# POWER INDICES AND PROBABILISTIC VOTING ASSUMPTIONS

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ABSTRACT:
This paper compares the theoretical bases of the Shapley-Shubik and Banzhaf indices of voting power for a legislature with weighted voting. Definitions based on probabilistic-voting assumptions, useful both as behavioral descriptions and for computation in empirical applications, are compared in terms of necessary and sufficient conditions on the choice of voting probabilities. It is shown that the Shapley-Shubik index requires stronger conditions than the Banzhaf index: the former that voting probabilities be chosen by all players from a common uniform distribution on the unit interval, the latter only that voting probabilities be selected independently from any set of distributions (on the unit interval) which have a common mean of 1/2. This result has a bearing on the theoretical criteria by which one may choose between the two indices in a voting context.

## 1. INTRODUCTION

Power indices for simple games have an important role in the empirical analysis of the distribution of voting power among individual members of a voting body. The two traditional and widely used power indices are those of Shapley and Shubik (1954) and Banzhaf (1965). Both employ a definition of voting power as the relative ability of each member of the voting body to change coalitions from losing to winning; both also have simple interpretations in terms of probabilistic voting. The voting models underlying the two indices, however, whether coalitional or probabilistic, are quite different; moreover, although in some empirical applications the indices do not differ too dramatically (for example, compare the results in Owen (1975a) and (1975b); also see Leech(1988) for a stockholder application), in many important cases they give significantly different results (for examples see Straffin(1977, 1978). There is a need, therefore, to choose between them on the basis of the theoretical plausibility of their respective assumptions in the particular voting situation under analysis.

This paper is concerned with the interpretation of the two power indices in terms of probability models: it shows that the Shapley-Shubik index requires much stronger assumptions about the manner in which the relevant probabilities are generated than does the Banzhaf index.

## 2. POWER INDICES AND PROBABILITY MODELS

A voting body, conceived as a weighted majority game, is a special case of a simple game (Shapley, 1962) defined for a set of players $N = \{1,2, . . . ,n\}$; player i casts a number of votes (or has weight) $w_i$. Coalitions of players are represented by subsets of N and are dichotomized into winning coalitions and losing coalitions on the basis of a quota q (conventionally $q \geq 1/2\sum w_i$, to rule out improper games). The game is generally represented by the notation: $\{q; w_1, \ldots, w_n\}$. A subset S corresponds to a winning coalition if it satisfies:

$$\sum_{i \in S} w_i \geq q ,$$

and is losing otherwise.

We can define a power index for each player as the relative number of times that that player, by adding his weight to a losing coalition, can change it into a winning coalition; that is, the power index is the relative number of swings for that player. A swing for player i occurs if a coalition $S_i$ exists such that:

$$\sum_{j \in S_i} w_j < q \quad \text{and} \quad \sum_{j \in S_i} w_j + w_i \geq q .$$

The Shapley-Shubik index for player i is defined as the number of swings relative to the number of orderings of the players, n!; the coalition model assumed is one where players are added sequentially in the build-up of the grand coalition, N. For a given swing, $S_i$, the index makes allowance for all possible permutations of the players conditional on that swing, $s_i!(n-s_i-1)!$, where $s_i$ is the number of members of the coalition $S_i$. Thus, the Shapley-Shubik index is a vector $\phi$ whose $i^{th}$ element is:

$$(1) \qquad \phi_i = \sum_{S_i} \frac{s_i!\,(n - s_i - 1)!}{n!} , \qquad i = 1, 2, \ldots, n .$$

The importance attached to each swing therefore depends on the number of members of the coalition $S_i$ and its complement $N - S_i$.

The Banzhaf index is the number of swings normalized by the total number of swings in some sense; the model of coalition formation assumed has no regard for orderings of the players, and therefore each swing receives equal importance. Let $\eta_i$ be the number of swings for i and let $\bar{\eta} = \sum \eta_i$ be the total number of swings for all players. Two versions of the Banzhaf index employ two normalizations:

The Normalized Banzhaf index is the vector $\beta$ whose $i^{th}$ element is the number of swings for i relative to the total number of swings for all players:

(2) $$\beta_i = \eta_i / \bar{\eta}, \qquad i = 1, 2, \ldots, n.$$

The Banzhaf Swing Probability is the vector $\beta'$ whose $i^{th}$ element is the number of swings for i relative to the total number of possible swings (the number of voting outcomes ignoring player i):

(3) $$\beta_i' = \eta_i / 2^{n-1}, \qquad i = 1, 2, \ldots, n.$$

We can obtain the Normalized Banzhaf index from equation (3) by the relation:

(4) $$\beta_i = \beta_i' / \sum \beta_i'.$$

The probabilistic nature of the normalization in equation (3) is obvious if we take all voting outcomes as being equally likely; likewise the Shapley-Shubik index has a direct probabilistic interpretation if we assume that all orderings are equiprobable.

Straffin (1977,1978) gives an alternative probabilistic basis for both the Shapley-Shubik index and the Banzhaf swing probability (hence the Normalized Banzhaf index from equation (4)), which does not depend on coalition models. This work is closely related to that of Owen (1972, 1982) on multilinear extensions of games as a means of simplifying the computation of the Shapley value. It is useful, in deriving the probabilistic representations of the indices, to begin by considering the general case before specializing to simple games.

A general n-person game has characteristic function v which associates a numerical value, v(S), with each coalition $S \subseteq N$. The characteristic function satisfies conventional monotonicity conditions: $v(\emptyset) = 0$; and $v(S \cup \{i\}) \geq v(S) + v(\{i\})$ if $i \notin S$. The multilinear extension of this game is a function $f(x)$ written:

$$(5) \qquad f(x) = \sum_{S \subseteq N} \prod_{j \in S} x_j \prod_{j \notin S} (1 - x_j)\ v(S) ,$$

where $x=(x_1, x_2, \ldots, x_n)$ and $0 \leq x_i \leq 1$ for all i. If we regard the elements of the vector x heuristically as independent probabilities attaching to each player, indicating the likelihood of his joining a coalition, then equation (5) is the expected value of the characteristic function given x.

The marginal value for player i, given the probabilities $x_j$ for $j \neq i$, is obtained on differentiating equation (5) with respect to $x_i$. Rewriting equation (5) as:

$$(6) \quad f(x) = \sum_{T; i \in T} \prod_{j \in T} x_j \prod_{j \notin T} (1 - x_j)\ v(T) + \sum_{S;\, i \notin S} \prod_{j \in S} x_j \prod_{j \notin S} (1 - x_j)\ v(S) ,$$

where $S, T \subseteq N$, and differentiating equation (6) gives:

$$(7) \quad f_i(x) = \sum_{T; i \in T} \prod_{j \in T; j \neq i} x_j \prod_{j \notin T} (1 - x_j)\ v(T) - \sum_{S; i \notin S} \prod_{j \in S} x_j \prod_{j \notin S; j \neq i} (1 - x_j)\ v(S) .$$

Letting $T=S \cup \{i\}$, we can rewrite equation (7) as:

$$(8) \quad f_i(x) = \sum_{S;\, i \notin S} \prod_{j \in S} x_j \prod_{j \notin S} (1 - x_j)\ [\, v(S \cup \{i\}) - v(S)] .$$

Expression (8) shows that $f_i(x)$ is the expected increment in the characteristic function by the addition of player i to a coalition, given the probability vector x.

For voting games the characteristic function is dichotomous, coalitions being either winning, with $v(S)=1$, or losing, with $v(S)=0$. In such cases the term in square brackets in expression (8) is zero except for a swing for player i, $S=S_i$. Thus, we can write expression (8) as the probability (for given x) of a swing as:

$$(9) \qquad f_i(x) = \sum_{S_i} \prod_{j \in S_i} x_j \prod_{j \notin S_i} (1 - x_j) .$$

This representation of the value of the game to player i assumes an arbitrary value for x corresponding to a particular voting issue ($x_j$ being the probability of player j voting "aye" on a particular issue before the voting body). An index of power in an abstract sense must be independent of issues and additional assumptions made to enable the elimination of x from equation (9).

Straffin (1977) shows that we can obtain both the Shapley-Shubik index and the Banzhaf swing probability from equation (9) by treating the $x_j$'s as random variables and making different assumptions about how they are generated. In particular, Straffin shows that we can obtain the respective indices by making these assumptions:

<u>Independence Assumption</u>: The $x_j$'s are selected independently from the uniform distribution on the interval [0,1].

<u>Homogeneity Assumption</u>: A number $\pi$ is selected from the uniform distribution on [0,1] and $x_j$ is set equal to $\pi$ for all j.

The independence assumption (which leads to the Banzhaf index) is conventionally interpreted as meaning that each voter will vote "aye" with a probability selected independently of the other voters. But the assumption of a uniform distribution means that the range of attitudes allowed for each voter is the same; each voter has the same likelihood of being committed (a value of $x_j$ close to 0 or 1) or uncommitted ($x_j$ close to 1/2) on any issue. On the other hand, the homogeneity assumption (which gives the Shapley-Shubik index) is equivalent to assuming all players to have a common set of values reflected in a common probability of voting "aye," $\pi$. The assumption of a uniform distribution means that all sets of values or attitudes receive equal importance. But we demonstrate here that it is not necessary to assume that attitudes follow a uniform distribution for the

Banzhaf index and that the independence assumption above is unduly strong. These findings affect the heuristic interpretation of the probability model assumed appropriate for the use of the index.

We can derive both indices from equation (9) by building in these assumptions respectively and taking its expectation. We may write the expectation in general as:

$$(10) \qquad E(f_i(x)) = \int_0^1\int_0^1 \cdots \int_0^1 f_i(x)\, g(x)\ dx_1\, dx_2 \ldots dx_n \ ,$$

where g(x) is the joint density of x.

The homogeneity assumption leads to the Shapley-Shubik index because in this case g(x)=1 for all x, and we can write equation (10) (as a path integral):

$$(11) \qquad \phi_i = E(f_i(\pi)) = \int_0^1 f_i(\pi)\, d\pi \ ,$$

where:

$$(12) \qquad f_i(\pi) = \sum_{S_i} \pi^{s_i} (1-\pi)^{n-s_i-1} .$$

Rewriting equation (11) as:

$$(13) \qquad \phi_i = \sum_{S_i} \int_0^1 \pi^{s_i} (1-\pi)^{n-s_i-1}\ d\pi \ ,$$

and using the Beta integral,

$$B(s_i + 1, n - s_i) = \frac{s_i\,!\,(n - s_i - 1)\,!}{n\,!} = \int_0^1 \pi^{s_i} (1-\pi)^{n-s_i-1}\ d\pi \ ,$$

gives the Shapley-Shubik index, equation (1). Straffin's homogeneity assumption is clearly necessary and sufficient.

Now considering the derivation of the Banzhaf index from equation (10), we show that Straffin's independence assumption, as stated above, is unduly strong. The condition that the probabilities are drawn from the uniform distribution is not necessary and that, in fact, nothing need be assumed about the form of the distribution; all that is required is the much weaker condition that the voting probabilities are selected independently from distributions with mean 1/2.

Assuming the $x_j$'s to be selected independently from distributions (not necessarily uniform) on [0,1] with $E(x_j)=1/2$, the joint density in equation (10) becomes:

$$(14) \qquad g(x) = \prod_{j \neq i} g_j(x_j) \ ,$$

where $g_j(x_j)$ is the marginal density for $x_j$.

Substituting equations (9) and (14) into equation (10) gives:

$$E(f_i(x)) = \sum_{S_i} \int_0^1 \cdots \int_0^1 \prod_{j \in S_i} x_j \prod_{j \notin S_i} (1 - x_j) \prod_{j \neq i} g_j(x_j)\, dx_1 \ldots dx_n$$

$$= \sum_{S_i} \prod_{j \in S_i} \int_0^1 x_j\, g_j(x_j)\, dx_j \prod_{j \notin S_i} \int_0^1 (1 - x_j)\, g_j(x_j)\, dx_j$$

$$= \sum_{S_i} \frac{1}{2^{n-1}} = f_i(1/2) = \frac{\eta_i}{2^{n-1}} = \beta_i' \ ,$$

which is the Banzhaf swing probability, equation (3).

Thus we can interpret the Banzhaf index as assuming that each player votes randomly and independently with a probability of 1/2. We have shown that this is equivalent to assuming that players select their voting probabilites randomly and independently from distributions with mean 1/2 without regard for the forms of those distributions. The fact that the index requires only stochastic independence and does not rely on a uniform distribution means its behavioral interpretation in terms of a probability model is much less restrictive

## 3. CONCLUSION

We have shown that the probabilistic-voting assumptions that Straffin proposes as the basis of the Banzhaf index are unduly strong. One need not assume <u>both</u> that voting probabilities be selected from a uniform distribution <u>and</u> that they be independent; it is sufficient that they be chosen independently from distributions that have a mean of 1/2.

This result implies that the distributional assumption underpinning the Shapley-Shubik index is much stronger than that behind the Banzhaf index. The Shapley-Shubik index requires the specification of the whole distribution from which voting probabilities are selected -- that it be uniform -- while for the Banzhaf index only the mean need be specified.

This result has a bearing on the heuristic criterion by which one may choose between the two power indices in a particular voting situation. The Shapley-Shubik index is appropriate in situations where voting reflects a common set of values, but all possible sets of values are of equal weight (the uniform distribution of the common voting probability). The Banzhaf index is appropriate if voters act independently of each other. But it does not require that they share the same range of values; only that, on an average issue, they be as likely to vote one way as the other. For example the Banzhaf index would be the appropriate power index for a voting body whose members vote independently and which contains some persons who tend to be committed (high chance of choosing a voting probability close to zero or unity) and some who were relatively uncommitted on most issues (voting probabilities clustered around 1/2).

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