PRICE CHANGES AND OPTIMUM TAXATION IN A MANY-CONSUMER ECONOMY

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I. INTRODUCTION

This paper follows recent work on the welfare effects of small perturbations from an initial general equilibrium with some distortions. Dixit (1975) and Hatta (1977) have analysed the one-consumer economy, and found several simple prescriptions for policy changes to improve welfare. Guesnerie (1977) has studied the many-consumer case when constant returns to scale prevail or profits are taxed at 100%. Diewert (1977) considers profits as returns to artificially defined fixed factors, and allows taxation of profits at any specified rates. In both of these last two papers it turns out that the simplicity of welfare-improving policies is lost in the many-consumer case. Increases in a Bergson type social welfare function depend very crucially on the distribution of ownership of fixed factors. General formulae mean little, and we must consider very special cases in order to obtain concrete results. Improvements in the Pareto sense are even harder to generate by simple formulae. In this paper I broadly follow the approach of Diewert, but try a different tack by concentrating on the question of when a Bergson or Pareto improvement is <u>not</u> possible, thus obtaining local necessary conditions for welfare optimality or Pareto efficiency of the initial equilibrium. This approach helps shed new light on a question that has been much discussed (e.g. Dasgupta and Stiglitz (1971) and (1972)):if profit taxation alone can yield enough revenue to finance government expenditures, is it optimal to rely on that alone and leave the commodities untaxed? The approach has added advantages of allowing a naturally parallel treatment of welfare optimality and Pareto efficiency,

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and of tying together diverse previous models and results.

2. THE MODEL

I begin by describing production. Let x be the vector of amounts of the commodities produced, and v the vector of fixed factor inputs. Factors like labour are included in the list of commodities with a negative sign as usual. The aggregate production possibilities are described by a set T of vectors (x,v); I shall not need to inquire into the separate technologies of private firms and public enterprises of which T is the sum. I shall assume that T is a closed convex cone. Thus there are constant returns to scale when fixed factors are taken into account. These may be genuine material factors, or artificially defined repositories for pure profits. The latter may be firm-specific, thereby allowing taxation of different firms' profits at different rates.

For most of the time, I shall suppose that T can be described by the functional inequality

$$G(x,v) \leq 0 \tag{1}$$

and an efficient production plan, i.e. one on the frontier of T, by the equation

$$G(x,v) = 0 , \tag{2}$$

where the scalar-valued function G is increasing in x, decreasing in v, concave, homogeneous of degree 1, and finally, differentiable. The last is problematic if the frontier of T has 'ridges', as in fact it is quite likely to do. But I shall assume that a differentiable approximation, including an associated approximation of the supporting prices, is possible, and work with that.

A vector (p,r) proportional to $(G_x,-G_v)$ can be interpreted as the vector of marginal costs and products. On the efficient frontier, these serve as producer prices. An inefficient production point can presumably be sustained by facing different producers with different market or shadow prices; again an explicit description is not necessary for my purpose. Unless explicitly stated otherwise, vectors are columns and ' denotes transposes. Then $x'G_x + v'G_v = G$ by Euler's Theorem, so

$$p'x - r'v \begin{cases} \leq & \text{for a feasible plan} \\ = 0 & \text{for an efficient plan.} \end{cases} \tag{3}$$

Consumers are indexed by h. Consumer h owns a vector v^h of fixed factors, and consumes a vector x^h of commodities. The budget constraint is

$$q'x^h = s'v^h \equiv m^h, \tag{4}$$

where q is the vector of consumer prices of commodities, s is the vector of prices received by the consumers for their fixed factors, and

m^h is transfer income in the units of account chosen. Thus no lump sum taxation is possible. The demand functions are

$$x^h = x^h(q, s'v^h) , \tag{5}$$

and utilities are

$$u^h = V^h(q, s'v^h) , \tag{6}$$

where V^h is the indirect utility function with the usual properties, in particular

$$V^h_q = -\lambda^h x^h , \quad V^h_m = \lambda^h . \tag{7}$$

Assuming non-satiation, the marginal utility of income λ^h is positive for each consumer. Note that the demands and utilities are homogeneous of degree zero in (q,s) jointly. Thus the joint vector (q,s) is indeterminate up to a scale factor, and this scale factor can be chosen independently of the similar choice for (p,r). The total private commodity demand vector will be written as x^p, and the total supply of fixed factors as v.

The government has a fixed commodity requirement x^g. It chooses (q,s), subject to the requirement that the resulting $x = x^p + x^g$ should be producible given the fixed factor inputs v. Then the vector (p,r) emerges up to scale, and in case of an efficient plan, the tax rates are implicitly defined. Similar implicit definitions of firm-specific taxes will emerge for an inefficient production plan. In particular, with production efficiency, we have

$$\Sigma_h \left\{ (q-p)'x^h + (r-s)'v^h \right\} = -p'x^p + r'v \quad \text{using (4)}$$
$$= p'x^g - (p'x - r'v)$$
$$= p'x^g \quad \text{using (3)}.$$

Thus the government's budget is automatically balanced, and need not be considered as another constraint. A similar but messier argument can be given in case of production inefficiency, allowing different production sets and prices for different firms.

Now consider an initial equilibrium with consumer prices (q,s), and producer prices (p,r) if production is efficient, each somehow normalised. Consider a small change (dq,ds) in consumer prices, with its implied producer price and tax changes left implicit. For part of the time I shall assume that unconstrained changes up or down in all consumer prices are conceivable; later I shall mention prior constraints and one-sided changes. We want to know whether such a change is feasible from the point of view of production, and whether it is an improvement in the sense of a Bergson social welfare function $W(u^1,u^2,...)$, or in the Pareto sense.

If the initial equilibrium is production inefficient, i.e. if (x,v) is in the interior of T, the feasibility of the change is guaranteed given only the continuity of demand functions. Note that this does not depend on the particular representation of T by the function G, and is thus free of any worries on that score. If the initial equilibrium is production efficient, feasibility of the change requires, using (1) and (2),

$$\Sigma_i \, (\partial G/\partial x_i) \left\{ \Sigma_j \, \Sigma_h \, (\partial x_i^h/\partial q_j) \, dq_j + \Sigma_k \, \Sigma_h \, (\partial x_i^h/\partial m^h) \, v_k^h \, ds_k \right\} \leq 0 \, .$$

In matrix notation,

$$(p'A \mid p'B) \begin{pmatrix} dq \\ \text{--} \\ ds \end{pmatrix} \leq 0 \qquad (8)$$

where the matrices A, B have elements

$$A_{ij} = \Sigma_h \, \partial x_i^h/\partial q_j \quad , \quad B_{ik} = \Sigma_h \, v_k^h \, \partial x_i^h/\partial m^h \, .$$

A Bergson welfare improvement amounts to

$$\Sigma_h \, (\partial W/\partial u^h) \left\{ \Sigma_j \, (\partial V^h/\partial q_j) \, dq_j + \Sigma_k \, (\partial V^h/\partial m^h) \, v_k^h \, ds_k \right\} > 0 \, ,$$

i.e. using (7),

$$(-\alpha' X \mid \alpha' V) \begin{pmatrix} dq \\ \text{--} \\ ds \end{pmatrix} > 0 \, , \qquad (9)$$

where $\alpha^h = (\partial W/\partial u^h).(\partial V^h/\partial m^h) > 0$, α is a vector with components

α^h, and X, V are matrices with elements

$$X_{hj} = x_j^h \quad , \quad V_{hk} = v_k^h \ .$$

A Pareto improvement requires that the expression

$$\Sigma_j \ (\partial V^h/\partial q_j) \ dq_j \ + \ \Sigma_k \ (\partial V^h/\partial m^h) \ v_k^h \ ds_k$$

be non-negative for all h and possitive for at least one h. Omitting the positive factors λ^h, this can be written

$$(-X \ \vdots \ V) \begin{pmatrix} dq \\ -- \\ ds \end{pmatrix} > 0 \ . \qquad (10)$$

where > 0 indicates a semi-positive vector.

Note that the change making (dq,ds) proportional to (q,s) changes nothing. The consumer demands and utilities remain unchanged, and feasibility is trivially satisfied. Using the homogeneity of demand functions and applying Euler's theorem, it can be verified that (8) holds with equality. Neither (9) nor (10) can be satisfied.

We will want to know when (8) and (9) can be satisfied together. For this we use the well known

FARKAS' LEMMA: Given a matrix C and a vector d with the number of columns of C equal to the dimension of d, exactly one of the following possibilities holds:

1) there exists an x such that $C x \leq 0$, $d' x > 0$,

2) there exists a $y \geq 0$ such that $y'C = d'$.

For the possibility of (8) and (10) being satisfied together, I use the

GENERALIZED FARKAS' LEMMA: Given two matrices C and D with the same number of columns, exactly one of the following possibilities holds:

1) there exists an x such that $C x \leq 0$, $D x > 0$,

2) there exists a $y \geq 0$ and $z >> 0$ such that $y' C = z' D$.

This generalization is proved in the Appendix. It is similar to, but in a sense stronger than, the theorem of Motzkin used by Diewert (1977). The basic lemma is a special case of the generalization, where D has only one row, and then z is a positive scalar that can be set equal to 1 without loss of generality.

We are now able to put this apparatus to use on the model.

3. PRODUCTION-INEFFICIENT EQUILIBRIA

If the initial equilibrium is production-inefficient, there is no impediment from the point of view of production feasibility for any small price changes. Thus (8) is irrelevant, and a welfare improvement is possible if and only if (dq, ds) can be found to satisfy (9). We can use Farkas' Lemma setting the matrix C equal to zero, so a welfare improvement is not possible if and only if $(- \alpha' X \,\vdots\, \alpha' V) = 0$, i.e.

$$\Sigma_h \ \alpha^h \ x_j^h = 0 = \Sigma_h \ \alpha^h v_k^h \quad \text{for all } j,k. \tag{11}$$

Thus we have found conditions for local optimality of the initial equilibrium. In this case these could have been written down directly by setting equal to zero all the partial derivatives of W with respect to all the q_j and s_k, but the approach used here has the advantage of relating more directly to the methods we will need for subsequent questions.

The focus here will be on showing that in several important cases it is impossible to satisfy (11), thus proving that if a welfare optimum exists, it must have aggregate production efficiency. Such cases are easy to spot when we recall that each α^h is positive. For example, if there is a j for which x_j^h is non-negative for all h and positive for at least one h, then (11) cannot be met for that j. This is the well-known Diamond-Mirrlees condition: if there is a commodity for which all consumers are either net buyers or net sellers in the weak sense, and at least one of them is active in trade, then production efficiency is desirable. The intuition is simple: welfare is clearly increased by lowering the price of such a commodity if all consumers are buyers and raising it if they are sellers, and this can be carried on so long as the production possibility frontier is not hit.

With primary factor or profit taxation there are further cases where (11) cannot hold. If there exists a k for which v_k^h is non-negative for all h and positive for at least one h, then (11) must fail for that k. This has the same intuitive explanation. Moreover, unless consumers can have short positions in their holdings in firms, the condition stated above is bound to be true for all k. Thus it seems that the introduction of profit taxation at variable rates greatly extends the presumption for production efficiency. The line of reasoning here is very similar to that of Hahn (1973) although the formal mathematics is developed somewhat differently.

Cases of production inefficient tax optima can arise when there are other constraints on admissible changes. One case is where some commodity price is zero in the initial equilibrium, and cannot be lowered any further. This is easily accomodated. If the relevant commodity is number i, we add a further constraint on feasible changes:

$$(-e_i \mid 0) \begin{pmatrix} dq \\ ds \end{pmatrix} \leq 0, \tag{12}$$

and use Farkas' Lemma to find when (12) and (9) can be satisfied together. Of course e_i is the vector with 1 as its i^{th} component and zeroes elsewhere. It is then easy to see that the conditions for a welfare improving admissible change not to exist are the same as (11) except that for commodity i we have

$$\Sigma_h \alpha^h x_i^h \geq 0 . \tag{13}$$

This is in keeping with the standard replacemtn of an equality by an inequality in writing the conditions for a corner optimum. A zero price can be optimum for i only if it is not desirable to raise the price, i.e.

if consumers in the appropriate welfare weighted average sense are buyers of this commodity. This suggests that such one-sidedness of admissible changes does not threaten production efficiency if it applies to commodities supplied by consumers like labour, and also to fixed factors. Incidentally, it is not so clear whether the consumer prices of fixed factors should be constrained to be non-negative; that will depend on whether taxes can be levied for possession of such factors as opposed to supply.

Other commonly imposed constraints tie together the rates of taxation on different commodities. This is more complicated since producer prices are inevitably brought in, and they depend on all consumer prices through demands, thus restraining independent variations of all components of (q,s). A detailed treatment seems too messy to be worthwhile, but some simple cases can be discussed. For example, suppose marginal costs are constant, and the first two commodities must satisfy $q_1/p_1 = q_2/p_2$. Then admissible changes must satisfy $dq_1/p_1 = dq_2/p_2$, and we can write dh for their common value and substitute in (9). The conditions for an admissible welfare-improving change to be impossible are again (11) for commodities 3, 4 etc. and for all fixed factors, but for the first two commodities we have one condition

$$\Sigma_h \ \alpha^h (p_1 x_1^h + p_2 x_2^h) = 0 . \tag{14}$$

If all consumers are either buyers of both commodities or else sellers of both, (13) cannot be satisfied and an optimum must be production efficient. Thus a constraint requiring equal tax rates on broad commodity groups bought by consumers does not threaten production efficiency, so long as variations in producer prices do not lead to serious complications elsewhere. The intuition behind this is again obvious. Production inefficient optima seem a more serious possibility if a commodity must be taxed at the same rate as labour, or equivalently on taking labour as numeraire one commodity is untaxable. The general problem and several cases are discussed by Munk (1976).

Now return to the case of an initial production inefficient equilibrium with no other constraints on permissible price changes, and ask whether a Pareto improvement is possible, i.e. if (10) can be satisfied. By the Generalized Farkas Lemma, (10) cannot be satisfied if and only if there exists a strictly positive vector α such that $\alpha' (- X \mid V) = 0$, or

$$\Sigma_h \alpha^h x_j^h = 0 = \Sigma_h \alpha^h v_k^h \text{ for all } j,k. \tag{15}$$

This looks remarkably like (11), but there is one point of difference. In (11) the α^h were given to us by the specified social welfare function, whereas in (15) they are a matter of choice. This makes economic sense. A Pareto improvement is a welfare improvement for any social welfare function of the Bergson type. Therefore if a Pareto improvement is not possible, a welfare improvement must be impossible for some welfare function. The α^h for this function will then satisfy (15).

Again we can find cases where it is impossible to satisfy (15) for any strictly positive α, i.e. conditions which ensure that any Pareto efficient equilibrium must be productively efficient. We can also modify (15) to allow one-sided changes or other constraints. I shall leave all this to the reader.

The general connection between welfare improvements and Pareto improvements is worth noting; similar relations will appear in the next section.

4. PRODUCTION-EFFICIENT EQUILIBRIA

Let us turn to the more interesting case where the initial equilibrium is production efficient. Assume that there are no binding constraints of sign or other restrictions on taxation. Production feasibility however does require that price changes (dq,ds) satisfy (8). We wish to know when (8) and (9) can be true together. Diewert (1977) states a general formula and considers some special examples. I shall concentrate on the question of when this is not possible, i.e. to derive conditions for local optimality of the tax structure implicit in the initial equilibrium. By Farkas' Lemma, the conditions are that there exists a non-negative number μ satisfying

$$\mu\, p'A = -\alpha'X \qquad (16)$$

$$\mu\, p'B = \alpha'V, \qquad (17)$$

or, written out in full,

$$\mu\, \Sigma_h \Sigma_i\, p_i\, \partial x_i^h/\partial q_j = -\Sigma_h\, \alpha^h x_j^h \quad \text{for all } j \qquad (16)$$

$$\Sigma_h \left\{ \alpha^h - \mu\Sigma_i\, p_i\, \partial x_i^h/\partial m^h \right\} v_k^h = 0 \quad \text{for all } k \qquad (17)$$

In (16) we have just the well known Diamond-Mirrlees conditions for optimum commodity taxation, while (17) yields similar conditions for optimum taxation of the fixed factors. A similar interpretation of keeping the marginal effects on welfare proportional to those on tax revenue can be given.

Using (8) and (10) and the Generalized Farkas Lemma, we see that the conditions for local Pareto efficiency of the initial equilibrium are that there exists a strictly positive vector α and a non-negative number μ such that (16) and (17) hold. Again the point of difference is that we can choose α.

These conditions can be interpreted in several cases to yield particular results. I shall first consider the case where the vectors v^h are linearly independent. For this it is necessary to have at least as many fixed factors as there are consumers, but this is not sufficient since it is always possible for a small number of consumers to have linearly dependent factor ownerships. Under such linear independence, (17) is equivalent to

$$\alpha^h = \mu \, \Sigma_i \, p_i \, \partial x_i^h / \partial m^h \quad \text{for all } h. \tag{18}$$

Substituting in (16)

$$\mu \, \Sigma_i \, p_i \left\{ \Sigma_h \, (\partial x_i^h / \partial q_j + x_j^h \, . \, \partial x_i^h / \partial m^h) \right\} = 0 \text{ for all } j \tag{19}$$

The possibility $\mu = 0$ can be ruled out under the same conditions as could guarantee production efficiency in the previous section. Then we note that the terms in the bracketed sum are simply the substitution effects

$$(\partial x_i^h / \partial q_j)_{u^h \text{const.}}$$

Therefore (19) is satisfied by making p proportional to q (and, incidentally, in no other way if at least one consumer's Slutsky matrix has the maximum possible rank). Keeping $p = q$ means taxing only the fixed factors, while making p proportional to q amounts to the same thing since an equiproportional tax on all commodities is just like a tax on transfer income. We can then assume the simple case without loss of generality. On setting $p = q$ and observing that

$$\Sigma_i \, q_i \, \partial x_i^h / \partial m^h = 1 \quad \text{for all } h, \tag{20}$$

we obtain from (18)

$$\alpha^h = \mu \quad \text{for all } h . \tag{21}$$

In this case, therefore, taxing fixed factors alone has led us to the first-best optimum. The point is that when enough consumer prices of fixed factors can be varied independently, we are able to vary all consumers' transfer incomes m^h independently of one another, i.e. we have instruments equivalent to lump sum transfers. It is then clearly best to use these alone. Of course all this assumes that in satisfying (18) no important sign constraints on the s_k will be violated, but there seems little one can say about this problem in general (cf. Dasgupta and Stiglitz (1972)). The formal analogies with spanning in portfolio theory and factor price equalization in trade theory are evident.

It turns out to be more useful to work from the opposite direction. Suppose it is productively feasible to rely on taxation of fixed factors alone. This amounts to assuming that such taxation can raise at least enough revenue to meet the government's requirements. Taking an initial equilibrium with $p = q$, we test whether it can satisfy the local optimality conditions, without confining ourselves to the first-best case. Using (20), and the similar result for price derivatives

$$\Sigma_i \, q_i \, \partial x_i^h / \partial q_j = - x_j^h \quad \text{for all } h, j \tag{22}$$

and substituting in (16) and (17), we see that the local optimality conditions are that there exists a non-negative μ satisfying

$$\Sigma_h \, (\alpha^h - \mu) x_j^h = 0 = \Sigma_h \, (\alpha^h - \mu) \, v_k^h \quad \text{for all } j,k \, . \tag{23}$$

These can obviously be satisfied if all the α^h happen to be equal, i.e. if the first-best optimum is attainable, for it suffices to choose μ equal to the common value of the α^h. However, it is of some interest that it is possible to satisfy (23) in some other cases, i.e. it is possible to have second-best optima relying on factor taxes alone. These are in a sense exceptional cases, but worthy of notice.

To see this, I begin by giving an interpretation to (23) which is available provided fixed factor taxation meeting the government's needs leaves a positive total of income to the consumers. Multiply the equation for commodity j by q_j and add, obtaining

$$\Sigma_h (\alpha^h - \mu) m^h = 0 ,$$

or defining the income shares $\theta^h = m^h/\Sigma_h m^h$,

$$\mu = \Sigma_h \alpha^h \theta^h . \qquad (24)$$

We can think of the θ^h as forming a probability distribution, and (24) says that μ is the average of the α^h under this distribution.

Now (23) can be written as

$$\Sigma_h (\alpha^h - \mu) (q_j x_j^h/m^h)\theta^h = 0 = \Sigma_h (\alpha^h - \mu) (s_k v_k^h/m^h) \theta^h$$

for all j and k. For each commodity j, the shares of it in the consumers' budgets should be uncorrelated with the marginal social valuations of their incomes. Similarly, for each factor k, the fractions of the consumers' transfer incomes accounted for by it should be uncorrelated with these marginal social valuations. For commodities, this leaves the desirability of changing the price of one commodity the same as that for another, leading to reliance on equiproportionate commodity taxation or fixed factor taxation.

Another way to look at (23) is to define commodity- and factor-wise shares for consumers:

$$\xi^h_j = x^h_j / x^p_j \quad \text{and} \quad \omega^h_k = v^h_k / v_k ,$$

so that

$$\Sigma_h \ \xi^h_j = 1 \text{ for all } j, \quad \Sigma_h \ \omega^h_k = 1 \text{ for all } k.$$

Then (23) becomes

$$\Sigma_h \ \alpha^h \ \xi^h_j = \mu = \Sigma_h \ \alpha^h \ \omega^h_k \text{ for all } j, k \qquad (25)$$

For each j and k, we can again think of the ξ^h_j and ω^h_k as probability distributions, and (25) says that (α^h) as a random variable should have the same mean according to all these distributions, i.e. they should be α-mean preserving spreads of one another.

Again (25) can be satisfied if all the α^h happen to be equal. But the discussion suggests another case. If all the individuals are scaled replicas of one another in the sense that $\xi^h_j = \omega^h_k = \theta^h$ for all j and k, then it suffices to set μ according to (24). The α^h need not be equal, so the optimum here will typically be a second-best. Note that we require replication only at the point in question, so homotheticity is not necessary.

In the two-person case there are no other possibilities. Consider any two commodities, say i and j, and write (25) for them as

$$\alpha^1 \xi_i^1 + \alpha^2 \xi_i^2 = \alpha^1 \xi_j^1 + \alpha^2 \xi_j^2 .$$

Hence

$$\alpha^1 \xi_i^1 + \alpha^2 (1 - \xi_i^1) = \alpha^1 \xi_j^1 + \alpha^2 (1 - \xi_j^1) ,$$

or

$$(\alpha^1 - \alpha^2) (\xi_i^1 - \xi_j^1) = 0 .$$

This can be true only in the first-best case $\alpha^1 = \alpha^2$, or the replica case $\xi_i^1 = \xi_j^1$. The simple point is that for distributions concentrated at two points there is nowhere to spread the weight, so that there are no non-trivial mean-preserving spreads. With more consumers there are such possibilities, but they look empirically implausible. This can also be said for the replica case, where a consumer owning a greater proportion of all fixed factors is also required to supply the same greater proportion of labour.

Conditions for local Pareto efficiency of an equilibrium relying on fixed factor taxes alone are that there exist positive α^h and a non-negative μ satisfying (23). This can always be achieved by setting all α^h and μ equal to 1. The result should be evident, since the equilibrium involves no distortions in the conventional sense.

Modifications to handle one-sided changes are easy. For example, consider the case where fixed factor taxation only just suffices to meet the government's needs. Now the initial equilibrium has $s = 0$, and if confiscatory taxation is impossible, changes are constrained by $ds \geq 0$. Setting this up to apply the Generalized Farkas Lemma, we find the local optimality conditions: there should exist a $\mu \geq 0$ such that

$$\Sigma_h (\alpha^h - \mu) x_j^h = 0, \quad \Sigma_h (\alpha^h - \mu) v_k^h \lesseqgtr 0 \quad \text{for all } j,k \tag{26}$$

For commodities, the idea of α-mean preserving spreads can be used and interpreted as before. The budget share approach is not available since all transfer incomes are zero. For fixed factors, we want the ownership of each by the consumers to be negatively correlated with the social marginal valuation of their incomes. If this were not true for any one factor, welfare could be increased by a slight increase in its consumer price with offsetting adjustments in other taxes to preserve feasibility. Thus it would be desirable to depart from pure factor taxation.

The general conclusion appears to be that reliance on fixed factor taxation alone, when productively feasible, is always Pareto efficient, but rarely welfare optimal whether in the first-best sense or the second-best.

5. BEST SMALL IMPROVEMENTS

When local welfare improvements are possible, there will often be several possibilities to choose from. Two interesting questions have been asked in this connection. Diewert (1977) and others have placed constraints on the magnitudes of possible changes, and looked for the best direction of local welfare improvement. Guesnerie (1977) has pointed out the possibility that it may be necessary or desirable to allow production inefficiency to secure a local improvement from a suboptimum starting point, even though a local optimum must be productively efficient. In this section I shall consider these issues in the framework of the model being used.

A geometric approach will help understand the problem more easily.

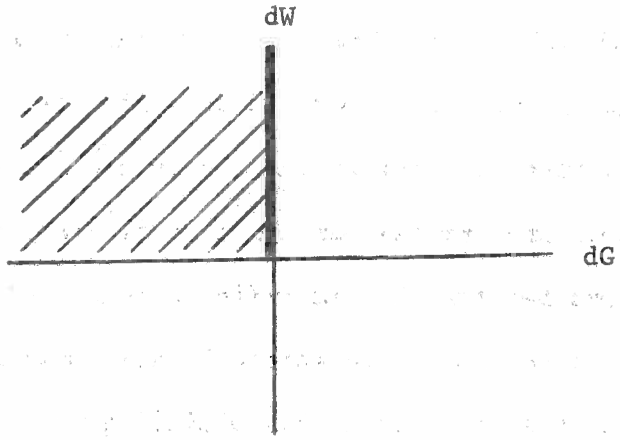

FIGURE 1

Figure 1 shows the changes dG in the economy's transformation function G, and dW in welfare. The initial equilibrium, assumed production efficient, is at the origin. Then productively feasible and welfare improving changes cover the north-west quadrant, including the vertical axis but excluding the horizontal.

Further constraints arise from limits on how rapidly the government can change tax rates. Choosing units to set all producer prices equal to 1 at the initial equilibrium, we can capture these limits by placing a bound on the magnitude of (dq,ds). Diewert stipulates that the Euclidean norm of this vector should not exceed 1. Under the local linear approximation the magnitude 1 involves no loss of generality, but the choice of the Euclidean norm has no clear economic appeal. It does illustrate the issue, so I shall work with it and later consider another possible norm. Now changes in the (dG,dW) space that meet this additional constraint are given by the image of the unit sphere in the (dq,ds) space under the locally linearized function relating them. To find the best local improvement, we look for a point that lies in the intersection of this image with the north-west quadrant, and has the highest ordinate among such points. The question is whether this occurs on the vertical axis, thus preserving production efficiency, or to the left, thus violating it.

To write the algebra of this more compactly, I rename the variables in the (dq,ds) space as $(z_1, z_2, ..)$, and dG and dW as y_1 and y_2 respectively. The linear functions linking the latter to the former are written as

$$y_i = \Sigma_j a_{ij} z_j \quad , \quad i = 1,2$$

This notation is for the purpose of this section only.

The maximisation problem is now

$$\text{maximise} \quad y_2 = \Sigma_j \, a_{2j} \, z_j$$

subject to

$$y_1 = \Sigma_j \, a_{1j} \, z_j \leq 0 \tag{27}$$

$$\Sigma_j \, z_j^2 \leq 1. \tag{28}$$

The Lagrangean is

$$L = \Sigma_j \left\{ (a_{2j} - \mu a_{1j}) \, z_j - \lambda z_j^{\,2} \right\} ,$$

and the first-order conditions are

$$a_{2j} - \mu a_{1j} - 2 \lambda z_j = 0.$$

Substituting in (28) and recalling complementary slackness, we find that the constraint must hold with equality, and

$$2\lambda = \left\{ \Sigma_j \, (a_{2j} - \mu a_{1j})^2 \right\}^{\frac{1}{2}} ,$$

$$z_j = (a_{2j} - \mu a_{ij}) \, / \left\{ \Sigma_j \, (a_{2j} - \mu a_{1j})^2 \right\}^{\frac{1}{2}} . \tag{29}$$

Using this in (27) yields

$$0 \gtrsim y_1 = \left\{ \Sigma_j\, a_{1j}\, a_{2j} - \mu \Sigma_j\, a_{1j}^{\ 2} \right\} \Big/ \left\{ \Sigma_j\, (\, a_{2j} - \mu a_{1j}\,)^2 \right\}^{\frac{1}{2}} , \qquad (30)$$

so that

$$\mu \gtrsim \Sigma_j\, a_{1j}\, a_{2j} \,/\, \Sigma_j\, a_{1j}^{\ \ 2} \qquad (31)$$

We now classify some possibilities. If $\Sigma_j\, a_{1j}\, a_{2j} > 0$, then $\mu > 0$, and by complementary slackness $y_1 = 0$. We can then find μ using (30), and z_j from (29). But the details are not of interest; the important conclusion is that the best small change preserves production efficiency.

If $\Sigma_j\, a_{1j}\, a_{2j} < 0$, then (30) shows $y_1 < 0$. By complementary slackness $\mu = 0$. Again the z_j can be found from (29), but the more intesting result is that production efficiency is violated in securing the best small change.

If $\Sigma_j\, a_{1j}\, a_{2j} = 0$, then (30) shows that either $\mu = 0$ and $y_1 = 0$, or $\mu_1 < 0$. The latter is impossible by complementary slackness, so we have the former, and production efficiency is preserved, but 'only just'.

These results can be interpreted as follows. For each j, a_{1j} is the relative marginal resource cost and a_{2j} the marginal welfare effect of the j^{th} price change. If welfare-increasing changes are costly in terms of resources on the average, then the best local improvement is compatible with maintained production efficiency. While this condition seems plausible it must not be forgotten that the initial equilibrium is typically suboptimal, and there may be single price changes which raise welfare at the same time as releasing resources. It is obvious how the results generalize to an elliptical norm $\Sigma_j k_j z_j^2 \leq 1$.

An even simpler treatment is possible for the norm $\Sigma_j |z_j|$. The image of the set where this norm does not exceed 1 is obtained simply by plotting the points $\pm(a_{1j}, a_{2j})$ in the y space, and forming their convex hull. Figure 2 shows the two possibilities that arise. If the

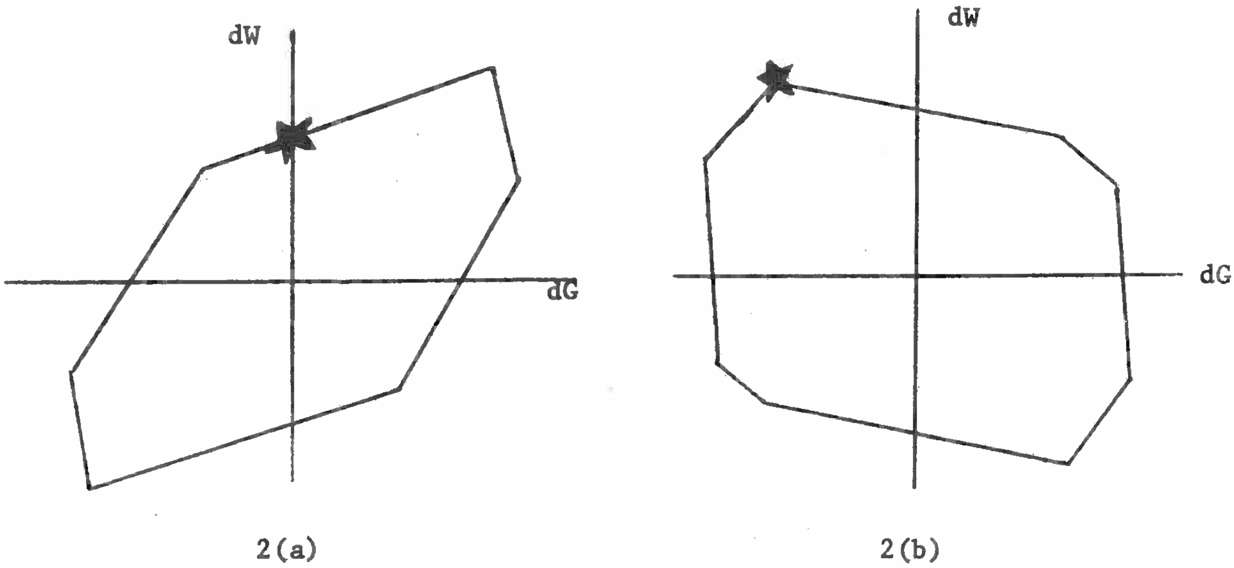

FIGURE 2 : in each case the best local improvement is shown by ★

single price change of unit magnitude which, among all such changes, yields the highest welfare improvement is resource-using, then the best local improvement occurs on the vertical axis and production efficiency is maintained. If this change is resource-releasing, then it is the best local change on its own, and production efficiency is violated. Needless to say, this best local change can then be a starting point for further tax changes when the constraints on permissible changes are relaxed, and the ultimate optimum will typically be production-efficient.

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APPENDIX

Proof the generalization of Farkas' Lemma :

It is easy to verify that the two alternatives cannot be true together. Now suppose that the first is false, i.e. $C\,x \leq 0$, $D\,x > 0$ cannot be met for any x. For any k, let d^k be the k^{th} row of D (a row vector), and let D^k be the matrix with the k^{th} row deleted. Then there does not exist an x satisfying

$$\begin{pmatrix} C \\ \hline -D^k \end{pmatrix} x \leq 0, \qquad d^k\, x > 0 \,.$$

By Farkas' Lemma, there exist row vectors η^k, ξ^k non-negative and satisfying

$$(\eta^k \,\vdots\, \xi^k) \begin{pmatrix} C \\ \hline -D^k \end{pmatrix} = d^k \,.$$

Some rearrangement of this yields

$$\eta^k\, C = \zeta^k\, D \,,$$

where ζ^k has its k^{th} component equal to 1, and its other components are the appropriate components of ξ^k.

This can be done for each k separately. Now define the column vectors y and z by

$$y' = \Sigma_k \eta^k , \quad z' = \Sigma_k \zeta^k .$$

These satisfy $y'C = z'D$ on summing the corresponding equations over k. Also y is non-negative, and z is strictly positive having picked up a 1 in each component and no negative contributions.