Courtship as a Waiting Game*

By

Ted Bergstrom and Mark Bagnoli
University of Michigan

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Abstract: In most times and places, women on average marry men who are older than themselves. We propose a partial explanation for this difference and for why it is diminishing. In a society where the economic roles of males are more varied and specialized than the roles of females, it may be that the relative desirability of females as marriage partners becomes evident at an earlier age for females than it does for males. We study an equilibrium model in which the males who regard their prospects as unusually good choose to wait until their economic success is revealed before choosing a bride. In equilibrium, the most desirable young females choose successful older males. Young males who do not believe that time will not treat them kindly will offer to marry at a young age. Although they are aware that young males available for marriage are no bargain, the less desirable young females will be offered no better option than the lottery presented by marrying a young male. We show the existence of equilibrium for models of this type and explore the properties of equilibrium.

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Address for Correspondence : Department of Economics, University of Michigan, Ann Arbor, Michigan 48109.

Courtship as a Waiting Game

Ted Bergstrom and Mark Bagnoli

In most times and places, women, on average, marry men who are older than themselves. A recent United Nations study[1] reports the average age of marriage[2] for each sex for more than 90 countries over the time interval between 1950 and 1985. In every country and in every time period reported, the mean age at marriage of males exceeded that of females. The smallest difference in mean ages was 1 year (Ireland) and the largest difference was 10.9 years (Mali). In 1985 in the United States, the difference was 1.9 years, in Western Europe about 2.5 years, and in Southern and Eastern Europe about 3.5 years. In Japan the difference was 3.7 years, in India, nearly 5 years and in the Middle East, about 4 years. In the Caribbean the age gap is about 5 years, in Central America, about 4 years and in South America, between 2 and 3 years In African countries, this gap ranges between 5 and 10 years. In most countries, the age difference between the sexes at marriage has diminished substantially between 1950 and 1985, but nowhere has it disappeared altogether.[3]

This paper proposes a partial explanation for this difference, for why it is diminishing over time, and for why it tends to be greater in traditional societies than in modern societies. We suggest that the difference between the marriage ages of males and females stems from the different economic roles of males and females and from a difference in the age at which about one's "quality" as a possible marriage partner becomes apparent.

In societies where male roles as economic providers are relatively varied and specialized, information about an individual male's economic capabilities may be revealed only gradually after he has spent time in the work force. In contrast, for a female who will be confined to childbearing, child care and traditional household roles, it may be that once she has reached physical maturity, the passage of time adds little information about her capabilities in her anticipated roles.[4]

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[1] *Patterns of First Marriage: Timing and Prevalence* (1990)

[2] The average computed is the "singulate mean age at marriage". This statistic estimates the average number of years spent in the single state by those who marry before age 50, and is computed from census statistics on the proportion of the population who have never married in each age group. See Hajnal (1953) for details.

[3] In 72 of the 91 countries listed in Tables 1-4, the age gap decreased and in 14 countries the gap increased. Exceptions to this pattern are Japan, Germany, and several countries in Southern and Eastern Europe.

[4] Although more information about her childbearing ability becomes available as she actually bears children, she and her partner must agree to a lifetime marriage before reproduction begins.

We assume that, other things being equal, both sexes prefer early marriage to late marriage. A male who expects to prosper can gain by delaying marriage until the evidence of his success allows him to attract a more desirable female. High quality females, on the other hand, have less to gain by postponing marriage, since the relevant information about their quality is available sooner. In the equilibrium that we study, all females marry young. The more desirable females marry successful older males and the less desirable females marry young males who believe their prospects for economic success are not very good.

Our model predicts that males who marry young are less likely to succeed in later life. There is evidence, at least for the United States, that this is the case. According to the 1980 U.S. census,[5] the fraction of males age 45-54 with annual incomes below $10,000 was 35% for those who married before age 18, 27% for those who married before age 20, and 17.5% for those who first married between ages 21 and 29. Median income of persons who married before 18 was $14,500, median income of those who married between 18 and 20 was $16,800, and median income of those who married between 22 and 29 was $19,000.[6]

We will present a starkly oversimplified theory of the timing of marriage and the acquisition of information. Certainly our model does not capture all of the forces that tend to produce a gender gap in age at first marriage. Our justification for studying this model is that we think it helps to clarify the logic of marital "lemons" models, and may provide a building block for more realistic and detailed theories.

1. Preferences, Information, and the Distribution of Quality

Consider a population of constant size, in which people are born, marry, and die. In every year, equal numbers of males and females reach maturity. People can choose to marry either in the first or the second year of maturity. Those who marry in the first year are said to marry at Age 1 and those who marry in the second year of maturity are said to marry at Age 2. Marriages are monogamous and there is no divorce or remarriage.

Some people are more desirable marriage partners than others. Assume that members of each sex agree in their rankings of the opposite sex. Indeed, let us make the stronger assumption that persons of each sex have identical von Neumann-Morgenstern utility functions over lotteries in which their marriage partners are randomly selected from the opposite sex. Let x_i be the von Neumann-Morgenstern utility that males assign to the prospect of marrying person female i and let y_i be the von Neumann-Morgenstern utility that females assign to the prospect of marrying male i. We will call x_i or y_i, the "quality" of individual i. The quality of females is distributed over an interval $[L_g, U_g]$

[5] Subject Report 4C, *Marital Characteristics*

[6] It is interesting to notice that males who first married after age 30 also do less well than those who marry in their mid 20's. In this group, 27% had incomes below $10,000 and median income was about $17,000.

with a cumulative distribution function, $F_g(x)$, and the quality of males is distributed over an interval $[L_b, U_b]$ with a c. d. f. , $F_b(y)$. Other things being equal, everyone would prefer marrying at Age 1 to marrying at Age 2. The utility cost of delaying marriage from Age 1 to Age 2 is c_b for males and c_g for females. Marrying even the least desirable member of the opposite sex is preferred to the prospect of remaining single.[7]

Full information about the quality of each female is publicly known she reaches Age 1. The quality of a male becomes public information only when he reaches Age 2. At Age 1, a male knows how well he will turn out,[8] but to the females, his prospects are indistinguishable from those of his contemporaries, except that his choice of when to marry may act as a signal of quality.

2. Marriage and Equilibrium

We model the marriage market as a game of incomplete information in which people choose to marry either at Age 1 or Age 2. Members of each generation make simultaneous choices about when to marry, without observing the choices made by their contemporaries. Thus each individual believes that changing his or her decision about when to marry will not alter the choices made by contemporaries. Given all of the individual choices of when to marry, the "payoff" to each individual is determined by a matching rule applied to the set of people who choose to marry in each time period. The population of males who choose to marry in any period is matched to the population of females who choose to marry in the same period in order of corresponding *expected* quality.

If the quality of all individuals in the marriage market were public information, then this matching would be entirely straightforward. The most desirable male would be matched to the most desirable female, the second most desirable male to the second most desirable female, and so on until the supply of persons of at least one sex has been exhausted. If the number of available persons of one sex exceeds that of the other, this process leaves some people from the "lower tail" of the quality distribution unmatched. Unmatched persons of Age 1 may reappear in the marriage market in the next period.

The actual matching rule is complicated by the fact that males of Age 1 are indistinguishable to females and hence are of equal expected quality. Applying the principle of matching by corresponding rank leads to the following assignment. At time period, t, the best unmarried male of Age 2 will be matched to the best female who chooses to marry at time t, the second best unmarried male will marry the second best unmarried female and so on until the supply of males whose quality exceeds the average of available Age 1 males is exhausted.

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[7] This assumption involves no real loss of generality. If there are persons so disagreeable that the prospect of being married to them is worse than remaining single, these persons will have no bearing on the outcomes and the model as formulated can be applied to the population that remains after excluding them.

[8] Formally, this model will apply if young males are not certain about how well they will turn out, but have some private information about their prospects. In this case, y_i can be interpreted as i's *expectation* of his quality in period 2.

The assignment of partners for the remainder of the population follows directly from the principal of matching by corresponding rank and from the fact that females cannot distinguish between males who choose to marry at Age 1.

Let $N_m(t)$ be the number of Age 1 males who choose to marry at time t. Let $N_f(t)$ be the numbers of females who choose to marry at time t and who are not matched to a male of who is better than the a random draw from the available Age 1 males. There are three possible cases.

(1) If $N_m(t) = N_f(t)$, then each of the males who chose to marry at Age 1 will be randomly assigned a partner from the set of $N_f(t)$ females who want to marry in this period and are not already taken by an Age 2 male.

(2) If $N_m(t) < N_f(t)$, then the best $N_m(t)$ of the $N_f(t)$ available females will be randomly matched to the males of Age 1. The remaining $N_f(t) - N_m(t)$ females will be matched in order of corresponding quality with any remaining males of Age 2 who are of lower quality than the average available male of Age 1. Females left over at the end of this process will be left unmatched. Those who are of Age 1 may reenter the marriage market in the next period at Age 2.

(3) If $N_m(t) > N_f(t)$, then a random draw of $N_f(t)$ males from the set of available males of Age 1 will be paired with the $N_f(t)$ females who are available and have not been matched with a male of higher quality. Males who chose to marry at Age 1 but did not receive partners in the random assignment will be able to reenter marriage market in the next period at Age 2.

With this set of matching rules, the assignment among the set of people who choose to marry in any given year has the *core* or *stable marriage assignment* property. Gale and Shapley (1962), Shapley and Shubik (1972).[9] That is to say, no two people of opposite sexes *who marry in the same year* would both get higher expected utility from marrying each other than they do from their actual choices.

Of course, specifying the matching rules for those who choose to marry in the same period does not constitute a full description of equilibrium, since the set of people who are in the marriage pool in any year depends on choices that individuals make about their age of marriage. We define equilibrium as a choice of age at marriage for each individual in each time period such that each person's choice of whether to marry at Age 1 or Age 2 maximizes his or her expected utility, given the choices of all other individuals.[10] The choices of others determine the quality distribution in the marriage pool in each year and thus determine the payoffs from marrying at Age 1 or Age 2. Given our definition of equilibrium, it will have to be that in equilibrium nobody who marries at Age 1 would

[9] For further references as well as a masterful treatment of this subject, see Roth and Sotomayor (1990).

[10] This is a Bayesian equilibrium of an agreeably simple nature. In this model, we do not have to wonder what inferences are to be drawn about a player's type if he or she deviates from equilibrium behavior. By the time that a deviation is observed by another player, the deviator will have reached Age 2. The type of every Age 2 person is common knowledge, so there is no mystery in how to regard someone who has in the past deviated from equilibrium strategies.

have a higher expected payoff from waiting to marry at Age 2 and nobody who marries at Age 2 would have a higher expected payoff by marrying at Age 1.

Study of this model is much simplified by the fact that equilibrium will always have the following property.

Proposition 1. *At any time t, the set of males who choose to wait until Age 2 to marry will be an "upper tail" of the quality distribution. That is, a set of the form* $\{y | y \geq y_t\}$ *for some* $y_t \in [L_b, U_b]$.

Proof: Consider two males, born at the same time, of quality y' and y where $y' > y$. If these males both marry at Age 1, then they face the same lottery and their expected payoff will be the same. If they wait until Age 2 to marry, then the male of quality y' will be matched to a female whose quality is at least as great as the quality of the female matched to the male of quality y. From this it follows that if it is worthwhile for a male of quality y to wait until Age 2 to marry, any male of higher quality than y will find it worthwhile to wait. ∎

3. Long-Run Stationary Equilibrium

Since we have assumed that the number of persons born in each year is constant and the quality distributions and preferences are the same in each generation, we can hope to find a long-run stationary equilibrium, in which each generation behaves in exactly the same way as all preceding generations.

It turns out that in long-run equilibrium, all females marry at Age 1. There is a threshold level of quality, y^*, such that in each time period, males of higher quality than y^* choose to marry at Age 2 and males of lower quality choose to marry at Age 1. The highest quality male from a generation will marry at Age 2 to the highest quality female from the next younger generation. The second highest quality male will marry at Age 2 to the second highest quality female of Age 1, and so on until the threshold quality y^* is reached. Males of quality lower than y^* will choose to marry at Age 1 and will receive a random assignment from the set of Age 1 females who were not of sufficiently high quality to be matched with the available males of Age 2.

In order to describe long-run equilibrium more explicitly, it is useful to define a "matching function" g such that $x = g(y)$ means that a female of quality x has the same ordinal rank among females as a male of quality y has among males. Thus $g(z)$ is the (unique) solution to the equation $F_b(z) = F_g(g(z))$. It will also be useful to have a notation $\mu_b(y)$ for the quality of the "average male who is no better than a male of quality y" and a similar notation $\mu_g(y)$ for females. Formally, $\mu_b(y) = \int_{L_b}^{y} z dF_b(z)/F_b(y)$ and $\mu_g(y) = \int_{L_g}^{y} z dF_g(z)/F_g(y)$.

Suppose that in every generation, males of higher quality than y^* choose to wait until Age 2 to marry, that all males of quality lower than y^* choose to marry at Age 1 and that all females choose to marry at Age 1. Then in every generation, all females of quality higher than $g(y^*)$ will be matched to males of Age 2. Males who choose to marry at Age 1 will be matched with a random draw from the set of females of Age 1. The expected quality of a random draw from this set is $\mu_g(g(y^*))$. If he waits to marry at Age 2, a male who is of the threshold quality y^* will be matched with a female of corresponding quality $g(y^*)$, and allowing for the cost of waiting, his utility will be $g(y^*)-c_b$. A male of threshold quality will be just indifferent between marrying at Age 1 and at Age 2. Therefore it must be that

$$\mu_g(g(y^*)) = g(y^*) - c_b. \tag{1}$$

We can fully characterize long-run stationary equilibrium with the following result.

Proposition 2. *If $y^* \in [L_b, U_b]$ satisfies Equation 1, then there is a long-run stationary equilibrium such that in every generation, each male of quality $y \geq y^*$ marries at Age 2 to a female of Age 1 whose quality is $g(y)$ and each male of quality $y < y^*$ marries at Age 1 to a female randomly selected from the set of females in his own generation of quality $x < g(y^*)$. Conversely every long-run stationary equilibrium is of this type.*

Proof: The assertion that the proposed arrangement is an equilibrium will be demonstrated if we can show that no individual can gain by deviating from the proposed equilibrium strategy. Consider a male of quality $y > y^*$. If he chooses to marry at Age 2, he will be matched to a female of quality $g(y)$ and his payoff will be $g(y) - c_b$. If he chooses to marry at Age 1, he will have an expected payoff of $\mu_g(g(y^*))$.[11] Since g is an increasing function of y, it follows from Equation 1 that he cannot gain by marrying at Age 1 rather than at Age 2.

Consider a male of quality $y < y^*$. If he marries at Age 1, he will have a random draw from the set of females of quality less than $g(y^*)$ and his expected payoff will be $\mu_g(g(y^*))$. If he waits until Age 2 to marry, then his quality will be common knowledge. All of the males from his own generation of quality $y \geq y^*$ will be in the marriage pool at this time and will be matched to all of the Age 1 females of quality $x \geq g(y^*)$. Therefore his payoff from marrying at Age 2 will be smaller than $g(y^*) - c_b$. From Equation 1, it follows that he cannot gain by marrying at Age 2 rather than at Age 1.

Consider any female. If she deviates from the strategy of marrying at Age 1, the expected quality of her partner will be no higher than the expected quality she can get at Age 1.[12] Since waiting is costly, she would not gain from choosing to marry at Age 2

[11] Since we have assumed that people in the same generation choose their age of marriage simultaneously, his choice to marry at Age 1 will not change the set of females who choose to marry at Age 1, nor will it change the set of unmarried Age 2 males. Therefore the pool of females who are available to marry Age 1 males does not change in response to his decision to marry at Age 1. It follows that the expected payoff from marrying at age 1 remains $\mu_g(g(y^*))$ whether or not he chooses to marry at Age 1.

[12] There is a slight complication. If she decided to delay marriage, then when her age is 1, the number of males in the marriage market would exceed the number of females by 1. Therefore a randomly selected male who chose to marry at Age 1 would not find a mate. He would reappear in the marriage market in the next year. But the addition of a randomly selected male from the set of males of quality $y < y^*$ will not improve the expected quality assignment for any female who waits until the next period to marry.

rather than at Age 1.

We have shown that if y^* satisfies Equation 1, no person can gain by deviating from the proposed equilibrium strategies. All that remains is to show that every long-run stationary equilibrium is of the type described in this proposition. From Proposition 1, it follows that in any equilibrium, the set of males divides into an upper quality interval who marry at Age 2 and a lower quality interval who marry at Age 1. If equilibrium is to be stationary, then the threshold quality at which these groups divide must be some constant y^*. If the pool of available males is the same in every period, then (since waiting is costly) it can never be worthwhile for females to choose to marry at Age 2 rather than at Age 1. Therefore in equilibrium all females must marry at Age 1. Finally it is straightforward to verify that males better than y^* will choose marriage at Age 2 and males worse than y^* will choose marriage at Age 1 only if Equation 1 is satisfied. ∎

4. Existence and Uniqueness of Long-run Equilibrium

The questions of existence and uniqueness of long-run equilibrium reduce to the question of whether Equation 1 has a solution and whether that solution is unique. Let us define the difference between a male or female's own quality, z, and the quality of the average male or female who is no better than z. Let $\delta_b(z) = z - \mu_b(z)$ and $\delta_g(z) = z - \mu_g(z)$. Then Equation 1 is equivalent to

$$\delta_g(g(y^*)) = c_b. \tag{2}$$

The following two assumptions will be sufficient for the existence and uniqueness, respectively of a solution to Equations 1 and 2.

Assumption 1. *The distribution function for quality of each sex is continuous and the difference between the quality of the most desirable female and the average quality of females exceeds the cost, c_b, to a male of waiting to marry at Age 2.*

Assumption 2. *The function, $\delta_g(x)$, (which is the difference between x and the average quality of females worse than x) is a monotone increasing function of x.*

Proposition 3. *If Assumption 1 holds, then there exists at least one long-run stationary state equilibrium where y^* solves Equation 1.*

Proof: By Assumption 1, $\delta(U_g) > c_b$. From the definition of the function $\delta()$, it follows that $\delta(L_g) = 0 < c_b$. The function $\delta(x)$ inherits continuity from the distribution function for x. Therefore from the intermediate value theorem, there must be at least one solution, x^*, to the equation $\delta(x^*) = c_b$. Let $y^* = g^{-1}(x^*)$. Then $\delta(g(y^*)) = c_b$. Therefore there exists a solution to Equations 1 and 2. From Proposition 2, it follows that there exists a long-run stationary equilibrium. ∎

Proposition 4. *If Assumption 2 holds, then any long-run stationary equilibrium is unique.*

Proof:

From Assumption 2 and the monotonicity of g, it must be that $\delta_g(g(y)) - c_b$ is a monotonic increasing function of y and hence there can be only one y^* for which $\delta(g(y^*)) = c_b$. From Proposition 2 it follows that every long-run stationary equilibrium must satisfy this equation. ∎

An Example

Suppose that the quality of females is uniformly distributed on an interval $[0, a]$ and the quality of males is uniformly distributed on the interval $[0, b]$. Then the function that maps males to females of corresponding quality rank is $g(y) = \frac{a}{b}y$. For the uniform distribution, the average quality of females worse than x is just $x/2$. Thus we have $\mu_g(x) = x/2$ and $\delta_g(x) = x - \mu_g(x) = x/2$. We see that $\delta_g(x)$ is an increasing function of x, so that Assumption 3 is satisfied. In fact we can readily solve for the unique equilibrium. The equilibrium condition, $\delta_g(g(y^*)) = c_b$ will be satisfied if $\frac{a}{2b}y^* = c_b$ or equivalently if $y^* = \frac{2bc_b}{a}$. Therefore if $0 < 2c_b < a$, there will exist a unique solution for y^* in the interval $(0, b)$. In long-run equilibrium all males of quality $y < y^* = \frac{2bc_b}{a}$ will choose to marry at Age 1. Males who marry at Age 1 will get a random draw from the population of females of Age 1 whose quality is lower than $g(y^*) = 2c_b$. The expected payoff of a draw from this pool will then be c_b. If a male of quality y^* marries at Age 2, he will be paired with a female of quality $g(y^*) = 2c_b$, but he has to bear the cost of waiting until Age 2. His utility payoff from waiting is $2c_b - c_b = c_b$, which is the same as the payoff from marrying at Age 1. Females of quality $x > 2c_b$, will marry Age 2 males of quality $\frac{bx}{a}$. Females of quality $x < 2c_b$ will get a random draw from the population of males who choose to marry at Age 1.

5. Log Concavity and Uniqueness of Equilibrium

As we have shown, monotonicity of the function $\delta_g(z)$ is sufficient for the uniqueness of equilibrium.[13] We found that if the quality of females is uniformly distributed, then $\delta_g(x)$ is strictly monotone increasing. It would be interesting to know more generally, what probability distributions have this property. It happens that there is a clearcut and elegant answer to this question.

The following lemma takes a first step toward this answer.

Lemma 1. *Let $F(x)$ be a cumulative distribution function defined on an interval $[L, U]$. Where $G(x) = \int_\ell^x F(t)dt$ and $\delta(x) = x - \int_\ell^x t\frac{f(t)}{F(x)}dt$, the function $\delta(x)$ is monotone increasing if and only if $G(x)/G'(x)$ is monotone increasing.*

Proof: Integrate the expression for $\delta(x)$ by parts. ∎

Definition. *A function $f : R^n \to R^+$ is log concave on the interval $[a, b]$ if $\log f$ is a concave function on $[a, b]$.*

[13] If $\delta_g(x)$ is not monotonic there will be multiple solutions to Equation 1 for at least some values of c_b.

By simple calculus, a twice differentiable function $f(x)$ will be log concave if and only if $f'(x)/f(x)$ is a *decreasing* function of f. Therefore we notice that the function $G(x)/G'(x)$ is monotone *increasing* if and only if $G(x)$ is a log concave function. This enables us to make the following observation.

Remark 1. *The function $\delta(x)$ monotone increasing if and only if $G(x)$ is log concave.*

In general it is not easy to verify directly whether common probability distributions have the property that, $G(x)$, the integral of the cumulative distribution function is log concave. Most of the familiar distributions have relatively simple closed-form *density* functions $f(x)$, but usually the cumulative distribution functions $F(x)$ and *a fortiori* the integrals of the cumulative density functions $G(x)$ do not have known closed form expressions.

As it happens, the property of log concavity is inherited from density functions by the corresponding cumulative distribution functions and from c.d.f.'s by their integrals. This is result has appeared in several places in the literature on operations research, statistics and economics. (See, for example, Prekova (1973), Pratt (1981), Goldberger (1983), Flinn and Heckman (1983), Caplin and Nalebuff (1988), and Dierker (1989).) The simple proof presented here is based on Dierker's proof of the same proposition.

Lemma 2. *If a function $f(x)$ is differentiable and log concave on $[a,b]$, then the function $F(x) = \int_a^x f(t)dt$ is also log concave on $[a,b]$.*

Proof. By elementary calculus, $F(x)$ will be log concave if $0 \geq (F'(x)/F(x)) = f'(x)F(x) - f(x)^2$. If f is log concave, then also by elementary calculus, it must be that for $x \leq t$, $f'(x)/f(x) \geq f'(t)/f(t)$. Therefore, for all $x \in [a,b]$,

$$\frac{f'(x)}{f(x)}F(x) = \frac{f'(x)}{f(x)}\int_a^x f(t)dt \leq \int_a^x \frac{f'(t)}{f(t)}f(t)dt.$$

But

$$\int_a^x \frac{f'(t)}{f(t)}f(t)dt = \int_a^x f'(t)dt = f(x) - f(a).$$

Therefore

$$\frac{f'(x)}{f(x)}F(x) \leq f(x) - f(a) \leq f(x).$$

and hence $0 \geq f'(x)F(x) - f(x)^2$. Therefore $F(x)$ is log concave. ∎

Applying Lemma 2 to a probability density function f and then once again to its integral F, we see that if f is log concave, then F is log concave and so also is G, where G is the integral of the probability density function. This fact, together with Remark 1, allows us to claim the following.

Lemma 3. *If the density function $f(x)$ is differentiable and log concave on $[a,b]$, then the function $\delta(x)$ is monotone increasing in x.*

Lemma 3 is very useful because it is possible to verify by simple calculations that many common probability distributions have log concave density functions. Consider, for example the standard normal distribution. This distribution has $\ln f(x) = \text{constant} - x^2/2$, which is obviously a concave function. Since the normal distribution is log concave, it follows that for the normal distribution, $\delta(x)$ is monotone increasing.

Proposition 4 and Lemma 3 allow us to state the following.

Proposition 5. *If the density function of the quality distribution of females is log concave, then long run stationary equilibrium is unique.*

In a recent study, Bagnoli and Bergstrom (1989) examine the log concavity of density functions, cumulative density functions, and their integrals for numerous common probability distributions. Remark 2 reports some of these results.

Remark 2. *The following probability distributions have log concave densities and hence monotone increasing $\delta(x)$ functions. Uniform, normal, logistic, extreme value, chi-squared, chi, exponential, and Laplace. The following probability distributions have log concave density functions for some but not all parameter values. Weibull, power function, gamma, beta.*

Log concavity of the density function is a sufficient, but not a necessary condition for $\delta(x)$ to be monotone increasing. Bagnoli and Bergstrom (1989) show that the log normal distribution and the Pareto distribution are examples of distribution functions which do not have log concave density functions but do have log concave cumulative density functions and monotone increasing $\delta(x)$.

Drawing on these result, we have the following.

Proposition 6. *Long run stationary equilibrium is unique if the density function of the distribution of female quality is of any of the following forms: Uniform, normal, logistic, extreme value, chi-squared, chi, exponential, Laplace, log normal, and Pareto.*

6. The Trajectory to Long-Run Equilibrium

If the population starts out in long run stationary equilibrium, it will remain there. But if initially the population is not in long run stationary equilibrium, it will *not* immediately jump to a stationary equilibrium. When the system does not start out in long run equilibrium, the dynamics are complicated by the fact that in some time periods, females will choose to delay their date of marriage because the supplies of available males and females males may be more favorable to them in the second period of their lives than in the first. A complete general characterization of the behavior of the system outside of long-run equilibrium appears to be very difficult. Here we settle for a pair of general results, one for each sex, and an example.

Proposition 1, which we proved earlier, is true whether or not the system is in long run stationary equilibrium. This result informs us that the set of males who choose to wait until Age 2 to marry is an "upper tail" of the distribution of males. While the behavior of females is much more complicated and not fully described here, we do have the following rather interesting general result.

Proposition 7. *In equilibrium, no female will ever marry a male of Age 2 whose quality rank is higher than her own.*

Proof: If in period t, some female marries a male of higher quality rank than her own, then there will have to be some young female of higher quality who does not marry a male of quality rank as high as her own in period t. This second female must, therefore, have voluntarily postponed marriage. But if she is willing to postpone her marriage, she must get a male whose quality exceeds that of her quality-match by at least c_b. This means that a third female of yet higher quality must be displaced one generation later. The process would have to continue, with females of ever higher quality in later generations being displaced. Eventually there would be no male sufficiently good to compensate the best displaced female for waiting until Age 2. ∎

Finally, we present a special example which is simple enough so that we can work out an exact solution for the pattern of marriage that starts out from a position off of the long run equilibrium path and moves gradually toward long run equilibrium.

An Example

For each sex, "quality" is uniformly distributed on the interval $[0, 1]$. Equal numbers of males and females are born in each period. In the initial period, there are no unmarried persons of Age 2 available from either sex. The utility cost of marrying at Age 2, rather than at Age 1 is $c < 1/2$ for members of either sex. A person's desirability to members of the opposite sex neither increases nor decreases between Ages 1 and 2.

In long run equilibrium, males of quality lower than $2c$ marry at Age 1, males of quality higher than $2c$ marry at Age 2, and all females marry at Age 1. But this population will not go all the way to long run equilibrium in a single step. If it did so, then no Age 1 males of quality $y > 2c$ would marry and since there are no males of Age 2 available, any female who marries in the first period would have to accept a random young male, whose expected quality would be c. But by waiting until the next period when some high quality Age 2 males become available, females of the highest quality could get spouses of nearly quality 1. Since by assumption, $1 - 2c > 0$, it must be that $1 - c > c$, so some of the best females will be better off waiting to marry at Age 2.

For this example, the pattern of ages at marriage converges to long run equilibrium in a simple, but rather surprising way. The proportion of males who choose to marry at Age 2 goes immediately to the equilibrium level and stays there. But females divide into four groups. In each period after the first, an interval of females at the top of the quality distribution marries at Age 1 to males of Age 2. An intermediate quality interval waits until Age 2 to marry at which time they marry males of Age 2. An interval of females just below these marries at Age 1 to males of Age 1. Finally at the bottom of the quality distribution of females is an interval of females who are left without partners.

Let X_t^1 denote the set of females born in year t who marry at Age 1 to males of Age 2. Let X_t^2 be the set of females born in year t who marry at Age 2 to males of Age 2, let X_t^3 be the set of females born in year t who marry at Age 2 to males of Age 1, let X_t^4 be the set of females born in year t who marry at Age 1 to males of Age 1 and let X_t^5 be the set of females born in year t who are left without mates. If initially there are no unmarried persons of Age 2, each of these sets is an interval. These intervals take the following form: $X_t^1 = (x_t^1, 1)$, $X_t^2 = (x_t^2, x_t^1)$, X_t^3 is empty, $X_t^4 = (x_t^4, x_t^2)$ and $X_t^5 = (0, x_t^4)$. Specifically, it turns out that

$$x_t^1 = 2c + (1-2c)\left(\frac{1}{2}\right)^{t-1}, \qquad x_t^2 = 2c + (1-2c)\left(\frac{1}{2}\right)^{t},$$

$$\text{and} \quad x_t^4 = (1-2c)\left(\frac{1}{2}\right)^{t}.$$

This means that x_t^1 starts out at 1 in the first period. In the second period, x_t^1 moves half way from 1 to the equilibrium value $2c$ and in each subsequent period again moves half way from its previous location to $2c$. Notice also that for all t, $x_t^2 = x_{t-1}^1$ and that the length of the interval X_t^2 of females who marry at Age 2 is halved in every period and is being squeezed asymptotically to $2c$. The interval set X_t^5 of females who are left unmatched is being halved in every period. In the limit, the behavior of females approaches the long run equilibrium in which all females of quality $x > 2c$ belong to X_t^1 and all females of quality $x < 2c$ belong to X_t^4.

7. A Case of "Pure Luck"

So far, we have assumed that although others can't tell one young male's prospects from another's, any male of Age 1 is, himself, fully informed about what his "quality" will be at Age 2. Suppose, instead, that young males are as ignorant as everyone else about how their fortunes will turn out and that each male's quality becomes publicly known when he reaches Age 2. As in the previous model, let us suppose that the quality of each female is public information when she reaches Age 1.

The shape of the distribution of quality of the females will determine whether males will be "risk-averse" or "risk-loving" with respect to gambles for marriage partners. For certain distributions, there is a relatively uninteresting equilibrium in which risk-aversion predominates and all males and all females marry at Age 1. But for some distributions in long-run equilibrium, some males may choose to "seek their fortunes" and wait to marry at Age 2, after their economic luck has been resolved. The most fortunate members of this group will be able marry the best young females. The least fortunate members of this group will be less desirable than young males whose fortunes are still unknown. Accordingly, they will be matched with the least desirable young females. The males who marry at Age 1 are matched to a set of females of intermediate quality. Since the tastes and prospects of Age 1 males are assumed to be identical, it must be that in equilibrium the two strategies yield equal expected utilities.

An Example

Suppose that the quality of females is distributed over the interval $[0, 1]$, where the density of females of quality x is $f(x)$. Let p be the fraction of the male population waits to marry at Age 2. Half of those who marry at Age 2 will turn out to be of lower quality and half will turn out to be of higher quality than the expected quality of a randomly chosen young male. Females whose quality lies in the interval $[0, g(p/2)]$ will marry males of Age 2 who turned out to be of lower than average quality. Females of whose quality lies in the interval $[g(1-p/2), 1]$ will marry males of Age 2 who turned out to be of higher than average quality. The remaining females will marry young males.

If a male marries at Age 1, he will get a random draw from the set of females whose quality is between $g(p/2)$ and $g(1-p/2)$. The expected quality of his mate will therefore be

$$\frac{1}{1-p}\int_{g(p/2)}^{g(1-p/2)} zf(z)dz.$$

If a male marries at Age 2, then with probability 1/2 he will get a random draw from the interval $[0, g(p/2)]$ and with probability 1/2 he will get a random draw from the interval $[1-g(p/2), 1]$. The expected quality of his mate will therefore be

$$\frac{1}{2p}\left(\int_0^{g(p/2)} zf(z)dz + \int_{g(1-p/2)}^{1} zf(z)dz\right).$$

Since at Age 1, all males view their prospects as equal, it must be that in equilibrium the strategy of waiting until Age 2 to marry has the same expected utility as the strategy of marrying at Age 1. Recalling that there is a cost of c_b to a male of waiting until Age 2 to marry, we have the equilibrium condition:

$$\frac{1}{1-p}\int_{g(p/2)}^{g(1-p/2)} zf(z)dz = \frac{1}{2p}\left(\int_0^{g(p/2)} zf(z)dz + \int_{g(1-p/2)}^{1} zf(z)dz\right) - c_b.$$

If the distribution function $f()$ is specified, then the above equation can be solved for p. Once p is known, the equilibrium is fully determined. We have computed explicit solutions for two simple special cases.

Case A: Let $f(y) = y^2/3$. If $c_b > 1/6$, then in long run equilibrium, everyone marries at Age 1. But if $c_b < 1/6$, then in long run equilibrium, the fraction p of the male population waits until Age 2 to marry, where $p = 2 - 12c_b$.

Case B: Let $f(y) = y^3/4$. If $c_b > 1/4$, then in long run equilibrium, everyone marries at Age 1. If $c_b < 1/4$, then in long run equilibrium, the fraction p of the male population waits until Age 2 to marry, where $p = 2 - 8c_b$.

In contrast to the case where males know their own quality at Age 1, this model predicts that a male's eventual economic success will be uncorrelated with his age at marriage. But like the earlier model, it predicts that in long run equilibrium, all females will marry young and that the most desirable females will marry successful males of Age 2.

8. Extensions

A better and more realistic model would have incomplete information about the quality of young females as well as of young males. For members of either sex, more information would become publicly available as they grow older. The rate at which the relevant information appears might be slower for males than for females, but in societies where females are allowed more varied economic roles, the relevant information about females would appear more slowly than in traditional societies.

Our model should also be enriched to allow for the possibility that persons will differ in inherited nonhuman wealth as well as in human capital. The value of a person's nonhuman wealth can be quite readily assessed at an early age. Therefore we might expect that heirs and heiresses, who can establish their desirability at an early age, would marry earlier than persons of equal expected lifetime income whose wealth is mostly anticipated earnings.

There is room for more systematic empirical testing, particularly if the model is enriched to allow varied roles for females and to take into account the role of nonhuman wealth. This, we hope will be accomplished in a later paper.

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