Ambiguity and rational expectations equilibria*

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Abstract

This paper demonstrates the existence and robustness of partially-revealing rational expectations equilibria (REE) when this equilibrium concept is expanded to allow for some traders to have ambiguity averse preferences. Further, the generic existence of fully-revealing REE is proven for a commonly-used subset of the class of ambiguity averse preferences. This finding illustrates that models with ambiguity aversion provide a relatively tractable framework through which partial information revelation may be studied in a general equilibrium setting without relying on particular distributional or von Neumann-Morgenstern utility assumptions or the presence of 'noise'.

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1 Introduction

Market prices convey information. In every market equilibrium, positive prices at least inform participants that there is some demand for all goods. In many markets, prices convey much more information, including market participants' private information about relevant economic variables. The concept of a rational expectations equilibrium (REE) formalized in Radner (1979) was formulated to study the dissemination of privately held information through market prices. One of the stark results of the pursuant literature (e.g. Radner (1979), Allen (1981), Allen (1982)) is that when the dimension of the space of private information is less than that of the space of prices in an exchange economy and if the market has no 'noise' (see for example Grossman and Stiglitz (1980)), equilibrium market prices almost surely reveal all agents' private information.[1]

The aim of this paper is to demonstrate that partial revelation can be a robust property of REE in standard heterogeneous information exchange economies when the class of preferences for market participants is expanded beyond the standard Savage (1954) subjective expected utility (SEU) model. Specifically, we demonstrate that when at least one investor has ambiguity averse (AA) preferences of the type discussed in Gilboa and Schmeidler (1989) or Schmeidler (1989), then the concept of REE permits robust equilibria that are partially revealing even in the 'lower-dimensional' case studied in Radner (1979). We explicitly construct partially-revealing REE for a set of economies that has positive Lebesgue measure in the space of economies.

Models with AA traders have proved useful in studying a variety of economic

[1]In the case of Allen (1981), the dimension of the space of private information must be half that of the space of prices.

phenomena. Epstein and Schneider (2008) find that ambiguous information may lead to assymetric portfolio reactions, equity premia dependent on idiosyncratic risk and skewness in returns, and persistent effects on prices. Other work on models with AA investors includes, inter alia, intertemporal asset pricing (Epstein and Wang 1994), incompleteness of financial markets (Mukerji and Tallon 2001), equity premia (Chen and Epstein 2002, Maenhout 2004, Ui 2008), portfolio home-bias (Epstein and Miao 2003, Uppal and Wang 2003), and limited market participation (Cao, Wang, and Zhang 2005, Ui 2008). This paper demonstrates that in addition to the pricing and portfolio properties discussed in the studies just cited, models with AA traders can also effect informational efficiency of prices.

This work differs from previous papers on the existence of partially-revealing REE. Prior work by Ausubel (1990), Pietra and Siconolfi (2008), Heifetz and Polemarchakis (1998), and Jordan (1982) require that there be more dimensions of uncertainty than the prices, i.e. are in the 'higher-dimensional' setting, and that utilities be state-dependent. In particular, Ausubel (1990) and Pietra and Siconolfi (2008) show that partial revelation can be robust in the classes of economies they consider. Citanna and Villanacci (2000), Polemarchakis and Siconolfi (1993), and Rahi (1995) permit only nominal assets, while Pietra and Siconolfi (1997) introduce extrinsic uncertainty to get non-revealing REE. Grossman and Stiglitz (1980) and Mailath and Sandroni (2003) impose specific structure on the von Neumann-Morgenstern utility and the uncertainty and make use of 'noise' traders – whose preferences are not explicitly modeled – to provide examples of partially-revealing REE.[2]

[2]Diamond and Verrecchia (1981) and Ganguli and Yang (2008) construct partially-revealing REE in the framework of Grossman and Stiglitz (1980) without the presence of noise traders. See also Dow and Gorton (2008) for a recent discussion on 'noise' traders.

Allen and Jordan (1998) provide an excellent survey of the literature on REE. In particular, they summarize results for the generic existence of fully-revealing REE in smooth economies satisfying additional dimensionality assumptions and discuss the possibilities for partial revelation within the SEU framework.

In the model presented here we work within the lower-dimensional setting, closely following that of Radner (1979). No assumptions beyond those necessary to ensure the existence of Arrow-Debreu equilibrium, after the realization of each signal, are placed on the preferences of the investors. The preferences of all investors are specified ex ante and each of these investors is rational in the sense that each makes optimal decisions given well-defined preferences and the constraints faced.[3] Indeed our method of constructing partially-revealing REE can be applied irrespective of the dimensional setting.

While we focus on the multiple-priors model developed by Gilboa and Schmeidler (1989), the methods we use to construct partially-revealing REE can be extended to other models of decision-making. Convex preferences represented by the Choquet expected utility model of Schmeidler (1989) or represented by the α-maxmin model of Ghirardato, Maccheroni, and Marinacci (2004) or convex preferences that exhibit first-order risk aversion, as developed by Segal and Spivak (1990), such as the rank-dependent probabilities model of Quiggin (1982) are obvious candidates.

Given the recent work on smooth models of decision-making under ambiguity by Klibanoff, Mukerji, and Marinacci (2005) and Maccheroni, Marinacci, and Rustichini (2006), a natural question arises as to the choice between these and the non-differentiable models of Gilboa and Schmeidler (1989), Ghirardato, Maccheroni, and

[3]See Blume and Easley (2008) and Gilboa, Postlewaite, and Schmeidler (2007) for insightful discussions on rationality.

Marinacci (2004), and Schmeidler (1989) in studying decision-making under ambiguity. The results of this paper indicate that the choice of preference representation can have consequences for the informational properties of prices.

Furthermore, there is evidence that non-differentiable ambiguity aversion is an appropriate preference representation for some decision-makers. Ahn, Choi, Kariv, and Gale (2007) report a laboratory portolio choice experiment which allows them to estimate the smooth and kinked models at the level of individual subjects. They find that there is a strong tendency to equate demands for securities that pay off in ambiguous states, a feature that is better accomodated by the kinked models than the smooth ones of decision-making under ambiguity. Bossaerts, Ghirardato, Guarneschelli, and Zame (2007) study the effects of heterogeneous attitudes toward ambiguity in competitive financial markets and provide experimental evidence which suggests that the kinked model of Ghirardato, Maccheroni, and Marinacci (2004) is a good description of decision-making under ambiguity.

In related work, Ozsoylev and Werner (2007) introduce an AA informed trader into the Grossman and Stiglitz (1980) set up and they achieve partial revelation without 'noise' in a manner analogous to the more general results here. Tallon (1998) shows by example that an AA uninformed trader would buy 'redundant' private information even when the REE price is fully-revealing.

The remainder of the paper proceeds as follows. We first present a leading parametric example in section 2. Section 3 outlines the model and the definition of an REE for the general model. Section 4 contains the main results on robust partial revelation for the general model and section 5 contains results on the generic existence of fully-revealing REE for a sub-class of AA preferences. Section 6 concludes. Proofs for ancillary results are contained in the appendix.

2 A leading parametric example

Two investors, labeled A and E, trade Arrow securities (state-contingent consumption) in a complete-market economy with payoff states $\{H, L\}$. The aggregate endowment of the securities is denoted $e = (e(H), e(L))$, with $e(H) = 2.2$ and $e(L) = 1.8$, i.e. L implies a low market outcome, and we assume investor $n \in \{A, E\}$ is endowed with $e^n = \frac{1}{2}e$. Both investors have logarithmic von Neumann-Morgenstern utility over consumption.

Investor A receives a signal s from the set $\{s_1, s_2, s_3\}$. These signals convey information to the market participants about the likelihood of state L being realized. When signal $s_i, i = 1, 2, 3$, is received, the 'true' conditional probability of state L being realized is $\rho(L|s_i)$. Assume that $\rho(L|s_1) = 0.8$, $\rho(L|s_2) = 0.5$, and $\rho(L|s_3) = 0.2$. Formally, ρ denotes the probability measure over $\{s_1, s_2, s_3\} \times \{H, L\}$ governing the resolution of uncertainty. We will use the qualifier *signal-contingent* to refer to the setting which describes trading after a signal has been received.

Investor A behaves as if he percieves ambiguity in the information that is conveyed by the signal s and is averse to this ambiguity.[4] In particular, for each s_i, he chooses consumption as if he wants to maximize the minimum expected utility over a closed, convex set of probability distributions $\gamma^A(s_i)$ over $\{H, L\}$. Thus, if signal s_i has been

[4]See for instance Epstein and Schneider (2008).

received, then A has preferences over consumption $x \in \mathbb{R}^2_+$ represented by

$$U^A(x) = \min_{\hat{\pi} \in \gamma^A(s_i)} \mathbb{E}_{\hat{\pi}}[ln(x)].^5 \tag{2.1}$$

We assume that under A's given signal s_1, the probability of state L ocurring $\hat{\pi}(L) \in [0.7, 0.9]$, i.e. $\gamma^A(L|s_1) = [0.7, 0.9]$ and similarly, beliefs when s_2 and s_3 have been received are $\gamma^A(L|s_2) = [0.4, 0.8]$ and $\gamma^A(L|s_3) = 0.2$, respectively. We postpone discussion of A's beliefs over $\{s_1, s_2, s_3\}$; these are not immediately relevant since A only makes decisions after receiving a signal.

Investor E receives no private signal about the asset market, but can infer information about market payoffs by observing market prices. Investor E does not perceive ambiguity in the signals and if prices reveal to her that the signal received is s_i then she assigns probability $\rho(L|s_i)$ to L ocurring, i.e. her beliefs coincide with objective conditional probability distrubtion ρ. Investor E believes that each of the three signals s_i has a $1/3$ chance of being realized. If prices do not reveal the realized signal, then she uses Bayes rule to update her beliefs over $\{H, L\}$. In other words, E is an SEU maximizer.

Using $\hat{\pi}^E$ to denote E's beliefs over $\{H, L\}$, her preferences over consumption $x \in \mathbb{R}^2_+$ are represented by

$$U^E(x) = \mathbb{E}_{\hat{\pi}^E}[ln(x)]. \tag{2.2}$$

The prices of the securities will be normalized so that they sum to one for each signal.[6] Although investor n's endowment is not signal contingent, her budget con-

[5] In this representation, A's attitude of aversion toward ambiguity can be formalized using Gajdos, Hayashi, Tallon, and Vergnaud (2008).

[6] Prices in this example can be shown to be strictly positive.

straint depends on prices $p = (p(L), p(H))$, which are signal contingent. The feasible set under this constraint is

$$\mathbb{B}(e^n, p) = \{x \in \mathbb{R}^2_+ : p \cdot x \leq p \cdot e^n\}. \tag{2.3}$$

A price function $\phi : \{s_1, s_2, s_3\} \to \mathbb{R}^2_+$, which defines a price p for every signal will be *partially revealing* if $\phi(s_i) = \phi(s_j)$ for some $i \neq j$ and will be *fully revealing* otherwise. An REE comprises a price function ϕ and allocations[7] $(x^A, x^E) \in \mathbb{R}^{12}_+$ for A and E which are feasible and utility-maximizing given beliefs that are consistent with any private information and the public information revealed by prices.

2.1 Partial revelation

Private information that is perceived to be ambiguous may not be revealed by asset prices. Since A acts as if the information conveyed in signals s_1 and s_2 is ambiguous, for a given price there is a set of priors that imply A will fully-insure. Suppose investor E knows only that the signal is in $\{s_1, s_2\}$, i.e. the price function ϕ^{PR} does not differ across the two signals. Assume also that A's beliefs are such that he can hold the same uncertainty-less allocation under both s_1 and s_2 at $\phi^{PR}(s_1) = \phi^{PR}(s_2)$.[8] Then E's and A's demands will not differ across the signals. If we can show that $\phi^{PR}(s_1) = \phi^{PR}(s_2)$ will clear markets, then ϕ^{PR} will be a partially-revealing REE price function. We now proceed to construct such an equilibrium.

Conditional on the signal s being in $\{s_1, s_2\}$, the probability of the state L being

[7] Allocations are an amount of consumption in states H and L for each investor under each signal.

[8] We use the umbrella term 'uncertainty-less' instead of the usual 'riskless' for a full-insurance allocation to accomodate the presence of ambiguity.

realized under E's beliefs is

$$\hat{\pi}^E(L|\{s_1, s_2\}) = \frac{3}{2}\left(\frac{1}{3}0.8 + \frac{1}{3}0.5\right) = 0.65 \tag{2.4}$$

Thus, if prices p do not reveal which of s_1 and s_2 realized, E's demand for the Arrow securities, with wealth $p \cdot e^E$ will be

$$x^E(p) = \left(\frac{0.65(p \cdot e^E)}{p(L)}, \frac{0.35(p \cdot e^E)}{p(H)}\right) \tag{2.5}$$

Investor A on the other hand, observes the private signal. Suppose that signal s_1 has been received. Then A's beliefs about the probability of state L ocurring are given by $\gamma^A(L|s_1) = [0.7, 0.9]$. Therefore, investor A's demand is given by[9]

$$x^A(p) = \begin{cases} \left(\frac{0.7(p \cdot e^A)}{p(L)}, \frac{0.3(p \cdot e^A)}{p(H)}\right) & p(L) < 0.7 \\ (p \cdot e^A, p \cdot e^A) & 0.7 \leq p(L) \leq 0.9 \\ \left(\frac{0.9(p \cdot e^A)}{p(L)}, \frac{0.1(p \cdot e^A)}{p(H)}\right) & p(L) > 0.9. \end{cases} \tag{2.6}$$

Analogously, A's beliefs when s_2 is received are $\gamma^A(L|s_2) = [0.4, 0.8]$. Therefore, A's demand will be

$$x^A(p) = \begin{cases} \left(\frac{0.4(p \cdot e^A)}{p(L)}, \frac{0.6(p \cdot e^A)}{p(H)}\right) & p(L) < 0.4 \\ (p \cdot e^A, p \cdot e^A) & 0.4 < p(L) < 0.8 \\ \left(\frac{0.8(p \cdot e^A)}{p(L)}, \frac{0.2(p \cdot e^A)}{p(H)}\right) & p(L) > 0.8 \end{cases} \tag{2.7}$$

[9]See the appendix for the optimality conditions for an AA trader.

Table 1: Equilibrium prices and market valuation (MV) in partially- and fully-revealing equilibria

<table>
<tr><th rowspan="2">s</th><th colspan="2">Partial Revelation</th><th colspan="2">Full Revelation</th><th rowspan="2">Change in MV</th></tr>
<tr><th>$\phi(L|s)$</th><th>MV</th><th>$\phi(L|s)$</th><th>MV</th></tr>
<tr><td>s_1</td><td rowspan="2">0.732</td><td rowspan="2">1.907</td><td>0.854</td><td>1.859</td><td>2.6%</td></tr>
<tr><td>s_2</td><td>0.598</td><td>1.961</td><td>-2.8%</td></tr>
<tr><td>s_3</td><td>0.234</td><td>2.106</td><td>0.234</td><td>2.106</td><td>–</td></tr>
</table>

Notice that A's demands under signals s_1 and s_2 are the same if price $p(L)$ under both signals is in the intersection of $\gamma^A(L|s_1) = [0.4, 0.8]$ and $\gamma^A(L|s_2) = [0.7, 0.9]$. In other words, if E's demand is such that when she knows only that $s \in \{s_1, s_2\}$, the market-clearing price of consumption in the low state satisfies $0.7 < p(L) < 0.8$, then there will exist a partially-revealing equilibrium where the price function ϕ^{PR} satisfies $\phi^{PR}(L|s_1) = \phi^{PR}(L|s_2) = p(L)$.

For the given parameter values, such a partially-revealing REE exists. Table 1 compares a fully- and partially-revealing equibrium for this numerical example. The fully-revealing equilibrium, i.e., where the price function ϕ^{FR} is injective, is constructed by considering an artificial full-communication economy where E also knows which signal realized. If the equilibrium price for this symmetric information economy under each signal is distinct across the signals, then ϕ^{FR} is constructed by setting $\phi^{FR}(s_i)$ equal to the equilibrium price for the full communication economy under s_i. For this example, it is then easy to verify that this price function and the corresponding full-communication economy allocations are a fully-revealing REE.[10]

Note that the partial revelation result does not rely on indeterminacy of equilibrium prices. Indeed the presence of E ensures that the price for each signal-contingent

[10] See also Radner (1979), p661-665 for an example and description.

trade is uniquely determined. As mentioned earlier, the partial-revelation construction uses the fact that for A, the full-insurance allocation may be optimal at a given price for different (sets of) beliefs. This of course would not be true for E.

The final column of table 1 also compares the aggregate market value ($p \cdot e$), denoted MV, under the partially-revealing and fully-revealing equilibria. It shows that under the partially-revealing equilibria, MV when signal s_1 has been received is 2.6% more than it would be if E knew A's privately held information. Likewise, if signal s_2 has been received, but prices do not reveal this to E, then the market will be undervalued by 2.8% relative to its value if E knew A's signal. These price differences occur because when E doesn't know whether signal s_1 or s_2 has been received, updating by Bayes rule implies that she will underweight the probability of the low state when signal s_1 has been received. Therefore, the cost of an asset that pays the market payoff in each state will increase relative to its cost when it is known that s_1 has been received because such an asset is believed to be more likely to pay a high amount. Similar reasoning applies to the case when s_2 has been received but E does not know whether s_1 or s_2 has been received.

It is also interesting to note that although this example has only two types of investors, the price differences due to partial revelation are of the type that need not disappear as the number of investors increases. Knowing the information held by a particular investor in this model generally will change all other investors' behavior and as such the qualititative properties of this equilibrium are not driven by the number of types of investors.

Table 2 compares allocations under the two REE. First, recall that neither of the endowments in this example fall on the full-insurance line. Hence, investors A and E

will trade to reach the equilibrium presented in the table. While the preferences of investor A imply that he will hold an uncertainty-less consumption allocation over a range of prices, none of the results in the paper rely on the AA investor having an uncertainty-less endowment.

Notice further that in this example A would hold a uncertainty-less allocation even if E knew A's information. However, the cost of this uncertainty-less allocation is very different under s_1 and s_2. When bad news has been received (signal s_1), the probability of a bad outcome is high. As such, consumption in the low state would be expensive and investor A would only be able to afford the constant consumption amount 0.929. Under the better signal (s_2), the price of consumption in the low state is lower, so investor A can affort more total consumption. In particular, investor A would be able to purchase the constant consumption 0.980. In the partially-revealing equilibrium, the consumption of investor A is between these two amounts.

Another interesting comparison is between the economy given here and one in which both investors are SEU decision-makers maximizers with common beliefs, but only one investor receives a signal. In this environment, private signals will generically be revealed through prices. In such a model, the consumption of both investors would be $(x(L), x(H)) = (0.9, 1.1)$ under any signal. The introduction of information that is perceived to be ambiguous by invester A causes the uninformed investor E to hold a consumption profile that is riskier (i.e. varies more across states) than she otherwise would hold. This arises because the introduction of ambiguous information causes investor A to fully-insure across states.

Table 2: Equilibrium allocations in partially- and fully-revealing equilibria

s	Partial Revelation		Full Revelation	
	$(x^A(L), x^A(H))$	$(x^E(L), x^E(H))$	$(x^A(L), x^A(H))$	$(x^E(L), x^E(H))$
s_1	(0.954,0.954)	(0.846,1.246)	(0.929,0.929)	(0.871,1.271)
s_2			(0.980,0.980)	(0.820,1.220)
s_3	(0.900,1.100)	(0.900,1.100)	(0.900,1.100)	(0.900,1.100)

2.2 Robustness

The partial revelation property in this example is robust in the space of (conditional) beliefs over $\{H, L\}$, which parametrize the space of economies.[11] Perturbing the beliefs of A and E shows that the conditions for partially-revealing REE to exist are satisfied for a non-empty and open set of parameters describing beliefs. In this finite-dimensional space, this set will have positive Lebesgue measure.

For any closed, convex set $\bar{\gamma}^A(s_i)$ of measures over $\{H, L\}$ given signal s_i, let $[\bar{\pi}_l(s_i), \bar{\pi}_u(s_i)] = \bar{\gamma}^A(L|s_i)$ denote the interval representing the probability of L occurring. For the purposes of this example, it suffices to work with the interval $\bar{\gamma}^A(L|s_i)$ for each signal s_i in considering perturbed beliefs of investor A.

For any economy with the endowments given above, if markets clear with A holding a uncertainty-less portfolio for any signal, the price $\bar{p} = (\bar{p}(L), \bar{p}(H))$ has to satisfy

$$\bar{p} \cdot e^A + \frac{\bar{\pi}^E(L)(\bar{p} \cdot e^E)}{\bar{p}(L)} = e(L) = 1.8 \tag{2.8}$$

and $\bar{\pi}_l(\cdot) \leq \bar{p}(L) \leq \bar{\pi}_u(\cdot)$, where E's belief is generically denoted $\bar{\pi}^E(L)$. Since the

[11] Radner (1979) established that full revelation is generic in the class of economies described by conditional beliefs over payoff states with only SEU maximizers in the market.

prices sum to one, this requirement simplifies to

$$0.2(\bar{p}(L))^2 + (0.7 + 0.2\bar{\pi}^E(L))\bar{p}(L) - 1.1\bar{\pi}^E(L) = 0. \tag{2.9}$$

This quadratic equation has a root between zero and one which implies that the price $p(L)$ is

$$\bar{p}(L) = \frac{((0.7 + 0.2\bar{\pi}^E(L))^2 + 0.88)^{\frac{1}{2}} - (0.7 + 0.2\bar{\pi}^E(L))}{0.4} \tag{2.10}$$

The solution, $\bar{p}(L)$ is continuous in $\bar{\pi}^E(L), \bar{\pi}_l(s_i)$ and $\bar{\pi}_u(s_i)$. So, for small enough perturbations of the parameters $\bar{\pi}^E(L)(s_i)$, $\bar{\pi}_l(s_i)$, and $\bar{\pi}_u(s_i)$, $i = 1, 2, 3$, around the values given before, the inclusion condition $-$ $\bar{\pi}_l(s_i) < \bar{p}(L) < \hat{\pi}_u(s_i)$ for all $i -$ holds for the solution $\bar{p}(L)$. This in turn provides a partially-revealing equilibrium price for each perturbation. Hence, the partial revelation property holds for an open set of parameters around the set originally given.

2.3 The set of prior beliefs

The previous section demonstrated that in the space of investors' conditional beliefs, partially-revealing REE exist for an open subset of beliefs. We now show that the result holds in terms of the space of prior beliefs, i.e., beliefs over $\{s_1, s_2, s_3\} \times \{H, L\}$.

Consider first the beliefs of the SEU investor E. Beliefs over $\{s_1, s_2, s_3\} \times \{H, L\}$

for this investor are

$$
\begin{array}{c|c|c|}
 & L & H \\
\hline
s_1 & 8/30 & 2/30 \\
\hline
s_2 & 5/30 & 5/30 \\
\hline
s_3 & 2/30 & 8/30 \\
\hline
\end{array}
\tag{2.11}
$$

These are calculated directly from the marginal over signals and the conditional beliefs over state H and L.

Under the assumption that investor A updates each element of the set of priors using Bayes' Rule[12], a set of beliefs over $\{s_1, s_2, s_3\} \times \{H, L\}$ that is consistent with the conditional sets given earlier is the set for which $\pi^A(s_1, L) \in [7/30, 9/30]$, $\pi^A(s_2, L) \in [4/30, 8/30]$ and $\pi^A(s_3, L) = 2/30$. Notice that the beliefs of SEU investor E are in this set.

In this space of prior beliefs one could then consider whether the partial revelation that occurs in this example remains upon perturbing beliefs. As an updating rule, Bayes' rule is a function that maps from the space of prior beliefs into conditional beliefs for each signal. Bayes' rule is a continuous function, so for any neighborhood B_{con} of conditional beliefs for a particular signal there exists a neighborhood B_{prior} of the current prior for which each prior belief in B_{prior} will lead to an updated belief that is in B_{con}. This idea has an analogue for investor A since each belief in the set of priors is updated by Bayes' rule.

The condition that allows partial revelation in this example is that $p(L) \in \gamma^A(L|s_1) \cap \gamma^A(L|s_2)$. This condition for the general model is stated in equation (4.3) but its interpretation can be seen here. Partial revelation will occur if the portfolio of investor

[12] Updating ambiguous beliefs is the topic of interesting ongoing research. Epstein and Schneider (2007) provides an explication of the issues involved.

A doesn't change across the two signals.[13] Since (as will be shown) this condition is robust to small perturbations of the parameters that define conditional beliefs and Bayes' rule implies that conditional beliefs are continuous in prior beliefs, this condition will also be robust to perturbations of the parameters describing prior beliefs.

Finally a note on the generality of this example. The body of the paper will show that although we have chosen numbers for concreteness in this example, the most significant assumption made in the example is that the investors have von Neumann-Morgenstern utility that is logarithmic. This assumption allows demand correspondences to be written concisely. The only properties needed in the more general analysis are that preferences be concave, von-Neumann Morgenstern utility functions be continuously differentiable and that they satisfy an Inada condition at 0.

3 The model

The market is populated by a finite set $\mathcal{N} = \{1, \ldots, n, \ldots, N\}$ of investors who live for 2 periods labeled 1 and 2. At the end of period 2, one of a finite set Ω of possible states of nature, denoted ω, is realized and investors in the economy consume.

In period 1, some information is revealed to each investor. Investor $n \in \mathcal{N}$ receives a private signal s^n from a finite set $\mathcal{S}^n = \{s_1^n, \ldots, s_S^n\}$. Let the set of all possible collections of private information that might be available to the market be labeled $\Sigma = \times_{n \in \mathcal{N}} \mathcal{S}^n$ with representative element σ and let $\mathcal{F}$ be the discrete algebra over Σ. The investors' private signals convey information about the likelihood of each outcome $\omega \in \Omega$ in period 2.

[13] Section 4.1 demonstrates that partial revelation can only occur if the portfolio of the AA investor has this property.

Each investor has an endowment $e^n \in \mathbb{R}^{|\Omega|}_{++}$ of the single consumption good and must choose a consumption allocation in $\mathbb{R}^{|\Omega|}_{+}$ for period 2. This allocation is financed by trading state-contingent consumption (Arrow securities) over Ω in the market that opens in period 2.

The market opens at the beginning of period 2 and in equilibrium, each investor derives information about the private signals of other investors by observing the prices of the contingent claims that are traded in the market as described in section 3.1. Let $P \subset \mathbb{R}^{|\Omega|}$ be the space of possible prices over contingent claims that can be purchased at the beginning of period 2. The conditions imposed on preferences and endowments ensure that P may be normalized so that its elements are non-negative and sum to one. This normalization will be assumed throughout the paper.

3.1 Preferences and beliefs

Investors in the market SEU decision-makers or AA decision-makers. The set of SEU maximizing investors is denoted $\mathcal{N}^E$ and has cardinality $N^E \geq 1$ while $\mathcal{N}^A$ denotes the set of AA investors and has cardinality $N^A \geq 1$. Preferences for the AA investors will be described first.

Let $\mathcal{C}(\Delta^{|\Omega|})$ denote the collection of non-empty, convex, closed subsets of $\Delta^{|\Omega|}$. Let $\gamma^n(f) \in \mathcal{C}(\Delta^{|\Omega|})$ denote the collection of probability distributions that AA investor n believes may govern the resolution of uncertainty over Ω when he knows that the joint signal $\sigma \in f$, i.e. it is the information conveyed by f. The tuple $(\gamma^n(f))_{f \in \mathcal{F}}$ is called a *belief system*.

While all AA investors have beliefs over $\Sigma \times \Omega$, we make no assumption about whether investors percieve any ambiguity over Σ. This is because we are concerned with the decisions made by the investors after they have received all possible information (from their private signals and from the prices). Hence, the presence or absence of ambiguity over Σ makes no difference to our results.[14]

For each information set $f \in \mathcal{F}$, $\pi^n(f) \in \Delta^{|\Omega|}$ denotes the updated beliefs of an SEU maximizing investor n if she knows that $\sigma \in f$. A belief system for investor n is given by $(\pi^n(f))_{f \in \mathcal{F}}$. The space of belief systems over Ω for AA investor $n \in \mathcal{N}^A$ is denoted Γ and that of belief systems over Ω for an SEU investor $n \in \mathcal{N}^E$ is denoted Π.

Investors utilize information from their private signal and from prices. Abusing notation, we let $f(s^n) \in \mathcal{F}$ be the set of joint signals σ that have $\sigma(n) = s^n$, where $\sigma(n)$ is the nth component of σ. Each investor n knows by her private signal that $\sigma \in f(s^n)$.

A price function $\phi : \Sigma \to P$ defines a price for every joint signal σ. In equilibrium, information is gathered from prices by using the equilibrium price function ϕ, so if the observed price is p, then $\phi^{-1}(p) \in \mathcal{F}$ is the information revealed by price p to all investors $n \in \mathcal{N}$. Combining the information derived from her personal signal and that inferred from prices, investor n in equilibrium has information $f(s^n) \cap \phi^{-1}(p)$, i.e. beliefs $\gamma^n(f(s^n) \cap \phi^{-1}(p))$ for AA investor $n \in \mathcal{N}^A$ (respectively, beliefs $\pi^n(f(s^n) \cap \phi^{-1}(p))$ for SEU investor $n \in \mathcal{N}^E$).

We make the following assumption about the preferences of the investors in the economy.

Assumption 1. *Given any $f \in \mathcal{F}$,*

[14] However, if one is interested in examining the decisions of investors before and after receiving private signal and price information then introducing ambiguity over Σ may be interesting as it would lead to questions of how beliefs are updated and whether decisions are dynamically consistent.

1. *Investor* $n \in \mathcal{N}^A$ *has preferences over* $x^n \in \mathbb{R}_+^{|\Omega|}$ *that are represented by the utility function*

$$U^n(x^n; f) = \min_{\hat{\pi} \in \gamma^n(f)} \mathbb{E}_{\hat{\pi}}[u^n(x^n)] \tag{3.1}$$

with $\gamma^n(f) \in \mathcal{C}(\Delta^{|\Omega|})$ *and* $\hat{\pi} >> 0$ *for all* $\hat{\pi} \in \gamma^n(f)$.

2. *Investor* $n \in \mathcal{N}^E$ *has preferences over* $x^n \in \mathbb{R}_+^{|\Omega|}$ *that are represented by the utility function*

$$U^n(x^n; f) = \mathbb{E}_{\pi^n(f)}[u^n(x^n)], \tag{3.2}$$

with $\pi^n(f) \in \Delta^{|\Omega|}$ *and* $\pi^n(f) >> 0$.

3. *For all* $n \in \mathcal{N}$, *the von Neumann-Morgenstern utility function* $u^n(\cdot)$ *satisfies* $u^n \in C^2, u'^n(\cdot) > 0, u''^n(\cdot) < 0$, *and* $\lim_{x \to 0} u'^n(x) = \infty$.

The aversion to ambiguity of the investor $n \in \mathcal{N}^A$ in this Gilboa and Schmeidler (1989) representation can be formalized using the results of Gajdos, Hayashi, Tallon, and Vergnaud (2008). It is also useful to note that the representation of AA investor n's preferences includes as a special case the situation in which n is an SEU maximizer.

3.2 Equilibrium

For any price vector $p \in P$, the set of feasible state-contingent consumption bundles or portfolios, called *the budget set* of investor n is

$$\mathbb{B}(e^n, p) = \{x \in \mathbb{R}_+^{|\Omega|} : p(e^n - x^n) \geq 0\}. \tag{3.3}$$

With this notation at hand, we can now define the equilibrium notion of interest.

Definition 1. *A pair* (x, ϕ)*, where* $\phi : \Sigma \to P$ *is a price function and* $x : \Sigma \to \mathbb{R}_+^{N|\Omega|}$ *is an allocation, is a rational expectations equilibrium (REE) if for all* n *and* σ*,* (x, ϕ) *satisfies*

1. $x^n(\sigma) \in \arg\max U^n(x^n(\sigma); f(\sigma(n)) \cap \phi^{-1}(\phi(\sigma)))$ *s.t.* $x^n \in \mathbb{B}(e^n, \phi(\sigma))$

2. $\sum_{n \in \mathcal{N}} (e^n(\sigma) - x^n(\sigma)) = 0$

Definition 2. *An REE price function* ϕ *is said to be fully-revealing if it is injective. It is said to be partially-revealing if it is not fully-revealing.*[15] *An REE is called fully-revealing if the corresponding price function is fully-revealing and is called partially-revealing otherwise.*

Following Radner (1979), we parametrize the space of economies by $\Gamma^{N^A} \times \Pi^{N^E}$, the space of belief systems of the investors over Ω. Also, note that $dim(\Sigma) < dim(P)$, where $dim(\Sigma)$ denotes the topological dimension of Σ and $dim(P)$ that of the price space P, as in Radner (1979).

4 Partial revelation

This section presents the core results of the paper on partial revelation. We first demonstrate the existence of belief systems which permit a partially-revealing REE in proposition 2. The main result on robustness of these partially-revealing REE will then be presented as theorem 1.

We start with the observation that since SEU preferences are a special case of AA preferences, applying Radner (1979) immediately gives the following result.

[15] Our notion of partially-revealing REE prices includes the case where the prices are non-revealing, i.e., ϕ is a constant function.

Observation 1. *There exists* $(\boldsymbol{\gamma}, \boldsymbol{\pi}) \in \Gamma^{N^A} \times \Pi^{N^E}$ *for which an REE exists.*

To demonstrate the existence of partially-revealing REE, we start by analyzing the behavior of one AA investor, who is called investor 1. To show that there exist some beliefs for which there is a partially-revealing REE, we need a few preliminary results.

For the remainder of this section, we restrict attention to the class of AA preferences where the set of beliefs satisfy the following assumption (see for example Siniscalchi (2006), who provides an axiomatization for this class of AA preferences).

Assumption 2. *There exists* $L < \infty$ *such that for all* $n \in \mathcal{N}^A$ *and any* $f \in \mathcal{F}$*, there exists* $(\pi^n_{1,f}, \ldots, \pi^n_{L,f}) \in \Delta^{|\Omega|L}$ *such that* $\gamma^n(f) = \bar{\text{co}}\{\pi^n_{1,f}, \ldots, \pi^n_{L,f}\}$*, where* $\bar{\text{co}}$ *denotes the closed convex hull.*

We call conditional beliefs $\gamma(\cdot)$ that satisfy assumption 2 *finitely-generated.* Define $\hat{\Gamma} \subseteq \mathcal{C}(\Delta^{|\Omega|})$ to be the space of all such finitely-generated conditional beliefs. That is, $\hat{\Gamma}$ is the collection of all polytopes in $\Delta^{|\Omega|}$ generated by at most L extreme points. Then $\hat{\Gamma}$ and $\hat{\Pi} \subseteq \Delta^{|\Omega|}$ denote the space of conditional beliefs for an AA investor and an SEU investor, respectively and are endowed with the respective Euclidean metrics.

Since Σ is finite, so is $\mathcal{F}$. We define $\Gamma \subseteq \hat{\Gamma}^{|\mathcal{F}|} \subseteq \Delta^{|\Omega|L|\mathcal{F}|}$ and $\Pi \subseteq \hat{\Pi}^{|\mathcal{F}|} \subseteq \Delta^{|\Omega||\mathcal{F}|}$ as the space of belief systems for an AA and an SEU investor, respectively and we endow $\Gamma^{N^A} \times \Pi^{N^E}$ with the Euclidean metric $||\cdot||$.[16]

[16]Given assumption 2, any conditional belief $\gamma(f)$ can be represented by its extreme points. Each possible set of extreme points can be represented by a point in $\Delta^{|\Omega|L}$ where L is the number of extreme points. Further, every point in $\Delta^{|\Omega|L}$ represents a possible set of conditional beliefs $\gamma(f)$. The mapping between these two sets is not bijective however, since each possible set of conditional beliefs $\gamma(f)$ may be represented by multiple points in $\Delta^{|\Omega|L}$ (by reordering the extreme points). In this sense, the space of belief systems Γ can be embedded in a subset of $\Delta^{|\Omega|L|\mathcal{F}|}$. Bayesian updating constrains the set Π. In particular, once $\hat{\pi}(\sigma)$ is defined for all $\sigma \in \Sigma$, then so is $\hat{\pi}(f)$ for all $f \in \mathcal{F}$ by Bayesian updating. A similar statement may apply to Γ. See the appendix for further discussion of this point.

For any information set N-tuple $\boldsymbol{f} = (f^n)_{n\in\mathcal{N}} \in \mathcal{F}^N$, with corresponding conditional beliefs $(\boldsymbol{\gamma}(\boldsymbol{f}), \boldsymbol{\pi}(\boldsymbol{f})) = ((\gamma^n(f^n))_{n\in\mathcal{N}^A}, (\pi^n(f^n))_{n\in\mathcal{N}^E}) \in \hat{\Gamma}^{N^A} \times \hat{\Pi}^{N^E}$, the investors trade in a standard complete-market economy with (a single consumption good and) heterogeneous beliefs. We use $\mathcal{E}(\boldsymbol{\gamma}(\boldsymbol{f}), \boldsymbol{\pi}(\boldsymbol{f}))$ to denote the Arrow-Debreu equilibria of this trading. That this set is non-empty for every $\boldsymbol{f} \in \mathcal{F}^N$ can be verified from Debreu (1959). The next result describes Arrow-Debreu equilibria of particular interest to our construction of partial revelation.

Proposition 1. *Let the investors' information be given by* $\boldsymbol{f} \in \mathcal{F}^N$ *with corresponding beliefs* $(\boldsymbol{\gamma}(\boldsymbol{f}), \boldsymbol{\pi}(\boldsymbol{f})) \in \hat{\Gamma}^{N^A} \times \hat{\Pi}^{N^E}$. *There exists* $\bar{\pi} \in \Delta^{|\Omega|}$ *such that if AA investor 1's beliefs satisfy* $\bar{\pi} \in \gamma^1(f^1)$ *then all equilibria in* $\mathcal{E}(\boldsymbol{\gamma}(\boldsymbol{f}), \boldsymbol{\pi}(\boldsymbol{f}))$ *satisfy* $x^1(\omega|\sigma) = x^1(\omega'|\sigma)$ *for all distinct* $\omega, \omega' \in \Omega$ *and all* $\sigma \in f^1$.

Proof. Consider a hypothetical economy in which investor 1 is replaced by an investor le with (Leontief) preferences of the form $U^{le}(x) = \min_{\omega\in\Omega}[x(\omega)]$ and endowment $e^{le} = e^1$, and all other investors have preferences as described by assumption 2, with beliefs given by $((\gamma^n(f^n))_{n\in\mathcal{N}^A-1}, (\pi^n(f^n))_{n\in\mathcal{N}^A})$.[17]

For any price vector $p \in P \cap \mathbb{R}_{++}$, investor le's demand is

$$x^{le} = pe^{le}\mathbf{1} \tag{4.1}$$

where $\mathbf{1}$ is the $|\Omega|$-dimensional vector of 1's.

An Arrow-Debreu equilibrium exists for this economy as can be verified from Debreu (1959). From the equilibrium in this economy one can derive conditions on the beliefs of investor 1 that ensure that her demand is equal to (4.1) at the equilibrium

[17] Such an investor could be construed as an AA decision-maker with beliefs $\gamma^{le}(f^1) = \Delta^{|\Omega|}$ and linear von-Neumann Morgenstern utility.

price. From the first-order conditions for the AA investor, it can be seen that if for the equilibrium price p, his beliefs are such that $p \in \gamma^1(f^1)$ then the full-insurance allocation x^{le} will be optimal for investor 1.[18] $\square$

The previous result implies that for any conditonal beliefs for the other investors in the economy, there is always a set of conditional beliefs for investor 1 with the property that if he holds these beliefs then he will hold an uncertainty-less portfolio in equilibrium.

The next step is to demonstrate that in this model from any REE one can construct an REE that reveals (possibly strictly) less information by allowing a single investor's preferences to demonstrate sufficient ambiguity aversion. For this result, we will need the following definition.

Definition 3. *An REE* $(\bar{x}, \bar{\phi})$ *reveals strictly less information than an REE* (x, ϕ) *if for all* $\sigma, \sigma' \in \Sigma, \phi(\sigma) = \phi(\sigma') \Rightarrow \bar{\phi}(\sigma) = \bar{\phi}'(\sigma')$ *and there exist distinct* σ, σ' *such that* $\phi(\sigma) \neq \phi(\sigma')$ *and* $\bar{\phi}'(\sigma) = \bar{\phi}(\sigma')$*.*

Proposition 2. *Let* σ' *and* σ'' *be two distinct signals that differ only in the private signal of AA investor 1. Let* (x, ϕ) *be an REE under (a belief system)* $(\boldsymbol{\gamma}, \boldsymbol{\pi}) \in \Gamma^{N^A} \times \Pi^{N^E}$ *in which* σ *and* σ' *are revealed, i.e.* $\phi(\sigma) \neq \phi(\sigma')$*. There exists* $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\pi}}) \in \Gamma^{N^A} \times \Pi^{N^E}$ *and an REE* $(\bar{x}, \bar{\phi})$ *under* $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\pi}})$ *that reveals strictly less information than* (x, ϕ)*.*

Proof. The proof is constructive. For all $\sigma \notin \{\sigma', \sigma''\}$, let $(\bar{x}(\sigma), \bar{\phi}(\sigma)) = (x(\sigma), \phi(\sigma))$ and $\bar{\gamma}^n(f^n) = \gamma^n(f^n)$ for f^n such that $\sigma', \sigma'' \notin f^n$.

[18]See corollary 2 in appendix A.2 for these conditions.

Let $(\bar{x}(\sigma'), \bar{\phi}(\sigma')) \in \mathcal{E}(\bar{\gamma}^1(\sigma'), (\gamma^n(\{\sigma', \sigma''\}))_{n\in\mathcal{N}^A-1}, (\pi^n(\{\sigma', \sigma''\}))_{n\in\mathcal{N}^E})$ where

$$\bar{\phi}(\sigma') \in \bar{\gamma}^1(\sigma'). \tag{4.2}$$

This implies that $\bar{x}^1(\omega; \sigma') = \bar{x}^1(\omega'; \sigma')$ for all $\omega, \omega' \in \Omega$. By Proposition 1 such beliefs exist. Define $(\bar{x}(\sigma''), \bar{\phi}(\sigma'')) = (\bar{x}(\sigma'), \bar{\phi}(\sigma'))$. The allocation and price functions $(\bar{x}, \bar{\phi})$ are a partially-revealing REE if

$$\bar{\phi}(\sigma') = \bar{\phi}(\sigma'') \in \mathrm{rint}[\bar{\gamma}^1(\sigma') \cap \bar{\gamma}^1(\sigma'')] \tag{4.3}$$

where $\mathrm{rint}[\gamma(f)]$ refers to the relative interior of $\gamma(f)$, i.e., its interior as a subset of $\Delta^{|\Omega|}$. Prices are the same across joint signals σ' and σ'' and these prices are Arrow-Debreu equilibrium prices given the beliefs of each trader $2, \ldots, N$ when they know only that $\sigma \in \{\sigma', \sigma''\}$. Hence, the investors' behavior is optimal under the price $\bar{\phi}(\sigma') = \bar{\phi}(\sigma'')$.[19]

By construction $(\bar{x}(\sigma), \bar{\phi}(\sigma))$ is an Arrow-Debreu equilibrium for each signal σ. So, $(\bar{x}, \bar{\phi})$ is an REE and because $\bar{\phi}(\sigma') = \bar{\phi}(\sigma'')$, it reveals less information than (x, ϕ). □

Nothing in the previous result requires that AA investors have beliefs that are finitely generated. The only constraint is that there exist an REE in the economy. Thus, if the preferences of all investors were generalized to display ambiguity aversion and it were known that a fully-revealing REE existed, then the method of constructing partially-revealing REE from fully-revealing REE used to prove proposition 2 would

[19] It is possible that the partially-revealing price $\bar{\phi}(\cdot)$ has the property that $\bar{\phi}(\sigma') = \bar{\phi}(\sigma''')$ for some $\sigma''' \neq \sigma''$. The proof of Theorem 1 shows that beliefs for other investors may be adapted in such a way that this does not occur.

apply equally well.

The inclusion condition (4.3) has economic content. For any signal σ, if an investor's beliefs γ satisfy $\phi(\sigma) \in \gamma(\sigma)$ then the investor believes that it is possible that the price vector represents the true probability distribution over states in Ω. That is, the investor believes that the market may have assigned relative prices to assets in these states that are exactly the likelihood of these assets paying off. Since investor 1 is ambiguity averse, he has a desire to fully insure as long as prices do not differ from what he thinks might be the true probabilities over states. Condition (4.3) implies that the information that he receives in his private signal under both σ' and σ'' does not give him any reason to bet against the odds that the market presents. This happens even though it can be the case that the set of distributions that he believes to be possible given σ' and σ'' respectively differ. The only requirement is that both $\gamma(\sigma)$ and $\gamma(\sigma')$ have the price vector $\phi(\sigma)$ as a common interior element.

Equation (4.3) can be interpreted as a restriction on the relative informativeness of an investor's signal, in this case of investor 1. On the other hand, (4.3) says nothing about the absolute informativeness of an investor 1's portion of the joint signals σ' and σ''. It is entirely consistent with the previous result to assume that if an SEU investor knew investor 1's private signal he would hold beliefs that are very different than he does when he cannot determine the signal. That is, generically in Π, $\pi^n(\sigma') \neq \pi^n(\sigma'') \neq \pi^n(\{\sigma', \sigma''\}), n \in \mathcal{N}^E$, meaning that beliefs under these three different states of information are likely to vary.

Condition (4.3) suggests that there may be a link between the partial revelation results we obtain and the result on full-insurance being Pareto optimal contained in Billot, Chateauneuf, Gilboa, and Tallon (2000). That result relates to symmetric information exchange economies with no aggregate uncertainty and characterizes full-insurance for all AA traders being Pareto optimal in terms of an intersection condition for the beliefs of *all* traders. Here, we show that the intersection across signals of the beliefs of a single AA trader may have negative implications for the informational efficiency of prices. There are no restrictions placed on how the beliefs of the other traders vary across the signals and the economy may have aggregate uncertainty. Similarly, unlike the results of Mukerji and Tallon (2001) or Epstein and Wang (1994), our construction does not rely on *all* AA investors choosing full-insurance.

As we also noted in the leading example, since we do not restrict the endowment distribution of any investor to be on the full-insurance subspace, the result does not rely on any 'no-trade' type arguments. Moreover, we are not relying on any indeterminacy of Arrow-Debreu equilibrium prices (given each signal) for this construction. The presence of SEU investors precludes indeterminacy (except on a measure-zero set of endowments) and indeed, the work of Rigotti and Shannon (2008) shows that Arrow-Debreu economies with AA traders are determinate except on a measure-zero set of endowments.

The *existence* of the partially-revealing REE we construct does not fall out of the realm of the results established by Radner (1979). In particular, investor 1 could be an SEU investor who may have different conditional beliefs under distinct signals σ' and σ'' and (possibly) different allocation choices, but these lead to the same (Arrow-Debreu) equilibrium price. However, it is apparent that in a world in which all investors are SEU maximizers this phenomenon is not robust (Radner (1979), Theorem). That is, perturbing the SEU investors' beliefs slightly must then induce differing Arrow-Debreu equilibrium prices across the signals and hence a fully-revealing REE.

In a world in which investors are ambiguity averse however, perturbation of beliefs does not necessarily remove the possibility of a partially-revealing REE as we now demonstrate.

As a step toward the main result, we now show for a special case that partial revelation is a robust property when there is an AA investor present. We make use of the following lemma which is proved in the appendix.

Lemma 4.1. *For $\gamma \in \hat{\Gamma}$, let $\hat{\pi} \in \text{rint}[\gamma(\sigma) \cap \gamma(\sigma')]$ for some $\sigma, \sigma' \in \Sigma$. Then there exist open neighborhoods $B(\gamma, \epsilon_\gamma) \subset \hat{\Gamma}$ and $B(\hat{\pi}, \epsilon_{\hat{\pi}}) \subset \Delta^{|\Omega|}$ such that for all $\gamma' \in B(\gamma, \epsilon_\gamma)$, $B(\hat{\pi}, \epsilon_{\hat{\pi}}) \subset \text{rint}[\gamma'(\sigma) \cap \gamma'(\sigma')]$.*

Proposition 3. *If there is a single AA investor ($N^A = 1$), then there exists a set $\tilde{PR} \in \Gamma \times \Pi^{N^E}$ with positive Lebesgue measure such that for each $(\boldsymbol{\gamma}, \boldsymbol{\pi}) \in \tilde{PR}$, there exists a partially-revealing REE.*

Proof. Let $(\boldsymbol{\gamma}, \boldsymbol{\pi}) \in \Gamma \times \Pi^{N^E}$ permit a partially-revealing REE. By proposition 2 such a $(\boldsymbol{\gamma}, \boldsymbol{\pi})$ exists. The proof will show that there exists an $\epsilon > 0$ such that for all $(\boldsymbol{\gamma'}, \boldsymbol{\pi'})$ that satisfy $||(\boldsymbol{\gamma'}, \boldsymbol{\pi'}) - (\boldsymbol{\gamma}, \boldsymbol{\pi})|| < \epsilon$, there is a partially-revealing REE.

The proof of proposition 2 illustrates that under our assumptions a sufficient condition for a partially-revealing REE to exist is condition (4.3). Let σ' and σ'' be two signals for which condition (4.3) holds.

The equilibrium price function ϕ is continuous at $(\boldsymbol{\gamma}, \boldsymbol{\pi})$. To see this, notice that for $\sigma \in \{\sigma', \sigma''\}$, $Z(p, \boldsymbol{\gamma}(\sigma), \boldsymbol{\pi}(\sigma))$ is differentiable at $p = \phi(\sigma')$. Further, by lemma 4.1, there exists a neighborhood $B_{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ in which the Jacobian $D_{\boldsymbol{\gamma}} Z(p, \boldsymbol{\gamma}, \boldsymbol{\pi})$ exists and is the zero matrix. Assume for now that $Z(\hat{p}, \boldsymbol{\gamma}(\sigma), \boldsymbol{\pi}(\sigma))$ is *regular*, i.e. $\det D_p Z \neq 0$, at the equilibrium price $\hat{p} = \phi(\sigma) = \phi(\sigma')$. Then there exists neighborhoods $B_{\hat{p}}$ of $\hat{p}$, $B_{\boldsymbol{\pi}}$

of $\boldsymbol{\pi}$, and $B_{\boldsymbol{\gamma}}$ of $\boldsymbol{\gamma}$ and a differentiable function $\hat{\phi}(\cdot)$ such that $Z(\hat{\phi}(\boldsymbol{\gamma'}, \boldsymbol{\pi'}), \boldsymbol{\gamma'}, \boldsymbol{\pi'}) = 0$ for all $(\boldsymbol{\gamma'}, \boldsymbol{\pi'}) \in B_{\boldsymbol{\gamma}} \times B_{\boldsymbol{\pi}}$. Thus, if $\hat{p}$ is a regular point then the equilibrium price function ϕ is continuous at $(\boldsymbol{\gamma}, \boldsymbol{\pi})$ and thus for any ϵ there exist $\delta_{\boldsymbol{\gamma}}$ and $\delta_{\boldsymbol{\pi}}$ such that if $\boldsymbol{\pi'} \in B(\boldsymbol{\pi}, \delta_{\pi})$ and $\boldsymbol{\gamma'} \in B(\boldsymbol{\gamma}, \delta_{\gamma})$ then $||\phi(\boldsymbol{\gamma'}, \boldsymbol{\pi'}) - \phi(\boldsymbol{\gamma}, \boldsymbol{\pi})|| < \epsilon$.[20]

Now, we verify that there exists a partially-revealing REE for which the equilibrium price p is a regular point. To see this, consider again an economy with an investor le who has Leontief preferences with $e^{le} = e^1$ and N^E SEU investors. The excess demand $Z(p, \boldsymbol{\pi})$ for this economy is differentiable for all prices $p \in P \cap \mathbb{R}^{|\Omega|}_{++}$ and is given by

$$Z(p, \boldsymbol{\pi}) = \sum_{n \in \mathcal{N}^E} (x^n(p) - e^n) + (pe^{le}\mathbf{1} - e^{le}). \tag{4.4}$$

Further, the Jacobian of excess demand is

$$DZ(p, \pi) = [D_p Z(\cdot), D_{\boldsymbol{\pi}} Z(\cdot)]. \tag{4.5}$$

The matrix $D_{\boldsymbol{\pi}} Z(\cdot)$ has rank $N^E(|\Omega| - 1)$, which implies that $DZ(p, \boldsymbol{\pi})$ always has at least rank $(|\Omega| - 1)$. Then, by the Transversality Theorem, for almost all $\boldsymbol{\pi}$, $\det D_p Z(p, \boldsymbol{\pi}) \neq 0$ when $Z(p, \boldsymbol{\pi}) = 0$. By proposition 2, for all possible beliefs $\boldsymbol{\pi'}$ there exists a $\boldsymbol{\gamma'}$ that generates a partially-revealing REE and as was just shown, for almost all of these $Z(p, \boldsymbol{\pi'})$ is determinate. Therefore, for almost all of these $\boldsymbol{\pi'}$, the partially-revealing REE price function $\hat{\phi}(\cdot)$ is continuous at $(\boldsymbol{\gamma'}, \boldsymbol{\pi'})$.

Now, by lemma 4.1, we may choose neighborhoods B_ϕ around $\phi(\sigma')$ and $B_{\boldsymbol{\gamma}}$ around $\boldsymbol{\gamma}$ for which $B_\phi \subset \text{rint}[\boldsymbol{\gamma'}(\sigma') \cap \boldsymbol{\gamma'}(\sigma'')]$ for all $\boldsymbol{\gamma'} \in B_{\boldsymbol{\gamma}}$. Since $\phi(\boldsymbol{\gamma}, \boldsymbol{\pi})$ is continuous at $(\boldsymbol{\gamma}, \boldsymbol{\pi})$, there exists a neighborhood $B((\boldsymbol{\gamma}, \boldsymbol{\pi}), \epsilon_{\boldsymbol{\gamma}, \boldsymbol{\pi}})$ such that $\phi(\boldsymbol{\gamma'}, \boldsymbol{\pi'}) \in B_\phi$ for all

[20] Though ϕ is defined earlier with domain Σ, the signals matter only in terms of the (conditional) beliefs. Hence we abuse notation and write $\phi(\boldsymbol{\gamma}, \boldsymbol{\pi})$ directly.

$(\boldsymbol{\gamma'}, \boldsymbol{\pi'}) \in B((\boldsymbol{\gamma}, \boldsymbol{\pi}), \epsilon_{\boldsymbol{\gamma},\boldsymbol{\pi}})$.

Combining the continuity of ϕ with lemma 4.1, choose an open rectangle $B(\boldsymbol{\gamma}, \boldsymbol{\pi})$, whose projection on Γ, denoted $B_\Gamma(\boldsymbol{\gamma}, \boldsymbol{\pi})$, satisfies $B_\Gamma(\boldsymbol{\gamma}, \boldsymbol{\pi}) \subset B_{\boldsymbol{\gamma}}$. This implies that for all $(\boldsymbol{\gamma'}, \boldsymbol{\pi'}) \in B((\boldsymbol{\gamma}, \boldsymbol{\pi}), \epsilon)$, $\phi(\boldsymbol{\gamma'}, \boldsymbol{\pi'}) \in \boldsymbol{\gamma'}(\sigma') \cap \boldsymbol{\gamma'}(\sigma'')$. This set contains an open ϵ-ball in $\Gamma \times \Pi^{N^E}$ and thus has positive measure. □

Before proceeding to show the generality of the preceding special case, we note the following result, which is proved in the appendix, and will be used in the proof of the main robustness result.

Lemma 4.2. *Let $x(p, \hat{\gamma}^A)$ represent demand for an AA investor (labeled A) with beliefs $\hat{\gamma}^A \in \hat{\Gamma}$, von Neumann-Morgenstern utility u^A, and endowment e^A. Suppose $\hat{\gamma}^A$ is finitely-generated by $(\hat{\pi}_l)_{l\in\{1,\ldots,L\}} \in (\Delta^{|\Omega|})^L$ and that for price p*

$$\hat{\pi}_1 = \arg\min_{\hat{\pi}\in\hat{\gamma}^A} \mathbb{E}_{\hat{\pi}}[u(x(p, \hat{\gamma}^A))] \tag{4.6}$$

(that is, $\hat{\pi}_1$ is the unique minimizing element of the linear functional $\mathbb{E}_{\hat{\pi}}[u(x(p, \hat{\gamma}^A))]$). Then $x(p, \hat{\gamma}^A)$ is differentiable at $(\hat{\pi}_l)_{l=\{1,\ldots,L\}}$.

And finally, we present the main result on robustness of partial revelation. In what follows, for $v, v' \in \mathbb{R}^{|\Omega|}$, $v \circ v' \in \mathbb{R}^{|\Omega|}$ denotes $(v_1 v'_1, \ldots, v_{|\Omega|} v'_{|\Omega|}) \in \mathbb{R}^{|\Omega|}$.

Theorem 1. *If assumptions 1 and 2 hold, there exists a set PR with positive Lebesgue measure in $\Gamma^{N^A} \times \Pi^{N^E}$ such that for each $(\boldsymbol{\gamma}, \boldsymbol{\pi}) \in PR$, a partially-revealing REE exists.*

Proof. Suppose (x, ϕ) is a partially-revealing REE, under the beliefs $(\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\pi}}) \in \Gamma^{N^A} \times \Pi^{N^E}$, that does not distinguish the distinct signals σ and σ'. Suppose further that

σ and σ differ only in the signal received by AA investor 1. By proposition 2 such REE exist. Then, for any AA investor n other than investor 1, one may assign beliefs $\gamma^n \in \Gamma$ such that (x, ϕ) is still an equilibrium, but for which x is differentiable at $\bar{\boldsymbol{\pi}}$ as follows.

For $n \in \mathcal{N}^A - \{1\}$, let $\hat{\pi}'_n \in \Delta^{|\Omega|}$ be the unique belief satisfying

$$\lambda^n p = \hat{\pi} \circ u^{n\prime}(x^n(\sigma)) \tag{4.7}$$

for $\hat{\pi} \in \Delta^{|\Omega|}$, where $p = \phi(\sigma)$ and λ^n is the multiplier associated with AA investor n's optimization problem.

Then we define a closed, convex set of beliefs $\gamma^n(\sigma) \in \hat{\Gamma}$ which has the property that

$$\mathbb{E}_{\hat{\pi}'_n}[u(x)] < \mathbb{E}_{\hat{\pi}}[u(x)] \text{ for all } \hat{\pi} \in \gamma^n(\sigma), \hat{\pi} \neq \hat{\pi}'_n. \tag{4.8}$$

To see that this can be done, notice that the set of beliefs satisfying equation (4.8) is the intersection of an $|\Omega|$-dimensional half space and the $(|\Omega|-1)$-dimensional simplex in $\mathbb{R}^{|\Omega|}$. Then select $(\hat{\pi}_l)_{l\in\{2,\ldots,L\}} \in (\Delta^{|\Omega|})^{L-1}$, such that for each $l \in \{2, \ldots, L\}$, equation (4.8) holds.

Using similar arguments as in lemma 4.2 we have that $x^n(p, \gamma^n(\sigma))$ is differentiable in the vertices that define $\gamma^n(\sigma)$ for all $n \in \mathcal{N}^A - \{1\}$. Thus, the result from proposition 3 applies and demonstrates the existence of an open ball in $\Gamma^{N^A} \times \Pi^{N^E}$ for each element of which there is a partially-revealing REE. □

Since the set of beliefs $(\boldsymbol{\gamma}, \boldsymbol{\pi})$ for which such a partially-revealing REE exists has positive Lebesgue measure, it is not an artifact of carefully chosen model parameters. In this sense, the inclusion of AA investors in a traditional REE framework provides for fundamentally different equilibria than those found generically when all investors have SEU preferences.

It is worth pointing out that assumption 2 is needed for the application of the Transversality Theorem. Although we restrict the attention to beliefs that satisfy this assumption, the existence of a partially-revealing REE can be established without it. It may also be possible to establish that the inclusion condition (4.3) is robust without it. To our knowledge, there is no version of the Transversality Theorem that can be applied when the space of parameters (the space of belief systems in our case) is infinite dimensional and the function of interest is not Frechet-differentiable in the parameter.[21]

5 Full revelation

This section proves the generic existence of fully-revealing REE in the model for a particular subclass of preferences displaying ambiguity aversion. This result is not surprising. The proofs of the generic existence of fully-revealing REE in Radner (1979) and subsequent work relied on constructions that first assumed that all investors were informed and then demonstrated that the subsequent prices would be consistent with this assumption except on a set of parameters of measure zero. The proof provided here is similar.

The previous results combined with the results in this section indicate that at least for the subclass of preferences considered herein, there will be multiple equilibria for some sets of parameters. One of these equilibria will be the partially-revealing

[21] The closest possibility to our knowledge is theorem 3.7 in Shannon (2006) which requires Frechet equi-differentiability.

equilibrium discussed in section 4 while another will be the fully-revealing equilibrium whose existence is proven here. The result is restricted to the space of beliefs that are generated as the core of a convex capacity because this class of preferences can be shown to lead to demand that is Lipschitz continuous, a requirement for the implicit function theorem that we employ in the proof.

The existence of multiple equilibria for some sets of parameters leads to the important question of whether one equilibrium is more reasonable than the other. While we feel that this topic is important, it seems that the appropriateness of one equilibrium over the other will be largely dependent on the application to which the REE equilibrium concept is applied. As such, we do not discuss equilibrium selection in this paper.

In what follows, for $m \geq 0$, $||\cdot||_m$ denotes the Euclidean metric on $\mathbb{R}^m$.

Definition 4. *Let X be a subset of $\mathbb{R}^m$. A function $f : X \to \mathbb{R}^k$ is Lipschitz continuous with Lipschitz constant $K \geq 0$ if for all $x, y \in X$,*

$$||f(y) - f(x)||_m \leq K||y - x||_k. \tag{5.1}$$

The next two definitions and the next lemma are from Clarke (1983). By Rademacher's Theorem, a function $F : \mathbb{R}^m \to \mathbb{R}^k$ that is Lipschitz continuous on an open subset of $\mathbb{R}^m$ is differentiable almost anywhere on that subset. Let Λ_F be the set of points in the domain for the function F at which F is not differentiable.

Definition 5. *The generalized Jacobian of F at x, denoted $\partial F(x)$ is given by*

$$\partial F(x) = \bar{\text{co}}\{\lim DF(x_i) : x_i \to x, x_i \notin \Lambda_F\}. \tag{5.2}$$

where $DF(x_i)$ is the Jacobian of F at the point of differentiability x_i.

The generalized Jacobian is a set of matrices (being a singleton if F is differentiable at x) defined for all x in the domain of a Lipschitz function F.

Definition 6. *Let $G : \mathbb{R}^k \to \mathbb{R}^k$. The generalized Jacobian $\partial G(x_0)$ at x_0 is said to be of maximal rank if every matrix $M \in \partial G(x_0)$ is non-singular.*

In order to state the next lemma and proposition, some notation must be clarified. Let $F : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$ and suppose $(\hat{x}, \hat{y}) \in \mathbb{R}^k \times \mathbb{R}^m$ solve $F(x, y) = 0$. Let $\partial_x F(x, y)$ be the set of all $k \times k$ matrices M such that for some $k \times m$ matrix M', the $k \times (k+m)$ matrix $[M, M'] \in \partial F(x, y)$.

Lemma 5.1 (Clarke, p256, Corollary)**.** *Suppose that $\partial_x F(\hat{x}, \hat{y})$ is of maximal rank. Then there exists a neighborhood Y of $\hat{y}$ and a Lipschitz function $\zeta : Y \to \mathbb{R}^k$ such that $\zeta(\hat{y}) = \hat{x}$ and for all $y \in Y$, $F(\zeta(y), y) = 0$.*

Recall from Radner (1979) that a full communication equilibrium (FCE) is a set of Arrow-Debreu equilibria (one for each joint signal σ) where in the equilibrium corresponding to each joint signal σ, investors hold the beliefs that they would hold if they knew all private information (i.e. they know that the joint signal is σ). An FCE is said to be confounding if $\phi(\sigma) = \phi(\sigma')$ for some distinct $\sigma, \sigma' \in \Sigma$, where with abuse of notation $\phi(\sigma)$ denotes the Arrow-Debreu equilibrium price under signal σ. An FCE is a fully-revealing REE if it is not confounding.

We make the following modification to assumption 2 regarding the beliefs of each AA investor (see again Siniscalchi (2006), p105) and denote by $\hat{\Gamma}_C$ (respectively Γ_C) the finite-dimensional Euclidean space (see below) of conditional beliefs (respectively, belief systems) for each AA investor that satisfy this assumption. We show that the set of $(\boldsymbol{\gamma}, \boldsymbol{\pi}) \in \Gamma_C^{N^A} \times \Pi^{N^E}$ for which an FCE is confounding is of measure zero.

Assumption 3. *For all $n \in \mathcal{N}^A$ and $\sigma \in \Sigma$, the conditional beliefs $\gamma^n(\sigma)$ are the core of a convex capacity.*[22]*.*

If each AA investor's beliefs can be generated by the core of a convex capacity ν, then each conditional belief can be generated by a set of the form

$$\gamma(\cdot) = \{\pi \in \Delta^{|\Omega|} : \pi(\omega) \geq \nu(\omega)\}. \tag{5.3}$$

where $(\nu(1), \ldots, \nu(|\Omega|)) \in [0,1]^{|\Omega|}$. Hence beliefs for each joint signal can be described by some vector $(\nu(\omega))_{\omega=1}^{|\Omega|} \in \mathbb{R}_+^{|\Omega|}$ under the restriction that the resulting set must be convex and non-empty. The set of such points is a subset of a Euclidean space, and so it makes sense to impose Lebesgue measure on this set. This clarification leads to the following analysis.

Let $Z^E : P \times \hat{\Pi}^{N^E} \rightarrow \mathbb{R}_+^{|\Omega|-1}$ represent the excess demand function (in $|\Omega| - 1$ markets) of all SEU investors and $Z^A : P \times \hat{\Gamma}_C^{N^A} \rightarrow \mathbb{R}_+^{|\Omega|-1}$ represent the excess demand for the AA investors (in $|\Omega| - 1$ markets). From theorem 3 of Rigotti and Shannon (2008), $Z^A(\cdot)$ is Lipschitz continuous in p. The price vector p is an equilibrium price vector given beliefs $(\hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\pi}}) \in \hat{\Gamma}_C^{N^A} \times \hat{\Pi}^{N^E}$ if and only if $Z(p, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\pi}}) = Z^A(p, \hat{\boldsymbol{\gamma}}) + Z^E(p, \hat{\boldsymbol{\pi}}) = 0$. To examine whether an FCE is confounding, we attempt to determine the size of the set of beliefs that will generate identical prices.

[22] A convex capacity is a set function $\nu : \mathcal{F} \rightarrow [0,1]$ such that (i)$B_1 \subseteq B_2 \Rightarrow \nu(B_1) \leq \nu(B_2)$, (ii)$\nu(\emptyset) = 0 = 1 - \nu(\Omega)$, and (iii)$\nu(B_1) + \nu(B_2) \leq \nu(B_1 \cap B_2) + \nu(B_1 \cup B_2)$ for all $B_1, B_2 \in \mathcal{F}$

Consider the following system of equations, $F : P^2 \times \hat{\Gamma}_C^{2NA} \times \hat{\Pi}^{2NE} \to \mathbb{R}^{3(|\Omega|-1)}$.

$$F(p_1, p_2, \boldsymbol{\hat{\gamma}_1, \hat{\gamma}_2, \hat{\pi}_1, \hat{\pi}_2}) = \begin{pmatrix} Z(p_1, \boldsymbol{\hat{\gamma}_1, \pi_1}) \\ Z(p_2, \boldsymbol{\hat{\gamma}_2, \pi_2}) \\ p_1 - p_2 \end{pmatrix} = 0. \tag{5.4}$$

For an FCE to be confounding it must be true that for some distinct $\sigma', \sigma'' \in \Sigma$, there exists a solution to the system $F(p_1, p_2, \boldsymbol{\gamma}(\sigma'), \boldsymbol{\gamma}(\sigma''), \boldsymbol{\pi}(\sigma'), \boldsymbol{\pi}(\sigma'')) = 0$.

Let B be the set of all $(p_1, p_2, \boldsymbol{\hat{\gamma}_1, \hat{\gamma}_2, \hat{\pi}_1, \hat{\pi}_2}) \in P^2 \times \hat{\Gamma}_C^{2NA} \times \hat{\Pi}^{2NE}$ that solve the system (5.4). Let $T(B)$ be the projection of this set into $\hat{\Gamma}_C^{2NA} \times \hat{\Pi}^{2NE}$. We show that the set $T(B)$ has measure zero in $\hat{\Gamma}_C^{2NA} \times \hat{\Pi}^{2NE}$. The following proposition provides the key result for this proof.

Proposition 4. *For any* $(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2) \in \hat{\Gamma}_C^{2NA}$, *let* $\partial F(p_1, p_2, \boldsymbol{\hat{\pi}_1, \hat{\pi}_2})$ *be the generalized Jacobian corresponding to* $(p_1, p_2, \boldsymbol{\hat{\pi}_1, \hat{\pi}_2})$. *Then, every* $M \in \partial F(p_1, p_2, \boldsymbol{\hat{\pi}_1, \hat{\pi}_2})$ *is of rank* $3(|\Omega| - 1)$.

Proof. To see this, note that for a fixed $(\boldsymbol{\hat{\gamma}_1, \hat{\gamma}_2}) \in \hat{\Gamma}_C^{2NA}$ the system is differentiable in $(\boldsymbol{\hat{\pi}_1, \hat{\pi}_2})$ but because of the possible non-differentiability of $Z^A(\cdot, \boldsymbol{\hat{\gamma}_1, \hat{\gamma}_2})$, it need not be differentiable in (p_1, p_2). However, it is true that every possible $M \in \partial F(p_1, p_2, \boldsymbol{\hat{\pi}_1, \hat{\pi}_2})$, will have the following form,

$$\begin{array}{ccccc} & p_1 & p_2 & \boldsymbol{\hat{\pi}_1} & \boldsymbol{\hat{\pi}_2} \\ 1: & A & 0 & C & 0 \\ 2: & 0 & B & 0 & D \\ 3: & I & -I & 0 & 0 \end{array} \tag{5.5}$$

where I is the $(|\Omega|-1)\times(|\Omega|-1)$ identity matrix. The $(|\Omega|-1)\times(|\Omega|-1)$ matrices A and B will vary across elements of $\partial F(\cdot)$, but the identity matrices and the matrices C and D will not. Now consider the $(|\Omega|-1)\times N^E(|\Omega|-1)$ matrix C. The form of matrix D is similar. Since only $Z^E(\cdot)$ depends on $\hat{\boldsymbol{\pi}}_{\mathbf{1}}$ and only SEU investor n's demand is affected by n's beliefs, C will have the form

$$C = \left(C^1, C^2, \cdots, C^{N^E}\right). \tag{5.6}$$

Applying the implicit function theorem to the SEU investor's first-order conditions reveals that each C^n is given by

$$C^n = \begin{pmatrix} -\frac{u^{n\prime}(x(1))}{u^{n\prime\prime}(x(1))} & 0 & \cdots & 0 \\ 0 & -\frac{u^{n\prime}(x(2))}{u^{n\prime\prime}(x(2))} & 0 & \cdots \\ & & \ddots & \\ 0 & \cdots & & -\frac{u^{n\prime}(x(|\Omega|-1))}{u^{n\prime\prime}(x(|\Omega|-1))} \end{pmatrix}. \tag{5.7}$$

By inspection the matrix C^n spans a space of dimension $(|\Omega|-1)$. Thus the columns corresponding to p_1, C^1, and D^1 will span a space of dimension $3(|\Omega|-1)$, regardless of the entries in the matrix A. Thus, all matrices in $\partial F(p_1, p_2, \boldsymbol{\pi}_{\mathbf{1}}, \boldsymbol{\pi}_{\mathbf{2}})$ span a space of dimension $3(|\Omega|-1)$. □

Proposition 5. *Let $\mu_{\hat{\Pi}^{2N^E}}$ denote the Lebesgue measure on $\hat{\Pi}^{2N^E}$ and let $T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))$ be the set of $(\hat{\boldsymbol{\pi}}_{\mathbf{1}}, \hat{\boldsymbol{\pi}}_{\mathbf{2}}) \in \hat{\Pi}^{2N^E}$ for which system 5.4 has a solution for a fixed $(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2) \in \hat{\Gamma}^{2N^A} - C$. Then for a fixed $\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2$, $\mu_{\hat{\Pi}^{2N^E}}(T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))) = 0$.*

Proof. By proposition 4, and lemma 5.1, the set of solutions $B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2)$ is a $2(|\Omega|-1)+$

$2N^E(|\Omega|-1)-3(|\Omega|-1)=(2N^E-1)(|\Omega|-1)$ dimensional manifold in $P^2 \times \Pi^{2N^E}$.[23] Let $\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}$ be the space corresponding to the exogenous variables in Proposition 4. Let $\{S_i\}_{i=1}^{\infty}$ be a countable covering of the space $\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}$. Proposition 4 and lemma 5.1 then tell us that there exist Lipschitz functions $\{\zeta_i\}_{i=1}^{\infty}$ such that $\cup_{i=1}^{\infty}\{\zeta_i(S_i)\}$ covers $T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))$.[24]

By Lemma A.10, the set $\zeta_i(S_i)$ has measure zero in $\hat{\Pi}^{2N^E}$ since the set $\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}$ is homeomorphic to the Euclidean space of dimension $(2N^E-1)(|\Omega|-1)$ while the dimension of $\hat{\Pi}^{2N^E}$ is $2N^E(|\Omega|-1)$.

Then since $\{\zeta_i(S_i)\}_{i=1}^{\infty}$ covers $T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))$,

$$\mu_{\hat{\Pi}^{2N^E}}(T(B(\hat{\boldsymbol{\gamma}}_1, \hat{\boldsymbol{\gamma}}_2))) \leq \mu_{\hat{\Pi}^{2N^E}}(\cup_{i=1}^{\infty}\zeta_i(S_i)) = \sum_{i=1}^{\infty} \mu_{\hat{\Pi}^{2N^E}}(\zeta_i(S_i)). \tag{5.8}$$

By Lemma A.10, $\mu_{\hat{\Pi}^2}(\zeta_i(S_i)) = 0$ for all i, which proves the result. □

We now state and prove the main result of this section.

Theorem 2. *The set of beliefs in $\Gamma_C^{N^A} \times \Pi^{N^E}$ for which there is not a fully-revealing REE has Lebesgue measure zero.*

[23] That is, every solution point is locally homeomorphic to a subset of $\mathbb{R}^{(2N^E-1)(|\Omega|-1)}$.

[24] To see how this is done, start with an arbitrary countable covering $\{A_i\}_{i=1}^{\infty}$ of $\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}$. For each A_i, let B_i be the set of points in B such that the projection of each point into $\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}$ is in S_i. For each of these points b, lemma 5.1 says that there is an open set $V_b \subseteq P^2 \times \hat{\Pi}^2 - (\tilde{P} \times \hat{\tilde{\Pi}}_1 \times \hat{\tilde{\Pi}}_2)$, an open set $W_b \in \tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}$ and a Lipschitz function ζ_{ib} that maps points in $(\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}) \cap W_b$ into points in $(B - (\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2})) \cap V_b$. By definition $\{W_b\}_{b \in B_i}$ is an open cover of A_i and $\{V_b\}_{b \in B_i}$ is an open cover of B_i. Since A_i and B_i are subsets of Euclidean space, each of these open covers has a countable subcover which we label $\{\mathcal{W}_b\}_{b=1}^{\infty}$ and $\{\mathcal{V}_b\}_{b=1}^{\infty}$ respectively. Let $\mathcal{B}_i$ be the collection of $b \in B_i$ such that either $V_b \in \{\mathcal{V}_b\}_{b=1}^{\infty}$ or $W_b \in \{\mathcal{W}_b\}_{b=1}^{\infty}$. Let us then replace each element A_i with the countable covering $\mathcal{W}_i = \{W_b\}_{b \in \mathcal{B}_i}$. The countable covering over $\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}$ formed in this way is countable since it is a countable collection of countable sets. The collection $\{\zeta_b\}_{b \in \mathcal{B}_i}$ is a countable collection of Lipschitz functions. Define another countable collection of Lipschitz function from $\tilde{P} \times \tilde{\Pi_1} \times \tilde{\Pi_2}$ into $\hat{\Pi}^2$, $\{\zeta_i\}_{i=1}^{\infty} = \{\{\zeta_{ib}\}_{b \in B_i}\}_{i=1}^{\infty}$ to be $\zeta_{ib}(p,c,d) = (p,c,d,T(\zeta_{ib}(p,c,d)))$ where $(p,c,d) \in W_b$ and $T(\zeta_b(p,c,d))$ is the component projection of $\zeta_{ib}(p,c,d)$ into $\hat{\Pi}^2$. The collection $\{\zeta_i\}_{i=1}^{\infty}$ along with the sets $\{S_i\}_{i=1}^{\infty} = \{\mathcal{W}_i\}_{i=1}^{\infty}$ have the stated properties.

Proof. Let $\mu_{(\hat{\Gamma}_C^{2N^A},\hat{\Pi}^{2N^E})}$ be the (product) Lebesgue measure over $\hat{\Gamma}_C^{2N^A} \times \hat{\Pi}^{2N^E}$, corresponding to the the Lebesgue measure $\mu_{\hat{\Gamma}_C^{2N^A}}$ over $\hat{\Gamma}_C^{2N^A}$ and Lebesgue measure $\mu_{\hat{\Pi}^{2N^E}}$ over $\hat{\Pi}^{2N^E}$. Define

$$T(B(\hat{\gamma}_1,\hat{\gamma}_2)) = \{(\hat{\boldsymbol{\pi}}_{\mathbf{1}},\hat{\boldsymbol{\pi}}_{\mathbf{2}}) \in \hat{\Pi}^{2N^E} : (\hat{\boldsymbol{\gamma}}_{\mathbf{1}},\hat{\boldsymbol{\gamma}}_{\mathbf{2}},\hat{\boldsymbol{\pi}}_{\mathbf{1}},\hat{\boldsymbol{\pi}}_{\mathbf{2}}) \in T(B)\}. \tag{5.9}$$

We now employ the properties of the product measure and note that

$$\mu_{\hat{\Gamma}_C^{2N^A},\hat{\Pi}^{2N^E}}(T(B)) = \int_{\hat{\Gamma}^{2N^A}} \mu_{\hat{P}i^{2N^E}}(T(B(\hat{\boldsymbol{\gamma}}_1,\hat{\boldsymbol{\gamma}}_2)))d\mu_{\hat{\Gamma}^{2N^A}} \tag{5.10}$$

From this it can be seen that if $\mu_{\hat{\Pi}^{2N^E}}(T(B(\hat{\boldsymbol{\gamma}}_1,\hat{\boldsymbol{\gamma}}_2))) = 0$ for $\mu_{\hat{\Gamma}_C^{2N^A}}$-almost all $\hat{\boldsymbol{\gamma}}_1,\hat{\boldsymbol{\gamma}}_2$, then $\mu_{\hat{\Gamma}_C^{2N^A},\hat{\Pi}^{2N^E}}(T(B)) = 0$. Thus, using the result of proposition 5 we get that in $\hat{\Gamma}_C^{2N^A} \times \hat{\Pi}^{2N^E}$, the set $T(B)$ of confounding beliefs is of $\mu_{\hat{\Gamma}_C^{2N^A},\hat{\Pi}^{2N^E}}$ measure zero.

As in Radner (1979), one may then extend this result to show that for any finite set of joint signals, the set of beliefs that lead an FCE to be confounding has measure zero, which in turns yields generic existence of fully-revealing REE. □

6 Conclusion

We show that when the REE concept is extended to include traders whose preferences are not of the SEU form, the information content of prices can vary significantly. That is, on a large set of economies equilibria can be partially revealing. These results come from the fact that the demand of an AA investor can be insensitive to changes in information when the investor is fully insuring. These results with non-differentiable utility representations can be contrasted with the work of Radner (1979), Allen (1981), and Allen (1982), which showed that at least in the lower-dimensional case, smooth preferences do not permit robust partial revelation.[25]

An aspect of our construction is that the AA trader's information is not revealed since he does not change his portfolio when his information changes, hence his utility remains the same across the signals. This raises the prospect of the 'Grossman and Stiglitz (1980) paradox'. While some information may be 'costly' to acquire, it is not clear generally that all information in markets would have such cost, for instance information obtained from expertise in one industry being applied to trading stocks in another. More generally, the presence of heterogeneous information does not seem to be predicated on information costs. Finally, recent work (Muendler 2007, Krebs 2008) suggests that the co-existence of informationally efficient prices and costly information is not paradoxical.

Another aspect is of course that we only address the possibility of partial revelation by showing it occurs for appropriate parameter sets. If the economy were somehow restricted so that all AA traders had a common set of prior beliefs (and all SEU traders had a common prior) over signals and states, then partially-revealing REE as constructed here may or may not exist depending on the endowments and von-Neumann Morgenstern utilities of the traders. That is, it is possible to construct examples where despite the presence of ambiguity aversion, no AA trader can fully-insure across at least two information signals, thereby precluding robust partial revelation. On the other hand, it is also possible to construct examples using the method described here so that such partial revelation exists. Unfortunately, these examples depend heavily on the specific parametrization – choice of von Neumann-

[25] In particular, Allen (1981), and Allen (1982) establish this for the case of smooth price functions under smooth preferences.

Morgenstern utility, endowment distribution, etc. – adopted. In sum, whether the ambiguity needed for partial revelation to exist is reasonable or not depends on the specific application at hand.[26]

In summary, the inclusion of AA investors implies that prices have a much richer set of possible information transmission capabilities. This finding suggests that the presence of ambiguity aversion in markets could have even broader implications than those in the studies cited in the introduction.

[26]Another way to think about the issue would be with a common set of priors, but differing ambiguity attitude such as in Ghirardato, Maccheroni, and Marinacci (2004) or Gajdos, Hayashi, Tallon, and Vergnaud (2008). However, here too the same problem regarding a general formulation seems to remain.

A Appendix

A.1 Beliefs

This section provides specific details on the space of beliefs for AA and SEU investors. First, we describe the beliefs of an AA investor and embed these beliefs in a finite-dimensional Euclidean space, using assumption 2, which allows us to use Lebesgue measure on the space of beliefs.

Given assumption 2, each of the conditional beliefs of AA investor $n \in \mathcal{N}^A$, $\{\gamma(f)\}_{f\in\mathcal{F}}$, can be characterized by a set of L probability distributions in $\Delta^{|\Omega|}$. Hence, the beliefs for AA investor n can be characterized by an element of the space $\Delta^{|\Omega|L|\mathcal{F}|}$. Note that although any point in $\Delta^{|\Omega|L|\mathcal{F}|}$ represents a set of beliefs that meets assumption 2, there are always multiple elements of $\Delta^{|\Omega|L|\mathcal{F}|}$ that represent the same beliefs over Ω since the order of the distributions $(\pi_1, \ldots, \pi_L)$ that define $\gamma(\sigma)$ doesn't matter when generating the convex hull of these points. The fact that the map from points in $\Delta^{|\Omega|L|\mathcal{F}|}$ into beliefs on Σ is not injective will not matter for the applications in this paper. Recall that $\Gamma \subseteq \Delta^{|\Omega|L|\mathcal{F}|}$.

This formulation of beliefs is used as it eases the discussion of convergence of beliefs. A sequence $\gamma^k \subseteq \Gamma$ is said to converge to some element $\gamma \in \Gamma$ if it converges in the (standard) Euclidean metric.

To obtain the robustness results which were used to prove proposition 3, we used lemma 4.1, which is proved below.

Proof of lemma 4.1. For any $\hat{\gamma} \in \hat{\Gamma}$ and $B \subseteq \Delta^{|\Omega|}$ closed, define the function

$$d(\hat{\gamma}, B) = \inf_{\hat{\pi}' \in \mathrm{rbd}\,\hat{\gamma}, \hat{\pi}'' \in B} ||\hat{\pi}' - \hat{\pi}''|| \tag{A.1}$$

where $\operatorname{rbd}\hat{\gamma}$ denotes the relative boundary of the closed set $\hat{\gamma}$.[27] Note that $d(\hat{\gamma}, B) = 0$ iff $\hat{\gamma} \cap B \neq \emptyset$, but $d(\cdot)$ is not a metric. We adopt it here to simplify working in the finite-dimensional space of extreme points though the result of the lemma holds more generally with the Hausdorff metric as discussed after this proof.

Let $B(\hat{\pi}, \epsilon_{\hat{\pi}})$ be an open ball containing $\hat{\pi}$ such that its closure $\bar{B}(\hat{\pi}, \epsilon_{\hat{\pi}}) \in \operatorname{int}\gamma(\sigma)$. The function $d(\gamma(\sigma), B)$ is continuous in γ if B is a compact set.[28] To see this, let $\{\pi_1, \ldots, \pi_L\}$ be the extreme points of the set $\gamma(\sigma)$ and Q be the finite set of facets ($L-1$ dimensional faces) of the convex polytope $\gamma(\sigma)$, equation (A.1) can be rewritten as

$$
\begin{aligned}
d(\gamma(\sigma), B) &= \min_{\pi' \in \operatorname{bd}\gamma(\sigma), \pi \in B} \|\pi' - \pi\| \\
= \min_{q \in Q, \pi \in B} \|q - \pi\| &= \min_{q \in Q} \min_{\pi' \in q, \pi \in B} \|\pi' - \pi\| \\
&= \min_{q \in Q} \min_{\alpha \in \Delta^L, \pi \in B} \left\| \sum_l \alpha_l \pi_l - \pi \right\|
\end{aligned}
$$

The facets of $\gamma(\sigma)$ are found by taking convex combinations of sets of $(L-1)$ vertices of $\gamma(\sigma)$. For each facet q, the function $\min_{\pi' \in q, \pi \in B} \|\pi' - \pi\| = \min_{\alpha \in \Delta^L, \pi \in B} \|\sum_l \alpha_l \pi_l - \pi\|$ is absolutely continuous over q in its vertices (since q is a compact set). Therefore the distance $\min_{q \in Q} ||q - \pi||$ is the minimum of a finite set of absolutely continuous functions and is therefore absolutely continuous in the set of vertices.

Since $d(\gamma(\sigma), B)$ is continuous in γ, there exists a neighborhood $B(\gamma, \epsilon^1_\gamma)$ such that for all $\gamma' \in B(\gamma, \epsilon^1_\gamma)$, $d(\gamma'(\sigma), \bar{B}(\pi, \epsilon^1_\pi)) > 0$. From this it can be shown that if

[27] That is, the boundary relative to its affine hull $\Delta^{|\Omega|}$.

[28] Recall that γ is an element of the finite-dimensional space Γ meaning that $\gamma(\sigma)$ and $\gamma(\sigma')$ belong to the finite dimensional space $\Delta^{|\Omega|L}$

$\gamma' \in B(\gamma, \epsilon^1_\gamma)$, then $B(\pi, \epsilon^1_\pi) \in \operatorname{int} \gamma'(\sigma)$.

This process is then repeated for $\gamma(\sigma')$ to show that there exists an ϵ^2_π and ϵ^2_γ such that $B(\pi, \epsilon^2_\pi) \subset \operatorname{int} \gamma'(\sigma')$ for all $\gamma' \in B(\gamma, \epsilon^2_\gamma))$.

Then, choosing ϵ_π and ϵ_γ so that $B(\pi, \epsilon_\pi) \subset B(\pi, \epsilon^1_\pi) \cap B(\pi, \epsilon^2_\pi)$ and $B(\gamma, \epsilon_\gamma) \subset B(\gamma, \epsilon^1_\gamma) \cap B(\gamma, \epsilon^2_\gamma)$ it is seen that for any $\gamma' \in B(\gamma, \epsilon_\gamma)$, $B(\pi, \epsilon_\pi) \subset \gamma'(\sigma) \cap \gamma'(\sigma')$.

□

An alternative proof of this lemma that imposes the Hausdorff metric topology directly on the space of beliefs Γ can be found in section A.3. Since this is an important result for the paper, the proof shows that these results can be generalized to the infinite-dimensional space $\mathcal{C}(\Delta^{|\Omega|})$ (i.e. beliefs that need not satisfy assumption 2). However, as we mentioned earlier, this more general space poses challenges for other proofs.

Characterizing the space of beliefs for SEU investors is more intuitive. The beliefs π^n for any SEU investor $n \in \mathcal{N}^E$ can be described by a marginal distribution $\pi^n_\Sigma \in \Delta^{|\Sigma|}$ over the set of joint signals Σ and a set of conditional distributions, one for each joint signal σ, denoted $(\hat{\pi}^n_\sigma)_{\sigma \in \Sigma} \in \Delta^{|\Omega||\Sigma|}$. The following result is used implicitly in proposition 3 and follows directly using Bayes rule.

Lemma A.1. *For any $\sigma, \sigma' \in \Sigma$ and $\epsilon > 0$ there exists a $\delta > 0$ such that if $||\pi'^n - \pi^n|| \leq \delta$ with $\pi'^n \in \Pi$ then $||\pi'^n(\{\sigma, \sigma'\}) - \pi^n(\{\sigma, \sigma'\})|| < \epsilon$.*

The proof of lemma 4.2 is presented next.

Proof of lemma 4.2. It suffices to see that $x(p, \hat{\gamma}^A)$ is differentiable in $\hat{\pi}_1$ since $x(p, \hat{\gamma}^A)$ is differentiable at the vertices $((\hat{\pi}_l)_{l \in \{2,\ldots,L\}})$ and its derivative is zero. This can be seen because local changes in these vertices do not change the minimizing distribution in $\hat{\gamma}^A$ and hence does not change demand.

Let $x(p, \hat{\pi})$ represent the demand of an SEU investor with belief $\hat{\pi} \in \Delta^{|\Omega|}$, endowment e^A, and von Neumann-Morgenstern utility u^A. With $\hat{\gamma}(\hat{\pi}) \in \hat{\Gamma}$ denoting the set of beliefs generated by $(\hat{\pi}, (\hat{\pi}_l)_{l\in\{2,\dots,L\}})$, let $x(p, \hat{\gamma}_{\hat{\pi}})$ denote the demand of an AA investor with beliefs $\hat{\gamma}_{\hat{\pi}}$.

Note that $x(p, \hat{\pi})$ is differentiable at $\hat{\pi}$ and $x(p, \hat{\pi}_1) = x(p, \hat{\gamma}^A)$ by construction. Since $\mathbb{E}_{\hat{\pi}}[u^A(x)]$ is continuous in $\hat{\pi}$ and x (by the continuity of u^A), we have that for every $\hat{\pi}$ in some open neighborhood $B_{\hat{\pi}_1}$ of $\hat{\pi}_1$

$$\mathbb{E}_{\hat{\pi}}[u^A(x(p, \hat{\pi}))] < \mathbb{E}_{\hat{\pi}_l}[u^A(x(p, \hat{\pi}))], \text{ for all } l \in \{2, \dots, L\}. \tag{A.2}$$

By the definition of $x(p, \hat{\pi})$, for all $\hat{\pi} \in B_{\pi_1}$

$$\begin{aligned} \hat{\pi} \circ u^A(x(p, \hat{\pi})) - \lambda^A p = 0 \\ p(e^A - x(p, \hat{\pi})) = 0 \end{aligned} \tag{A.3}$$

where λ^A denotes the corresponding Lagrange multiplier.

The demand $x(p, \hat{\pi})$ satisfies the optimality conditions for AA investor A, for all $\hat{\pi} \in B_{\pi_1}$.[29] Thus, $x(p, \gamma_{\hat{\pi}}) = x(p, \hat{\pi})$ in $B_{\hat{\pi}_1}$ and the result follows. □

A.2 Optimization for AA investors

This section collects results on optimization for the AA investors. Some of the results may be found in Aubin (1979) and in cases where the proof is not expositionally

[29] The first-order optimality conditions are given in appendix A.2.

important and can be found elsewhere we have cited an appropriate reference. In what follows, we consider any AA investor $n \in \mathcal{N}^A$ whose preferences satisfy assumptions 1 and 2. Recall that for $v, v' \in \mathbb{R}^{|\Omega|}$, $v \circ v' \in \mathbb{R}^{|\Omega|}$ denotes $(v_1 v'_1, \ldots, v_{|\Omega|} v'_{|\Omega|}) \in \mathbb{R}^{|\Omega|}$.

The set $\partial U^n(x_0)$ for $x_0 \in \mathbb{R}^{|\Omega|}_+$ is defined as

$$\partial U^n(x_0) = \{p \in \mathbb{R}^{|\Omega|} : U^n(x) - U^n(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in \mathbb{R}^{||}\} \tag{A.4}$$

and is called the *superdifferential* of U^n at x_0.[30]

If $x_0 >> 0$ maximizes U^n over $R^{|\Omega|}_+$, then

$$0 \in \partial U^n(x_0). \tag{A.5}$$

If U^n is strictly concave then condition (A.5) is also sufficient for x_0 to be a maximum.

The next two results on AA preference representations that satisfy assumptions 1 and 2 (that they are monotonic, concave and satisfy an Inada condition at zero) have proofs that are straightforward and hence ommitted.

Lemma A.2. *If investor n satisfies assumptions 1 and 2 then $U^n(x)$ is strictly increasing in each of its arguments and strictly concave.*

Lemma A.3. *For any price vector $p \in \mathbb{R}^{|\Omega|}_{++}$, if*

$$x_0 \in \arg\max_{x \in \mathbb{B}(e^n, p)} U^n(x) \tag{A.6}$$

[30] The superdifferential and generalized Jacobian of U^n coincide since it is concave (lemma A.2 below), see Clarke (1983).

then $x_0 >> 0$.

In what follows, let $g_p^n(x) \equiv p(e^n - x)$ and for $\lambda^n \in \mathbb{R}_+, \mathcal{L}^n(x,\lambda) \equiv U^n(x) + \lambda^n g_p^n(x)$.

Lemma A.4. *There there exists a Lagrange multiplier* $\lambda^n \in \mathbb{R}_+$ *such that*

$$\max_{x \in \mathbb{B}(e^n, p)} U^n(x) = \max_{x \in \mathbb{R}_+^{|\Omega|}} [U^n(x) + \lambda^n g_p^n(x)] \tag{A.7}$$

and $\mathcal{L}^n(x, \lambda^n)$ *is strictly concave in* x.

Proof. U^n is continuous and concave and meets the necessary constraint qualification in (Aubin 1979)[5.3.1, Theorem 1]. □

Given the strict concavity of $\mathcal{L}^n(x,\lambda)$ in x, a necessary and sufficient condition for a solution to $\max_{x \in \mathbb{R}_+^{|\Omega|}} \mathcal{L}^n(x,\lambda)$ for a given λ is that

$$0 \in \partial \mathcal{L}^n(x, \lambda) \tag{A.8}$$

To derive the first order conditions for an AA investor the following lemma is needed.

Lemma A.5. *For* $\mathcal{L}^n(\cdot)$,

$$\partial \mathcal{L}^n(x) = \partial U^n(x) + \partial g_p^n(x) = \partial U^n(x) + g_p'^n(x). \tag{A.9}$$

Proof. It is straightforward to show that $\partial U^n(x) + \partial g_p^n(x) \subseteq \partial \mathcal{L}^n(x)$. We show that $\partial \mathcal{L}^n(x) \subseteq \partial U^n(x) + \partial g_p^n(x)$.

Suppose that there exists $p \in \partial \mathcal{L}^n(x)$ such that $p \notin \partial U^n(x) + \partial g_p^n(x)$. By assumption

$$U^n(y) + g_p^n(y) - U^n(x) - g_p^n(x) \leq p(y-x) \text{ for all } y \in \mathbb{R}_+^{|\Omega|}. \tag{A.10}$$

Since g_p^n is affine, $\partial g_p^n(x) = \{g_p'^n(x)\}$ and one may define $p = g_n'^n(x) + p'$. Since $p' + g_p'^n(x) \notin \partial U^n(x) + \partial g_p^n(x) = \partial U^n(x) + g_p'^n(x)$ this implies that $p' \notin \partial U^n(x)$. Therefore, there exists $y^* \in \mathbb{R}_+^{|\Omega|}$ such that

$$U^n(y^*) - U^n(x) > p'(y^* - x). \tag{A.11}$$

and since g_p^n is affine,

$$g_p^n(y^*) - g_p^n(x) = g_p^{'n}(x)(y^* - x). \tag{A.12}$$

Summing (A.11) and (A.12) yields

$$U^n(y^*) + g_p^n(y^*) - U^n(x) - g_p^n(x) > (p' + g_p^{'n}(x))(y^* - x) \tag{A.13}$$

which contradicts (A.10). □

Applying lemma A.5 implies the following necessary condition for AA investors.

$$0 \in \partial U^n(x_0) - \lambda^n p. \tag{A.14}$$

Assumption 1 ensures that all optimal allocations are interior. This fact, combined with the previous lemma implies the following result.

Proposition 6. *If* $x_0 >> 0$ *solves the problem*

$$\max_{x \in \mathbb{B}(e^n, p)} U^n(x) \tag{A.15}$$

then

$$\begin{aligned} &\lambda^n p \in \partial U^n(x_0) \\ &p(e^n - x_0) = 0 \end{aligned} \tag{A.16}$$

The conditions (A.16) are necessary and sufficient for utility maximization since $U^n(\cdot)$ is strictly concave. The next corollary provides a sufficient condition for two AA investors to behave identically in an Arrow-Debreu equilibrium, which underlies the robustness of the partially-revealing REE discussed in the paper.

Corollary 1. *For two investors* $m, n \in \mathcal{N}^A$*, suppose* $e^m = e^n$ *and that for the allocation* $x_0 \in \mathbb{R}_+^{|\Omega|}$*,* $\partial U^m(x_0) \subseteq \partial U^n(x_0)$*. Then if* x_0 *solves*

$$\max_{x \in \mathbb{B}(e^m, p)} U^m(x) \tag{A.17}$$

then it is also solves

$$\max_{x \in \mathbb{B}(e^n, p)} U^n(x) \tag{A.18}$$

Proof. If (A.16) holds for investor m then it must hold for investor n as well since the budget constraints are the same and the FOCs given in equation (A.14) for investor m imply that equation (A.14) holds for investor n. □

The following characterization is useful to obtain an idea of the geometry of the set $\partial U^n(x)$.

Definition 7. *Define the derivative from the right of* U^n *at* x_0 *in the direction* y *to* *be*

$$D_+U^n(x_0)(y) = \lim_{\alpha\to 0+} \frac{U^n(x_0+\alpha y) - U^n(x_0)}{\alpha} \tag{A.19}$$

Note that y need not be in $\mathbb{R}_+^{|\Omega|}$, although for sufficiently small α it must be true that $x_0 + \alpha y \in \mathbb{R}_+^{|\Omega|}$ since U^n is only defined over $\mathbb{R}_+^{|\Omega|}$. The next result then follows from Aubin (1979)[Section 4.3.2, proposition 4].

Lemma A.6.

$$D_+U^n(x_0)(y) = \min_{p\in\partial U^n(x_0)} py \tag{A.20}$$

Therefore we have that for x_0,

$$\partial U^n(x_0) = \{p \in \mathbb{R}_{++}^{|\Omega|} : py \geq D_+U^n(x_0)(y) \text{ for all } y \in \mathbb{R}^{|\Omega|}\} \tag{A.21}$$

$D_+U^n(x)(\cdot)$ is the support function of $\partial U^n(x)$ and we turn to it better understand $\partial U^n(x)$. The following is a consequence of Aubin (1979)[Section 4.3.3, proposition 6].

Lemma A.7. *For $\hat{\gamma}^n \in \hat{\Gamma}$, let $\hat{\gamma}_0^n(x) = \{\hat{\pi} \in \hat{\gamma}^n : U^n(x) = \mathbb{E}_{\hat{\pi}}[u^n(x)]\}$.*

$$D_+U^n(x_0)(y) = \min_{\hat{\pi}\in\hat{\gamma}_0^n(x)} D_+(\mathbb{E}_{\hat{\pi}}[u^n(x)])(y) \tag{A.22}$$

Standard calculus shows that

$$\begin{aligned} D_+(\mathbb{E}_{\hat{\pi}}[u^n(x)])(y) &= \lim_{\alpha\to 0^+} \frac{\mathbb{E}_{\hat{\pi}}[u^n(x+\alpha y)] - \mathbb{E}_{\hat{\pi}}[u^n(x)]}{\alpha} \\ &= \frac{d}{d\alpha}\mathbb{E}_{\hat{\pi}}[u^n(x+\alpha y)]|_{\alpha=0^+} \\ &= \mathbb{E}_{\hat{\pi}}[u'^n(x)y] \end{aligned} \tag{A.23}$$

Applying this to the definition of $\partial U^n(x_0)$ gives

$$\partial U^n(x_0) = \{p \in \mathbb{R}^{|\Omega|}_{++} : py \geq \min_{\hat{\pi} \in \hat{\gamma}_0(x_0)} \mathbb{E}_{\hat{\pi}}[u'^n(x)](y) \text{ for all } y \in \mathbb{R}^{|\Omega|}\}. \tag{A.24}$$

From this, the superdifferential for some allocations can be calculated directly.

Lemma A.8. *Let $x \in \mathbb{R}^{|\Omega|}_{+}$ be a consumption allocation for AA investor n such that $x(\omega) = x(\omega')$ for all $\omega, \omega' \in \Omega$.*

$$\partial U^n(x) = u'^n(x) \circ \hat{\gamma}^n. \tag{A.25}$$

Proof. Noting that $u'^n(x(\omega)) = u'^n(x(\omega'))$ for all ω, ω' and applying equation (A.24) gives

$$\partial U^n(x) = \{p \in {}^{|\Omega|}_{++} : py \geq u'^n(x) \min_{\hat{\pi} \in \hat{\gamma}^n_0(x)} \mathbb{E}_{\hat{\pi}}[y] \text{ for all } y \in \mathbb{R}^{|\Omega|}\}. \tag{A.26}$$

As defined in lemma A.7, $\hat{\gamma}^n_0(x) = \hat{\gamma}^n$ for the allocation x since all probability distributions in $\hat{\gamma}^n$ give the same expected value for the constant random variable $u^n(x)$. One can rewrite equation (A.26) as

$$\partial U^n(x) = \{p \in \mathbb{R}^{|\Omega}_{++} : \frac{py}{u'^n(x)} \geq \min_{\hat{\pi} \in \hat{\gamma}^n} \hat{\pi} y \text{ for all } y \in \mathbb{R}^{|\Omega|}\}. \tag{A.27}$$

The right hand side of the inequality in the definition is the support function for the set $\hat{\gamma}$, hence $pu'^n(x) \in \hat{\gamma}$. □

Alternatively, if for a particular allocation x, $\hat{\gamma}^n_0(x)$ is a singleton, equation (A.24) reduces to

$$\partial U^n(x) = \{[\hat{\pi}(\omega)u'^n(x(\omega))]_{\omega\in\Omega}\}. \tag{A.28}$$

and the general case follows as below.

Lemma A.9. *The superdifferential of U^n at $x \in \mathbb{R}_+^{|\Omega|}$ is given by*

$$\partial U^n(x) = u'^n(x) \circ \hat{\gamma}_0^n(x) \tag{A.29}$$

Proof. The following manipulation of the definition of $\partial U^n(x)$ gives the result.

$$\begin{aligned} \partial U^n(x) = & \quad \{p \in \mathbb{R}_{++}^{|\Omega|} : py \geq \min_{\hat{\pi}\in\hat{\gamma}_0^n(x)} \mathbb{E}_{\hat{\pi}}[u'^n(x)y] \text{ for all } y \in \mathbb{R}^{|\Omega|}\} \\ = & \quad \{p \in \mathbb{R}_{++}^{|\Omega|} : py \geq \min_{q\in u'^n(x)\cdot\hat{\gamma}_0^n(x)} qy \text{ for all } y \in \mathbb{R}^{|\Omega|}\} \end{aligned} \tag{A.30}$$

□

Corollary 2. *An allocation $x_0 \in \mathbb{R}_+^{|\Omega|}$ is a solution to*

$$\max U^n(x) \;\; s.t. \;\; x \in \mathbb{B}(e^n, p) \tag{A.31}$$

if and only if

$$\begin{aligned} \lambda^n p \in \hat{\gamma}_0^n(x_0) \circ u'^n(x_0) \\ p(e^n - x_0) = 0 \end{aligned} \tag{A.32}$$

Corollary 3. *Let $n, m \in \mathcal{N}^A$ be two different AA investors with beliefs $\hat{\gamma}^m$ and $\hat{\gamma}^n$, but identical von Neumann-Morgenstern utility functions $u^n = u^m$ and endowments*

($e^m = e^n$). If $\hat{\gamma}_0^m(x) \subseteq \hat{\gamma}_0^n(x)$ then if $x \in \mathbb{R}_+^{|\Omega|}$ solves

$$\max U^m(x) \;\; s.t. \;\; x \in \mathbb{B}(e^m, p) \tag{A.33}$$

then x also solves

$$\max U^n(x) \;\; s.t. \;\; x \in \mathbb{B}(e^n, p) \tag{A.34}$$

Proof. Inspecting equations (A.16) shows that any solution to these equations for beliefs $\hat{\gamma}^m$ is also a solution for beliefs $\hat{\gamma}^n$. □

Proposition 7. *Let $(x, p) \in \mathbb{R}_+^{N|\Omega|} \times P$ be an Arrow-Debreu equilibrium for the economy characterized by $\hat{\boldsymbol{\gamma}} = (\hat{\gamma}^n) \in \hat{\Gamma}^N$. If $\hat{\boldsymbol{\gamma}}' = (\hat{\gamma}'^n)$ satisfies $\hat{\gamma}_0^n(x^n) \subseteq \hat{\gamma}_0'^n(x^n)$ then (x, p) is an equilibrium for the economy described by $\hat{\boldsymbol{\gamma}}$ as well.*

Proof. Applying corollary 3 for each investor shows that the allocation x continues to be utility maximizing for all investors. Markets must also clear since (x, p) is an equilibrium for the economy $\hat{\boldsymbol{\gamma}}$. □

A.3 The Hausdorff metric

First we show that if the Hausdorff metric is applied to the the space of AA beliefs, then the result of lemma A.1 holds.

Proof of lemma A.1. For any two compact and convex sets $K, K' \subseteq \Delta^{|\Omega|-1}$, let $d_H(A, B)$ be the distance between these two sets as defined by the Hausdorff metric,

$$d_H(K, K') = \max\{\sup_{\hat{\pi} \in K} \inf_{\hat{\pi}' \in K'} ||\hat{\pi} - \hat{\pi}'||, \sup_{\hat{\pi}' \in K'} \inf_{\hat{\pi} \in K} ||\hat{\pi} - \hat{\pi}'||\} \tag{A.35}$$

For u in the $|\Omega|-1$ dimensional unit circle $S^{|\Omega|-1}$, define $h(K,u) = \sup_{\hat{\pi}\in K} K \cdot u$ to be the support function of the set K at u. Then by Schneider (2003)[Theorem 1.8.11],

$$d_H(K,K') = \sup_{u\in S^{|\Omega|-1}} |h(K,u) - h(K',u)|. \tag{A.36}$$

Recall that if the set $K \subset K'$ then for all $u \in S^{|\Omega|-1}$, $h(K,u) \leq h(K',u)$ and if the inclusion becomes strict then the inequality also becomes strict. To begin the proof, define $\epsilon_{\hat{\pi}}$, such that the closure $\bar{B}(\hat{\pi}, \epsilon_{\hat{\pi}}) \subset \hat{\gamma}$ of an open ball $B(\hat{\pi}, \epsilon_{\hat{\pi}})$ and let

$$\epsilon = \min_{u\in S^{|\Omega|-1}} |h(B(\pi, \epsilon_{\hat{\pi}}), u) - h(\gamma, u)| > 0. \tag{A.37}$$

Let $B(\hat{\gamma}, \epsilon/2)$ be the set of polytopes generated by at most L extreme points that have Hausdorff distance from $\hat{\gamma}$ less than $\epsilon/2$. Then as we show next, for each $\hat{\gamma}' \in B(\hat{\gamma}, \epsilon/2)$, $\bar{B}(\hat{\pi}, \epsilon_{\hat{\pi}}) \subset \hat{\gamma}'$.

Suppose to the contrary that there exists some $\hat{\gamma}' \in B(\hat{\gamma}, \epsilon/2)$ with the property that $\bar{B}(\hat{\pi}, \epsilon_{\hat{\pi}})$ is not a subset of $\hat{\gamma}'$. Then it must be true that for some $\hat{\pi}' \in \bar{B}(\hat{\pi}, \epsilon_{\hat{\pi}})$ there exists a $\bar{u}' \in S^{|\Omega|-1}$ such that

$$h(\{\hat{\pi}'\}, u') > h(\hat{\gamma}', u'). \tag{A.38}$$

Given that $|h(\gamma, u') - h(\gamma', u')| < \epsilon/2$, if $-\epsilon/2 < h(\gamma, u') - h(\gamma', u') < 0$, then

$$h(\{\hat{\pi}'\}, u') - h(\hat{\gamma}, u') > \epsilon/2 > 0. \tag{A.39}$$

which implies that $\hat{\pi}' \notin \gamma$, yielding a contradiction.

On the other hand, if $\epsilon/2 > h(\gamma, u') - h(\gamma', u') > 0$, then

$$h(\hat{\gamma}, u') - h(\{\hat{\pi}'\}, u') < \epsilon/2. \tag{A.40}$$

Since $\{\hat{\pi}'\} \subset \bar{B}(\pi, \epsilon_{\hat{\pi}}) \subset \hat{\gamma}$ this contradicts the definition of ϵ given in (A.37). □

A.4 Ancillary results for generic full revelation

The next results follow from the properties of Lipschitz functions and the Lebesgue measure on $\mathbb{R}^m$.

Lemma A.10. *A set $A \subset \mathbb{R}^m$ has Lebesgue measure zero iff for each $\epsilon > 0$ there exists a countable set of cubes $\{S_i^\epsilon\}_{i=1}^\infty$ such that for each ϵ, (i) $A \subseteq \cup_{i=1}^\infty S_i^\epsilon$ and (ii)$\sum_{i=1}^\infty \mu(S_i^\epsilon) < \epsilon$.*

Lemma A.11. *Let $A \subset \mathbb{R}^m$ and suppose that A has Lebesgue measure 0 in $\mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}^k$ is a Lipschitz continuous function then $f(A)$ has Lebesgue measure 0 in $\mathbb{R}^k$.*

Proof. Let μ_m represent Lebesgue measure in $\mathbb{R}^m$. Since $\mu_m(A) = 0$, for any $\epsilon > 0$ there exists a set of cubes $\{S_i^\epsilon\}_{i=1}^\infty$ that satisfies (i) $A \subseteq \cup_{i=1}^\infty S_i^\epsilon$ and (ii) $\sum_{i=1}^\infty \mu(S_i^\epsilon) < \epsilon$.

Let δ_i^ϵ be side length of the cube S_i^ϵ. The volume of each cube S_i^ϵ is given by $V(S_i^\epsilon) = (\delta_i^\epsilon)^m$. By the Lipschitz continuity of f, there exists $K > 0$ such that for each $x, y \in A$, $||f(x) - f(y)||_k < K||x - y||_m$. Hence, denoting the center of the cube S_i^ϵ by a_i^ϵ, it follows that $f(S_i^\epsilon)$ must be contained in a cube R_i^ϵ with center $f(a_i^\epsilon)$ and side length less than or equal to $K\delta_i^\epsilon$.

The volume of $f(S_i^\epsilon)$ satisfies $V(f(S_i^\epsilon)) \leq V(R_i^\epsilon) = (K\delta_i^\epsilon)^k = K^k V(S_i^\epsilon)$. Finally, for any $\epsilon > 0$, one may select a set of cubes $\{R_i^\epsilon\}_{i=1}^\infty$ that satisfies the conditions of lemma A.10 by using the above procedure and selecting a cover of A that has total volume less than or equal to $\epsilon/(K^k)$. □

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