CONSISTENT INTERTEMPORAL DECISION-MAKING UNDER UNCERTAINTY*

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Paul Weller

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* I owe a substantial debt to Peter Hammond, who first aroused my interest in this area, for many valuable suggestions. All remaining errors are my own.

Introduction

Strotz (6) seems to have been the first to provide a formal analysis of the problem of inconsistency in intertemporal decision-making. An individual who plans for the future may find that as time elapses he wishes to revise his plan. Strotz examined the form of the discount function as a possible source of inconsistency, when there was no uncertainty facing the individual. His conclusions have subsequently been refined in work by Pollak (4), Heal (3) and Burness (1).

Hammond (2) has examined the problem in a more general context, and it is essentially his framework I use in what follows to characterise consistent planning under uncertainty. I prove that under certain assumptions, consistency is equivalent to maximising expected utility on the set of feasible plans, with a restricted set of utility functions and a tree of subjective probability distributions which satisfy a Bayesian updating rule.

The Model

Time is assumed to be discrete, and in each period, one of a possible set of states of the world is realised. The state will in general depend on those states which have preceded it. An event will be a sequence of states $\langle s_1, s_2 \ldots \rangle$, possibly infinite. S is the set of possible events at the initial planning date, $t = 0$. The choices open to the agent are represented by a decision tree A, which is a set of branches $a = \langle a(o), a(1), \ldots \rangle$ possessing the requisite tree structure (v. Hammond (3)). For each branch $a \in A$, $a(t)$, $t = o,1,2 \ldots$, is a node of the tree. I write n for an arbitrary node, and N for the set of all nodes belonging to A.

In reduced form the agent faces a set X of strategies which map S into Y, the set of outcomes.[1] At each node $n \in N$ the agent faces the same set of outcomes[2], a set $S(n)$ of events still possible, and a set $X(n)$ of strategies $x(n)$, which map $S(n)$ into Y.

If $N(n)$ is the set of possible nodes following n (including n) then the intertemporal decision problem in extensive form is described by the collection $\langle X(n), S(n), N(n) \rangle_{n \in N}$ with the following properties :

(1) $X(n_o) = X$; $S(n_o) = S$; $N(n_o) = N$

where n_o is the initial node of the tree.

(2) If $n' \in N(n)$, then $S(n') \subseteq S(n)$, $N(n') \subseteq N(n)$ and for all $x'(n') \in X(n')$, there exists $x(n) \in X(n)$ such that $x(n',s) = x(n,s) : S(n')$, where $x(n,s)$ is the value of the mapping $x(n)$ at s, and $x(n) : S(n')$ is its restriction to $S(n')$.

I now make the following assumptions :

A.1. There exists a preference ordering $R(n)$ on $X(n)$ for all $n \in N$.

A.2. Each $R(n)$ satisfies the conditions necessary for the Expected Utility theorem to hold.

A.2.1. Each $X(n)$ contains all functions from $S(n)$ to Y.

A.1. is only really restrictive in the case of an infinite horizon. It is known from the literature on optimal capital accumulation that it is not always possible to construct orderings on infinite sequences of outcomes. I rule out such complications. A.2. is an intratemporal consistency condition which guarantees that

$$(3) \quad x_1(n)\ R(n)\ x_2(n) \quad \text{iff} \quad \int_{S(n)} U_n\ (x_1(n,s))\ \ d\Pi(n) \geq \int_{S(n)} U_n(x_2(n,s))\ d\Pi(n)$$

where (i) $U_n\ (\cdot)$ is a real-valued function defined on Y, and unique up to a positive linear transformation.

(ii) $\Pi(n)$ is a unique probability measure on $\mathcal{S}(n)$, a σ-algebra for $S(n)$ [3].

The assumption A.2.1. is included in A.2. but is stated explicitly since it is referred to below.

The following definitions will also be necessary.

D.1. The set $\langle R(n) \rangle_{n \in N}$ will be called a dynamic ordering.

D.2. The dynamic ordering $R(n)_{n \in N}$ is consistent if whenever $n' \in N(n)$ and $x_1(n,s) = x_2(n,s)$ for all $s \in S(n) - S(n')$, then

$$x_1(n)\ R(n)\ x_2(n) \text{ iff } x_1(n) : S(n')\ R(n')\ x_2(n) : S(n')$$

D.3. The set of utility functions $\langle U_n(\cdot) \rangle_{n \in N}$ is consistent if whenever $n' \in N(n)$

$$(4) \quad U_{n'}(y) = a + b\,U_n(y) \qquad b > 0$$

or $U_{n'}$ is equivalent to U_n, $(U_{n'} \sim U_n)$.

D.4. If $\langle (S(n), \mathcal{S}(n), \Pi(n)) \rangle_{n \in N}$ are a set of probability spaces, then the set of probability measures $\langle \Pi(n) \rangle_{n \in N}$ is Bayesian if wherever $n' \in N(n)$

(i) $\mathcal{S}(n') \subseteq \mathcal{S}(n)$

(ii) $\Pi(n')(A) = \dfrac{\Pi(n)(A)}{\Pi(n)(S(n'))}$ for all $A \in \mathcal{S}(n')$

In order to provide some justification for using the term Bayesian, I could write, somewhat less precisely

(ii) $\Pi(n')(A) = \dfrac{\Pi(n)(A \cap S(n'))}{\Pi(n)(S(n'))}$ for all $A \in \mathcal{S}(n)$

Of course $\Pi(n')(A)$ is not strictly speaking defined unless $A \in \mathcal{S}(n')$. But it then becomes clearer that $\Pi(n_o)$ is to be regarded as the prior measure, and $\Pi(n)$ the posterior measure at n.

Result

I shall now state and prove the following result :

If A.1. - 2.1. hold, a dynamic ordering $\langle R(n) \rangle_{n\in N}$ is consistent if and only if there exists some consistent set $\langle U_n(\cdot) \rangle_{n\in N}$ and a unique set of Bayesian probability measures $\langle \Pi(n) \rangle_{n\in N}$ which satisfy the Expected Utility theorm for each $n \in N$ [4]

Proof

(a) Assume that $\langle R(n) \rangle_{n\in N}$ is consistent. Then in order to demonstrate the existence of a consistent set $\langle U_n \rangle_{n\in N}$ and unique Bayesian $\langle \Pi(n) \rangle_{n\in N}$ I need to show that for any arbitrary n, and $n' \in N(n)$

(i) $U_{n'} \sim U_n$

(ii) $\Pi(n')(A) = \dfrac{\Pi(n)(A)}{\Pi(n)(S(n'))}$ $A \in \mathcal{S}(n')$

Select an arbitrary N and $n' \in N(n)$. For any strategies $x'_1(n'), x'_2(n') \in X(n')$ it is possible to construct strategies $x_1(n), x_2(n) \in X(n)$ such that

$$\text{(5)} \qquad \left.\begin{aligned} x_1(n,s) &= x'_1(n',s) \\ x_2\ (n,s) &= x'_2(n',s) \end{aligned}\right\} \qquad s \in S(n')$$

$$\text{(6)} \qquad x_1(n,s) = x_2(n,s) \qquad\qquad s \in S(n) - S(n')$$

A.2.1. is required here to ensure that such strategies can be constructed.

A.2. guarantees that

$$\text{(7)} \quad x_1(n)\ R(n)\ x_2(n) \quad \text{iff} \quad \int_{S(n)} U_n(x_1(n,s))\ d\Pi(n) \geq \int_{S(n)} U_n(x_2(n,s))\ d\Pi(n)$$

$$\text{(8)} \quad x'_1(n')R(n')x'_2(n')\text{iff} \quad \int_{S(n')} U_{n'}(x',s))\ d\Pi(n') \geq \int_{S(n')} U_{n'}(x'_2(n',s))\ d\Pi(n')$$

Since the dynamic ordering is consistent

$$\text{(9)} \quad \int_{S(n)} U_n(x_1(n,s))\ d\Pi(n) \geq \int_{S(n)} U_n(x_2(n,s))\ d\Pi(n) \quad \text{iff}$$

$$\int_{S(n')} U_{n'}(x'_1(n',s))\ d\Pi(n') \geq \int_{S(n')} U_{n'}(x'_2(n',s))\ d\Pi(n')$$

Using (5) and (6)

$$(10) \quad \int_{S(n')} U_n(x'_1(n',s))\ d\Pi(n) \geq \int_{S(n')} U_n(x'_2(n',s))\ d\Pi(n) \quad \text{iff}$$

$$\int_{S(n')} U_{n'}(x'_1(n',s))\ d\Pi(n') \geq \int U_{n'}(x'_2(n',s))\ d\Pi(n')$$

But

$$(11) \quad \int_{S(n')} U_n(n'_1(n',s))\ d\Pi(n) \geq \int_{S(n')} U_n(x'_2(n',s))\ d\Pi(n) \quad \text{iff}$$

$$\frac{1}{\Pi(n)(S(n'))} \int_{S(n')} U_n(x'_1(n',s))\ d\Pi(n) \geq \frac{1}{\Pi(n)(S(n'))} \int_{S(n')} U_n(x'_2(n',s))\ d\Pi(n)$$

and it is clear that $\Pi(n)/\Pi(n)(S(n'))$ is a probability measure on $\mathcal{S}(n')$. In other words, strategies in $X(n)$ are ordered equivalently by $\int_{S(n')} U_{n'}(\cdot)\ d\Pi(n')$ and $\int_{S(n')} U_n(\cdot)\ \frac{d\Pi(n)}{\Pi(n)(S(n'))}$. But the Expected Utility theorem states that :

(i) $\Pi(n)$ is unique

(ii) $U_{n'}$ is unique up to a positive linear transformation.

Therefore $\Pi(n')(A) = \frac{\Pi(n)(A)}{\Pi(n)(S(N'))}$ $A \in \mathcal{A}(n')$

and $U_{n'} \sim U_n$. This completes the proof of necessity.

To demonstrate sufficiency, one need note only that (10) will hold if $\langle U_n \rangle_{n \in N}$ is consistent and $\langle \Pi(n) \rangle_{n \in N}$ Bayesian. (9) then follows given (5) and (6), and consistency follows from A.2.

Footnotes

[1] These may be infinite sequences of single period consequences.

[2] This is not in reality true, but since at each node n the set of single period consequences which have already occurred is uniquely determined, there is no loss of generality and notation is simplified by representing the problem in this way.

[3] In the original Savage theorem (5), Π was a finitely additive measure. To avoid certain technical complications which do not affect the results, I assume that Π is countably additive and use Lebesgue integration.

[4] A Bayesian set of measures $\langle \Pi(n) \rangle_{n \in N}$ is unique if there exists no other Bayesian set defined on the same $\langle \mathcal{E}(n) \rangle_{n \in N}$ also satisfying the Expected Utility theorem for each $n \in N$

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