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On Measuring Inefficiency with Public Goods: an Input-Oriented Approach+

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Abstract *Extending Kay and Keen's (1988) work, Debreu's coefficient of resources utilization is applied to economies with public goods and it is used to discuss the issue of measuring welfare loss in the standard case where government imposes distortionary commodity taxation in order to finance the supply of a single public good. It is then argued that such a measure can help to cast light on the issue of over-under provision of public goods in second best economies.*

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1.Introduction

Discussing the notion of deadweight loss, Kay and Keen (1988) have recently argued that some of the ambiguities surrounding this notion depend on how inefficiency is assessed. Economy can be thought of as a process converting, through production and exchange, a vector of initial resources (the "inputs") in a vector of utility values (the "outputs"). Thus, in principle, one could think of measuring inefficiency either in terms of the extra "outputs" which could be produced with the same initial resources or in terms of the "inputs" which could be saved in order to produce the same outputs. Economists have generally used output measures; but, as originally suggested by Debreu (1951; 1954), in welfare economics output measures have a clear disadvantage. As utility functions are only ordinal concepts, measuring inefficiency in terms of outputs requires one to impose some cardinalization on the utility functions of individuals. But selecting a particular cardinalization is equivalent to choose a particular set of distributional weights. Thus, all output measures are inevitably blend with distributional issues. In contrast, an input measure avoids this problem by keeping each individual at the same utility level of the inefficient state. While this is in a sense equivalent to recognize a special status to the original distribution of utilities, this is exactly what should be done if one is unconcerned with distributional issues and the analysis is only aimed to assess allocative inefficiency.

Kay and Keen (1988) did not consider public goods and did not discuss the issue of assessing whether public goods are over or under provided with respect the efficient level. In this paper, by using Lindahl personalized prices, we extend the Debreu's measure of inefficiency to economies with public goods and we discuss its links with the traditional measures of deadweight loss. In a companion paper (Bordignon, 1993), by using an input measure of inefficiency we discuss the issue of over /under provision of public goods in the context of an optimal tax-expenditure exercise.

This paper is organized as follows. In section 2 we introduce the economy where we wish to carry on our analysis. In section 3 we build the associated economy which is needed in order to compute the Lindahl prices. In section 4 we derive Debreu's coefficient of resource utilization in the associated economy and we discuss its implications for the original economy. In section 5 we interpret Debreu's measure in terms of the traditional notions of consumers' and producers' surplus. In section 6 we close the paper by suggesting the application which is carried on in the companion paper.

2. Analyzing inefficiency in an economy with public goods

Our main task in this paper is to extend Debreu's (1951) measure to economies with public goods[1]. In these economies, the counter-factual Pareto-efficient state to be derived must not only be characterized by competitive prices for the private goods but also by a vector of personalized Lindahl prices for the public goods. The strategy chosen to tackle this problem is that of enlarging the space of commodities of the original economy by considering an associated economy where the consumption of the public good by each individual is treated as the consumption of a different private commodity. For simplicity, we will consider a simple competitive economy where government imposes a set of taxes on private commodities to finance the supply of a single pure public good. We will also impose a set of assumptions in order to guarantee that public good is produced on the frontier; thus, the only sources of inefficiency in this economy derive from commodity taxation and from the fact that the public good may not be supplied at the optimal level. Extensions to more complex cases should be straightforward.

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[1] A clear and detailed presentation of Debreu's measure and of its properties is in Kay and Keen (1988:269-273).

In the original economy, that we indicate with $\mathcal{E}$, there are H consumers, indexed by h, F private firms, indexed by f, n private commodities, indexed by i, and a public sector which uses private goods as inputs to produce a single public good, G. Following the convention of measuring outputs positively and inputs negatively, a production plan for firm f is $\mathbf{y}^f$, a vector in R^n. The production possibilities set for firm f is $\mathbf{Y}^f \subset R^n$. It is assumed that, for each firm f, $\mathbf{Y}^f$ is: A.1) closed; A.2) convex; A.3) satisfies free disposal; A.4) contains the origin. A.1 and A.2 are essential in order to employ the second welfare theorem[2]; A.3 guarantees non-negative competitive prices; A.4 guarantees that competitive profits are not negative. It is also assumed that there are no externalities in production (so that production possibility sets of firms can be summed together) and that each firm behaves as price-taker. Firm f behaviour is characterized by the profit function $\Pi^f(\mathbf{p})$:

$$(1) \quad \Pi^f(\mathbf{p}) = \max \mathbf{p}\mathbf{y}^f \quad \text{s.t. } \mathbf{y}^f \in \mathbf{Y}^f$$

where $\mathbf{p}$ indicates producers prices (appropriate transpositions are taken as read). There is also a public sector which can be thought of as composed by a single firm.

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[2]Debreu's measure of inefficiency could also be defined for non-convex economies, but the interpretation of the results could not be sustained. See the discussion in Debreu (1951).

A production plan for the public firm is $(\mathbf{r},G)$, a vector in R^{n+1}. $\mathbf{r}$ indicates the vector of private goods used as inputs in the production of G. The production possibilities set for the public firm, $\mathbf{Y}_G \subset R^{n+1}$, is assumed to satisfy assumptions A.1-A.4 above. Government owns and perfectly controls the public firm, which is forced to "sell" the public good to the public sector at the marginal cost p_G. All possible profits deriving from the production of G are then absorbed by the public sector. These assumptions ensure that there is no technological inefficiency in the production of the public good. The public firm behaviour can then be described by the profit function $\Pi^g(p_G,\mathbf{p})$:

(2) $$\Pi^g(p_G,\mathbf{p}) = \max\ p_G G + \mathbf{pr} \quad \text{s.t. } (G,\mathbf{r}) \in \mathbf{Y}_G$$

A consumption "plan" for household h is indicated by $(\mathbf{x}_h,G)$, a vector in R^{n+1}. Here usual terminology is misleading, because in the original economy G is not chosen by the consumers, but supplied by the state. The consumption set of each consumer h, $\mathbf{X}_h \subset R^{n+1}$, is assumed closed, convex and bounded below; preferences defined over this set are continuous and strictly convex and are represented by the strictly quasi-concave utility function $U^h=U^h(\mathbf{x}_h,G)$, which we assume strictly increasing in each element (labor supply enter negatively as a component of x_h). Each consumer h is entitled with an initial endowment of private goods $\mathbf{w}_h \in R^n_{++}$ which is assumed strictly positive. Following Kay and

Keen (1988) we define here endowment as traded endowment, thus avoiding any need to consider unobserved parts of endowment. Consumption by household h is then equal to traded endowment plus net trades, $\mathbf{x}_h-\mathbf{w}_h$, (i.e. $\mathbf{x}_h=\mathbf{w}_h+(\mathbf{x}_h-\mathbf{w}_h)$). Consumers own private firms and all the original endowment of private goods of the economy. To maintain generality we also assume that each h receives a lump sum transfer b_h from the state. Under this set of assumptions, consumer h's maximizing problem is:

$$(3)\quad \max_{\mathbf{x}_h} U^h(\mathbf{x}_h,G) \text{ s.t. } \mathbf{q}\mathbf{x}_h = b_h + \mathbf{p}\mathbf{w}_h + \Sigma_f\phi_{hf}\Pi^f \equiv I_h$$

where $\mathbf{q}$ are consumers prices and ϕ_{hf} is h's share of profits of firm f ($\Sigma_h\phi_{hf}=1$). $\mathbf{q}-\mathbf{p}=\mathbf{t}$ are commodities taxes. By substituting the optimal choices to (3) in the utility function, we can alternatively express consumer h's behaviour with the indirect utility function $V^h(\mathbf{q},G,I_h)$ or with the expenditure function $E^h(\mathbf{q},G,V^h(\mathbf{q},G,I_h))=I_h$. For convenience, we express the solutions to problem (8) in terms of the corresponding compensated demand functions $\mathbf{x}_h=\mathbf{x}_h(\mathbf{q},G,U^h)$.

A competitive equilibrium in this economy is characterized by an n+1 vector of prices, n prices for private goods ($\mathbf{p}$) plus the "price" for the public good (p_G), which clears the N equations for private goods plus the government budget constraint. Taxes and lump-sum transfers are selected exogenously. Using (1), (2) and (3) a competitive equilibrium in the private markets entails:

$$(4)\quad \Sigma_h \mathbf{x}_h(\mathbf{q},G,U^h) - \mathbf{w}^* = \Sigma_f \mathbf{y}^f(\mathbf{p}) + \mathbf{r}(p_G,\mathbf{p})$$

where $\mathbf{w}^*=\Sigma_h \mathbf{w}_h$. Summing over the budgets constraints of the consumers, and substituting (4) in this summation, we obtain:

$$(5)\quad (\mathbf{q}-\mathbf{p})\Sigma_h \mathbf{x}_h(\mathbf{q},G,U^h) + \Pi^g(p_G,\mathbf{p}) = \Sigma_h b_h + p_G G$$

and by Walras's law, government's budget also clears. One of the n+1 equations in (4) and (5) is then redundant; we drop (5) and arbitrarily select the price of good 1 as the numeraire. For convenience, we also assume that good 1 is the untaxed good so that $p_1=q_1=1$[3]. For future reference, let us indicate with an asterisk the equilibrium values of quantities demanded and supplied in the original economy. Thus, in equilibrium, $\mathbf{x}^*_h$ is h's consumption, G^* is government supply of the public good, $\mathbf{y}^{f*}$ is the netput supply of firm f and $U^{h*}=U^h(\mathbf{x}^*_h,G^*)$.

3. The associated economy

[3] One good must be assumed to be untaxed, because otherwise a proportional tax on all goods is equivalent, from the consumer's budget constraint, to a tax on endowment: i.e. a lump-sum tax.

Following Milleron (1971) and Roberts (1974)[4], we then associate to the original economy a new economy, identical to the first but with a suitably redefined commodity space. The new economy, that we will indicate with $\mathcal{E}'$, has the same set of consumers and the same set of firms of the original economy. Each consumer h in $\mathcal{E}'$ has a consumption set:

$$\hat{\mathbf{X}}_h = \{\hat{\mathbf{x}}_h \in R^{n+H} \mid \hat{\mathbf{x}}_h = [\mathbf{x}_h; 0, 0, .., G^h, 0, 0...0], \ (\mathbf{x}_h, G) \in \mathbf{X}_h\}$$

That is, to each point in the original consumption set in n+1 dimensions, we associate a point in the new space in n+H dimensions. The consumption of public good by consumer h is then treated as the consumption of a private good, identified by the n+h dimension. Similarly, the production possibilities set of firm f in $\mathcal{E}'$ is :

$$\hat{\mathbf{Y}}_f = \{\hat{\mathbf{y}}^f \in R^{n+H} \mid \hat{\mathbf{y}}^f = [\mathbf{y}^f; 0, 0, 0......0], \ \mathbf{y}^f \in \mathbf{Y}_f\}$$

As private firms do not produce or use as inputs public goods in the original economy, their production of the H "private" goods in the new economy is a vector of zeros. The production possibilities set of the public firm in $\mathcal{E}'$ is:

$$\hat{\mathbf{Y}}_G = \{\hat{\mathbf{y}}_G \in R^{n+H} \mid \hat{\mathbf{y}}_G = [\mathbf{r}, G^1, G^2, ...G^H], \ G^h = G, \ \forall h, \ (\mathbf{r}, G) \in \mathbf{Y}_G\}$$

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[4]The original idea of expanding the commodity space to prove the existence of a Lindahl equilibrium is due to Foley (1970).

Thus, the production of the public good by the public firm in the original economy is treated as the production of H different private goods in the associated economy. The equality in the second part of the definition guarantees that the quantity of these H private goods is the same in each dimension. Finally, let us also extend the original vector of endowment as follows:

$$\hat{w}^* = \{w^*;0,0,0,0...0\}$$

It is easy to check that the properties of the original consumption and production sets will carry on to the newly defined sets. In particular, $\hat{X}_h$ is closed, convex and bounded below, while $\hat{Y}_f$ and $\hat{Y}_G$ satisfies properties A.1-A.4. Preferences defined over $\hat{X}_h$ also satisfy the same properties of preferences defined over X_h. We then use the same utility indicator to represent preferences over the two sets. A feasible state in $\mathcal{E}'$ must satisfy the n+H equations

$$(6) \qquad \Sigma_h \hat{x}_h - \hat{w}^* = \Sigma_f \hat{y}^f + \hat{y}_G$$

It is immediate to establish the following:

Proposition 1

"If a state is feasible in $\mathcal{E}$, its image in $\mathcal{E}'$ is feasible in $\mathcal{E}'$ and vice versa"

Proof See Milleron, 1971; lemma 2.1, p.429

That is, there is an one-to-one correspondence between the feasible states in the original economy and the feasible states in the associated economy. This implies that the image in $\mathcal{E}'$ of the equilibrium state in the original economy must be feasible in $\mathcal{E}'$:

$$(7) \qquad \Sigma_h \hat{\mathbf{x}}_h^* - \hat{\mathbf{w}}^* = \Sigma_f \hat{\mathbf{y}}^{f*} + \hat{\mathbf{y}}_G^*$$

where $\hat{\mathbf{x}}_h^*$ is the image in $\mathcal{E}'$ of $(\mathbf{x}_h^*, G^*)$ in $\mathcal{E}$, and similarly for the other terms in (7). From the definition of the associated economy and proposition 1 it also follows that a Pareto efficient state in $\mathcal{E}$ is also Pareto efficient in $\mathcal{E}'$ and vice versa (see Milleron, 1971:427).

<u>4. Debreu's measure for an economy with public goods</u>

Armed with the associated economy, we can apply Debreu's measure of inefficiency to the associated economy $\mathcal{E}'$ and analyze its implications for the original economy. To this aim let us define the set:

$$(8) \quad \Omega_h(U^{h*}) = \{\hat{\mathbf{x}}_h | \ U^h(\hat{\mathbf{x}}_h) \geq U^{h*}\}$$

$\Omega_h(U^{h*})$ is the set of all bundles of goods which would give to individual h at least as much utility as in the equilibrium of the original economy. Note that $\hat{\mathbf{x}}_h$ includes also the consumption by individual h of G^h. The set $\Omega_h(U^{h*})$ is clearly strictly convex, closed and bounded from below. In the associated economy, the consumption bundles of individuals can be summed together: let us then define the set $\Gamma(\mathbf{U}^*)$ as

$$(9)\quad \Gamma(\mathbf{U}^*) = \{\Sigma_h \hat{\mathbf{x}}_h \mid \hat{\mathbf{x}}_h \in \Omega_h(U^{h*}) \text{ and } G^1{=}G^2{=}\ldots G^H\}$$

$\Gamma(\mathbf{U}^*)$ is then the set of all consumption bundles which would give to each individual at least as much utility as in the original state. The equality in the definition of the set is needed in order to avoid that the input requirement set (to be defined below) contains different endowments of the public goods for the different individuals. $\Gamma(\mathbf{U}^*)$ is also clearly strictly convex, closed and bounded below. The input requirement set for the associated economy is:

$$(10)\quad \hat{W}(\mathbf{U}^*) = \Gamma(\mathbf{U}^*) - \hat{\mathbf{Y}}_G - \Sigma_f \hat{\mathbf{Y}}_f$$

$\hat{W}(\mathbf{U}^*)$ is the set of resources which would allow each consumer h to reach at least utility level U^{h*}, taking into account the production possibilities of the economy. Using standard assumptions plus the ones explicitly introduced before it can be shown that $\hat{W}(\mathbf{U}^*)$ is closed, bounded below, strictly convex and satisfies free-disposal[5]. Let us then consider the lower boundary of this set, $\delta\hat{W}(\mathbf{U}^*)$. This boundary, roughly analogous to a Scitovsky curve, is composed by all physical resources which would allow consumption units to reach at most utility levels $\mathbf{U}^*$. Thus, the allocations corresponding to this lower boundary are Pareto efficient. As $\hat{W}(\mathbf{U}^*)$ is convex and preferences exhibit no satiation we can apply the Second Welfare Theorem for economies with public goods (see Milleron, 1971:430-431, Theorem 2.1) to establish that there is a n+H price vector, different from zero, which would support each allocation belonging to $\delta\hat{W}(\mathbf{U}^*)$ as a Lindahl equilibrium. That is, associated to each $\hat{\mathbf{w}}' \in \delta\hat{W}(\mathbf{U}^*)$ there is a n+H price vector $\mathbf{p}(\hat{\mathbf{w}}')$ (with n prices for the private goods and H personalized prices for the public good) such that

$$(11) \qquad \mathbf{p}(\hat{\mathbf{w}}')\ [\hat{\mathbf{w}} - \hat{\mathbf{w}}'] \geq 0 \quad \forall\ \hat{\mathbf{w}} \in \hat{W}(\mathbf{U}^*)$$

where strict monotonicity of preferences and free-disposal of the set $\hat{W}(\mathbf{U}^*)$ imply that each component of $\mathbf{p}(\hat{\mathbf{w}}')$ is strictly positive. From (11), Debreu's (1951) measure of welfare loss, $D(\hat{\mathbf{z}},\mathbf{U}^*)$, can be defined as follows:

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[5] $\hat{W}(\mathbf{U}^*)$ is closed because the sum of closed sets is closed (allowing the condition in Debreu (1951:279, note 7); it is strictly convex because the sum of convex sets is convex and preferences exhibit strict convexity; it is bounded below because $\Gamma(\mathbf{U}^*)$ is bounded below and no positive outputs can be produced from zero inputs; and it satisfies free-disposal (i.e. $\mathbf{w}' > \mathbf{w} \in W(\mathbf{U}^*)$ implies $\mathbf{w}' \in W(\mathbf{U}^*)$) because free-disposal is allowed in production.

$$(12)\quad D(\hat{w}^*,U^*) = \min_{\hat{w}' \in \delta\hat{W}(U^*)} p(\hat{w}')\,[\hat{w}^* - \hat{w}']/p(\hat{w}')\hat{w}^*$$

i.e. $D(\hat{w}^*,U^*)$ is the least distance between the original vector of utilizable resources $\hat{w}^*$ and the Pareto optimal set $\hat{W}(U^*)$, using the competitive prices associated with each element of the set $\hat{W}(U^*)$ as weighting factors. The term on the denominator of the RHS of (12) is simply a scaling factor needed to account for the fact that competitive prices are only defined up to an arbitrarily positive scalar. Given the properties of the set $\hat{W}(U^*)$, we can directly appeal to Debreu (1951:284) to establish the following:

Proposition 2

" Let $\hat{w}^0 \in \delta\hat{W}(U^*)$ be such that $p(\hat{w}^0)\hat{w}^0/p(\hat{w}^0)\hat{w}^* = \max\,[p(\hat{w}')\hat{w}'/p(\hat{w}')\hat{w}^*,\ \hat{w}' \in \delta\hat{W}(U^*)]$. Then $\hat{w}^0 = \rho\hat{w}^*$, where $\rho \leq 1$."

The scalar ρ is Debreu's coefficient of resources utilization: ρ is the smallest fraction of actually utilizable physical resources which would permit the achievement of U^*, given the production possibilities of the economy. From (12), proposition 2 entails $D(\hat{w}^*,U^*)=1-\rho$, so that the Debreu's measure of inefficiency is simply equal to the proportion of initial resources which could be thrown out of the economy still reaching the same level of utilities as in the original state. Also note that

$\hat{w}^0=\rho\hat{w}^*=[\rho w^*,0,0\ldots 0]'$; i.e. since there is no public good in the input endowment of the original economy, there is also no public good endowment in the counter-factual Pareto-efficient state.

The fact that n+H price vector $\mathbf{p}(\hat{w}^0)=[p_1^0,..,p_n^0;p^1,..,p^H]$ supports a Lindahl competitive equilibrium at $\hat{w}^0$ implies the existence of following relationships between the original and the associated economy:

1) there is a consumption plan for each consumer h, $\hat{\mathbf{x}}_h^0=[\mathbf{x}_h^0,0,0,..,G^{h0},..0]$, such that $\hat{\mathbf{x}}_h^0$ minimize the total expenditure needed at prices $\mathbf{p}(\hat{w}^0)$ to attain utility level U^{h*}. In the original economy, $(\mathbf{x}_h^0,G^0)$ solves:

$$(13)\quad \min \quad \mathbf{p}^0\mathbf{x}_h + p^hG \quad \text{s.t.} \quad U^h(\mathbf{x}_h;G)=U^{h*}$$

2) There is a production plan for each private firm, $\hat{\mathbf{y}}^{f0}$, such that $\hat{\mathbf{y}}^{f0}=[\mathbf{y}^{f0},0,0\ldots 0]'$ maximize profits at prices $\mathbf{p}(\hat{w}^0)$; i.e. $\mathbf{y}^{f0}$ solves:

$$(14)\qquad \max\ p^0\mathbf{y}^f \quad \text{s.t.} \quad \mathbf{y}^f\in\mathbf{Y}^f$$

3) There is a production plan for the public firm, $\hat{\mathbf{y}}_G^0$, such that $\hat{\mathbf{y}}_G^0=[\mathbf{r}^0,G^0,G^0,\ldots G^0]$ maximize profits at prices $\mathbf{p}(\hat{w}^0)$. From the definition of $\hat{\mathbf{Y}}_G$, $(\mathbf{r}^0,G^0)$ solves:

(15) max $\Sigma_h p^h G + p^0 \mathbf{r}$ s.t. $(G,\mathbf{r}) \in \mathbf{Y}_G$

4) At prices $\mathbf{p}(\hat{\mathbf{w}}^0)$ net demand equals net supply in each market:

$$(16) \qquad \Sigma_h \hat{\mathbf{x}}_h^0 - \hat{\mathbf{w}}^0 = \Sigma_f \hat{\mathbf{y}}^{f0} + \hat{\mathbf{y}}_G^0$$

Note that (16) implies $G^{10}=G^{20}=...=G^{H0}=G^0$; in a Lindahl equilibrium, at the personalized prices each individual h demands the same quantity of public good. By substituting (7) and (16) in (12):

$$(17) \quad D(\hat{\mathbf{w}}^*,\mathbf{U}^*) = \mathbf{p}(\hat{\mathbf{w}}^0)[\hat{\mathbf{w}}^*-\hat{\mathbf{w}}^0]/\mathbf{p}(\hat{\mathbf{w}}^0)\hat{\mathbf{w}}^* =$$
$$=\mathbf{p}(\hat{\mathbf{w}}^0)[(\Sigma_h \hat{\mathbf{x}}_h^* - \Sigma_f \hat{\mathbf{y}}^{f*} - \hat{\mathbf{y}}_G^*) - (\Sigma_h \hat{\mathbf{x}}_h^0 - \Sigma_f \hat{\mathbf{y}}^{f0} - \hat{\mathbf{y}}_G^0)]/\mathbf{p}(\hat{\mathbf{w}}^0)\hat{\mathbf{w}}^*$$

Thus, Debreu's measure of efficiency is simply equal to the difference between aggregate consumption and production between the two states, evaluated at the (normalized) supporting prices $\mathbf{p}(\hat{\mathbf{w}}^0)/\mathbf{p}(\hat{\mathbf{w}}^0)\hat{\mathbf{w}}^*$.

5. Interpreting Debreu's measure

Let us note that, as only relative prices matter, if $\mathbf{p}(\hat{\mathbf{w}}^0)$ support a Lindahl equilibrium at $\hat{\mathbf{w}}^0$, so do the normalized prices $\mathbf{p}(\hat{\mathbf{w}}^0)/\mathbf{p}(\hat{\mathbf{w}}^0)\hat{\mathbf{w}}^*$. To save notation, let us then maintain the symbols $\mathbf{p}^0$ for the (normalized) prices of private goods and $\mathbf{p}^h$ for the (normalized) personal prices for the public good. We can then rewrite (17) as:

$$(18)\quad D(\hat{w}^*,U^*) = \Sigma_h\{[\mathbf{p}^0,\mathbf{p}^h]\ (\hat{\mathbf{x}}_h^* - \hat{\mathbf{x}}_h^0)\} + \Sigma_f\{[\mathbf{p}^0,\mathbf{p}^h](\hat{\mathbf{y}}^{f0} - \hat{\mathbf{y}}^{f*})\} + \\ + [\mathbf{p}^0,\mathbf{p}^h](\hat{\mathbf{y}}_G^0 - \hat{\mathbf{y}}_G^*)$$

From (13) to (15) each of the three elements in the RHS of (18) is non negative. The first term in the RHS of (18) is the summation across all consumers of the extra income spent by each individual h to attain utility U^{h*} by consuming at prices $[\mathbf{p}^0,\mathbf{p}^h]$ the consumption bundle $\hat{\mathbf{x}}_h^*$ rather then the cost-minimizing bundle $\hat{\mathbf{x}}_h^0$. Similarly, the second and third term are the summation across all firms, private and public, of the profit losses due to the fact of producing, at prices $[\mathbf{p}^0,\mathbf{p}^h]$, the bundles $\hat{\mathbf{y}}^{f*}$ and $\hat{\mathbf{y}}_G^*$ rather than the profit maximizing bundles $\hat{\mathbf{y}}^{f0}$ and $\hat{\mathbf{y}}_G^0$. We will now manipulate each of the terms in the RHS of (18) so as to express them in terms of the familiar notions of consumers and producers loss. Writing the first term in the RHS of (18) in full:

$$(19)\quad \mathbf{p}^0\Sigma_h\mathbf{x}_h^* + \Sigma_h p^h G^* - \mathbf{p}^0\Sigma_h\mathbf{x}_h^0 - \Sigma_h p^h G^0 \geq 0$$

Recall, from section 2, that $\mathbf{x}_h^* = \mathbf{x}_h(\mathbf{q},G^*,U^{h*})$ and $\mathbf{q}\mathbf{x}_h^* = E^h(\mathbf{q},G^*,U^{h*})$. Note further that $\mathbf{p}^0\mathbf{x}_h^0 = \mathbf{p}^0\mathbf{x}_h(\mathbf{p}^0,p^h,U^{h*}) = \mathbf{p}^0\mathbf{x}_h(\mathbf{p}^0,G^0,U^{h*}) = E^h(\mathbf{p}^0,G^0,U^{h*})$. This follows from the fact that $\mathbf{x}_h^0$ is still the cost minimizing choice to attain utility U^{h*} if the prices for the private goods are $\mathbf{p}^0$, and individual h is forced to consume the public good at the level that he would have chosen anyhow, G^0. Then, by summing and subtracting $\Sigma_h(\mathbf{q}\mathbf{x}_h^* + E^h(\mathbf{p}^0,G^*,U^{h*}))$ in (19), this can be rewritten as:

$$(20)\quad \Sigma_h(E^h(\mathbf{q},G^*,U^{h*}) - E^h(\mathbf{p}^0,G^*,U^{h*}) - (\mathbf{q}-\mathbf{p}^0)\mathbf{x}_h^*) + \\ + \Sigma_h(E^h(\mathbf{p}^0,G^*,U^{h*}) - E^h(\mathbf{p}^0,G^0,U^{h*}) + (G^*-G^0)p^h) \geq 0$$

The first three terms in (20) represent a general equilibrium equivalent variation measure of consumers deadweight loss from commodity taxation. They are identical to the terms derived in Kay and King (1998:278, eq.26); we then refer the reader to their paper for a discussion. The second group of terms in (20) is an equivalent variation measures of the welfare loss induced by distortion in public good supply with respect to the optimal level. The sum of these terms is also certainly non negative, as strict convexity of preferences implies that each $E^h(\mathbf{p}^0,G,U^{h*})$ is convex in G. Their sum is approximately equal to the area below the compensated demand curve for the public good evaluated in the interval (G^*-G^0). To see this, let us express the difference $E^h(\mathbf{p}^0,G^*,U^{h*}) - E^h(\mathbf{p}^0,G^0,U^{h*})$ as a second order Taylor approximation around the point $(\mathbf{p}^0,G^0,U^{h*})$; doing this and substituting we get:

$$(21)\quad \Sigma_h(E^h(\mathbf{p}^0,G^*,U^{h*}) - E^h(\mathbf{p}^0,G^0,U^{h*}) + (G^*-G^0)p^h) \simeq \\ \Sigma_h E^h_{GG}(G^*-G^0)^2/2 \geq 0$$

which is approximately the area indicated above. (In (21) we use the fact $-E^h_G(\mathbf{p}^0,G^0,U^{h*})=p^h$; E^h_G and E^h_{GG} are respectively the first and the second order partial derivative of the expenditure function $E^h(.)$ with respect to the public good G). Let us then turn to the second and third term in (18). Writing them in full:

$$(22)\quad \Sigma_f \mathbf{p}^0(\mathbf{y}^{fo}-\mathbf{y}^{f*}) + p^0_G\ (G^0-G^*) + \mathbf{p}^0(\mathbf{r}^0-\mathbf{r}^*) \geq 0$$

where $p^0_G=\Sigma_h p^h$ and $\mathbf{r}^0,\mathbf{r}^*$ are the vectors of private goods used respectively in the production of G^0 and G^*. By the previous argument, summing and subtracting $\mathbf{p}\Sigma_f\mathbf{y}^{f*} + \mathbf{pr}^*$ $+p_G G^*$ to these terms and rearranging, (22) can be written as:

$$(23)\quad \Pi^g(p^0_G,\mathbf{p}^0) - \Pi^g(p_G,\mathbf{p}) + \Sigma_f(\Pi^f(\mathbf{p}^0) - \Pi^f(\mathbf{p})) + (\mathbf{p}-\mathbf{p}^0)\Sigma_f\mathbf{y}^{f*} + (\mathbf{p}-\mathbf{p}^0)\mathbf{r}^* + (p_G-p^0_G)G^* \geq 0$$

The terms in (23) capture the value, at the shadow prices $(p^0_G,\mathbf{p}^0)$, of the distortions induced in the production decisions of firms by commodity taxation and by public expenditure at G^*. In terms of the standard presentation, these elements represent producers losses: i.e. they are the sum of all areas above the supply curves, evaluated in the intervals $(\mathbf{p}-\mathbf{p}^0)$ and $(p_G-p^0_G)$. Except for the terms related to the production of G they are also discussed in Kay and

Keen (1988; 278, eq.27); we then again refer the reader to their paper.

Thus, Debreu's measure is basically the sum of consumers and producers losses, as usually presented in public finance textbooks. A serious complication with respect to this presentation is due to the presence of varying producer prices across equilibria. As this is likely to be an insuperable difficulty for any empirical application of the Debreu's measure, it is worth analyzing Debreu's measure for the case of constant producer prices for the private goods[6]. In this case, to the Debreu's measure of inefficiency with public good can be given a nice interpretation. To see this, note that from (18), (19), (22) and (23), at $\mathbf{p}=\mathbf{p}^0$, Debreu's measure simply reduces to:

$$(24)\quad D(\hat{\mathbf{w}}^*,\mathbf{U}^*) = \mathbf{p}\Sigma_h\mathbf{x}_h^* - \mathbf{p}\Sigma_h\mathbf{x}_h^0 + \mathbf{pr}^0 - \mathbf{pr}^*$$

As producer prices for private goods are fixed we can write $-\mathbf{pr}^0=C(\mathbf{p},G^0)=C(G^0)$ and $-\mathbf{pr}^*=C(\mathbf{p},G^*)=C(G^*)$, where $C(G)$ is the cost of producing public good G, assumed to be differentiable in G. Then, as above, by summing and subtracting $\Sigma_h(\mathbf{qx}_h^* + E^h(\mathbf{p},G^*,U^{h*}))$ in (24) we can rewrite it as:

[6] Fixed producer prices for the private goods would be guaranteed, for example, by a linear technology for the private goods and a convex technology for the public good. Note that this is the case usually discussed in the literature on optimal provision of public good; see for example, Wilson (1991).

$$(25) \quad D(\hat{\mathbf{w}}^*,\mathbf{U}^*) = \Sigma_h[E^h(\mathbf{q},G^*,U^{h*})-E^h(\mathbf{p},G^*,U^{h*})-(\mathbf{q}-\mathbf{p})\mathbf{x}_h^*] + \Sigma_h[E^h(\mathbf{p},G^*,U^{h*})-E^h(\mathbf{p},G^0,U^{h*})] + C(G^*) - C(G^0)$$

The first three terms are again an equivalent variation measure of the deadweight loss from taxation, this time evaluated with respect to the actual revenue from taxation. The second group of terms is simply the Haberger's triangle for the public good: it is the intersection of area beneath the (compensated) demand curve for the public good and the area above the supply curve for the public good in the interval (G^0,G^*). To see this even more clearly, let us express the last four terms in (25) in integral form:

$$(25) \quad \Sigma_h[E^h(\mathbf{p},G^*,U^{h*})-E^h(\mathbf{p},G^0,U^{h*})] + C(G^*) - C(G^0) = \int_{G^*}^{G^0}[-\Sigma_h E^h_G(\mathbf{p},G,U^{h*}) - C_G(G)]dG = \int_{G^*}^{G^0}[\Sigma_h MRS^h(\mathbf{p},G,U^{h*}) - MRT(G)]dG$$

where MRS^h is the marginal rate of substitution for individual h and MRT is the marginal rate of transformation of the public good into the private good chosen as numeraire. $\Sigma_h MRS^h(\mathbf{p},G,U^{h*})$ is the compensated demand curve for the public good, derived by summing together vertically the individual demands. Thus, with fixed prices in production for private goods, Debreu's measure of inefficiency is simply the sum of the deadweight loss from taxation plus the deadweight loss deriving from the supply of the public good at the inefficient level.

If the public good were financed through a system of lump-sum taxes, **q=p**; the first three terms in (25) would then disappear and Debreu's measure of inefficiency would be simply equal to (26). For future reference, note that by adopting an input measure it is then immediate to establish the direction of inefficiency in this case (see also Bordignon, 1990): at unchanged utility and prices for the private goods, strict convexity of preferences implies that the each MRS^h must be falling in G while convexity of C(G) implies that the MRT must be non-decreasing in G; thus, as $\Sigma_h MRS^h(\mathbf{p},G^0,U^{h*}) = MRT(G^0)$ it follows immediately that $\Sigma_h MRS^h(\mathbf{p},G^*,U^{h*}) \gtrless MRT(G^*)$ implies $G^0 \gtrless G^*$. Note that if we had allowed utility to change across states, adopting an output measure of efficiency, not even this simple result could have been achieved. In fact, unless individual preferences have some very peculiar structure (roughly corresponding to Gorman's polar form; see Bergstrom and Cornes, 1983), the optimal level of public good supply depends on utility distribution. Thus, even in a first-best economy, the adoption of an output measure of efficiency makes in general impossible to establish the direction of inefficiency by examination of the marginal conditions.

6. Conclusions

In this paper we have extended Debreu's coefficient of resource utilization to an economy with a public good and interpreted it in terms of the standard notions of consumers' and producers' loss. The results confirm the existence of a complete duality between prices of private goods and quantities of public goods and of the corresponding measures of inefficiency. In the case of private goods, taxation distorts prices; in the case of public good, government "distorts" quantities by supplying the public good at an inefficient level. In both cases welfare loss is measured in terms of the area between (compensated) demand and supply, having as one of the side of the Haberger's triangle the distance between "consumers price" ($\Sigma_h MRS^h$) and "producers price" (MRT).

An obvious extension of the present work is in term of assessing the direction of inefficiency in public good supply in second best economies. The input approach to inefficiency suggest that the counter-factual optimal levels of public goods to be compared with the actual supply of public goods should be chosen so as to guarantee individuals the same level of utility of the inefficiency state. In contrast, all the literature which has attempted to amend Samuelson's rule to second best economies and to infer from this amended rule the direction of inefficiency has worked by having implicitly in mind an "output" measure of inefficiency (see, for example, Atkinson and Stern (1974), King (1986), Batina (1990a, 1990b), Wilson (1991a, 1991b) and Ballard and Fullerton (1992)). In our view, the resulting general failure to come up with clear-cut results is partly a consequence of the failure of distinguishing between distributive and efficiency issues.

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