COHERENT SPECIFICATION OF DEMAND SYSTEMS WITH CORNER SOLUTIONS AND ENDOGENOUS REGIMES

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Coherent specification of demand systems with corner solutions and endogenous regimes

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Abstract

We seek to provide fairly general conditions that may be imposed on parameters in order that any empirical consumer demand system is consistent with utility maximization. In standard demand systems the imposition of these regularity conditions is less essential than in more complex situations, like when there is rationing or when there are endogenously switching regimes. The paper starts out by giving a number of examples where the failure to properly take into account restrictions following from neoclassical theory leads to models that are not internally coherent.

Let Θ be the space of parameters which generate internally coherent models. We show that even if the true parameter vector belongs to Θ, failure to constrain the parameter estimates to Θ in "maximum likelihood" estimation may yield inconsistent estimates outside of Θ.

Next a general framework is provided in which it is possible to formulate parameter restrictions which guarantee utility consistent models. For various cases (standard demand systems, rationing, endogenous regimes) we suggest general conditions that can be imposed in order to guarantee coherency of the empirical model. Since random parameter variation is allowed for, to capture non-systematic differences in preferences across individuals, the conditions also imply restrictions on the stochastic specification of an empirical model. For a number of familiar demand systems we show what the parameter restrictions amount to in practice.

1. Introduction

Empirical researchers in the field of demand theory are becoming increasingly aware of the tight structure that may be imposed on their models by neoclassical theory. In the somewhat older literature on demand systems a typical approach would be to choose a particular representation of preferences and derive the corresponding demand functions. After tacking on an error term, the system would next be estimated. In the estimation, restrictions from neoclassical theory might or might not be imposed. In either case authors often have tested the various Slutsky conditions for their particular empirical specification, with mixed results. As noted by McElroy (1987) the attention for consistency with neoclassical theory has mostly been limited to the systematic part of the demand equations, with a rather cavalier treatment of the error structure. Her own work is a notable exception in this respect.

Whether or not authors would severely test neoclassical restrictions for their data set, it seems fair to say that in a standard demand system the empirical specification is rather loosely connected with the underlying theory. If the estimation results turn out to be inconsistent with a utility maximization hypothesis, one can still regard the empirical model as an adequate description of reality. This is no longer true in more complicated situations where the theory is used more intensively. In models dealing with rationing, corner solutions, nonlinear budget constraints, or endogenously switching regimes, utility theory plays a more crucial role than in traditional demand systems. If in these models Slutsky restrictions are violated they will in general not be coherent, in the sense that probabilities of mutually exclusive events do not sum to unity or that an endogenous variable is not determined unambiguously by the model (c.f. e.g. Gourieroux et al., 1980 and Schmidt, 1981).

To avoid these types of problems one has to place appropriate restrictions on the parameters appearing in the model. Furthermore, special care has to be taken with the specification of the error structure. In this paper we provide various examples of problems arising if neoclassical restrictions are not imposed properly. Next we propose conditions that guarantee coherency of the model given the phenomenon one wants to model (a standard demand equation, rationing, or switching regimes). We apply the conditions to a number of frequently used specifications of preferences and show how the conditions can be imposed in practice.

In most of our analysis concavity of the expenditure function plays an important role. (We will alternatively denote this as concavity of the cost function or simply as "negativity"). The vast majority of the specifications of preference structures considered in the literature only satisfy concavity of the expenditure function locally. Those that are capable of satisfying concavity globally, are in general quite restrictive. The only flexible form that can satisfy negativity globally is the generalized McFadden cost function proposed by Diewert and Wales (1987). See also Barnett and Lee (1985) and Barnett (1983).

Of course, the importance of concavity for the behavior of a neoclassical model in non-standard situations has been noted before. For example, Ransom (1987a) has noted that the Wales and Woodland (1983) model of a stochastic consumer demand system with binding non-negativity constraints is well-behaved if the parameters satisfy certain regularity conditions, which are closely connected to concavity of the cost function. However, as will be seen in Section 4 below, the imposition of concavity for the quadratic utility specification of Wales and Woodland in a relevant area of the price space requires a rather intricate stochastic specification. Similarly, Hausman (1985) notes the importance of negativity to have a well-behaved model of behavior under a kinked budget constraint. In his case imposition of concavity is rather simple. See Section 2. Van Soest and Kooreman (1986, 1987) have noted that the approach of Lee and Pitt (1986) to stochastic consumer demand systems via the use of shadow prices leads in general to incoherent models unless certain conditions are imposed on the parameters. These conditions are closely related to concavity of the cost function.

In Section 2 we give three examples of, respectively, kinked budget constraints, rationing, and endogenous regimes, which all highlight the crucial role played by concavity to obtain an internally coherent model. To stress the importance of coherency we conclude Section 2 with an example of a simultaneous probit model where failure to impose coherency conditions in "maximum likelihood" estimation yields inconsistent estimates of the parameters, even though the true parameters satisfy coherency conditions. Section 3 introduces notation and some basic concepts. It also provides three conditions that can be imposed on standard demand systems. The conditions are applied to some often used preference specifications. It is argued however that such conditions can often be ignored if one is willing to make sufficiently general assumptions about the errors in the model.

Section 4 considers the case of rationing and provides conditions that can be imposed on the parameters in order to guarantee an internally coherent model. The conditions are elaborated for the same specifications of preferences as considered in Section 3. In Section 5 we consider internal coherency of models with endogenous regimes. The conditions are quite similar to the ones given in Section 4. Again applications to some popular demand systems are given. Section 6 concludes.

2. Concavity and Coherency. Four examples

2.1 Kinked budget constraints

Figure 2.1 illustrates the simplest possible case of a standard model of individual labor supply in the presence of kinked budget constraints, as developed by Hausman in numerous papers (see, e.g., Hausman, 1981,1985). Given the budget constraint, the individual chooses the number of hours which maximizes utility (h^* in the figure).

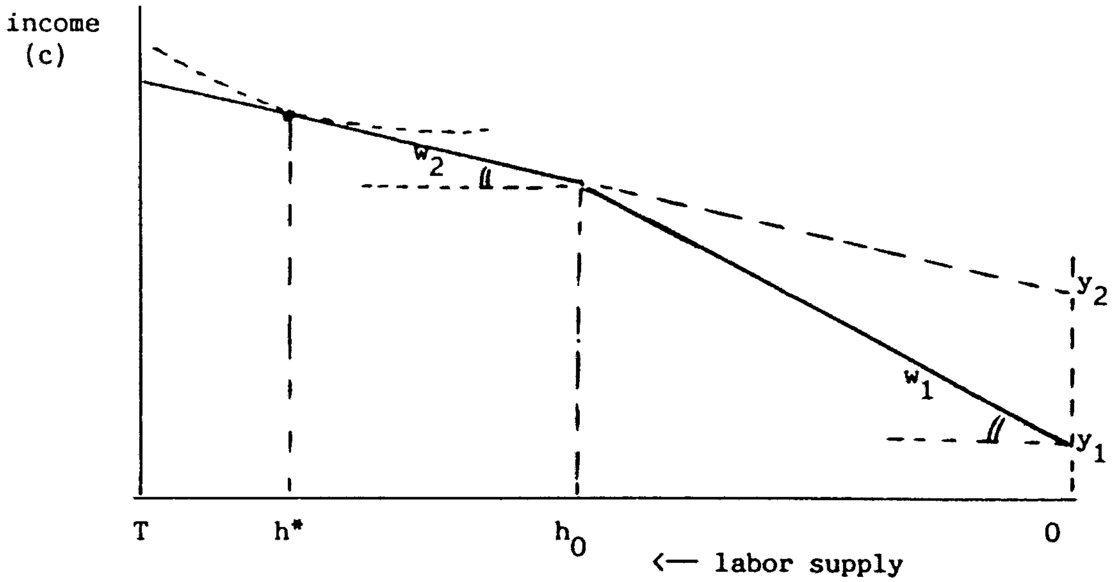

Figure 2.1 Individual labor supply and kinked budget constraints

A typical utility specification used in this kind of work is

$$U(h,c) = (\beta h-\alpha)\exp\{\beta(h-\gamma-\beta c)/(\alpha-\beta h)\}\ , \qquad (2.1)$$

where h is the number of hours worked per period and c is total consumption. Note that U is an increasing function of c if $\beta\neq 0$. Along each linear segment of the budget curve, this utility function implies linear labor supply functions of the form

$$h_j = \alpha w_j + \beta y_j + \gamma \qquad (j=1,2), \qquad (2.2)$$

where w_j is (minus) the slope of the j-th segment, y_j is the intercept of the j-th segment with the line h=0, and h_j is the desired number of hours if the wage rate is w_j and non-labor income is y_j.
With (2.2) corresponds an indirect utility function of the form

$$V(w_j,y_j) = \exp(\beta w_j)\{\beta^2 y_j + \alpha\beta w_j - \alpha + \gamma\beta\}$$

If the direct utility function is strictly quasi-concave on the whole budget set, then the optimum h* can easily be found with (2.2). There are five possibilities:

$$\begin{array}{lll} \text{A. } h_1 \leq 0 & h^* = 0 & \\ \text{B. } 0 < h_1 \leq h_0 & h^* = h_1 & \\ \text{C. } h_2 \leq h_0 \leq h_1 & h^* = h_0 & \qquad (2.3) \\ \text{D. } h_0 < h_2 \leq T & h^* = h_2 & \\ \text{E. } T < h_2 & h^* = T. & \end{array}$$

To allow for unobserved preference variation, assume that β is a random variable defined on the real line. Hausman (1981) and Blomquist (1983) assume that β is negative with probability one. For $\alpha>0$ this guarantees quasi-concavity of U at all points of the budget set. In this example we show what can happen if the concavity problem is neglected and β is allowed to be positive but nevertheless (2.3) is applied. The following probabilities to the five cases in (2.3) can be assigned:

$$\begin{aligned} Pr[A] &= Pr[\beta \leq (-\alpha w_1 - \gamma)/y_1] \\ Pr[B] &= Pr[(-\alpha w_1 - \gamma)/y_1 < \beta \leq (-\alpha w_1 - \gamma + h_0)/y_1] \\ Pr[C] &= Pr[(-\alpha w_1 - \gamma + h_0)/y_1 < \beta \leq (-\alpha w_2 - \gamma + h_0)/y_2] \\ Pr[D] &= Pr[(-\alpha w_2 - \gamma + h_0)/y_2 < \beta \leq (-\alpha w_2 - \gamma + T)/y_2] \\ Pr[E] &= Pr[(-\alpha w_2 - \gamma + T)/y_2 < \beta] \end{aligned}$$

Let the parameter values be $\alpha=20$, $\gamma=0$ and let the budget constraint be characterized by $w_1=1$, $w_2=1/2$, $y_1=1$, $y_2=11$, $T=40$, so $h_0=20$ and $c_0=21$. Then we can identify the five cases with segments of the β-axis as follows.

```
      A                  B                C      D           E
------------->|<------------------->|<---->|<------->|<---------------
--------------|---------------------|------|---------|----------------
             -20                    0    10/11     30/11           β
```

Thus, given a distributional assumption regarding β, the calculation of the probabilities is straightforward.

Now consider a second set of parameter values: $\alpha=10$, $\gamma=0$ with the same budget constraint as in the example above. The corresponding segments of the β-axis are as follows.

```
                            B
  A   |<------------------------------------------->|
---->|
     |            |   D    |                       |   E
     |            |<------>|<------------------------------------
-----|------------|--------|-----------------------|--------------------
    -10         15/11    35/11                    10                 β
```

(C = $\emptyset$)

Clearly, the model is now not coherent, in the sense that probabilities do not sum to unity. In other words: From (2.3) it is not possible to write h^* as a function of β. To see the root of the trouble, notice that quasi-concavity of the direct utility function (or equivalently negativity, i.e. concavity of the expenditure function) requires $\alpha \geq \beta h_j$. Using (2.2) it is easy to verify that concavity is satisfied at the kink point for both h_1 and h_2 for the first set of parameter values but not for the second. In fact, the inequality

$$(h_0 - \alpha w_1 - \gamma)/y_1 < (h_0 - \alpha w_2 - \gamma)/y_2 \ , \tag{2.4}$$

which is required to avoid the incoherency, is equivalent to concavity at h_0 for both line segments. This can be shown straightforwardly as follows:

Assume that y_1 and y_2 are both positive and use the relation $y_1 + h_0 w_1 = y_2 + h_0 w_2 = c_0$. Then (2.4) can be simplified to

$$\alpha y_2 > (h_0 - \alpha w_2 - \gamma) h_0 \ . \tag{2.5}$$

If, being on segment 2, the individual chooses h_0, this implies $\beta = (h_0 - \alpha w_2 - \gamma)/y_2$. Inserting this in (2.5) yields $\alpha > \beta h_0$, which is the concavity condition at h_0 (with β such that h_0 is the optimum in case of the linear budget constraint characterized by w_2 and y_2). A completely similar argument shows that if β is such that h_0 is the preferred point along segment 1, the same condition follows from (2.4). One can also work backwards from the concavity condition to (2.5) which establishes the equivalence. In conclusion, concavity at the kink point is necessary and sufficient to avoid problems of incoherency.

This example can be seen as a simple example of a model with endogenous regimes due to a set of inequality constraints. The general framework is discussed in Section 5. One of the goals of this paper is to discuss methods of avoiding the problems with internal coherency as encountered in the example. In this specific example there are two apparent ways of avoiding problems.

The first option is the restriction of the range of possible realizations of the random variable β and the value of the fixed parameter α. If β is negative with probability one and α is nonnegative, the problems do not arise. The reason for this is the fact that in this case the concavity condition $\alpha \geq \beta h$ is satisfied for all nonnegative h.

Another possibility, which avoids truncation of the distribution of β, is to impose (2.5) for all 'relevant' values of h_0 and c_0. Notice that (2.5) can be rewritten as

$$\alpha c_0 > (h_0 - \gamma) h_0 \tag{2.5'}$$

Thus, if the fixed parameters α and γ are restricted such that (2.5') holds for all relevant (h_0, c_0) (e.g. all (h_0, c_0) in the sample), the coherency problem is avoided. In a sense, the latter method is less restrictive than the first one, because it does not necessarily imply quasi-concavity of the direct utility function at all points of the budget set.

2.2 Non-negativity constraints in the Translog demand system

Lee and Pitt (1986) consider a Translog demand system with binding non-negativity constraints:

$$s_i = \{\alpha_i + \sum_{j=1}^{n} \beta_{ij} \log v_j\} / D \qquad (i=1,\ldots,n), \tag{2.6}$$

where

$$D = -1 + \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \log v_j$$

β_{ij} : parameters $(i,j=1,\ldots,n)$

n : number of goods

v_j : p_j/y with p_j the price of j-th good $(j=1,\ldots,n)$ and y income

s_i : budget share of good i

α_i : random parameters $(i=1,\ldots,n)$, representing random preferences, $\alpha_1+\ldots+\alpha_n = -1$.

The demand regime where the first ℓ goods are not consumed is characterized by the conditions

$$\begin{aligned} &\pi_i(\bar{v}) \leq v_i && (i=1,\ldots,\ell) \\ &x_i > 0 && (i=\ell+1,\ldots,n), \end{aligned} \tag{2.7}$$

where

$\pi_i(\bar{v})$: virtual price (or 'shadow price') of the i-th good

$\bar{v}$: vector of market prices of the goods consumed in positive amounts

x_i: demand for the i-th good given that the first ℓ goods are not consumed.

The various regimes correspond to different values of the α_i $(i=1,\ldots,n)$. Lee and Pitt (1986) characterize the regimes by solving the α_i from (2.6) and (2.7).

Van Soest and Kooreman (1987) construct examples for n=3. Figure 2.2 gives one such example.

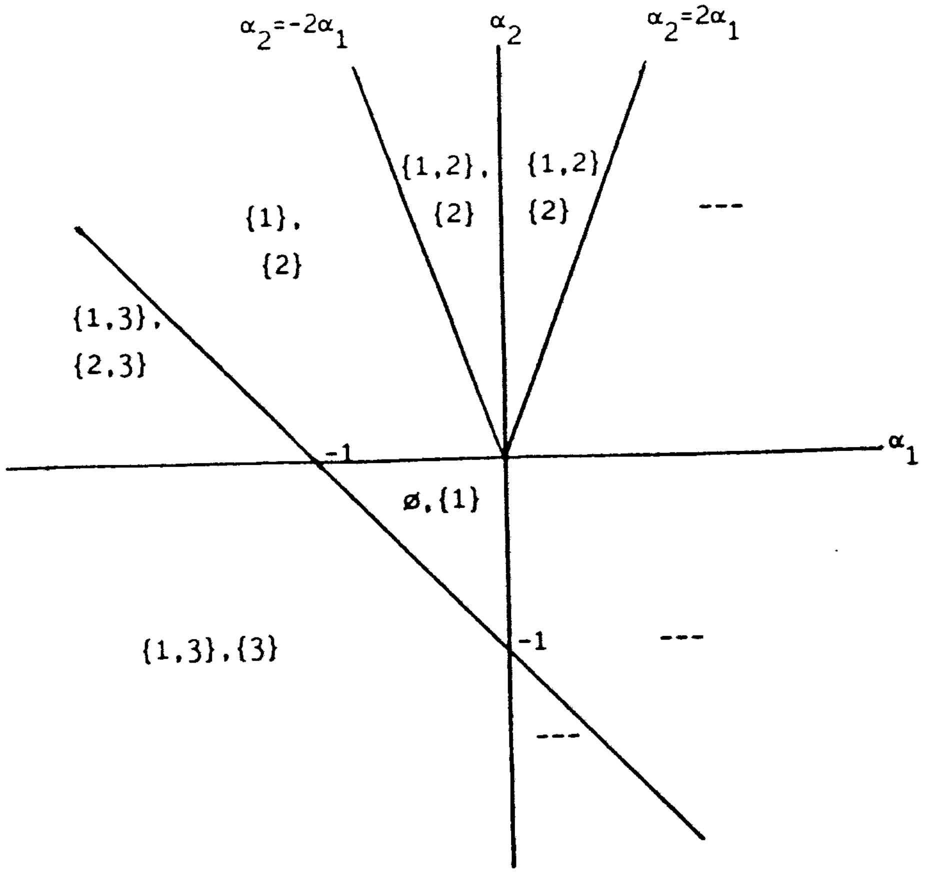

Figure 2.2 Incoherency in the Translog demand system with binding non-negativity constraints

Let $B = [\beta_{ij}]_{i,j=1,...,3} = \begin{bmatrix} -1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $v = (1,1,1)'$.

Figure 2.2 shows the number of solutions of (2.7), for each realization of the vector $\alpha = (\alpha_1, \alpha_2, -1-\alpha_1-\alpha_2)'$ of random variables. Each solution is characterized by some regime, i.e. a subset of {1,2,3} indicating which constraints are binding.

The figure indicates for each vector α those regimes that yield a solution. For α with $\alpha_1>0$ and $\alpha_2<2\alpha_1$, no solution is found and for other α's (except for some set of probability zero) there are two solutions, implying that different regimes occur simultaneously. This would imply that the probabilities with which each regime occurs do not sum to unity.

For other parameter values, i.e. different values of β_{ij}, such problems need not arise. Van Soest and Kooreman give sufficient conditions to avoid the incoherency. It turns out that these same conditions also guarantee concavity of the cost function for all feasible values of the budget shares of the n goods. Thus, a strong connection is suggested between concavity of the cost function and internal coherency of the demand system.

2.3 Rationing

Consider the following Gorman Polar Form cost function for a case with three goods.

$$C(u,p_1,p_2,p_3)=-1/2(p_2^2/p_3)\exp(p_1/p_3)-p_3\exp(p_2/p_3)+\sum_{i=1}^{3} a_i p_i + up_3 \tag{2.8}$$

$$(p_i>0, a_i>0).$$

The 2x2 submatrix of second order derivatives with respect to p_1 and p_2 is

$$-\frac{1}{p_3}\begin{bmatrix} 1/2\, v_2^2 \exp(v_1) & v_2\exp(v_1) \\ v_2 \exp(v_1) & \exp(v_1)+\exp(v_2) \end{bmatrix}$$

where $v_1=p_1/p_3$, $v_2=p_2/p_3$.
This matrix is negative definite for $v_1<v_2$.
The demand functions, derived by application of Shephard's Lemma, are

$$q_1 = -1/2\, v_2^2 \exp(v_1) + a_1 \tag{2.9a}$$
$$q_2 = -v_2\exp(v_1) - \exp(v_2) + a_2 \tag{2.9b}$$

Suppose now that q_1 is rationed at $q_1=\bar{q}_1$. We know from rationing theory (c.f., e.g., Deaton and Muellbauer, 1980), that q_2 is then obtained by first solving v_1 from (2.9a), for given v_2 and $q_1=\bar{q}_1$, and inserting the solution ($\bar{v}_1$, say) in (2.9b). Let us assume that $\bar{q}_1= -1+ a_1$. This is a perfectly feasible value; it is generated by, for instance, $v_1=\log 2$, $v_2=1$, so that $v_1<v_2$ and negativity is satisfied.

Now assume however that $\bar{q}_1=-1+a_1$ and $v_2=1/2$. Then $\bar{v}_1=\log 8 > v_2$. Hence there does not exist a shadow price $\bar{v}_1$ for which the cost function is concave, even although both $\bar{q}_1$ and v_2 are perfectly feasible. It is the combination $\bar{q}_1=-1+a_1$ and $v_2=1/2$ which causes problems.

This example shows that the relationship between negativity and the existence of a well-behaved solution of the rationing problem is not straightforward. The basic reason is that concavity is defined on a certain set of prices and income. But in order to solve the rationing problem one first has to compute the shadow prices, and it is only after the solution has been obtained that one can check whether concavity is satisfied at the rationing point.

□

At this point we want to make a number of observations. First of all, note that if the usual conditions for utility maximization (convex budget sets, convex preferences) hold, then endogenous variables are uniquely determined. This follows from standard Kuhn-Tucker theory. Thus, 'regularity' implies coherency. Secondly, the reverse does not hold. An example is given in Figure 2.3.

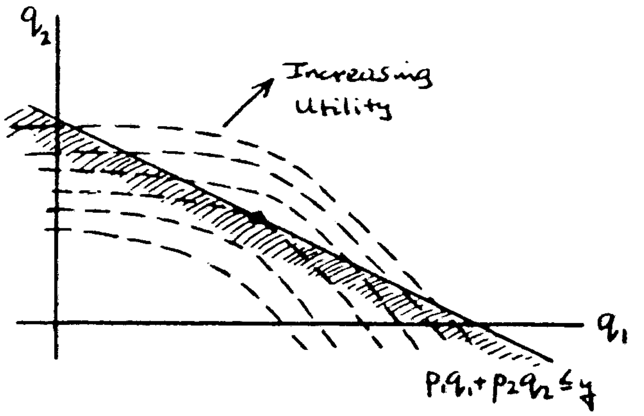

Figure 2.3. Coherency without regularity

Th unique point of tangency with the budget line satisfies first order conditions for utility maximization, but obviously does not represent a utility maximum. Yet, demand functions, being the solutions of first order conditions may very well be coherent.

In the third place, almost any specification used in practice will satisfy regularity conditions for utility maximization only locally. The main reason for this is the quest for flexible forms. Usually, flexibility is only possible if global concavity properties are sacrificed (e.g. Diewert and Wales, 1987). In itself this is reasonable, as generally economic models only aim to describe behavior of agents for a certain range of exogenous variables. See for instance Figure 2.4, where the indifference curves are convex in a certain part of the commodity space, but not everywhere. As long as we restrict attention to this 'regular area' no problems arise. Alternatively, and this is the approach in this paper, preference parameters can be restricted in such a way that indifference curves are convex in a given area of interest.

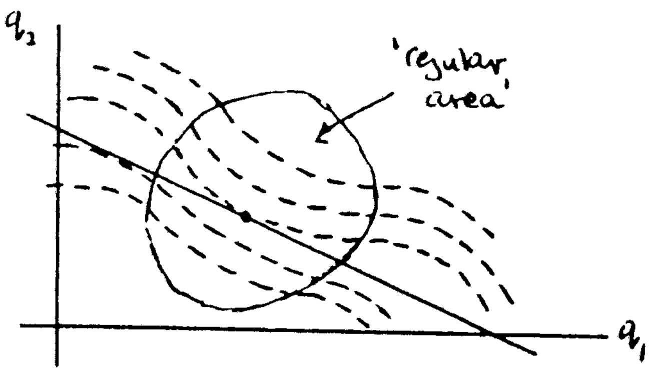

Figure 2.4. Locally well-behaved preferences

Although the requirement that a model should be coherent may appear self-evident, one may still ask whether imposition of coherency conditions is strictly necessary. After all, if the true data generating process is coherent then parameter estimates might converge to values which satisfy coherency conditions automatically. Also, one may ask whether it is possible to test coherency conditions imposed on parameters. The next subsection addresses these issues by means of yet another example.

2.4 Incoherency and ML-estimation

We consider the following simultaneous Probit-model (See e.g. Schmidt, 1981).

$$y_1^* = \beta_1 x + \gamma_1 y_2 + \varepsilon_1$$

$$y_2^* = \beta_2 x + \gamma_2 y_1 + \varepsilon_2 \tag{2.10}$$

$$y_i = 1 \text{ if } y_i^* > 0 \text{ and } y_i = 0 \text{ if } y_i^* \leq 0 \quad (i=1,2)$$

Here x denotes an (observable) exogenous variable, y_1^* and y_2^* are latent endogenous variables, y_1 and y_2 are observed endogenous variables and ε_1 and ε_2 are random variables following a bivariate normal distribution:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

It is straightforward to derive the probabilities of the four different outcomes which are possible:

$$\begin{aligned}
&\Pr[y_1=0,\ y_2=0] = \Phi(-\beta_1 x)\ \Phi(-\beta_2 x), \\
&\Pr[y_1=0,\ y_2=1] = \Phi(-\beta_1 x-\gamma_1)\ \Phi(-\beta_2 x), \\
&\Pr[y_1=1,\ y_2=0] = \Phi(-\beta_1 x)\ \Phi(-\beta_2 x-\gamma_2), \text{ and} \\
&\Pr[y_1=1,\ y_2=1] = \Phi(-\beta_1 x-\gamma_1)\ \Phi(-\beta_2 x-\gamma_2),
\end{aligned} \tag{2.11}$$

where Φ denotes the standard normal distribution function.

In general, these four probabilities do not sum to one. In fact, their sum equals

$$1 + [\Phi(\beta_1 x+\gamma_1) - \Phi(\beta_1)]\ [\Phi(\beta_2 x+\gamma_2) - \Phi(\beta_2)] \tag{2.12}$$

so that a condition for coherency is that $\gamma_1\gamma_2=0$. This renders the model recursive (cf. Schmidt, 1981). The condition given here is just a special case of the coherency conditions given by Gourieroux et al. (1980).

Let us now assume that the 'true' parameters of the data generating process are given by

$$\beta_1=1,\ \gamma_1=-1,\ \beta_2=0, \text{ and } \gamma_2=0 \tag{2.13}$$

Note that $\gamma_2=0$ implies that the true model is coherent. Furthermore, we assume that the exogenous variable x is a dummy variable with value 1 for half of the observations and value 0 for the other half. Inserting the true parameter values in (2.11) yields

$$\begin{aligned}
&Pr[y_1=0,y_2=0|x=0]=0.250; \; Pr[y_1=0,y_2=0|x=1]=0.079;\\
&Pr[y_1=0,y_2=1|x=0]=0.421; \; Pr[y_1=0,y_2=1|x=1]=0.250;\\
&Pr[y_1=1,y_2=0|x=0]=0.250; \; Pr[y_1=1,y_2=0|x=1]=0.421;\\
&Pr[y_1=1,y_2=1|x=0]=0.079; \; Pr[y_1=1,y_2=1|x=1]=0.250;
\end{aligned} \tag{2.14}$$

For a sample of 2T observations, T with x=0 and T with x=1, let K(i,j,k) be the number of observations with $y_1=i$, $y_2=j$ and x=k (i,j,k $\in$ {0,1}). Note that

$$\operatorname*{plim}_{T\to\infty} \{K(i,j,k)/T\} = Pr[y_1=i,y_2=j|x=k] \quad (i,j,k \in \{0,1\}).$$

Although the meaning of the concept 'Maximum Likelihood Estimator' is cumbersome in a model which is not coherent, application of the ML-technique is possible without imposing coherency. Our purpose is to show that the resulting estimator for the parameter β_2 is inconsistent. Because of the assumption that ϵ_1 and ϵ_2 are independent, the log-'likelihood' function based on (2.11) can be written in the following form:

$$L(\beta_1,\beta_2,\gamma_1,\gamma_2) = L_1(\beta_1,\gamma_1) + L_2(\beta_2,\gamma_2).$$

where

$$L_1(\beta_1,\gamma_1) = \sum_{j,k=0}^{1} K(0,j,k) \log \Phi(-\beta_1 k-\gamma_1 j) + \sum_{j,k=0}^{1} K(1,j,k) \log \Phi(\beta_1 k+\gamma_1 j)$$

and

$$L_2(\beta_2,\gamma_2) = \sum_{i,k=0}^{1} K(i,0,k) \log \Phi(-\beta_2 k-\gamma_2 i) + \sum_{i,k=0}^{1} K(j,1,k) \log \Phi(\beta_2 k+\gamma_2 j).$$

Maximization of L is thus achieved by maximizing L_1 (with respect to β_1 and γ_1) and L_2 (with respect to β_2 and γ_2) separately. This means that the two simultaneous Probit-equations are treated as if they were separate Probit-equations, i.e. application of the ML-technique implicitly assumes that y_2 is exogenous in the first equation and y_1 is exogenous in the second equation. Since the true value of γ_2 is 0, y_2 is independent from ϵ_1 and there is nothing wrong in estimating the parameters of the first equation in this way, i.e. the estimates for β_1 and γ_1 are consistent. The estimates for β_2 and γ_2 however are inconsistent, as can be shown by a straightforward computation of their probability limits: In the limiting.

case $(T\to\infty)$, the sample fractions $K(i,j,k)/T$ equal the probabilities given by (2.14) and L_2/T can be written as

$$L_2(\beta_2,\gamma_2)/T = 0.250 \log \Phi(0) + 0.079 \log \Phi(-\beta_2) + 0.250 \log \Phi(-\gamma_2) + 0.421 \log \Phi(-\beta_2-\gamma_2) + 0.421 \log \Phi(0) + 0.250 \log \Phi(\beta_2) + 0.079 \log \Phi(\gamma_2) + 0.250 \log \Phi(\beta_2+\gamma_2).$$

Since L_2 has the form of a Probit-likelihood, it is globally concave. Its unique maximum can easily be found numerically; it is attained for $\beta_2=0.5726$ and $\gamma_2=-0.8405$. Thus we have

$$\operatorname*{plim}_{T\to\infty} \hat{\beta}_2 = 0.5726 \neq 0 = \beta_2 \quad \text{and} \operatorname*{plim}_{T\to\infty} \hat{\gamma}_2 = -0.8405 \neq 0 = \gamma_2.$$

Finally, note that if the restriction $\gamma_2= 0$ is imposed (such that internal coherency is guaranteed) then the estimate for β_2 is consistent: $L_2(\beta_2,0)$ is maximized for

$$\hat{\beta}_2= -\Phi^{-1}(\{K(0,0,1)+K(1,0,1)\}/\{K(0,0,1)+K(0,1,1)+K(1,0,1)+K(1,1,1)\})$$

so $\operatorname*{plim}_{T\to\infty} \hat{\beta}_2 = -\Phi^{-1}(0.5) = 0 = \beta_2.$

The example shows that Maximum Likelihood estimation is not appropriate if coherency is not guaranteed for all values in the parameter space on which the likelihood function is to be maximized. Even if the model is coherent for the true parameter values, the ML-technique may yield inconsistent estimates and can lead to the conclusion that the model is not coherent: For a large enough sample, the null hypothesis $\gamma_1\gamma_2= 0$ would be rejected using standard methods of statistical inference. Moreover, the example shows that even the estimates of parameters which have no direct relation to the coherency condition (β_2 in the example) can be inconsistent. It thus makes clear that coherency is a conditio sine qua non for the use of Maximum Likelihood.

□

3. Well-behaved demand functions

This section provides a description of a general framework for the estimation of a demand system which is consistent with utility maximizing behavior. We first introduce some notation and standard regularity conditions. Next we consider restrictions on the parameter space that may be imposed in estimation in order to ensure that the estimated system satisfies the regularity conditions. Finally, examples are given of the imposition of the parameter restrictions in the estimation of some well-known demand systems.

3.1 Regularity conditions

We assume that each individual maximizes some direct utility function subject to a linear budget constraint. Topics such as rationing and non-negativity constraints are discussed in later sections. We start from an indirect utility function v_ϑ given by

$$u = v_\vartheta(p,y) \qquad ((p,y) \in \tilde{V}_\vartheta \subset \mathbb{R}^n \times \mathbb{R}),$$

where $p = (p_1,\dots,p_n)'$ is a vector of prices of n commodities,
y denotes income (or total expenditures on the n commodities),
u is the utility level, and
$\vartheta \in \Theta \subset \mathbb{R}^m$ is a vector of (fixed or random) parameters.

Standard regularity conditions for <u>given</u> $\vartheta \in \Theta$ are:

A1. v_ϑ is homogeneous of degree 0:
for all $(p,y) \in \tilde{V}_\vartheta$ and $\lambda \in \mathbb{R}^+$, $(\lambda p,\lambda y) \in \tilde{V}_\vartheta$ and $v_\vartheta(\lambda p,\lambda y) = v_\vartheta(p,y)$.

A2. v_ϑ is twice continuously differentiable with respect to prices and income and for all $(p,y) \in \tilde{V}_\vartheta$, $(\partial v_\vartheta/\partial y)(p,y) > 0$.

Assumption A2 implies that v_ϑ is strictly increasing in y and allows for the introduction of the expenditure or cost function e_ϑ on the set $\tilde{E}_\vartheta = \{(p,v_\vartheta(p,y)); (p,y) \in \tilde{V}_\vartheta\}$. e_ϑ is implicitly defined by

$$v_\vartheta(p,e_\vartheta(p,u)) = u \qquad ((p,u) \in \tilde{E}_\vartheta).$$

The dual approach is only consistent with utility maximizing behavior if 'strict' concavity is guaranteed. More precisely: e_ϑ is said to be <u>regular</u> at given $(p,u) \in \tilde{E}_\vartheta$ if the nxn matrix $(\partial^2 e_\vartheta/\partial p \partial p')(p,u)$ is negative semi-definite and of rank n-1. v_ϑ is said to regular at $(p,y) \in \tilde{V}_\vartheta$ if e_ϑ is regular at $(p,v_\vartheta(p,y))$. With these definitions the third regularity condition can be formulated:

A3. v_ϑ is regular at all $(p,y) \in \tilde{V}_\vartheta$.

In what follows we work with a convex subset V_ϑ of $\tilde{V}_\vartheta$, where for all points in V_ϑ the conditions A1-A3 are satisfied. V_ϑ is referred to as the regular set in (p,y)-space.

Uncompensated demand functions on V_ϑ are derived using Roy's identity:

$$q = F_\vartheta(p,y) \qquad ((p,y) \in V_\vartheta).$$

Here $q=(q_1,\dots,q_n)'$ is a vector of (not necessarily non-negative) quantities and the components of the vector-valued function F_ϑ are given by

$$F_{\vartheta,i}(p,y) = -(\partial v_\vartheta/\partial p_i)(p,y) \;/\; (\partial v_\vartheta/\partial y)(p,y) \qquad (i=1,\dots,n).$$

The regular set in q-space, $Q_\vartheta \subset R^n$, is defined as

$$Q_\vartheta = \{F_\vartheta(p,y);\ (p,y) \in V_\vartheta\}.$$

If v_ϑ satisfies A1 and A3, then F_ϑ is homogeneous of degree zero and one-to-one from $\{(p,1) \in V_\vartheta\}$ onto Q_ϑ. (See, e.g., Gale and Nikaido, 1965)

3.2 Parameterization and restrictions in the parameter space

Preference variation between different individuals (or households) can be incorporated in the parameter vector ϑ. For each individual t, we write

$$\vartheta_t = g_t(\nu, \eta_t) \quad .$$

Here ν is a vector (or matrix) of fixed parameters (with the same value for all individuals) and the vectors η_t are independent drawings from some probability distribution which does not depend on t. The (vector-valued) function g_t may depend on t through a vector x_t of observed individual characteristics. The most obvious example is

$$\vartheta_t = g_t(\nu, \eta_t) = \nu\, x_t + \eta_t \ ,$$

where ν is a matrix of ν_{ij}'s. Thus, systematic preference variation is allowed for if g_t depends on t, whereas the presence of the η_t's implies random variation of preferences.

In estimating the system of demand equations, the following conditions may be imposed on the admissible values of ν and/or on the set Ω of all possible realizations of the η_t's.

B1. (Regularity in a minimal subset of (p,y)-space)
For all t, for all admissible ν and $\eta \in \Omega$: $V_{g_t(\nu,\eta)} \supset V_{min}$.

This condition states that for all parameter values (and thus for all possible individual preference structures) the model must be able to explain behavior for at least some minimal subset of (p,y)-space. It implies that the parameter space Θ cannot be too large; otherwise there might be values of ν or η_t such that at some points of V_{min} the regularity conditions A1-A3 are not all satisfied. V_{min} can, for instance, be rectangular:

$$V_{min} = \{(p,y) \in R^n x\ R\ ;\ (\underline{p},\underline{y}) \leqq (p,y) \leqq (\bar{p},\bar{y})\},$$

for given values $\underline{p},\underline{y},\bar{p},\bar{y}$.

This condition is illustrated in Figure 3.1. Here a chosen set V_{min} in (p,y)-space and (for some given ν and t) the regular area's $V_{g_t(\nu,\eta)}$ are given for two different values η_1 and η_2 of η. If preferencesw are characterized by η_1, behavior cannot be explained if, e.g. $(p,y)=(p_0,y_0)\in V_{min}$. Therefore η_1 is excluded, i.e. $\eta_1 \notin \Omega$. For $\eta=\eta_2$, the model can explain behavior for all $(p,y)\in V_{min}$, so η_2 may be allowed. Thus for given ν and t condition B1 implies a restriction on Ω. Together these restrictions imply that Ω and Θ cannot be too large.

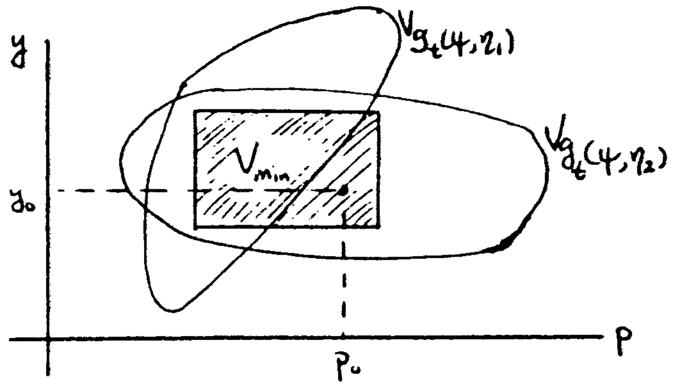

Figure 3.1. Condition B1 in (p,y)-space

B2. ('External coherency')

Let, for all t, VQ_t be a given subset of $\{(p,y,q) \in R^n x R x R^n;\ p'q = y\}$. Then for all t, for all admissible ν and $(p,y,q) \in VQ_t$, there is an $\eta \in \Omega$ such that $F_{g_t(\nu,\eta)}(p,y) = q$.

One can think of VQ_t as the set of prices, incomes and quantities which may arise for observation t. For example, if no measurement or optimization errors are involved, VQ_t must at least contain the observed (p,y,q)-vector for individual t. In fact, VQ_t may then consist of just one point. Condition B2 states that the parameter space cannot be too small. The random preferences must allow so much flexibility that for all admissible ν and at least one possible value of η a given (observed) quantity vector is optimal for given prices and income. This motivates the term "external coherency": The model has to be coherent with available data, or in other words the likelihood contribution of any given data point should be strictly positive. If the model allows for measurement errors, the condition may be omitted.

Figure 3.2 gives an example for one data point (i.e. a particular budget line and a point A on it). For each value of ψ there must be at least one value of $\eta \in \Omega$ which produces an indifference curve tangent to the budget line at A.

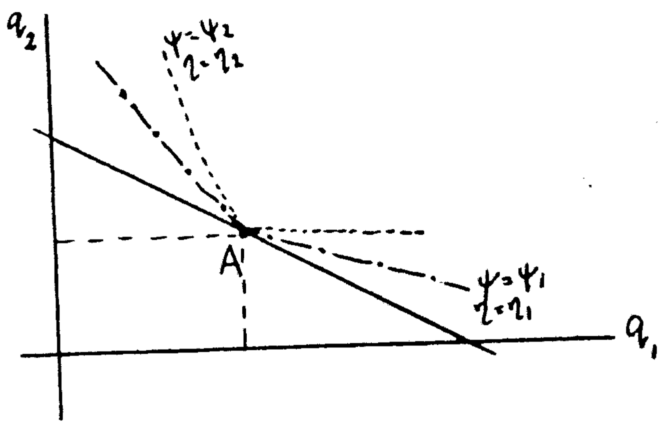

Figure 3.2. Condition B2 in q-space

B3. (Regularity in a minimal subset of q-space)

For all t, for all admissible ψ and $\eta \in \Omega$: $Q_{g_t(\psi,\eta)} \supset Q_{min}$.

This condition states that, for given (fixed and random) parameter values, certain quantity vectors must be optimal for some prices and income. As with B1 this means that the parameter space cannot be too large. As long as the issue of rationing (and shadow prices) is not addressed, we might do without this condition.

This condition is illustrated in Figure 3.3. For given parameter values $g_t(\psi,\eta)$, the commodity space consists of three parts: The area where the direct utility function is not defined (because shadow prices do not exist) (QN), the area where indifference curves exist but are not convex (QI) and the regular area $Q_{g_t(\psi,\eta)}$. The condition states that parameters have to be restricted such that Q_{min} is contained in $Q_{g_t(\psi,\eta)}$.

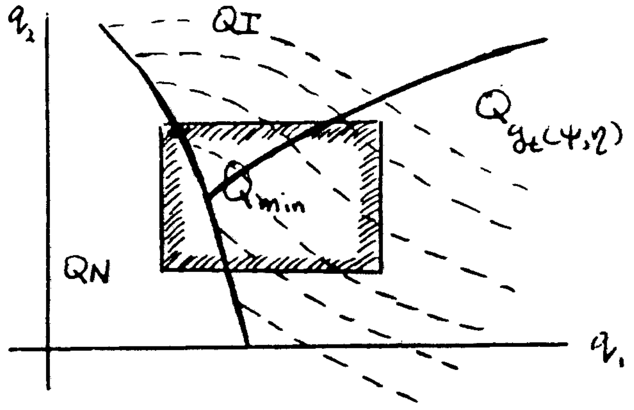

Figure 3.3 Condition B3 in q-space

Conditions B1 and B3 look similar since they both appear to define an area in q-space where indifference curves are convex. However, since restrictions are imposed a priori, before parameters are estimated, it is not possible to tell which point in q-space corresponds to a particular (p,y)-combination. Thus, if we choose a particular V_{min} and estimate the parameters imposing B1 it may turn out that indifference curves are convex in an area quite different from the Q_{min} we had in mind. Similarly, a choice of Q_{min} and imposition of B3 may actually imply concavity on an area in (p,y)-space quite different from V_{min}. This point is illustrated below with respect to the quadratic utility function.

Even if V_{min} and Q_{min} are chosen such that they contain all observed price-income combinations and quantities, condition B2 cannot be dispensed with. B1 and B3 may for instance place such severe restrictions on Ω that a particular point in (p,y)-space is mapped on the corresponding observed vector in q-space for no $\eta \in \Omega$. This is why it has been said that for B1 and B3 the parameter space should not be too large, whereas for B2 it should not be too small.

3.3 Examples

Example 1: Linear Expenditure System (LES)

The indirect utility function is defined for positive prices:

$$v_\vartheta(p,y) = (y - p'\gamma) \prod_{k=1}^{n} p_k^{-\alpha_k} , \qquad (3.1)$$

with $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i > 0$ $(i=1,\ldots,n)$, $p = (p_1,\ldots,p_n)'$, $\gamma = (\gamma_1,\ldots,\gamma_n)'$.

The expenditure function is

$$e_{\alpha,\gamma}(p,u) = p'\gamma + u \prod_{k=1}^{n} p_k^{\alpha_k} .$$

A typical element of the matrix of second order derivatives is

$$\{(\partial^2 e_\vartheta/\partial p \partial p')(p,u)\}_{i,j} = \frac{u}{p_i p_j} \{ \prod_{k=1}^{n} p_k^{\alpha_k} \} \{\alpha_i \alpha_j - \delta_{ij}\alpha_i\} ,$$

with $\delta_{ij} = 1$ if $i=j$, $\delta_{ij} = 0$ if $i \neq j$.

Given the assumptions on the α_i, this matrix is negative semi-definite if and only if $u > 0$. In view of (3.1) this requires $y - p'\gamma > 0$ so that a maximal choice for the regular set is given by

$$V_{\alpha,\gamma} = \{(p,y);\ p > 0 \text{ and } y - p'\gamma > 0\}.$$

The uncompensated demand functions are

$$q_i = F_{\vartheta,i}(p,y) = \gamma_i + (\alpha_i/p_i)(y - p'\gamma) \qquad (3.2)$$

From the definition of Q_ϑ it is easy to see that the corresponding regular region in q-space is given by

$$Q_{\alpha,\gamma} = \{q \in \mathbb{R}^n;\ q > \gamma\}.$$

Random preferences can be incorporated as follows:

$$\gamma_t = \gamma_{t,0} + \eta_t \ ,$$

where $\gamma_{t,0} = (\gamma_{t1,0}, \ldots, \gamma_{tn,0})'$ is fixed (and may depend on personal characteristics of individual t) and $\eta_t = (\eta_{t1}, \ldots, \eta_{tn})'$ is random (with a distribution that does not depend on t).

We elaborate the conditions B1 through B3:

B1. The random variable η_t should be restricted such that $y - \gamma_t' p > 0$ for all $(p,y) \in V_{min}$. If, for instance, we take V_{min} to be the rectangle

$$V_{min} = \{(p,y);\ 0 \leqq p \leqq \bar{p},\ 0 \leq y \leq \bar{y}\},$$

then the η's have to satisfy

$$\max_{0 \leqq p \leqq \bar{p}} p'\eta < -\max_t \max_{0 \leqq p \leqq \bar{p}} p'\gamma_{t,0} \ , \tag{3.3}$$

which then defines the largest possible Ω for given $\gamma_{t,0}$'s.

B2. Solving γ from (3.2) yields

$$\gamma_i = q_i - \lambda\, \alpha_i/p_i \quad (i=1,\ldots,n), \text{ for some arbitrary } \lambda > 0.$$

If we define VQ_t as $\{(p_t, y_t, q_t)\}$ (with $p_t' q_t = y_t$), then Ω has to be large enough to contain for each t at least one value of η in the set

$$\{\eta \in \mathbb{R}^n\ ;\ \eta_i = q_{ti} - \lambda\, \alpha_i/p_{ti} - \gamma_{ti,0}\ (i=1,\ldots,n) \text{ for some } \lambda > 0\}\ .$$

Note that by choosing λ large enough we can always guarantee that η will be in Ω according to (3.3). Hence, conditions B1 and B2 can be satisfied simultaneously. As said before, if we allow for measurement or optimization errors, condition B2 need not be imposed.

B3. We may want to define Q_{min} as the set $\{q;\ q > \bar{q}\}$, with $\bar{q}$ some given vector, so that the γ's have to satisfy $\gamma \leqq \bar{q}$, or equivalently $\eta \leqq \bar{q} - \gamma_{t,0}$ for all t. If one wants to impose B1 and B3 simultaneously, this condition has to be imposed jointly with (3.3).

It may be illuminating to discuss the role of the three conditions in the estimation of the Linear Expenditure System a little bit further. Suppose we do not allow for measurement errors. Then B1 and B3 imply restrictions on the range of the random variables η which depend on parameters implicit in $\gamma_{t,0}$. This in itself may give rise to non-standard estimation problems. For the rest, however, the imposition of B1 and B3 (and the fact that B2 can always be satisfied) makes sure that whatever our estimates will be, the resulting model is always consistent with neoclassical theory. Also, since B2 is satisfied the likelihood is well-defined for all data points.

□

Example 2: Quadratic Direct Utility Function (QDU)

The direct utility function is given by

$$U(q) = \gamma'q - 1/2\ q'B\ q \tag{3.4}$$

where $\gamma = (\gamma_1,\ldots,\gamma_n)'$ and $B = \begin{bmatrix}\beta_{11}\cdots\beta_{1n}\\ \cdots\cdots\\ \beta_{1n}\cdots\beta_{nn}\end{bmatrix}$ is positive definite.

The utility function has a satiation point at $q = B^{-1}\gamma$, with corresponding utility level $u = 1/2\ \gamma'B^{-1}\gamma$. The demand functions are given by

$$q = B^{-1}\gamma - (p'B^{-1}p)^{-1}\{\gamma'B^{-1}p - y\}\ B^{-1}p$$

and the indirect utility function is thus given by

$$v(p,y) = 1/2\ \{\gamma'B^{-1}\gamma - (p'B^{-1}p)^{-1}[\gamma'B^{-1}p - y]^2\}.$$

The indirect utility function is increasing in y as long as the satiation point is not in the budget set, i.e. as long as $y < \gamma' B^{-1} p$. Homogeneity of degree zero is satisfied automatically.

The expenditure function is given by

$$e_{B,\gamma}(p,u) = \gamma' B^{-1} p - (p' B^{-1} p)^{1/2} [\gamma' B^{-1} \gamma - 2u]^{1/2}.$$

The Hessian of the expenditure function is

$$(\partial^2 e_\theta / \partial p \partial p')(p,u) = (p' B^{-1} p)^{1/2} [\gamma' B^{-1} \gamma - 2u]^{1/2} [(p' B^{-1} p)^{-1} (B^{-1} p)(B^{-1} p)' - B^{-1}].$$

As one would expect, $e(p,u)$ is only defined for $u \leq 1/2\ \gamma' B^{-1} \gamma$ (i.e. for u less than or equal to the satiation level) and $p \neq 0$. It is easy to show that, since B is positive definite, the matrix

$$(p' B^{-1} p)^{-1} (B^{-1} p)(B^{-1} p)' - B^{-1}$$

is negative semi-definite and of rank n-1. Hence, the cost function is concave for $u \leq 1/2\ \gamma' B^{-1} \gamma$.

In what follows, we assume that there is one commodity, say the n-th, for which the price is always positive, i.e. $p_n > 0$.
This suggests the following choice for $V_{B,\gamma}$:

$$V_{B,\gamma} = \{(p,y);\ y < \gamma' B^{-1} p,\ p_n > 0\}.$$

Let us consider the following stochastic specification (see, e.g., Ransom, 1987b) :

$$\gamma_t = \gamma_{t,0} + \eta_t \ ,$$

where $\gamma_{t,0} = (\gamma_{t1,0}, \ldots, \gamma_{tn,0})'$ is fixed, and $\eta_t = (\eta_{t1}, \ldots, \eta_{tn})'$ is a vector of random variables with $\eta_{tn} = 0$.
The elaboration of B1 - B3 is as follows.

B1. $(\gamma_{t,0} + \eta)' B^{-1} p - y > 0$ for all t, $\eta \in \Omega$, and $(p,y) \in V_{min}$.

If we define V_{min} as for LES, this condition turns into

$$\min_{0 \leq p \leq \bar{p}} \eta' B^{-1} p \geq \bar{y} - \min_t \min_{0 \leq p \leq \bar{p}} \gamma'_{t,0} B^{-1} p \tag{3.5}$$

which is then again the definition of Ω. It is quite similar to the corresponding condition (3.3) for LES.

B2. Again, let us define VQ_t as $\{(p_t, y_t, q_t)\}$. Solving η from the demand functions yields

$$\eta_i = \frac{p_{ti}}{p_{tn}} \gamma_{tn,0} - \gamma_{ti,0} + \sum_{j=1}^{n} \{\beta_{ij} - \frac{p_{ti}}{p_{tn}} \beta_{nj}\} q_{tj}$$

and Ω should be big enough to contain the η's obtained in this way for all t, and for all values of $\gamma_{t,0}$ and β in the admissible parameter space.

B3. Inversion of the demand system for given parameter values (including γ) yields shadow prices and corresponding virtual income as a function of q:

$$p = \lambda \ (\gamma - B\ q) \text{ and } y = p'q \ ,$$

where λ can be chosen arbitrarily. The solution (p,y) is a point in $V_{B,\gamma}$ iff $\lambda > 0$ and $\gamma_n - (B\ q)_n > 0$. Thus, imposition of regularity in a given region Q_{min} in q-space yields $\gamma_n - (Bq)_n > 0$ for all $q \in Q_{min}$. This can be achieved by restricting the values of fixed parameters only, since we have assumed that γ_n is non-random. Truncation of the distribution of η is unnecessary. If, for instance, Q_{min} is some rectangle, 2^n simultaneous linear inequality restrictions on the coefficients in B and $\gamma_{tn,0}$ result for each individual t. The conditions to be imposed in estimation are then obtained as the intersection of the inequalities for each individual.

To get some more feeling for these conditions, we look at a simple numerical example for two commodities. Let n = 2, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and assume that $\gamma_{t,0} = \gamma_0$, fixed and independent of t.

B1. Let V_{min} be a 'rectangle in (p,y)-space', i.e.

$$V_{min} = \{(p,y);\ 0 < v_\ell \le p_1/y \le v_u \text{ and } v_\ell \le p_2/y \le v_u\}.$$

Since it is assumed that p_2 always exceeds 0 it is convenient to work with the normalization $p_2 = 1$. V_{min} can then be written as

$$V_{min} = \{(p_1,y);\ v_\ell y \le p_1 \le v_u y \text{ and } v_u^{-1} \le y \le v_\ell^{-1}\}$$

A feasible $\gamma = (\gamma_1,\gamma_2)'$ has to satisfy

$$\gamma_1 p_1 + \gamma_2 - y > 0 \quad \text{for all } (p_1,y) \in V_{min}.$$

Thus, γ is feasible iff

$$\gamma_2 > v_\ell^{-1} - \gamma_1,\ \gamma_2 > v_u^{-1} - \gamma_1 v_\ell/v_u \text{ and } \gamma_2 > v_\ell^{-1} - \gamma_1 v_u/v_\ell.$$

Figure 3.4 presents the feasible area (FA) in (γ_1,γ_2)-space. In this example (i.e. for this choice of B) the feasible area is non-empty for every $v_u \ge v_\ell > 0$.

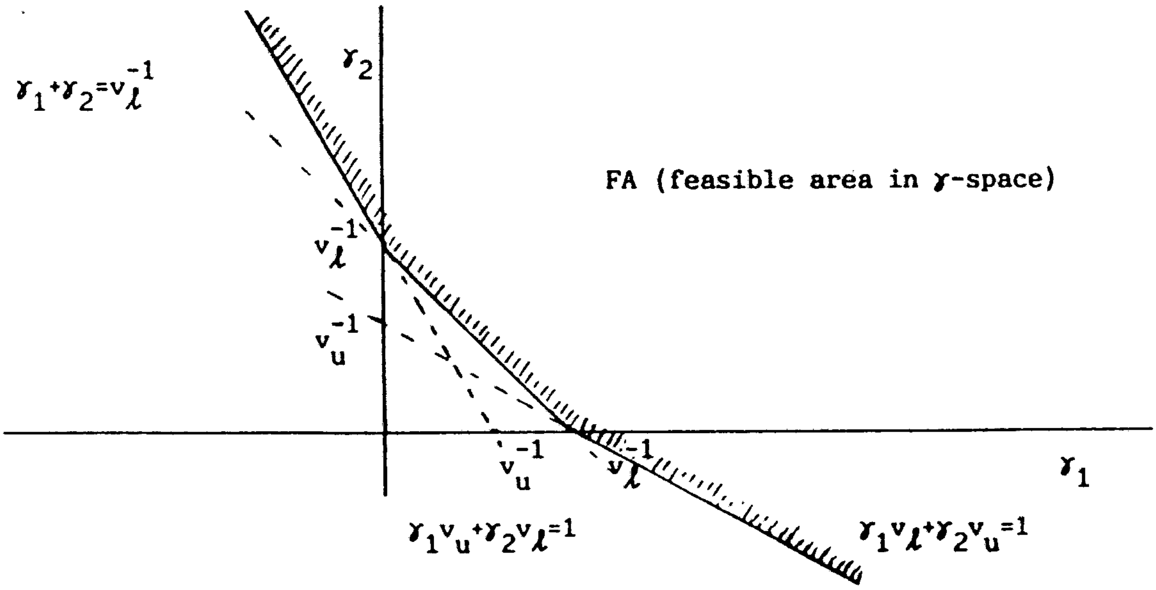

Figure 3.4 The feasible area in γ-space

For each feasible γ it is possible to derive the regular area in (p_1, y)-space:

$$V_\gamma = \{ (p_1, y) \; ; \; \gamma_1 p_1 + \gamma_2 - y > 0 \}.$$

The intersection of these V_γ's is the region in (p_1, y)-space, where the indirect utility function behaves well for <u>all</u> $\gamma \in FA$:

$$V = \bigcap_{\gamma \in FA} V_\gamma .$$

In Figure 3.5 V and V_{min} are presented. Note that automatically $V \supset V_{min}$, but the figures show that V is much larger than V_{min}. (One could have chosen V instead of V_{min} to begin with; this yields the same region FA in γ - space)

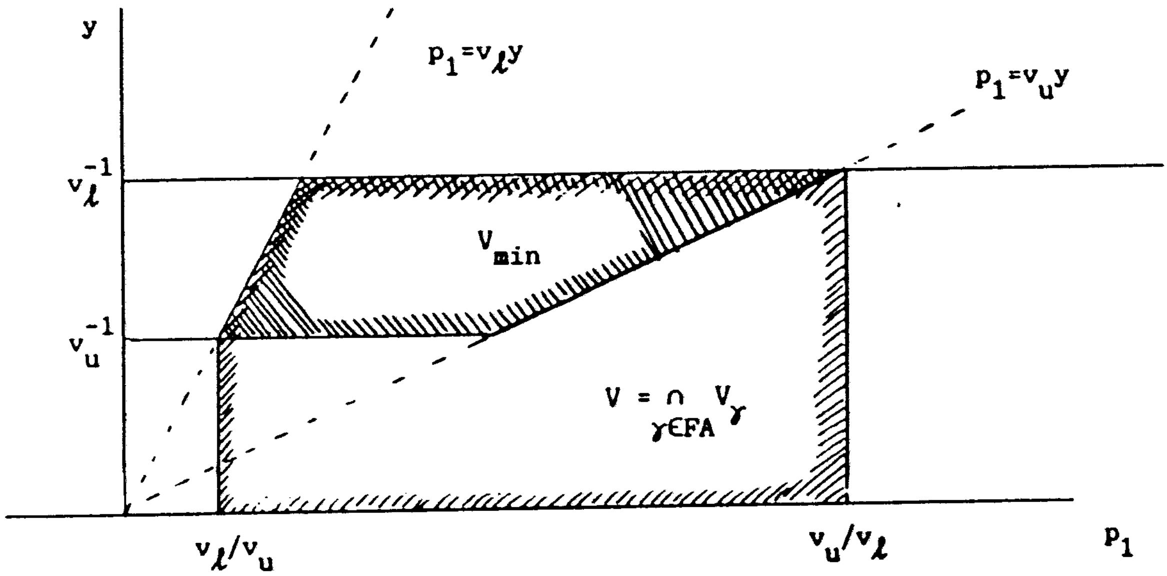

Figure 3.5 The minimal and the actual regular region in (p_1, y)-space

B2. For fixed $\gamma_2 = \gamma_{2,0}$ and given p_1, y, $(p_2 = 1)$, q_1 and q_2 with $y = p'q$, we must find a feasible solution for γ_1 from the demand system

$$q_1 = \gamma_1 - (1+p_1^2)^{-1} \{\gamma_1 p_1 + \gamma_2 - y\} \, p_1$$

$$q_2 = \gamma_2 - (1+p_1^2)^{-1} \{\gamma_1 p_1 + \gamma_2 - y\} \, p_2$$

This is a system of two linearly dependent equations in γ_1 with a unique solution:

$\gamma_1 = q_1 + p_1(\gamma_2 - q_2)$.

The solution is feasible iff

$$(p_1 + 1)\gamma_2 > v_\ell^{-1} - q_1 + p_1 q_2 \text{ , and}$$

$$(p_1 + v_u/v_\ell)\gamma_2 > v_\ell^{-1} - q_1 + p_1 q_2 \text{ , and}$$

$$(p_1 + v_\ell/v_u)\gamma_2 > v_u^{-1} - q_1 + p_1 q_2 \text{ .}$$

If sample prices p_1 always exceed $-v_\ell/v_u$ then it is possible to guarantee the existence of a feasible solution for all (q_1,q_2,y) in the sample by restrictng γ_2 to be large enough.

B3. For given γ, the regular region in q-space is given by

$$Q_\gamma = \{(q_1,q_2) \in \mathbb{R}^2 \ ; \ q_2 < \gamma_2\}.$$

Thus, regularity on some region $Q_{min} \subset \{(q_1,q_2) \in \mathbb{R}^2 \ ; \ q_2 \leq \bar{q}_2\}$ is guaranteed if γ_2 is restricted to values larger than $\bar{q}_2$. Note that γ_1 is not restricted. If once again FA is defined as the feasible area in (p,y)-space, i.e. the area where concavity is satisfied for all γ_1 and for all $\gamma_2 \geq \bar{q}_2$, then it is easy to see that this feasible area is empty!

□

<u>Example 3: Almost Ideal Demand System (AIDS)</u>

Let $v = (\log p_1,\dots,\log p_n)'$, $\alpha = (\alpha_1,\dots,\alpha_n)'$, $\beta = (\beta_1,\dots,\beta_n)'$, Γ a symmetric nxn-matrix with typical element γ_{ij}. The expenditure function is

$$e_{\alpha,\beta,\Gamma}(p,u) = \exp\{a(p) + u\, b(p)\}, \tag{3.5}$$

where $a(p) = \alpha_0 + \alpha' v + 1/2\, v'\Gamma v$, and

$$b(p) = \exp\ (\beta' v)\ .$$

The expenditure function is homogeneous in p if $\alpha'\iota=1$, $\Gamma\iota=0$ and $\beta'\iota=0$, where ι is an n-dimensional vector with unit elements.
The uncompensated demands are given by

$$s = \alpha + \Gamma v + \beta\ \{\log y - a(p)\} \tag{3.6}$$

where $s = (s_1,\ldots,s_n)'$, s_i being the i-th budget share.
The concavity condition for the expenditure function is

$$C \leqq 0\ , \tag{3.7}$$

where $C = \Gamma + \beta\ \beta'\ \{\log y - a(p)\} - \Delta + s\ s'$,with $\Delta = \text{diag}(s)$. <u>Sufficient</u> conditions for C to be negative semi-definite are

$$\text{(a) } \Gamma \leqq 0\ ; \quad \text{(b) } \log y \leq a(p)\ ; \quad \text{(c) } 0 \leq s_i \leq 1 \quad (i=1,\ldots,n). \tag{3.8}$$

We introduce random preferences as follows:

$$\alpha_t = \alpha_{t,0} + \eta_t$$

with $\eta_t'\iota = 0$, η_t random.

B1. Suppose V_{min} is defined by $V_{min} = \{(p,y)\ ;\ 0 \leqq v \leqq \bar{v},\ 0 \leq y \leq \bar{y}\ \}$.
Condition (b) requires the following restriction on fixed parameters and on the range of the random variables:

$$\max_{0 \leqq v \leqq \bar{v}} (-\eta' v) \leq \alpha_0 - \log \bar{y} + \min_t \min_{0 \leqq v \leqq \bar{v}} (\alpha_{t,0}' v + 1/2\ v'\Gamma v)$$

We also have to impose condition (c). Rewrite (3.6) as follows:

$$s = (I - \beta v')\eta + \alpha_{t,0} + \Gamma v + \beta\{\log y - \alpha_0 - \alpha_{t,0}' v - 1/2\ v'\Gamma v\} \tag{3.9}$$

From this expression it is clear that (c) imposes a number of additional linear restrictions on the range of the η's. Thus, as with

LES and QDU, we find that the η's are confined to a polyhydron, although it is a bit more difficult to characterize than previously.

B2. Let $VQ_t=\{(p_t,y_t,q_t)\}$ with $p_t'q_t=y_t$, $p_t>0$ and $y_t>0$. η can be solved from the linear system

$$s_t= [I-\beta v_t']\eta + \alpha_{t,0}+ \Gamma v_t+ \beta\{\log y_t- \alpha_0- \alpha_{t,0}'v_t- \frac{1}{2} v_t'\Gamma v_t\} \tag{3.10}$$

$$\eta'\iota=0$$

where $s_t=(s_{t1},\ldots,s_{tn})'$, $v_t=(v_{t1},\ldots,v_{tn})'$, $s_{ti}=p_{ti}q_{ti}/y_t$, $v_{ti}=\log p_{ti}$ $(i=1,\ldots,n)$. The solution must satisfy condition (3.7). If $0\leqq s_t\leqq\iota$ and $\Gamma\leqq 0$, a sufficient condition for this can be derived from (3.6):

$$-\eta'v_t \leq \alpha_0- \log y_t+ \alpha_{t,0}'v_t+ \frac{1}{2} v_t'\Gamma v_t \tag{3.11}$$

Substituting the solution for η obtained from (3.10) in (3.11) yields an intricate condition on (p_t,y_t,q_t). The fixed parameters must be chosen so that this condition is satisfied for all t.

B3. The characterization of the area in quantity space where the system can be well-behaved appears to be extremely difficult. In terms of budget shares, such a characterization is substantially more straightforward. For some purposes this may suffice.

□

4. Rationing if the regime is exogenous

In this section, we consider the problem of an individual who maximizes utility, facing not only the budget constraint, but also a given set of equality constraints. This can be seen as an introduction to the more realistic case of endogenous regimes: the individual faces a set of inequalities; in the point of highest utility some will be binding and others will not. In this section we assume that it is known in advance, that the constraints on goods 1 through k are binding and the other constraints are not. The number k and the order of the goods may vary across individuals, i.e. different individuals may be rationed with respect to different goods.

The individual solves the problem

$$\operatorname*{Max}_{q_{II}} U_\theta(\bar{q}_I, q_{II}) \qquad \text{s.t.} \quad y = p_I'\bar{q}_I + p_{II}'q_{II}.$$

Here we have written $q = (q_I', q_{II}')'$, $p = (p_I', p_{II}')'$ and the constraints are given by $q_I = \bar{q}_I$.

Starting from the indirect utility function v_θ and corresponding demand system F_θ as in Section 3, the solution of this maximization problem is found using the notion of shadow (or virtual) prices (see, e.g., Neary and Roberts, 1980):

Find $\tilde{p}_I \in R^k$, $\tilde{y} \in R$ and $q_{II} \in R^{n-k}$ such that

$$\begin{cases} ((\tilde{p}_I, p_{II}), \tilde{y}) \in V_\theta \\ F_\theta((\tilde{p}_I, p_{II}), \tilde{y}) = (\bar{q}_I, q_{II}) \\ \tilde{y} = y + (\tilde{p}_I - p_I)'\bar{q}_I \end{cases} \tag{4.1}$$

The optimal quantities, taking into account the constraints $q_I = \bar{q}_I$, are then given by q_{II}.

For individual t with income y_t, confronted with prices p_t and rationed quantities $\bar{q}_{It}$, we want problem (4.1) to yield a unique and feasible solution. This suggests the following three conditions.

C1. (Regularity in a minimal subset of (p,y)-space)
For all admissible γ and all $\eta \in \Omega$: $V_{g_t(\gamma,\eta)} \supset V_{min}$.

This condition is exactly identical to condition B1 in the previous section.

C2. ('External coherency')

Let VQ_t be a given subset of $\{(p,y,q) \in R^n \times R \times R^n;\ p'q=y\}$.
Then for all admissible γ and all $(p,y,q) \in VQ_t$, there exists at least one $\eta \in \Omega$ such that there are $\tilde{p}_I \in R^k$ and $\tilde{y} \in R$ with

$$\begin{cases} ((\tilde{p}_I,p_{II}),\tilde{y}) \in V_{g_t(\gamma,\eta)} \\ F_{g_t(\gamma,\eta)}((\tilde{p}_I,p_{II}),\tilde{y}) = q \\ \tilde{y} = y + (\tilde{p}_I - p_I)'q_I \end{cases}$$

This condition states that certain quantity vectors q_{II} can be optimal for given prices, income and rationed quantities q_I. If no measurement errors are present, VQ_t must at least contain the observed vector $(p_t,y_t,(\bar{q}_{It},q_{IIt}))$ for the individual t. This condition is similar to condition B2 in the previous section. It guarantees, once again, that each data point has a non-zero likelihood contribution. C2 is weaker than B2 since quantities $\bar{q}_{It}$ do not have to be rationalized.

C3. ('Solvability')
Let VQ_t^I be a given subset of $\{(p,y,q_I) \in R^n \times R \times R^k\}$.
Then for all admissible γ and all $\eta \in \Omega$ and all $(p,y,q_I) \in VQ_t^I$, there exist $\tilde{p}_I \in R^k$, $\tilde{y} \in R$ and $q_{II} \in R^{n-k}$ such that

$$\begin{cases} ((\tilde{p}_I, p_{II}), \tilde{y}) \in V_{g_t(\nu,\eta)} \\ F_{g_t(\nu,\eta)}((\tilde{p}_I, p_{II}), \tilde{y}) = (q_I, q_{II}) \\ \tilde{y} = y + (\tilde{p}_I - p_I)' q_I \end{cases}$$

Condition C3 states that for each admissible vector of parameters, (4.1) must have a solution that is well-behaved. It is imposed to avoid the problem encountered in Example 2.3. If there are no measurement errors on prices, income or <u>rationed</u> quantities, then the set VQ_t^I must at least contain the observed vector $(p_t, y_t, \bar{q}_{It})$. There is no similarity between the conditions C3 and B3. The reader must be well aware of the difference between C3 and C2: C3 states that for <u>each</u> $\eta \in \Omega$ the rationed utility maximization problem has <u>some</u> regular solution. C2 states that the model must be able to explain a given observation, i.e. to each observed optimum there must correspond <u>some</u> $\eta \in \Omega$.

The conditions have to be imposed for all individuals simultaneously. Condition C1 is not necessary for internal coherency of the model with rationing because it involves restrictions on actual prices, whereas for the rationed commodities only shadow-prices matter. The condition is important however when the model is used for simulations in which the rationing is relaxed. Condition C3 is strongly related to the internal coherency problem: It states that to each possible realization of η there corresponds <u>at least</u> one vector q_{II} of endogenous variables. Together with the concavity of the expenditure function and convexity of V_θ this implies internal coherency of the model.

<u>Example 1: LES</u>

Solving

$$\max_{q_{II}} U_\theta(\bar{q}_I, q_{II}) = \prod_{i=1}^{k} (\bar{q}_i - \gamma_i)^{\alpha_i} \prod_{i=k+1}^{n} (q_i - \gamma_i)^{\alpha_i} \qquad \text{s.t.} \quad y = p_I' \bar{q}_I + p_{II}' q_{II},$$

yields

$$q_i = \gamma_i + \alpha_i(y - p_I'\bar{q}_I - p_{II}'\gamma_{II}) \,/\, \{p_i \sum_{j=k+1}^{n} \alpha_j\} \qquad (i=k+1,\ldots,n), \qquad (4.2)$$

where $\gamma_{II} = (\gamma_{k+1},\ldots,\gamma_n)'$.

Alternatively, (4.2) can be obtained by first solving the shadow prices $(\tilde{p}_1,\ldots,\tilde{p}_k)'$ from the first k unconditional demand equations, with p_I replaced by $\tilde{p}_I$ and y replaced by $\tilde{y}$, yielding

$$\tilde{p}_i = \alpha_i(y - p_1'\bar{q}_1 - p_2'\gamma_2) \,/\, \{(\bar{q}_i - \gamma_i) \sum_{j=k+1}^{n} \alpha_j\} \qquad (i=1,\ldots,k),$$

and

$$\tilde{y} = y + \sum_{j=1}^{k} (\tilde{p}_j - p_j)'\bar{q}_j \qquad (4.3)$$

and next substituting the solution into the notional demand equations for goods k+1 through n, again with p_I replaced by $\tilde{p}_I$ and y replaced by $\tilde{y}$. The solution is feasible iff $\bar{q}_I > \gamma_I$ and $q_2 > \gamma_{II}$, or, equivalently, $\tilde{p}_I > 0$, $p_{II} > 0$, and $y - \tilde{p}_I'\gamma_I - p_{II}'\gamma_{II} > 0$.

Note that equation (4.1) has exactly the same functional form as the notional demand functions (3.2), the only difference being that α_i is replaced by $\alpha_i / \{\alpha_{k+1} + \ldots + \alpha_n\}$ and y is replaced by $y - p_I'\bar{q}_I$, and that (4.1) does not depend on $(\alpha_1,\ldots,\alpha_k)$ nor on $(\gamma_1,\ldots,\gamma_k)$. If all individuals are rationed with respect to the same goods, one might simply impose the conditions B1 - B3 described in the previous section, with y replaced by $y - p_I'\bar{q}_I$ and γ replaced by $\gamma_{II} = (\gamma_{k+1},\ldots,\gamma_n)'$.

Conditions C2 and C3 can be elaborated as follows:

C2. Let $VQ_t = \{(p_t, y_t, q_t)\}$, with $p_t > 0$ and $y_t = p_t'q_t$.
Solving γ from (4.2) yields

$$\gamma_{it} = q_{it} - \lambda\alpha_i / p_{it} \quad (i=k+1,\ldots,n) \text{ for some arbitrary } \lambda > 0. \qquad (4.4)$$

The solution is feasible if $\gamma_i < q_i$ $(i=1,\ldots,k)$.
Ω has to be large enough to contain at least one value of η such that $\gamma_t = \gamma_{t,0} + \eta$ satisfies (4.4) and is feasible.

Note that this condition is similar to the corresponding condition B2 for LES in Section 3. It is weaker because the quantities $q_1,\dots,q_k$ do not have to be rationalized.

C3. Let $VQ_t^I = \{(p_t, y_t, q_{It})\}$. Existence of a feasible solution for given $\gamma = (\gamma_I, \gamma_{II})$ means

$$q_{It} > \gamma_I \text{ and } y - p_{It}'q_{It} - p_{II}'\gamma_{II} > 0.$$

Substitution of $\gamma = \gamma_{t,0} + \eta$ yields k+1 inequality restrictions on η that restrict the set Ω.

□

Example 2: QDU

We assume that no rationing applies to the quantity of the n-th commodity, i.e. the commodity which was treated differently from the other commodities in Section 3.

Solving

$$\max_{q_{II}} U_9(\bar{q}_I, q_{II}) = (\gamma_I', \gamma_{II}') \begin{bmatrix} \bar{q}_I \\ q_{II} \end{bmatrix} - 1/2\, (\bar{q}_I', q_{II}') \begin{bmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{bmatrix} \begin{bmatrix} \bar{q}_I \\ q_{II} \end{bmatrix}$$

$$\text{s.t.} \quad y = p_I'\bar{q}_I + p_{II}'q_{II} \quad ,$$

yields

$$q_{II} = B_{22}^{-1}(\gamma_{II} - B_{12}\bar{q}_I) - (p_{II}'B_{22}^{-1}p_{II})^{-1}[(\gamma_{II} - B_{12}\bar{q}_I)'B_{22}^{-1}p_{II} - y - p_I'\bar{q}_I]B_{22}^{-1}p_{II} \qquad (4.5)$$

with obvious partitioning of γ and B.

The solution is feasible iff $(\bar{q}_I, q_{II})$ is in the regular area of q-space, i.e. iff

$$\gamma_{II}'B_{22}^{-1}p_{II} - y + \bar{q}_I' \{p_I - B_{12}B_{22}^{-1}p_{II}\} > 0 \ .$$

Note that (4.5) has exactly the same functional form as the notional demand functions, the only difference being that y is replaced by $y - p_I'\bar{q}_{II}$, γ by $(\gamma_{II} - B_{12}\bar{q}_I)$, B by B_{22} and p by p_{II}.

The elaboration of conditions C2 and C3 is similar to the previous example:

C2. Let $VQ_t = \{(p_t, y_t, q_t)\}$, with $p_{nt} > 0$ and $y_t = p_t' q_t$.
Solving γ from (4.5) yields

$$\gamma_{II} = B_{12} q_{It} + B_{22} q_{IIt} - \lambda_t p_{IIt}, \tag{4.6}$$

where

$$\lambda_t = \{-\gamma_{tn,0} + [B_{12} q_{It} + B_{22} q_{IIt}]_n\} / p_{tn}. \tag{4.7}$$

The solution is feasible iff $\lambda_t > 0$.
Ω has to be large enough to contain at least one value of η such that $\gamma_t = \gamma_{t,0} + \eta$ is feasible and satisfies (4.6), with λ given by (4.7).

C3. Let $VQ_t^I = \{(p_t, y_t, q_{It})\}$.
Existence of a feasible solution for given $\gamma = (\gamma_I, \gamma_{II})$ means

$$\gamma_{II}' B_{22}^{-1} p_{IIt} - y_t + \bar{q}_{It}' \{p_{It} - B_{12} B_{22}^{-1} p_{IIt}\} > 0.$$

Substitution of $\gamma = \gamma_{t,0} + \eta$ yields an inequality restriction on η that restricts the set Ω.

□

Example 3: AIDS

The shares for this specification are given by (3.9). The equation

$$F_{g(\nu,\eta)}((\tilde{p}_I, p_{II}), \tilde{y}) = q$$

can be written as follows:

$$\tilde{s} = [I - \beta\tilde{v}']\eta + \alpha_{t,0} + \Gamma\tilde{v} + \beta\{\log\tilde{y} - \alpha_0 - \alpha_{t,0}'\tilde{v} - \tfrac{1}{2}\tilde{v}'\Gamma\tilde{v}\}, \tag{4.8}$$

where $\tilde{v} = (\tilde{v}_I', v_{II}')'$, $\tilde{v}_I = (\tilde{v}_1, \ldots, \tilde{v}_k)'$, $\tilde{v}_i = \log\, \tilde{p}_i$ $(i=1,\ldots,k)$, $\tilde{s} = (\tilde{s}_I', s_{II}')'$, $\tilde{s}_I = (\tilde{s}_1, \ldots, \tilde{s}_k)'$, $\tilde{s}_i = \tilde{p}_i \bar{q}_i / \tilde{y}$ $(i=1,\ldots,k)$. It is impossible to derive an analytical expression for shadow prices $\tilde{p}_I$ from (4.8). Numerical methods have to be used. As a consequence, the elaboration of conditions C2 and C3 seems extremely difficult.

C2. Let $VQ_t=\{(p_t,y_t,q_t)\}$ as in Example 3.3. The condition states that (4.8) (with (p,y,q) replaced by (p_t,y_t,q_t)) must yield at least one feasible solution for $(\eta,\tilde{p}_I)$. Since (4.8) does not permit an analytic solution for $\tilde{p}_I$, this condition can only be checked numerically, but not imposed in any obvious way. Note again that it is weaker than B2 because the quantities $\bar{q}_I$ do not have to be rationalized.

C3. Because of the intricate way in which $\tilde{p}_I$ enters (4.8), virtually nothing can be said about this condition analytically. In specific examples, for given values of the fixed parameters, numerical methods might prove useful, but imposition a priori seems to be impossible.

5. Rationing if the regime is endogenous

In this section we consider the problem of an individual who maximizes utility subject to a set of linear inequality constraints. Common examples are the case of non-negativity constraints (see, e.g., Wales and Woodland, 1983, Lee and Pitt, 1986, Ransom, 1987, Van Soest and Kooreman, 1987) and the kinked budget set in labor supply models (Hausman (1981, 1985), Moffitt, 1986, Blomquist, 1983). In contrast to the discussion in the previous section, we now assume it is not known in advance which constraints are binding and which are not. The 'regime', i.e. the way constraints are split up between binding and non-binding ones, is therefore endogenous.

The utility maximization problem in its primal form can be written as

$$\max_{q} \; U_{\theta}(q) \text{ s.t. } R\,q \leq r\,, \tag{5.1}$$

where R is a kxn-matrix and $r \in \mathbb{R}^k$.
Here k is the number of restrictions, including the budget constraint.
Specific choices of R and r yield the examples referred to above:

<u>Example a:</u> non-negativity constraints: $q \geq 0$,
budget constraint: $p'q \leq y$.
So $k=n+1$, $R=(p,-I)'$ and $r=(y,0,\ldots,0)'$.

□

<u>Example b:</u> kinked budget constraint: $c \leq w_j h + y_j$ $(j=1,\ldots,m)$,
time constraints: $h \geq 0$ and $h \leq T$.
(notation as in Section 2.1; note that $q = (c,h)'$)

So $k=m+2$, $R' = \begin{bmatrix} 1 & \ldots & 1 & 0 & 0 \\ -w_1 & \ldots & -w_m & -1 & 1 \end{bmatrix}$, $r = (y_1,\ldots,y_m,0,T)'$

□

If the utility function is strictly quasi-concave on the convex set $\{q \in \mathbb{R}^n;\; Rq \leq r\}$, the solution of the maximization problem can be found using the Kuhn-Tucker theorem. The Kuhn-Tucker conditions for the maximization problem are as follows:

If q is optimal, then there exists a vector $\tilde{\lambda} \in R^k$ such that

$$\begin{aligned}
&\tilde{\lambda} \geqq 0 , \\
&R\, q \leqq r , \\
&\tilde{\lambda}'(R\, q - r) = 0 , \text{ and} \\
&(\partial U_\vartheta/\partial q)(q) = R'\tilde{\lambda} .
\end{aligned} \tag{5.2}$$

This can be rewritten employing the corresponding (homogeneous of degree zero) demand system $F_\vartheta(p,y)$. This demand system has the properties

$$(\partial U_\vartheta/\partial q)(F_\vartheta(p,y)) = \mu p \quad \text{for some } \mu > 0, \text{ and}$$

$$p'F_\vartheta(p,y) = y.$$

Making use of these properties and substituting $\lambda = \tilde{\lambda}/\mu$, (5.2) can be written as

$$\begin{aligned}
&\lambda \geqq 0 , \\
&R\, q \leqq r , \text{ and} \\
&q = F_\vartheta(R'\lambda, r'\lambda) .
\end{aligned} \tag{5.3}$$

(As the demand system is homogeneous of degree zero and $\lambda \neq 0$ (non-satiation), some normalization on λ may be added). $R'\lambda$ and $r'\lambda$ can be interpreted as a vector of shadow prices and shadow income respectively.

To illustrate the general nature of (5.3), we elaborate (5.3) for the two examples given above.

<u>Example a</u> (continued)

(5.3) yields $\lambda \geqq 0$, $p'q \leq y$, $-q \leqq 0$, and
$q = F_\vartheta(\{\lambda_1 p - (\lambda_2, \ldots, \lambda_{n+1})'\}, \lambda_1 y)$.

Monotonicity of the utility function in at least one of the goods implies that the budget constraint is binding, so $\lambda_1 > 0$. We can choose the normalization $\lambda_1 = 1$ and this yields, with $\tilde{\lambda} = (\lambda_2,\ldots,\lambda_{n+1})'$:

$$\tilde{\lambda} \geqq 0,\ p'q = y,\quad q \geqq 0, \text{ and}$$
$$q = F_\vartheta(p-\tilde{\lambda},\ y).$$

This is the well-known result that shadow-prices $(p-\tilde{\lambda})$ cannot exceed real prices (p).

□

<u>Example b</u> (continued)

(5.3) yields $\lambda \geqq 0$, $c \leq w_j h + y_j$ $(j=1,\ldots,m)$, $-h \leq 0$, $h \leq T$ and

$$(h,c)' = F_\vartheta((-\sum_{j=1}^{m} w_j\lambda_j - \lambda_{m+1} + \lambda_{m+2},\ \sum_{j=1}^{m}\lambda_j)',\ \sum_{j=1}^{m}\lambda_j y_j + T\lambda_{m+2})$$

Monotonicity in c implies that $\lambda_1+\ldots+\lambda_m$ is positive and allows the normalization $\lambda_1+\ldots+\lambda_m = 1$. Thus we have

$$\lambda \geqq 0,\ \lambda_1+\ldots+\lambda_m = 1,\ c \leq w_j h + y_j\ \ (j=1,\ldots,m),\ 0 \leq h \leq T \text{ and}$$

$$(h,c)' = F_\vartheta((-\sum_{j=1}^{m} w_j\lambda_j - \lambda_{m+1} + \lambda_{m+2},\ 1)',\ \sum_{j=1}^{m}\lambda_j y_j + T\lambda_{m+2})$$

If all tax-brackets consist of more than a single point, then at most two restrictions can be binding at the same time. This means that there are only 2m+1 regimes: m regimes with one binding constraint and m+1 regimes with two binding constraints (m-1 kink points and two corners).

In the case of one binding constraint, say the j-th $(j\in\{1,\ldots,m\})$, we have

$$(h,c)' = F_\vartheta((-w_j,1)',y_j) \qquad (\lambda_j=1)$$

and in case of a kink point, say between brackets j and j+1 $(j\in\{1,\ldots,m-1\})$, we have

$$(h,c)' = F_g((-w_j\lambda_j - w_{j+1}[1-\lambda_j],1)', y_j\lambda_j + y_{j+1}[1-\lambda_j]) = F_g((-\tilde{w},1)',\tilde{y})$$

This is a familiar result: The shadow wage $\tilde{w}$ lies somewhere between the wage rates w_j and w_{j+1} and shadow income $\tilde{y}$ satisfies $\tilde{y}+\tilde{w}h=y_j+w_jh=y_{j+1}+w_{j+1}h$, where h is the number of hours at the kink point. The two corners yield similar results.

□

Let us now consider conditions one may want to impose, in order to guarantee that (5.3) yields well-defined solutions. First of all, B1 may be imposed once again, although it should be realized that in the present context not only actual prices but also shadow prices matter.

Correspondingly to B2 we can impose a condition stating that it must at least be possible to rationalize a given set of restrictions and quantities per individual:

D2. ('External Coherency') Let RQ_t be a given set of restrictions (including the budget constraint) and quantities that satisfy these restrictions. For all t, ν and $(R,r,q) \in RQ_t$ there exists some $\eta \in \Omega$ such that there is a vector $\lambda \in \mathbb{R}^k$ with $\lambda \geqq 0$ and $q = F_{g_t(\nu,\eta)}(R'\lambda, r'\lambda)$.

Operationalization of this condition for a given demand system may be difficult. It is essentially the same condition as B2 (see Figure 3.2) and C2.

The most important condition, of course, is a condition which guarantees that problem (5.3) has a well-defined solution. It is well-known that this is the case if the objective function maximized is strictly quasi-concave and the constraints define a convex set. Convexity of the choice set is already clear from the general set-up in (5.1). Quasi-concavity of the utility function on the budget set is easily guaranteed by imposing condition B3 and taking Q_{min} convex and large enough to contain (the upper edge of) any budget set over which utility is to be maximized. We will refer to this as condition D3. Thus, D3 is the same as B3 (see Figure 3.3) for a sepecific choice of Q_{min}.

Example 1: LES

We confine ourselves to the discussion of condition D3. Bearing in mind that the condition must hold for each individual t, we omit the subscripts t. In Section 3.3 it was shown that regularity at a given point q in quantity-space means $q > \gamma$. Condition D3 therefore implies

$$\gamma < q \text{ for each } q \text{ with } R\,q \leqq r\ .$$

This implies, for given systematic part γ_0 of γ, truncation of the distribution of η. In case of non-negativity constraints as well as in case of a kinked budget constraint, this leads to imposition of negativity of the γ_i's.

□

Example 2: DQU

As in the previous example, we confine ourselves to the discussion of condition D3 and omit subscripts t. From Section 3.3 we know that regularity at a given point q in quantity space is equivalent to

$$\gamma_n - (B\,q)_n > 0. \tag{5.4}$$

Thus, condition D3 implies that (5.4) must hold for all q in the budget set $Q_{min} = \{q \in \mathbb{R}^n;\ Rq \leqq r\}$. This is achieved by restricting the fixed parameter γ_n:

$$\gamma_n > \max_q \{(B\,q)_n;\ R\,q \leqq r\}. \tag{5.5}$$

The maximum of the right hand side of (5.5) can be found by linear programming. (The maximum exists if Q_{min} is compact).

In the special case of non-negativity constraints, assuming that all prices are strictly positive, (5.5) yields

$$\gamma_n > y \max_{1 \leq j \leq n} (\beta_{nj}/p_j).$$

In case of the kinked budget constraint (5.5) yields

$$\gamma_n > \max_{0 \le j \le m} (\beta_{21} h_j + \beta_{22} c_j),$$

where (h_j, c_j) $(j=1,\ldots,n)$ are the corners $(h_0, c_0) = (0, y_1)$ and $(h_m, c_m) = (T, w_m T + y_m)$ and the kink points $(h_j, c_j) = (\{y_{j+1} - y_j\}/\{w_j - w_{j+1}\}, w_j h_j + y_j)$ $(j=1,\ldots m-1)$.

□

Example 3: AIDS

In the previous sections it has been shown that regularity conditions in some region Q_{min} in q-space for this demand system are very intricate because shadow prices cannot be derived in closed form. Thus, in general, no analytical results can be derived.

□

Example 4: Translog

In general, it is not possible to derive analytical expressions for shadow prices for the Translog specification, so problems arise which are similar to those encountered with AIDS. In the special case of non-negativity constraints however, there is a way to avoid these problems. In this case shadow prices corresponding to the optimal quantity vector are either real prices (if $q_i > 0$ then $\tilde{p}_i = p_i$) or can be obtained from a system of linear equations (see e.g. Lee and Pitt, 1986). This result implies that it is possible to guarantee coherency of the model without solving the problem of deriving shadow prices at each point in some region in q-space. This issue is discussed in Van Soest and Kooreman (1987), where sufficient conditions for internal coherency are given which imply restrictions on fixed parameters only (and no truncation of the distribution of η). These conditions are weaker than D3.

□

6. Conclusions

The examples in Section 2 underline the necessity of the imposition of coherency conditions in practice. Not only do we need parameter restrictions to make sure that the model is coherent, we have also seen that even if the true data generating process is coherent, failure to impose appropriate conditions may yield inconsistent ML-estimates of the parameters. These estimates would then make us believe that the model is misspecified. This also illustrates the fact that it is impossible to test the coherency conditions. The requirement of coherency is after all a logical one and not an empirical one.

If we would have tractable, flexible and globally concave functional specifications for our demand systems the treatment of coherency conditions would be straightforward. Since tractable, flexible systems only have local concavity properties (the only globally concave flexible system suggested by Diewert and Wales (1987) does not permit explicit expressions for the demand functions), the formulation and implementation of parameter restrictions that guarantee regularity in some sense becomes quite intricate.

There are two basic reasons for this. First of all the analysis in Section 5 makes clear that we can guarantee well-behaved demand systems if we can guarantee that the direct utility function is quasi-concave on the budget set of an individual. Generally we do not want to impose this, because in practice most of the budget set is irrelevant for the individual anyway. Thus we are satisfied if the utility function is quasi-concave in a part of the budget set where we most likely observe the individual to be (so we can for instance generally ignore all interior points of the budget set). By making the area where regularity conditions are imposed as small as possible we maintain as much flexibility of the functional form as we can. At the same time this complicates the analysis because we have to think more carefully about the area where regularity should hold. This for instance explains why conditions under exogenous rationing may be different from the conditions under endogenous regimes.

The second essential complication arises because the budget set <u>and</u> the parameters may differ across individuals. We have seen that certain conditions, like "external coherency", suggest that the parameter space should not be too small, whereas other conditions suggest that it should not be too large. These conditions may easily be conflicting.

Somewhat related to the previous points, the stochastic specification tends to be difficult. In the examples considered the random variables were usually constrained to a polyhedron. If, for instance, we would specify a normal distribution for the random preferences, this would lead to complicated truncations.

Another implication of the analysis appears to be that for models with endogenous regimes or corner solutions, one needs in general the direct utility function in closed form. This is rather clear from the analysis in Section 5, but also under exogenous rationing, conditions like C2 or C3 require knowledge of shadow prices in a rationing point. Although in principle one could compute shadow prices numerically whenever given in implicit form, it is next to impossible to impose conditions like C2 or C3 when no closed form expressions for shadow prices are available. And, of course, knowing shadow prices corresponding to given quantities amounts to knowing the direct utility function. As a result, many of the popular flexible forms like AIDS or Indirect Translog cannot be used in general. In this paper we have illustrated the imposition of the various conditions for some direct utility functions. There is one flexible form proposed by Hausman and Ruud (1984), which has not been dealt with here. In a separate paper (Kapteyn et al, 1988), we have used this system in a non-linear and non-convex budget application and we have imposed concavity restrictions along the lines set out in Section 4.

Altogether, the treatment of endogenous regimes or corner solutions appears to require rather tedious procedures for the imposition of regularity conditions and it severely limits the number of functional forms that can be considered. Despite these difficulties, it should be clear that without the imposition of regularity conditions one will often end up with a nonsensical model. Thus the choice appears between complexity and incoherency.

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