_i(x_i, y_{-i})$ in a game $G = (X_i, u_i)_{i\in N}$.

3 Existence of Nash equilibrium

In this section, we establish a general existence result (Subsection 3.1) and use it, in the case where players' action spaces are metric, to derive the main results of Reny (1999) and Carmona (2009), and the generalization of Reny's Theorem obtained by Barelli and Soza (2009) (Subsection 3.2).

3.1 A general existence result

Our general existence result (Theorem 1 below) states that all compact and quasiconcave games satisfying a weak form of better-reply security have a Nash equilibrium.

Our notion of weak better-reply security guarantees that any game satisfying it can be approximated by a well-behaved game (in the sense that a fixed point argument can be used to establish the existence of approximate equilibria) and that the approximation can be done in such a way that limit points of approximate equilibria of the approximating game, with the level of approximation suitably converging to zero, are themselves a Nash equilibria of the original game.

Weak better-reply security combines two properties. The first is entitled better-reply closeness. This property is the one responsible for the limit result and is formally defined as follows. Let $\underline{u}$ be a $\mathbb{R}^n$-valued function on X, $\underline{v}_i(x_{-i}) = \sup_{x_i \in X_i} \underline{u}(x_i, x_{-i})$ for all $i \in N$ and $x_{-i} \in X_{-i}$, $G = (X_i, u_i)_{i \in N}$ be a game and Γ be the closure of the graph of $u = (u_1, \ldots, u_n)$. We say that G is *better-reply closed relative to* $\underline{u}$ if $(x^*, u^*) \in \Gamma$ and $u_i^* \geq \underline{v}_i(x_{-i}^*)$ implies that x^* is a Nash equilibrium of G.

The second property that is part of weak better-reply security is named approximate payoff security. This property is responsible for the existence of an approximating game and of approximate equilibria for it with a level of approximation suitably converging to zero. A game $G = (X_i, u_i)_{i \in N}$ is *upper (resp. lower) payoff secure* if for all $i \in N$, $\varepsilon > 0$ and $x \in X$ there exists an open neighborhood $V_{x_{-i}}$ of x_{-i} and a nonempty, closed, convex valued, upper (resp. lower) hemicontinuous correspondence $\varphi_i : V_{x_{-i}} \rightrightarrows X_i$ such that $u_i(x') \geq u_i(x) - \varepsilon$ for all $x' \in \text{graph}(\varphi_i)$. We note that upper payoff security was introduced by Barelli and Soza (2009) under the name of generalized payoff security. Given a $\mathbb{R}^n$-valued function $\underline{u}$ on X, a game $G = (X_i, u_i)_{i \in N}$ is *approximately payoff secure relative to* $\underline{u}$ if, for all $i \in N$, $\underline{u}_i \leq u_i$, $\underline{u}_i(\cdot, x_{-i})$ is quasiconcave for all $x_{-i} \in X_{-i}$ and $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ is either upper or lower payoff secure. In this case, we say that $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ is an approximating game of G.

The notion of weak better-reply combines approximate payoff security and better-reply closeness. Formally, a game G is *weakly better-reply secure* if there exists a $\mathbb{R}^n$-valued function $\underline{u}$ on X such that G is both approximately payoff secure and better-reply closed relative to $\underline{u}$.

Our existence result states that compact, quasiconcave games that are weakly better-reply secure have a Nash equilibrium.

Theorem 1 *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave and weakly better-reply secure, then G has a Nash equilibrium.*

Theorem 1 is established with the help of three lemmas. The first of these lemmas states that if G is compact, quasiconcave and approximately payoff secure relative to $\underline{u}$, then the approximating game $\underline{G}$ has an f-equilibrium provided that f is continuous. Note that, under those assumption, $\underline{G}$ is a compact, quasiconcave game, satisfying either upper or lower payoff security. Thus, Lemma 1 can be understood as stating that every compact, quasiconcave game satisfying either upper or lower payoff security has a f-equilibrium for all continuous f. When stated in this way it becomes clear that Lemma 1 is a generalization of Reny (1996, Theorem 3) and Prokopovych (2009, Theorem 2).

Lemma 1 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and quasiconcave game. If G is approximately payoff secure relative to $\underline{u}$, then $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ has an f-equilibrium for all continuous $f \in F(\underline{G})$.*

Proof. Define $B : X \rightrightarrows X$ by $B(x) = \{y \in X : \underline{u}_i(y_i, x_{-i}) > f_i(x_{-i})$ for all $i \in N\}$. Note that B is nonempty-valued since $f_i < \underline{v}_i$ for all $i \in N$ and is convex-valued since $\underline{u}_i(\cdot, x_{-i})$ is quasiconcave for all $i \in N$ and $x_{-i} \in X_{-i}$.

Next, we show that for all $x \in X$, there exist an open neighborhood V_x of x and a nonempty, closed, convex valued correspondence $\varphi_x : V_x \rightrightarrows X$ such that $\varphi_x(x') \subseteq B(x')$ for all $x' \in V_x$ and φ_x is upper (resp. lower) hemicontinuous if $\underline{G}$ is upper (resp. lower) payoff secure.

In order to establish the above claim, let $x \in X$ and consider $y \in B(x)$. Then, for all $i \in N$, $\underline{u}_i(y_i, x_{-i}) > f_i(x_{-i}) + 2\eta$ for some $\eta > 0$ sufficiently small. Since $\underline{G}$ is either upper or lower payoff secure and f is continuous, it follows that for all $i \in N$, there exist an open neighborhood $V_{x_{-i}}$ of x_{-i} and a nonempty, convex valued correspondence $\varphi_i : V_{x_{-i}} \rightrightarrows X_i$ such that $\underline{u}_i(x') \geq \underline{u}_i(y_i, x_{-i}) - \eta$ for all $x' \in \text{graph}(\varphi_i)$, $f_i(x_{-i}) > f_i(x'_{-i}) - \eta$ for all $x'_{-i} \in V_{x_{-i}}$ and φ_i is upper (resp. lower) hemicontinuous if $\underline{G}$ is upper (resp. lower) payoff secure. Define, for all $i \in N$, $V_i = X_i \times V_{x_{-i}}$, $V = \cap_{i \in N} V_i$ and $\varphi : V \rightrightarrows X$ by $\varphi(x') = \prod_{i \in N} \varphi_i(x'_{-i})$ for all $x' \in V$. Let $x' \in V$ and $y' \in \varphi(x')$. Then, for all $i \in N$, it follows that $x'_{-i} \in V_{x_{-i}}$ and $y'_i \in \varphi_i(x'_{-i})$. Then $\underline{u}_i(y'_i, x'_{-i}) \geq \underline{u}_i(y_i, x_{-i}) - \eta > f_i(x_{-i}) + \eta > f_i(x'_{-i})$ for all $i \in N$. Hence, $y' \in B(x')$ and so $\varphi(x') \subseteq B(x')$.

Due to the above claim, we obtain a family $\{V_x\}_{x \in X}$ where V_x is an open neighborhood of x, and a family $\{\varphi_x\}_{x \in X}$ where $\varphi_x : V_x \rightrightarrows X$ is an upper (resp. lower) hemicontinuous correspondence with nonempty, convex, compact values satisfying $\varphi_x(x') \subseteq B(x')$ for all $x' \in V_x$ if $\underline{G}$ is upper (resp. lower) payoff secure. Since X is compact, there exists a finite open cover $\{V_{x_j}\}_{j=1}^m$ and a partition of the unity $\{\beta_j\}_{j=1}^m$ subordinate to $\{V_{x_j}\}_{j=1}^m$. Define $\phi : X \rightrightarrows X$ by $\phi(x) = \sum_{j=1}^m \beta_j(x)\varphi_{x_j}(x)$. Then, ϕ is an upper (resp. lower) hemicontinuous correspondence with nonempty, convex, compact values if $\underline{G}$ is upper (resp. lower) payoff secure. In both cases, ϕ has a fixed point x^* either by Glicksberg's fixed point theorem or by Michael's selection theorem and Brouwer's fixed point theorem.

Let $1 \leq j \leq m$ be given and note that if $x^* \in V_{x_j}$, then $\varphi_{x_j}(x^*) \subseteq B(x^*)$ while if $x^* \notin V_{x_j}$, then $\beta_j(x^*) = 0$. Since B is convex-valued, then $x^* \in B(x^*)$. Hence, for all $i \in N$, $\underline{u}_i(x^*) > f_i(x^*)$ and so x^* is a f-equilibrium of $\underline{G}$. ■

Lemma 2 also concerns approximately payoff secure games. It states that if a game G is approximately payoff secure relative to $\underline{u}$, then the approximating game $\underline{G}$

is such that players' value functions are lower semicontinuous.

Lemma 2 *If $G = (X_i, u_i)_{i \in N}$ is approximately payoff secure relative to $\underline{u}$, then $\underline{v}_i$ is lower semicontinuous for all $i \in N$.*

Proof. Let $i \in N$, $x_{-i} \in X_{-i}$ and $\varepsilon > 0$ be given. Let $0 < \eta < \varepsilon$ and let $x_i \in X_i$ be such that $\underline{u}_i(x_i, x_{-i}) - \eta > \underline{v}_i(x_{-i}) - \varepsilon$. By approximate payoff security, there exists an open neighborhood $V_{x_{-i}}$ of x_{-i} and a nonempty-valued correspondence $\varphi_i : V_{x_{-i}} \rightrightarrows X_i$ such that $\underline{u}_i(x') \geq \underline{u}_i(x) - \eta$ for all $x' \in \text{graph}(\varphi_i)$. Then, for all $x'_{-i} \in V_{x_{-i}}$, letting $x'_i \in \varphi_i(x'_{-i})$, we obtain that $\underline{v}_i(x'_{-i}) \geq \underline{u}_i(x') \geq \underline{u}_i(x) - \eta > \underline{v}_i(x_{-i}) - \varepsilon$. Hence, $\underline{v}_i$ is lower semicontinuous. ■

Since $\underline{v}_i$ is lower semicontinuous for all $i \in N$, it follows by Reny (1999, Lemma 3.5) that there exists a sequence $\{v_i^k\}_{k=1}^{\infty}$ of continuous real-valued functions on X_{-i} such that $v_i^k(x_{-i}) \leq \underline{v}_i(x_{-i})$ and $\liminf_k v_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i})$ for all $k \in \mathbb{N}$, $i \in N$, $x_{-i} \in X_{-i}$ and all sequences $\{x_{-i}^k\}_{k=1}^{\infty}$ converging to x_{-i}. When G is a compact game, this form of approximation together with better-reply closeness is enough for the limit points of approximate equilibria of $\underline{G}$ to be Nash equilibria of G.

Lemma 3 *Let $G = (X_i, u_i)_{i \in N}$ be a compact and better-reply closed game relative to $\underline{u}$. If $x^* \in X$, $\{x_k\}_{k=1}^{\infty} \subseteq X$ and $\{f_k\}_{k=1}^{\infty} \subseteq F(\underline{G})$ are such that $x^* = \lim_k x_k$, $\liminf_k f_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i}^*)$ for all $i \in N$ and x_k is a f_k-equilibrium of $\underline{G} = (X_i, \underline{u})_{i \in N}$ for all $k \in \mathbb{N}$, then x^* is a Nash equilibrium of G.*

Proof. Since u is bounded, we may assume that $\{u(x_k)\}_{k=1}^{\infty}$ converge. Let $u^* = \lim_k u(x_k)$ and note that $(x^*, u^*) \in \Gamma$.

Let $i \in N$. Since x_k is a f_k-equilibrium of $\underline{G}$ for all $k \in \mathbb{N}$, then $u_i(x_k) \geq \underline{u}_i(x_k) \geq f_i^k(x_{-i}^k)$ for all $k \in \mathbb{N}$, and so $\liminf_k f_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i}^*)$ implies that $u_i^* \geq \underline{v}_i(x_{-i}^*)$. Since $u_i^* \geq \underline{v}_i(x_{-i}^*)$ for all $i \in N$ and G is better-reply closed, it follows that x^* is a Nash equilibrium of G. ■

We finally turn to the proof of Theorem 1, which is obtained easily from Lemmas 1—3.

Proof of Theorem 1. Let $\underline{u}$ be such that G is both approximately payoff secure and better-reply closed relative to $\underline{u}$.

For all $k \in \mathbb{N}$ and $i \in N$, let $f_i^k : X_{-i} \to \mathbb{R}$ be defined by $f_i^k(x_{-i}) = v_i^k(x_{-i}) - 1/k$ for all $x_{-i} \in X_{-i}$, where $\{v_i^k\}_{k=1}^{\infty}$ is a sequence of continuous real-valued functions on X_{-i} such that $v_i^k(x_{-i}) \leq \underline{v}_i(x_{-i})$ and $\liminf_k v_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i})$ for all $k \in \mathbb{N}$, $i \in N$, $x_{-i} \in X_{-i}$ and all sequences $\{x_{-i}^k\}_{k=1}^{\infty}$ converging to x_{-i} (as remarked above, the existence of this sequence follows from Lemma 2 and Reny (1999, Lemma 3.5)). Since f_k is continuous and $f_i^k < \underline{v}_i$ for all $i \in N$, Lemma 1 implies that $\underline{G} = (X_i, \underline{u}_i)_{i \in N}$ has a f_k-equilibrium, x_k, for all $k \in \mathbb{N}$.

Since X is compact, we may assume that $\{x_k\}_{k=1}^{\infty}$ converges. Letting $x^* = \lim_k x_k$, we have that $\liminf_k f_i^k(x_{-i}^k) \geq \underline{v}_i(x_{-i}^*)$ for all $i \in N$ and Lemma 3 implies that x^* is a Nash equilibrium of G. ■

3.2 Corollaries of the general existence result

Theorem 1 allows us to obtain, when players' action spaces are metric, the existence results of Reny (1999, Theorem 3.1), Carmona (2009, Corollary 2) and Barelli and Soza (2009, Corollary 6.8). In all of these cases, we need to find a $\mathbb{R}^n$-function $\underline{u}$ on X such that the game is weakly better reply-secure relative to $\underline{u}$.

Reny's existence theorem establishes the existence of Nash equilibria in better-reply games, which are defined as follows. A game $G = (X_i, u_i)_{i \in N}$ is *better-reply secure* if whenever $(x^*, u^*) \in \Gamma$ and x^* is not a Nash equilibrium, there exists a player $i \in N$, a strategy $\bar{x}_i \in X_i$, an open neighborhood U of x_{-i}^* and a number $\alpha_i > u_i^*$ such that $u_i(\bar{x}_i, x'_{-i}) \geq \alpha_i$ for all $x'_{-i} \in U$.

In order to obtain Reny's theorem, we let, for all $i \in N$ and $x_{-i} \in X_{-i}$, $N(x_{-i})$ denote the set of all open neighborhoods of x_{-i} and define (as Reny did) the function $\underline{u}_i : X \to \mathbb{R}$ by

$$\underline{u}_i(x) = \sup_{U \in N(x_{-i})} \inf_{x'_{-i} \in U} u_i(x_i, x_{-i}). \tag{1}$$

As Reny (1999) has argued, it is immediate that if G is better-reply secure, then G is better-reply closed relative to $\underline{u}$. It is also clear from the definition that $\underline{u}_i \leq u_i$ for all

$i \in N$. Hence, it suffices to show that $\underline{u}_i$ is quasiconcave in the owner's strategy for all $i \in N$ and that players' can secure payoffs using a well-behaved correspondence. In this case, it turns out that each player $i \in N$ can use the constant, single valued correspondence $\varphi_i \equiv \{x_i\}$ to secure $\underline{u}_i(x) - \varepsilon$, for all strategies x and $\varepsilon > 0$.

Corollary 1 (Reny) *If G is compact, quasiconcave and better-reply secure, then it has a Nash equilibrium.*

Proof. Let $\underline{u}$ be defined as in (1). Then, G is is better-reply closed relative to $\underline{u}$. Hence, we are left to show that G is approximately payoff secure.

It is clear that $\underline{u}_i(x) \leq u_i(x)$ for all $i \in N$ and $x \in X$ since for all $U \in N(x_{-i})$, we have that $x_{-i} \in U$ and so $\inf_{x'_{-i} \in U} u_i(x_i, x'_{-i}) \leq u_i(x)$.

We next show that $\underline{u}_i(\cdot, x_{-i})$ is quasiconcave for all $i \in N$ and $x_{-i} \in X_{-i}$. Let $\alpha \in \mathbb{R}$, $x_i, x'_i \in \{z_i \in X_i : \underline{u}_i(z_i, x_{-i}) > \alpha\}$ and $\lambda \in (0, 1)$. Then, there exist U and U' in $N(x_{-i})$ such that $\inf_{z_{-i} \in U} u_i(x_i, z_{-i}) > \alpha$ and $\inf_{z_{-i} \in U'} u_i(x'_i, z_{-i}) > \alpha$.

Define $\bar{x}_i = \lambda x_i + (1 - \lambda)x'_i$ and $\bar{U} = U \cap U'$. It follows that $\bar{U}$ is an open neighborhood of x_{-i}. Let $z_{-i} \in \bar{U}$. The quasiconcavity of $u_i(\cdot, z_{-i})$ implies that

$$u_i(\bar{x}_i, z_{-i}) \geq \min\{u_i(x_i, z_{-i}), u_i(x'_i, z_{-i})\} \geq \min\left\{\inf_{y_{-i} \in U} u_i(x_i, y_{-i}), \inf_{y_{-i} \in U'} u_i(x_i, y_{-i})\right\} > \alpha.$$

Hence, $\underline{u}_i(\bar{x}_i, x_{-i}) > \alpha$ and so $\bar{x}_i = \lambda x_i + (1 - \lambda)x'_i \in \{z_i \in X_i : \underline{u}_i(z_i, x_{-i}) > \alpha\}$, proving that $\underline{u}_i(\cdot, x_{-i})$ is quasiconcave.

Finally, we show that $\underline{G}$ is both upper and lower payoff secure. Let $i \in N$, $\varepsilon > 0$ and $x \in X$. Then, there exists $U \in N(x_{-i})$ such that $\inf_{z_{-i} \in U} u_i(x_i, z_{-i}) > \underline{u}_i(x) - \varepsilon$. Let $\varphi_i : U \rightrightarrows X_i$ be defined by $\varphi_i(x'_{-i}) = \{x_i\}$ for all $x'_{-i} \in U$. Then, for all $x' \in \text{graph}(\varphi_i)$ we have that $x'_i = x_i$, $x'_{-i} \in U$ and so $U \in N(x'_{-i})$, which implies that $\underline{u}_i(x') = \underline{u}_i(x_i, x'_{-i}) \geq \inf_{z_{-i} \in U} u_i(x_i, z_{-i}) > \underline{u}_i(x) - \varepsilon$. ∎

Barelli and Soza (2009) have extended Reny's Theorem by considering generalized better-reply secure games. A game $G = (X_i, u_i)_{i \in N}$ is *generalized better-reply secure* if whenever $(x^*, u^*) \in \Gamma$ and x^* is not a Nash equilibrium, there exists a player $i \in N$, an open neighborhood U of x^*_{-i}, a nonempty, convex, closed valued, upper hemicontinuous correspondence $\varphi_i : U \rightrightarrows X_i$ and a number $\alpha_i > u^*_i$ such that $u_i(x') \geq \alpha_i$ for all $x' \in \text{graph}(\varphi_i)$.

The result of Barelli and Soza (2009) can be obtained from Theorem 1 by considering the following function $\underline{u}$. For all $i \in N$, $x \in X$ and $U \in \mathcal{N}(x_{-i})$, let $W_U(x)$ be the set of all nonempty, convex, closed valued upper hemicontinuous correspondences $\varphi_i : U \rightrightarrows X_i$ satisfying $x \in \text{graph}(\varphi_i)$. Then, for all $i \in N$, define $\underline{u}_i : X \to \mathbb{R}$ by

$$\underline{u}_i(x) = \sup_{U \in \mathcal{N}(x_{-i})} \sup_{\varphi_i \in W_U(x)} \inf_{z \in \text{graph}(\varphi_i)} u_i(z). \tag{2}$$

The idea of generalized better-reply security is that, instead of being forced to use a particular strategy, players are allowed to use a well-behaved correspondence to secure payoffs. This generalization of better-reply security can be seen by noting that the function $\underline{u}$ defined in (2) coincides with the one defined in (1) when the set $W_U(x)$ is replaced by the singleton set consisting of the constant, single valued correspondence $\varphi_i \equiv \{x_i\}$. Thus, the proof that a generalized better-reply secure game $G = (X_i, u_i)_{i \in N}$ is weakly better-reply secure relative to $\underline{u}$ is similar to that of Corollary 1 and requires only to replace constant, single valued correspondences with upper hemicontinuous correspondences having nonempty, convex and closed values.

Corollary 2 (Barelli and Soza) *If G is compact, quasiconcave and generalized better-reply secure, then it has a Nash equilibrium.*

Proof. Let $\underline{u}$ be defined as in (2). We first show that G is better-reply closed relative to $\underline{u}$. Let $(x^*, u^*) \in \Gamma$ be such that $\underline{v}_i(x^*_{-i}) \leq u^*_i$ for all $i \in N$, and, in order to reach a contradiction, suppose that x^* is not a Nash equilibrium of G. By generalized better-reply security, there exist $i \in N$, $U \in \mathcal{N}(x^*_{-i})$, a nonempty, convex, closed valued, upper hemicontinuous correspondence $\varphi_i : U \rightrightarrows X_i$ and $\alpha_i > u^*_i$ such that $u_i(z) \geq \alpha_i$ for all $z \in \text{graph}(\varphi_i)$. Let $x_i \in \varphi_i(x^*_{-i})$. Then, $\varphi_i \in W_U(x_i, x^*_{-i})$ and so $\underline{v}_i(x^*_{-i}) \geq \underline{u}_i(x_i, x^*_{-i}) \geq \inf_{z \in \text{graph}(\varphi_i)} u_i(z) \geq \alpha_i > u^*_i$, a contradiction. Hence, x^* is a Nash equilibrium of G.

We next show that G is approximately payoff secure relative to $\underline{u}$. First, note that $u_i(x) \geq \underline{u}_i(x)$ for all $i \in N$ and $x \in X$. In fact, for all $U \in \mathcal{N}(x_{-i})$ and $\varphi_i \in W_U(x_i, x_{-i})$, we have that $x \in \text{graph}(\varphi_i)$. Thus, $\inf_{z \in \text{graph}(\varphi_i)} u_i(z) \leq u_i(x)$ and so $\underline{u}_i(x) \leq u_i(x)$.

Next, we show that $\underline{u}_i(\cdot, x_{-i})$ is quasiconcave for all $i \in N$ and $x_{-i} \in X_{-i}$. Let $\alpha \in \mathbb{R}$, $x_i, x_i' \in \{z_i \in X_i : \underline{u}_i(z_i, x_{-i}) > \alpha\}$ and $\lambda \in (0, 1)$. Then, there exist $U, U' \in N(x_{-i})$, $\varphi_i \in W_U(x_i, x_{-i})$ and $\varphi_i' \in W_{U'}(x_i', x_{-i})$ such that $\inf_{z \in \text{graph}(\varphi_i)} u_i(z) > \alpha$ and $\inf_{z \in \text{graph}(\varphi_i')} u_i(z) > \alpha$.

Define $\bar{x}_i = \lambda x_i + (1-\lambda)x_i'$, $\bar{U} = U \cap U'$ and $\bar{\varphi}_i = \lambda\varphi_i + (1-\lambda)\varphi_i'$ in $\bar{U}$. It follows that $\bar{U}$ is an open neighborhood of x_{-i} and $\bar{x}_i \in \bar{\varphi}_i(x_{-i})$. Furthermore, $\bar{\varphi}_i$ is well-defined: it is clearly nonempty and convex valued and it follows by Aliprantis and Border (1999, Theorem 17.32, p. 571) that it is compact-valued and upper hemicontinuous. Hence, $\bar{U} \in N(x_{-i})$ and $\bar{\varphi}_i \in W_{\bar{U}}(\bar{x}_i, x_{-i})$.

Let $\bar{z} \in \text{graph}(\bar{\varphi}_i)$. Then, there exist $z_i \in \varphi_i(\bar{z}_{-i})$ and $z_i' \in \varphi_i'(\bar{z}_{-i})$ such that $\bar{z}_i = \lambda z_i + (1-\lambda)z_i'$. The quasiconcavity of $u_i(\cdot, \bar{z}_{-i})$ implies that

$$u_i(\bar{z}) \geq \min\{u_i(z_i, \bar{z}_{-i}), u_i(z_i', \bar{z}_{-i})\} \geq \min\left\{\inf_{y \in \text{graph}(\varphi_i)} u_i(y), \inf_{y \in \text{graph}(\varphi_i')} u_i(y)\right\} > \alpha.$$

Hence, $\underline{u}_i(\bar{x}_i, x_{-i}) \geq \inf_{y \in \text{graph}(\bar{\varphi}_i)} u_i(y) > \alpha$. Thus, $\bar{x}_i = \lambda x_i + (1-\lambda)x_i' \in \{z_i \in X_i : \underline{u}_i(z_i, x_{-i}) > \alpha\}$, proving that $\underline{u}_i(\cdot, x_{-i})$ is quasiconcave.

Finally, we show that $\underline{G}$ is upper payoff secure. Let $i \in N$, $\varepsilon > 0$ and $x \in X$. Then, there exists $U \in N(x_{-i})$ and $\varphi_i \in W_U(x)$ such that $\inf_{z \in \text{graph}(\varphi_i)} u_i(z) > \underline{u}_i(x) - \varepsilon$. Then, for all $x' \in \text{graph}(\varphi_i)$ we have that $x_{-i}' \in U$ and $x_i' \in \varphi_i(x_{-i}')$, that is, $U \in N(x_{-i}')$ and $\varphi_i \in W_U(x')$. Thus, $\underline{u}_i(x') \geq \inf_{z \in \text{graph}(\varphi_i)} u_i(z) > \underline{u}_i(x) - \varepsilon$. ■

Carmona (2009) established an existence result independent of those of Reny (1999) and Barelli and Soza (2009), which is valid for weakly upper semicontinuous and weakly payoff secure games. Formally, a game $G = (X_i, u_i)_{i \in N}$ is *weakly payoff secure* if v_i is lower semicontinuous for all $i \in N$. Furthermore, G is *weakly upper semicontinuous* if, defining $\bar{u} : X \times X \to \mathbb{R}^n$ by $\bar{u}(x, y) = (u_1(x_1, y_{-1}), \ldots, u_n(x_n, y_{-n}))$, then for all (x, y, α) in the frontier of the graph of $\bar{u}$, there exists $i \in N$ and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, y_{-i}) > \alpha_i$.

The result in Carmona (2009) is obtained from Theorem 1 by using $\underline{u} = u$. In fact, in compact, quasiconcave, weakly upper semicontinuous and weakly payoff secure games, the best-reply correspondence is upper hemicontinuous with nonempty, convex and closed values and it can used to secure players' payoffs.

Corollary 3 (Carmona) *If G is compact, quasiconcave, weakly upper semicontinuous and weakly payoff secure, then it has a Nash equilibrium.*

Proof. We first show that G is better-reply closed relative to u. Let $(x^*, u^*) \in \Gamma$ be such that $v_i(x^*_{-i}) \leq u^*_i$ for all $i \in N$. We claim that $u^*_i = u^*_i(x)$ for all $i \in N$. Indeed, (x^*, u^*_i) belongs to the closure of the graph of u_i and if $(x^*, u^*_i) \notin \text{graph}(u_i)$, then there exists $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x^*_{-i}) > u^*_i$ (Carmona (2009, Theorem 1)). But then, $v_i(x^*_{-i}) \geq u_i(\hat{x}_i, x^*_{-i}) > u^*_i \geq v_i(x^*_{-i})$, a contradiction. Thus, $u^*_i(x) = u^*_i \geq v_i(x^*_{-i})$ for all $i \in N$ and x^* is a Nash equilibrium.

Finally, we show that G is upper payoff secure. Let $i \in N$, $\varepsilon > 0$ and $x \in X$. Let $BR_i : X_{-i} \rightrightarrows X_i$ be defined by $BR_i(x_{-i}) = \{x_i \in X_i : u_i(x_i, x_{-i}) = v_i(x_{-i})\}$ for all $x_{-i} \in X_{-i}$ and let $\hat{x}_i \in BR_i(x_{-i})$. Hence, $u_i(\hat{x}_i, x_{-i}) = v_i(x_{-i}) > u_i(x) - \varepsilon/2$. Since v_i is lower semicontinuous, let $U \in N(x_{-i})$ be such that $v_i(x'_{-i}) > v_i(x_{-i}) - \varepsilon/2$ and define $\varphi_i(x'_{-i}) = BR_i(x'_{-i})$ for all $x'_{-i} \in U$. Then, φ_i is a upper hemicontinuous correspondence with nonempty, convex, closed values (Carmona (2009, Theorem 1)) and, for all $x' \in \text{graph}(\varphi_i)$, we have that $u_i(x') = v_i(x'_{-i}) > v_i(x_{-i}) - \varepsilon/2 > u_i(x) - \varepsilon$. ■

The case of weakly payoff secure and weakly upper semicontinuous games is particularly simple because there is no need to approximate the original game, i.e., we can have $\underline{u} = u$. Corollary 4 below shows that the same is true when G is either upper or lower payoff secure and weakly reciprocally upper semicontinuous. Recall that a game $G = (X_i, u_i)_{i \in N}$ is *weakly reciprocal upper semicontinuous* if for all (x, α) in the frontier of the graph of u, there exists $i \in N$ and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > u_i$ (this notion was introduced by Bagh and Jofre (2006)).

Corollary 4 (Bagh and Jofre, Barelli and Soza) *If $G = (X_i, u_i)_{i \in N}$ is compact, quasiconcave, upper or lower payoff secure and weakly reciprocally upper semicontinuous, then G has a Nash equilibrium.*

Proof. We show that G is weakly better-reply secure relative to u. Clearly, it suffices to show that G is better-reply closed relative to u.

Let $(x^*, u^*) \in \Gamma$ be such that $u_i^* \geq v_i(x_{-i}^*)$ for all $i \in N$. If $(x^*, u^*) \notin \text{graph}(u)$, then weak reciprocal upper semicontinuity implies that there exists $i \in N$ and $\hat{x}_i \in X_i$ such that $v_i(x_{-i}^*) \geq u_i(\hat{x}_i, x_{-i}^*) > u_i^* \geq v_i(x_{-i}^*)$, a contradiction. It follows that $(x^*, u^*) \in \text{graph}(u)$, implying that $u_i(x^*) = u_i^* \geq v_i(x_{-i}^*)$ for all $i \in N$. Hence, x^* is a Nash equilibrium of G. ■

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