Core-stable bidding rings*

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Abstract

We propose a semi-cooperative game theoretic approach to check whether a given coalition is stable in a Bayesian game with independent private values. The ex ante expected utilities of coalitions, which are achieved at an incentive compatible (noncooperative) coalitional equilibrium, describe a (cooperative) partition form game. A coalition is core-stable if the core of a specific characteristic function, derived from the partition form game, is not empty. As an application, we study collusion in auctions in which the bidders' final utility possibly depends on the winner's identity. We show that such direct externalities offer a possible explanation for cartels' structures (not) observed in practice.

Keywords: Auctions, Bayesian game, collusion, core, partition function game.

Classification JEL: C71, C72, D44

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1 Introduction

Collusion in auctions has been mostly studied as a mechanism design problem for a given ring (see, e.g., Graham and Marshall (1987), Mailath and Zemsky (1991), McAfee and McMillan (1992) for early references). In this approach, every member of the ring is submitted to an individual participation constraint. We also consider a given ring but ask whether it is stable, in the sense that no subgroup of bidders would like to secede from the ring. Such collective participation constraints are traditionally captured by core-like solution concepts. However, two difficulties arise when trying to define the core of an auction game or, more generally, of a (non-cooperative) Bayesian game.

A first difficulty, which already appears under complete information, is that every coalition faces strategic externalities, so that it must make conjectures on the behavior of the players who are outside the coalition. A second difficulty is that, if the players possess private information, every coalition faces incentive constraints. Up to now, to the best of our knowledge, the two issues have been explored independently of each other. In particular, the role of (the absence of) externalities in the formulation of incentive constraints has been neglected.[1]

In this paper, we shall establish that, in a class of Bayesian games which includes standard auctions (namely, games with independent private values and quasi-linear utilities), the second difficulty can basically be ignored. Then, we shall show how to use the core concept to check the stability of coalitions which can commit to an incentive compatible mechanism ex ante, i.e., before their members get their private information. We shall illustrate the solution concept in standard first price and second price auctions, in which losing bidders do not care on the winner's identity. In this framework, the core-stability of coalitions is widely guaranteed. Finally, we shall turn to a model proposed by Jehiel and Moldovanu (1996) in order to provide examples where direct external effects make coalitions unstable.

In the next paragraphs, we briefly survey the literature and explain how the present paper builds on it.

[1]This is not to say that cooperation under incomplete information has not been investigated. However the complex interference of incentive compatibility and externalities has not yet been studied. For instance, the problem does not appear in two-person games, a model that is still under study (see A. Kalai and E. Kalai (2009)).

Strategic externalities under complete information

The core was originally defined in games described by a characteristic function. Aumann (1961) applied the solution concept to strategic form games by measuring the worth of a coalition as the amount that it can guarantee whatever the complementary coalition does. The corresponding core is known as the $\alpha-$core. This solution concept has been criticized on the grounds that it involves incredible threats from the complementary coalition. As a remedy, Ray and Vohra (1997) and Ray (2007) construct a partition form game (as defined by Lucas and Thrall (1963)) in which, given a partition of the players, coalitions evaluate their worth at a Nash equilibrium of an auxiliary game between the coalitions. They call such a Nash equilibrium a "coalitional equilibrium". Maskin (2003) and Hafalir (2007) propose definitions of the core for partition form games. The core with "cautious expectations" is one of them. In the strategic framework, this solution concept refines the $\alpha-$core in order to account for a natural form of sequential rationality.

Starting from our original noncooperative Bayesian game, we construct a partition form game with complete information by extending Ray (2007)'s coalitional equilibrium to games with incomplete information. We then apply the core with "cautious expectations". Under complete information, our core is included in the $\alpha-$core of the original strategic form game.

Incomplete information without externalities

If the members of a coalition are not affected by the actions of players who are outside the coalition, incentive compatible mechanisms for that coalition can be defined as in standard mechanism design, even if the coalition's members care about the information of outside players. This assumption is typically made in Myerson (e.g., 1984, 2007), who refers to orthogonal coalitions[2]. An assumption of purely informational externalities is also fulfilled in exchange economies with differential information. In that framework, Forges

[2]Myerson (1984) considers interim binding agreements in Bayesian games. Orthogonal coalitions only appear in the course of his analysis. He focuses on the generalization of two fundamental solution concepts for cooperative games with complete information, namely the Nash bargaining solution and the Shapley value with nontransferable utility. According to this objective, only the mechanism chosen by the grand coalition is implemented. Mechanisms for other coalitions are regarded as rational threats and "are not required to satisfy any equity or incentive compatibility conditions". An interpretation of the construction is given in the case of orthogonal coalitions.

and Minelli (2001) and Forges, Mertens and Vohra (2002) construct a characteristic function by associating, with every coalition, the set of all ex ante expected utility vectors that the coalition can achieve by relying on some incentive compatible mechanism. They give sufficient conditions for the nonemptiness of the core of that characteristic function, which they call the "ex ante incentive compatible core" and illustrate that, as opposed to what happens under complete information, the ex ante incentive compatible core can be empty. Forges, Minelli and Vohra (2002) survey other core notions[3].

The basic solution concept in this paper can be seen as an ex ante incentive compatible core for Bayesian games, a model that involves strategic externalities by definition.

Ex ante commitment in auctions

In order to associate a standard cooperative game (namely, a characteristic function or a partition form) to a noncooperative game with incomplete information, we assume that coalitions can commit to an incentive compatible mechanism at the ex ante stage, i.e., before their members get their private information. This assumption first requires that an ex ante stage can be identified, which is true in many economic applications, like auctions, in which private information reduces to the value of some parameter, like a valuation or a cost. According to empirical data (see, e.g., Porter and Zona (1993), Pesendorfer (2000)), bidding rings often consist of well-identified groups (e.g., "incumbents", as opposed to "newcomers") whose characteristics do not depend on particular information states. Such bidding rings typically form at an early stage. For instance, local suppliers may be aware that a procurement auction will take place and consider to collude before the precise project specifications are published. At the time they commit to a collusion mechanism, they do not figure out their exact valuations, i.e., the costs incurred by the project.

A number of papers studying collusion in auctions with independent private values (Graham and Marshall (1987), McAfee and McMillan (1992), Marshall et al. (1994), Waehrer (1999), ...) implicitly assume that rings are formed at the ex ante stage. Indeed, they investigate the strategy of rings within an a priori given coalition structure, namely a partition of the bidders,

[3]As a noticeable reference posterior to Forges, Minelli and Vohra (2002)'s survey, Myerson (2007) establishes the nonemptiness of an appropriately defined interim incentive compatible core in cooperative games with incomplete information and orthogonal coalitions.

which does not depend on the bidders' private information. If the bidders gather into rings, the game is still tractable in the case of a second price auction, since possibly asymmetric players are not an issue in that case. However, as is well-known, first price auctions between asymmetric bidders are difficult to handle (see Krishna (2002)). In fact, first price auctions between bidding rings were partly studied for the specific asymmetries that they generate (see Lebrun (1991, 1999), Marshall et al. (1994), Waehrer (1999), ...). In these studies, rings operate as single entities, which automatically share their information, without relying on any (incentive compatible) mechanism. This simplifying assumption is founded if bidding rings can make arbitrary inside transfers: our proposition 1 below states that, in any Bayesian game with private independent values, every coalitional equilibrium can be made incentive compatible thanks to appropriate balanced transfers. This allows us to use the results of McAfee and McMillan (1992), Marshall et al. (1994), Lebrun (1999) and Waehrer (1999) on Nash equilibria of first price auctions between asymmetric bidders to assess the stability of bidding rings.

Some papers (Mailath and Zemsky (1991), Caillaud and Jehiel (1998), Marshall and Marx (2007), ...) focus on a specific bidding ring, for instance, the grand coalition, or an arbitrary coalition facing single individual bidders, and formulate participation constraints for the individual members of the ring at the interim stage. More precisely, every member of the ring can decide to leave the ring once he knows his private information. The precise form of the participation constraints depends on the way in which the ring members are supposed to react when one of them secedes from the ring. Interim participation constraints for coalitions raise a number of conceptual issues, already in the absence of externalities (see, e.g., Forges, Minelli and Vohra (2002), Myerson (2007)) and we will not address the question in this paper.

Outline of the paper and further relationships with the literature

Section 2 is devoted to the model and solution concept. In subsection 2.1, we fix a Bayesian game Γ with independent, private values. For every partition P of the players, we define an auxiliary Bayesian game $\Gamma(P)$ in which the players are the coalitions in P. A coalitional equilibrium w.r.t. P is defined as a Nash equilibrium of $\Gamma(P)$. This solution concept does not involve explicit incentive constraints inside coalitions and is thus a direct extension of the coalitional equilibrium proposed by Ray and Vohra (1997) for games with complete information. Coalitional equilibria have been considered in auctions (e.g., in Lebrun (1991), McAfee and McMillan (1992), Marshall et al. (1994), Waehrer (1999)). We generate a partition form game by defining $v_\sigma(S; P)$ as the sum of the expected payoffs of the players in coalition $S \in P$ at the equilibrium $\sigma(P)$, assuming that such an equilibrium exists and is unambiguously selected. In subsection 2.2, we address the issue of incentives. Every coalition chooses a mechanism which determines actions and balanced transfers for its members as a function of the information of its members. Proposition 1 shows that every coalitional equilibrium can be made incentive compatible thanks to appropriate balanced transfers for every coalition. In section 2.3, we propose a notion of core-stability for a bidding ring, which does not necessarily gather all the bidders. As in Marshall and Marx (2007), we focus on a single ring and assume that the bidders outside the ring do not collude. This assumption is consistent with well documented cases (see, e.g., Porter and Zona (1993), Pesendorfer (2000)). To test the stability of a ring R, we look at some members, $S \subseteq R$, who contemplate leaving the ring. The players in S can plan on acting jointly and must anticipate any form of collusion by the remaining members of the ring, namely $R \setminus S$. Indeed, the potential colluders in R have met each other, so that subcoalitions of R may form, even if the whole group does not. By contrast, the secession of some bidders has no effect on the absence of collusion outside the initially planned ring, namely in $N \setminus R$. Hence, if $S \subseteq R$ leaves R, it should expect a coalitional equilibrium relative to a coalition structure P of the form $\left\{S, \Pi, \{k\}_{k \in N \setminus R}\right\}$, where Π is any partition of $R \setminus S$. We say that the ring R is core-stable if no subcoalition $S \subseteq R$ can guarantee a better expected payoff to its members by leaving the ring (whatever the partition Π formed by the players in $R \setminus S$). Stability of the grand coalition amounts to the non-emptiness of the core with cautious expectations (as defined by Hafalir (2007)) of the partition form game.

In section 3, we apply core-stability to auctions. As a benchmark, we consider standard auctions. In the case of second price auctions (subsection 3.1), our partition form game reduces to a characteristic function already derived by Mailath and Zemsky (1991), who formulated collusion as a mechanism design problem for every ring. In second price auctions, strategic externalities have thus no effect on collusion. We also recover (proposition 2) that all rings are core-stable in that model. In subsection 3.2, thanks to results of Lebrun (1999) and Waehrer (1999), we establish that the grand coalition is always core-stable in a first price auction (proposition 3). In the absence of general, analytical solutions for first price auctions with asymmetric bidders, we only check that all coalitions are core-stable in two specific examples, borrowed from McAfee and McMillan (1992) and Marshall et al. (1994). In subsection 3.3, we investigate the effects of direct externalities on collusion. We first assume, as Jehiel and Moldovanu (1996), that a bidder suffers more if a competitor wins the auction than if the object is not sold at all ("negative externalities"). Proposition 4 states that the grand coalition is core-stable under appropriate assumptions. Example 1 illustrates some limits of this result. In example 2, we propose a three person first price auction game in which a two bidder cartel is not stable. In example 3, we assume that direct externalities can possibly be positive. We show that the grand coalition is not core-stable and that there exist non-singleton rings which are core-stable. These example confirm that direct externalities make cooperative behavior difficult, which was already suggested in Jehiel and Moldovanu (1996), but give a more precise content to that phenomenon. Indeed, Jehiel and Moldovanu (1996) only show that, under reasonable assumptions, no agreement between (some of) the buyers and/or the seller can be stable. They thus depart from collusion of the bidders in the original auction game. We focus on the latter form of collusion and show, in Jehiel and Moldovanu (1996)'s framework, that the grand coalition is always stable (proposition 4) but that small coalitions may not be stable (example 2). Caillaud and Jehiel (1998) point out that direct externalities may prevent the grand coalition from being ex post efficient but do not address the question of its ex ante stability.

2 Model and solution concept

2.1 From Bayesian games to cooperative games

Let us fix a Bayesian game with independent, private values $\Gamma \equiv \left[N, \{T_i, q_i, A_i, u_i\}_{i \in N}\right]$, namely a set of players N and for every player i, $i \in N$,

- a set of types T_i
- a probability distribution q_i over T_i
- a set of actions A_i
- a utility function $u_i : T_i \times A \to \mathbb{R}$, where $A = \prod_{i \in N} A_i$.

Let P be a coalition structure, namely a partition of N. From Γ and P, we construct an auxiliary Bayesian game $\Gamma(P) \equiv \left[P, \{T_S, q_S, A_S, U_S\}_{S\in P}\right]$, in which the players are the coalitions S, $S \in P$, and

- $T_S = \prod_{i\in S} T_i$, $q_S = \bigotimes_{i\in S} q_i$, $A_S = \prod_{i\in S} A_i$
- $U_S(t_S, (a_K)_{K\in P}) = \sum_{i\in S} u_i(t_i, (a_K)_{K\in P})$, where $t_S = (t_i)_{i\in S}$, $a_K = (a_i)_{i\in K}$

A strategy of S in $\Gamma(P)$ is a mapping $\sigma_S : T_S \to A_S$. Such a definition makes sense if the members of coalition S share their information in T_S before jointly deciding on an action profile in A_S. We show in the next subsection that such strategies are derived from appropriate coalitions' mechanisms, which allow for transfers between the coalitions' members. Thanks to these mechanisms, utilities become fully transferable. Furthermore, incentive compatibility conditions are automatically satisfied.

As in Ray and Vohra (1997) and Ray (2007), we define a coalitional equilibrium relative to P as a Nash equilibrium $(\sigma_S)_{S\in P}$ of $\Gamma(P)$. We assume that for every P, there exists a coalitional equilibrium relative to P and in case of multiple equilibria, we fix a mapping σ associating a coalitional equilibrium $\sigma(P)$ with every P.[4] We denote as $v_\sigma(S;P)$ the expected utility of S at $\sigma(P)$, for every $S \in P$, namely

$$v_\sigma(S;P) = E\left[\sum_{i\in S} u_i\left(\widetilde{t}_i, \sigma(P)(\widetilde{t})\right)\right] \tag{1}$$

where r.v.'s are denoted with a $\widetilde{}$ and $\sigma(P)(\widetilde{t}) = \left(\sigma(P)_K(\widetilde{t}_K)\right)_{K\in P}$. (1) defines a partition form game, which is constructed from Γ and σ, with $\Gamma(P)$ as an intermediary step.

Let $T = \prod_{i\in N} T_i$. By evaluating (1) at the grand coalition N, we get

$$v_\sigma(N;P) \equiv v(N) = \max_{\tau\in A^T} E\left[\sum_{i\in N} u_i(\widetilde{t}_i, \tau(\widetilde{t}))\right] \quad \text{for every } \sigma \text{ and } P. \tag{2}$$

[4] Ray and Vohra (1997) give sufficient conditions for the existence of a coalitional equilibrium but their result is not useful in our applications to auctions. However, many specific results are available in this context. Ray (2007) argues that the partition form game only makes sense if a unique coalitional equilibrium can be associated with every partition (possibly up to transfers). We rather take the view that in case of multiple equilibria, some "standard of behavior" allows us to select among them.

Given a coalitional equilibrium mapping σ and a partition P of N, $\sigma(P)$ is a feasible strategy for N (i.e., $\sigma(P) \in A^T$). Hence, v_σ is "grand coalition superadditive", or, according to an equivalent terminology, N is efficient in v_σ:

$$v(N) \geq \sum_{S \in P} v_\sigma(S; P) \qquad \text{for every } P \tag{3}$$

2.2 Coalitions' mechanisms

Let us fix a coalition S. A mechanism μ_S for S is a pair of mappings $\mu_S = (\tau_S, m_S)$:

$$\begin{aligned} \tau_S &: T_S \to A_S \\ m_S &: T_S \to \left\{ z \in \mathbb{R}^S : \sum_{i \in S} z_i \leq 0 \right\} \end{aligned}$$

τ_S is S's decision scheme and m_S is a balanced transfer scheme[5]. As usual, the interpretation is that members of S are invited to report their types to a planner who then chooses a profile of actions and transfers.

We assume that utilities over mechanisms are quasi-linear. More precisely, the utility of μ_S for player $i \in S$, given his type t_i, reported types $r_S = (r_j)_{j \in S}$, a "strategy" $\sigma_{N \setminus S} : T_{N \setminus S} \to A_{N \setminus S}$ for the players outside S (e.g., $\sigma_{N \setminus S} = (\sigma_K)_{K \in P, K \neq S}$, for some partition P of N) and types $t_{N \setminus S}$ for the players outside S is

$$u_i(t_i, \tau_S(r_S), \sigma_{N \setminus S}(t_{N \setminus S})) + m_S^i(r_S)$$

As this expression explicitly shows, every member i of S incurs an externality from the strategic choices of the players in $N \setminus S$ but, thanks to the private value assumption, does not face any direct informational externality. We define the incentive compatibility (I.C.) of the mechanism μ_S given a mapping $\sigma_{N \setminus S} : T_{N \setminus S} \to A_{N \setminus S}$. More precisely, μ_S is I.C. given $\sigma_{N \setminus S}$ iff for every $i \in S$, every type t_i and reported type r_i,

[5]We restrict ourselves to interim transfers, which are defined over T_S. They can be interpreted as the expectations of more general ex post transfer schemes, which are defined over $T_S \times A_S$ if actions are eventually observable. If the players' utilities only depend on actions through some outcome function $\theta : A_S \to \Theta$, where Θ denotes the set of observable outcomes, ex post transfers are better defined over $T_S \times \Theta$.

$$
\begin{aligned}
& E\left[u_i(t_i, \tau_S(t_i, \widetilde{t}_{S\setminus i}), \sigma_{N\setminus S}(\widetilde{t}_{N\setminus S})) + m_S^i(t_i, \widetilde{t}_{S\setminus i})\right] \\
\geq \quad & E\left[u_i(t_i, \tau_S(r_i, \widetilde{t}_{S\setminus i}), \sigma_{N\setminus S}(\widetilde{t}_{N\setminus S})) + m_S^i(r_i, \widetilde{t}_{S\setminus i})\right]
\end{aligned}
$$

This definition makes sense because coalition S must take account of the behavior of the players in $N \setminus S$ in elaborating its own strategy. In the case of complete information, S just looks for a best reply to $N \setminus S$'s action profile. In the case of incomplete information with private values, S looks for an I.C. best reply to $N \setminus S$'s strategy $\sigma_{N\setminus S}$, without entering the details of $\sigma_{N\setminus S}$ (whether the players lie or not, how they possibly gather into subcoalitions, etc.). The next proposition justifies the coalitions' strategies in the auxiliary Bayesian game; in particular, we show that explicit I.C. conditions are not necessary. The construction, which goes back to Arrow (1979) and d'Aspremont and Gérard-Varet (1979, 1982), has been widely used in economic frameworks which do not involve externalities (see, e.g., Forges et al. (2002)).

Proposition 1 *Let $S \subseteq N$; let $\sigma_{N\setminus S} : T_{N\setminus S} \to A_{N\setminus S}$ be an arbitrary strategy of $N \setminus S$ and let σ_S be a best response of S to $\sigma_{N\setminus S}$ in $\Gamma(\{S, N \setminus S\})$. There exists a transfer scheme m_S such that*

1. *$\sum_{i\in S} m_S^i(r_S) = 0$ for every $r_S \in T_S$*
2. *The mechanism (σ_S, m_S) is I.C. given $\sigma_{N\setminus S}$.*

Proof: Let us fix S, $\sigma_{N\setminus S}$ and σ_S as in the statement. For every $i \in S, t_i \in T_i, a_S \in A_S$ let us set

$$h_i(t_i, a_S) = E[u_i(t_i, a_S, \sigma_{N\setminus S}(\widetilde{t}_{N\setminus S}))]$$

Since σ_S is a best response to $\sigma_{N\setminus S}$,

$$\sum_{i\in S} h_i(t_i, \sigma_S(t_S)) \geq \sum_{i\in S} h_i(t_i, a_S) \qquad \forall t_S \in T_S, a_S \in A_S \tag{4}$$

Let $\widehat{m}_S^i(r_S) = \sum_{j\in S\setminus i} h_j(r_j, \sigma_S(r_S))$. For every $i \in S$, type $t_i \in T_i$, reported type $r_i \in T_i$ and reported types $r_{S\setminus i} \in \prod_{j\in S\setminus i} T_j$ of the other members of S,

$$\begin{aligned} h_i(t_i,\sigma_S(r_S)) + \widehat{m}_S^i(r_S) &= h_i(t_i,\sigma_S(r_S)) + \sum_{j\in S\setminus i} h_j(r_j,\sigma_S(r_S)) \\ &\leq h_i(t_i,\sigma_S(t_i,r_{S\setminus i})) + \sum_{j\in S\setminus i} h_j(r_j,\sigma_S(t_i,r_{S\setminus i})) \\ &= h_i(t_i,\sigma_S(t_i,r_{S\setminus i})) + \widehat{m}_S^i(t_i,r_{S\setminus i}) \end{aligned} \tag{5}$$

where the inequality is due to (4) w.r.t. the type vector $(t_i, r_{S\setminus i})$.

Hence, the mechanism $(\sigma_S, \widehat{m}_S)$ is I.C. given $\sigma_{N\setminus S}$, but not yet balanced. Let $\overline{m}_S^i(r_i) = E[\widehat{m}_S^i(r_i, \widetilde{t}_{S\setminus i})]$. By taking expectations in (5) we conclude that $(\sigma_S, \overline{m}_S)$ is I.C. given $\sigma_{N\setminus S}$.

Finally, let $m_S^i(r_S) = \overline{m}_S^i(r_i) - \dfrac{1}{|S|-1}\displaystyle\sum_{j\in S\setminus i}\overline{m}_S^j(r_j)$. Then (σ_S, m_S) is I.C. given $\sigma_{N\setminus S}$ and $\sum_{i\in S} m_S^i(r_S) = 0$ for every $r_S \in T_S$. ■

A direct consequence of this proposition is that, under our assumptions, every coalitional equilibrium can be made I.C. More precisely, let P be a partition of N and σ be a coalitional equilibrium relative to P; for every $S \in P$, there exists m_S such that (σ_S, m_S) is I.C. given $(\sigma_K)_{K\in P, K\neq S}$ and

$$v_\sigma(S;P) = E\left[\sum_{i\in S}\left(u_i(\widetilde{t}_i, \sigma_S(\widetilde{t}_S), (\sigma_K(\widetilde{t}_K))_{K\in P, K\neq S}) + m_S^i(\widetilde{t}_S)\right)\right]$$

2.3 Core-stability of a (single) ring

Let us denote as $\mathcal{P}(K)$ the set of all partitions of K, for $K \subseteq N$. Let $R \subseteq N$; from $v_\sigma(S;P)$, we derive the following characteristic function over R

$$w_\sigma^R(S) = \min_{\Pi\in\mathcal{P}(R\setminus S)} v_\sigma(S; \left\{S, \Pi, \{k\}_{k\in N\setminus R}\right\})$$

In particular, for the grand coalition N,

$$w_\sigma^N(S) = \min_{\Pi\in\mathcal{P}(N\setminus S)} v_\sigma(S;\{S,\Pi\}) \tag{6}$$

We say that R is *core-stable* (w.r.t. σ) iff the (standard) core of w_σ^R, $C(w_\sigma^R)$, is not empty. The interpretation is the following:

- The coalitional equilibrium mapping σ is given.

- The ring R considers to form; the players outside R are supposed to act individually. R proposes to every $i \in R$ a share x_i of the total expected payoff $w_\sigma^R(R) = v_\sigma(R; \left\{R, \{k\}_{k \in N \setminus R}\right\})$, to be achieved by means of an I.C. mechanism $\mu_R = (\sigma_R, m_R)$.

- *Every* subcoalition S of R considers non-participation; if S does not participate, the players outside R remain singletons, the players in $R \setminus S$ partition themselves as they wish. Hence S can guarantee the total expected payoff $w_\sigma^R(S)$ to its members.

- If the participation constraint of every $S \subseteq R$ is satisfied, R forms; every player observes his type; R implements μ_R.

Basic properties

- Every singleton $\{k\}$, $k \in N$, is core-stable.

- Recalling (2), for every σ, $w_\sigma^N(N) = v(N)$; by (3) and (6), w_σ^N is grand coalition superadditive (N is efficient in w_σ^N). This property does not necessarily hold for w_σ^R, $R \subsetneq N$ (see example 2 in section 3.3).

- $C(w_\sigma^R)$ corresponds to cautious expectations of the subcoalitions of R. In particular, $C(w_\sigma^N)$ contains the usual variants of the core of the partition form game v_σ (see Hafalir (2007)). For instance, the core with singleton expectations, or $s-$core, of v_σ, denoted as $C_s(v_\sigma)$, is defined as the standard core $C(f_\sigma^s)$ of the characteristic function

$$f_\sigma^s(S) = v_\sigma\left(S; \left\{S, \{j\}_{j \in N \setminus S}\right\}\right) \tag{7}$$

  Similarly, the core with merging expectations, or $m-$core, of v_σ, $C_m(v_\sigma)$, is defined as $C_m(v_\sigma) = C(f_\sigma^m)$, where

$$f_\sigma^m(S) = v_\sigma(S; \{S, N \setminus S\}) \tag{8}$$

  It readily follows from the definitions that $C(f_\sigma^s)$ and $C(f_\sigma^m)$ are subsets of $C(w_\sigma^N)$. Unlike w_σ^N, the characteristic functions f_σ^s and f_σ^m are not necessarily grand coalition superadditive (see example 1 in section 3.3).

- w_σ^N can be defined in terms of the conjecture of every coalition S on the partition to be formed by the players of $N \setminus S$ if S secedes from the grand coalition N. For every coalition S, let $B(S)$ be a partition of N which contains S as a cell. Given a partition form game v, let $f^B(S) = v(S; B(S))$. The B-core of v is defined as the core of the characteristic function game f^B. The $s-$core and the $m-$core correspond respectively to $B(S) = \{S; \{j\}, j \in N \setminus S\ \}$ and $B(S) = \{S, N \setminus S\}$. The grand coalition N is then core-stable (w.r.t. σ) if, for some specification of the conjecture $B(S)$ of every coalition S, the $B-$core of v_σ is not empty.

- If Γ is a game with complete information, let

$$v_\alpha(S) = \max_{a_S \in A_S} \min_{a_{N\setminus S} \in A_{N \setminus S}} \left[\sum_{i \in S} u_i(a_S, a_{N\setminus S}) \right] \tag{9}$$

In particular, $v_\alpha(N) = v(N)$. The $\alpha-$core of Γ is defined as $C(v_\alpha)$ (see Aumann (1961)). It is easily checked that, for every σ and every $S \subsetneq N$, $w_\sigma^N(S) \geq v_\alpha(S)$. Hence, $C(w_\sigma^N) \subseteq C(v_\alpha)$.[6] The extension of the definition of the $\alpha-$core to incomplete information may be delicate in the presence of incentive constraints. In particular, our previous construction of transfers, which made any coalitional equilibrium incentive compatible (see proposition 1), cannot be used for the maxmin, since the latter solution concept requires that coalition S considers *any* possible strategy of coalition $N \setminus S$. However, in the framework of standard auctions, the difficulties disappear. Indeed, every coalition S guarantees itself a total expected payoff of 0, whatever the mechanism adopted by $N \setminus S$, by having all its members bidding 0 independently of their types, a strategy that is clearly I.C. for S. Furthermore, S cannot guarantee more than 0, since the members of $N \setminus S$ can all bid the maximal possible amount, which is I.C. for $N \setminus S$. Hence, the $\alpha-$core is well-defined and not empty in standard auctions. But the usual objection against maxmin strategies applies: why should S fear costly overbidding from $N \setminus S$?

[6] Hafalir (2007) focuses on abstract partition form games, which are not necessarily generated by a strategic form game. Hence he does not distinguish the core with cautious expectations from the $\alpha-$core. In our framework, at least under complete information, Aumann (1961)'s original definition of the $\alpha-$core can be used.

3 Applications

In this section, we apply our solution concept, in which coalitions play best replies to each other, to auctions with independent private values. In the first two subsections, we consider standard auctions, that is, without direct externalities. We check the core-stability of coalitions in several specific auction models which have been proposed in the literature. In subsections 3.1 and 3.2, we illustrate that, in absence of direct externalities, coalitions are core-stable. In subsection 3.3, we allow for direct negative externalities and show that the grand coalition is still core-stable in this case. However, the $s-$core and the $m-$core of the underlying partition form game can be empty (example 1) and small coalitions may not be core-stable (example 2). Finally, if externalities are possibly positive, the $\alpha-$core may be empty (example 3).

3.1 Standard second price auctions

Let player i's type $\widetilde{t}_i$ be a continuous random variable over $\left[\underline{t}_i, \overline{t}_i\right]$, $0 \leq \underline{t}_i \leq \overline{t}_i$, to be interpreted as his valuation for a single object. $A_i = [0, M]$ is the set of possible bids, where $M \geq \max_{i \in N} \overline{t}_i$. Let $a = (a_k)_{k \in N}$ be an $n-$tuple of bids. A second price auction is defined by the following utility functions

$$\begin{aligned} u_i(t_i, a) &= t_i - \max_{j \neq i} a_j \quad \text{if } a_i > \max_{j \neq i} a_j \\ &= \frac{1}{\eta(a)}(t_i - a_i) \quad \text{if } a_i = \max_{j \neq i} a_j \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $\eta(a) = \left| \{k \in N : a_k = \max_{j \in N} a_j\} \right|$.

As is well-known, this game has an equilibrium in weakly dominant strategies. More generally, let P be a partition of N. The auxiliary Bayesian game $\Gamma(P)$ has a coalitional equilibrium in weakly dominant strategies described by $\sigma_S^k(t_S) = t_k$ for some $k \in S$ such that $t_k = \max_{j \in S} t_j$ and $\sigma_S^i(t_S) = 0$ for $i \in S$, $i \neq k$, for every $S \in P$ and $t_S = (t_j)_{j \in S}$. It is easily checked that for every P and $S \in P$,

$$v_\sigma(S; P) = v_\sigma(S; \{S, N \setminus S\}) = E\left[\left(\max_{i \in S} \widetilde{t}_i - \max_{j \in N \setminus S} \widetilde{t}_j\right)^+\right] \equiv \varphi(S)$$

where $f^+ = \max\{f, 0\}$. The previous expression shows that, at the equilibrium in weakly dominant strategies, the external effects disappear, so that v_σ reduces to a plain characteristic function. In particular, for every $S \subseteq R \subseteq N$ and every $\Pi \in \mathcal{P}(R \setminus S)$, $v_\sigma(S; \left\{S, \Pi, \{k\}_{k \in N \setminus R}\right\}) = \varphi(S)$ and a ring R is core-stable iff $C(\varphi|_R)$ is not empty, where $\varphi|_R(S) = \varphi(S)$ for every $S \subseteq R$.

Proposition 2 (Mailath and Zemsky (1991), Barbar and Forges (2007)) *In a standard second price auction, all rings are core-stable.*

Proof: Mailath and Zemsky (1991) establish that φ is balanced. Barbar and Forges (2007) further show that φ is supermodular (convex). If the bidders are symmetric, namely if the types $\widetilde{t}_i$, $i = 1, ..., n$, are i.i.d., an easy direct argument shows that giving the same amount $\frac{\varphi(N)}{|N|}$ to every member of N defines a payoff $n-$tuple in $C(\varphi)$: first, I denoting the indicator function

$$\varphi(S) \leq E\left[\left(\max_{i \in S} \widetilde{t}_i\right) I\left[\max_{i \in S} \widetilde{t}_i \geq \max_{j \in N \setminus S} \widetilde{t}_j\right]\right] \tag{10}$$

Further, it is easily checked that

$$\begin{aligned} & P\left(\left\{\max_{i \in S} \widetilde{t}_i \leq t\right\} \cap \left\{\max_{i \in S} \widetilde{t}_i \geq \max_{j \in N \setminus S} \widetilde{t}_j\right\}\right) \\ = \; & P\left(\left\{\max_{i \in N} \widetilde{t}_i \leq t\right\} \cap \left\{\max_{i \in S} \widetilde{t}_i \geq \max_{j \in N \setminus S} \widetilde{t}_j\right\}\right) \\ = \; & F^n(t)\frac{|S|}{|N|} \end{aligned}$$

where F is the distribution function of any $\widetilde{t}_i$. It follows then from (10) that $\varphi(S) \leq \varphi(N)\frac{|S|}{|N|}$. ■

3.2 Standard first price auctions

In this subsection, we assume that the n initial bidders are symmetric, namely that the valuations $\widetilde{t}_i$, $i = 1, ..., n$ are i.i.d. Let $a = (a_k)_{k \in N}$ be an $n-$tuple of bids. A first price auction is defined by the following utility functions

$$\begin{aligned} u_i(t_i, a) &= t_i - a_i \quad \text{if } a_i > \max_{j \neq i} a_j \\ &= \frac{1}{\eta(a)}(t_i - a_i) \quad \text{if } a_i = \max_{j \neq i} a_j \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $\eta(a)$ is defined as for the second price auction.

Obviously, given a nontrivial partition P of N, the players of the auxiliary Bayesian game $\Gamma(P)$ are not symmetric. By Lebrun (1999), $\Gamma(P)$ has a unique equilibrium, for every partition P. In other words, there exists a unique coalitional equilibrium mapping σ. However, no general analytical solution is available.

Waehrer (1999, proposition 2) shows that for every partition P and every coalitions R, $S \in P$ such that $|R| \leq |S|$

$$\frac{v_\sigma(S;P)}{|S|} \leq \frac{v_\sigma(R;P)}{|R|} \tag{11}$$

In words, at a first price auction, the per capita expected payoff of a cartel's member is greater in small cartels[7]. This result enables us to deduce the following

Proposition 3 *In a standard first price auction with symmetric bidders, the grand coalition is core-stable.*

Proof: We will show that the vector payoff allocating the amount $\frac{v(N)}{|N|}$ to every member of N is in the $s-$core of the underlying partition game v_σ. Let $S \subsetneq N$ and $P = \left\{S, \{k\}_{k \in N \setminus S}\right\}$. Recalling the definition of the $s-$core (see (7)), we have to show that

$$\frac{v(N)}{|N|} \geq \frac{v_\sigma(S;P)}{|S|} \tag{12}$$

From (11), we deduce that for every $j \in N \setminus S$,

$$v_\sigma(\{j\};P) \geq \frac{v_\sigma(S;P)}{|S|}$$

while, from the grand coalition superadditivity of v_σ (recall (3)),

$$v(N) \geq v_\sigma(S;P) + \sum_{j \in N \setminus S} v_\sigma(\{j\};P)$$

The latter two inequalities yield (12). ■

[7] Waehrer (1999) also shows that for second price auctions, the inequality goes the other way round.

The previous reasoning can be applied to establish the stability of a bidding ring $R \subsetneq N$ if v_σ is superadditive on R. Such a property indeed holds in examples proposed by McAfee and McMillan (1992) and Marshall et al. (1994).

McAfee and McMillan (1992) assume that $\widetilde{t}_i \in \{0,1\}$, $i = 1, ..., n$. They show (in inequality (13)) that, for every coalition $S \subsetneq N$ and $j \in N \setminus S$,

$$\begin{aligned} & v_\sigma \left(S \cup \{j\} ; \left\{ S \cup \{j\} , \{k\}_{k \in N \setminus (S \cup \{j\})} \right\} \right) \\ \geq \quad & v_\sigma \left(S; \left\{ S, \{k\}_{k \in N \setminus S} \right\} \right) + v_\sigma \left(\{j\} ; \left\{ S, \{k\}_{k \in N \setminus S} \right\} \right) \end{aligned}$$

or, equivalently, recalling our notation f_σ^s (see (7))

$$f_\sigma^s \left(S \cup \{j\} \right) \geq f_\sigma^s(S) + v_\sigma \left(\{j\} ; \left\{ S, \{k\}_{k \in N \setminus S} \right\} \right)$$

One can also check that (11) holds in their framework so that

$$v_\sigma \left(\{j\} ; \left\{ S, \{k\}_{k \in N \setminus S} \right\} \right) \geq \frac{f_\sigma^s(S)}{|S|}$$

Hence

$$\frac{f_\sigma^s(S \cup \{j\})}{|S| + 1} \geq \frac{f_\sigma^s(S)}{|S|}$$

and, by induction, for every coalitions R, S such that $S \subseteq R$,

$$\frac{f_\sigma^s(R)}{|R|} \geq \frac{f_\sigma^s(S)}{|S|} \tag{13}$$

Since $f_\sigma^s(S) \geq w_\sigma^R(S)$ for $S \subseteq R$, with equality if $S = R$, the latter inequality implies that every ring R is core-stable in McAfee and McMillan (1992)'s example.

Marshall et al. (1994) compute f_σ^s by numerical methods in the case of five initial bidders uniformly distributed over $[0,1]$. Their table III shows that $\frac{f_\sigma^s(S)}{|S|}$ is increasing with the size of S (i.e., (13) holds) so that, in their example too, all rings are core-stable.

3.3 First price auction with complete information and direct externalities

Jehiel and Moldovanu (1996) (henceforth, JM) introduced, and extensively studied, first price auctions in which every bidder suffers an externality if a competitor acquires the object. In order to concentrate on the externality effects, they assume complete information. The basic game reduces to $\Gamma \equiv \left[N, \{A_i, u_i\}_{i\in N}\right]$, which describes a first price auction between the agents in N. $A_i = \{0, \epsilon, 2\epsilon, ...\}$ is the set of possible bids[8]. The utility functions are described by an $n \times n$ matrix $E = [e_{ij}]$; for every i, $e_{ii} \equiv t_i$ is agent i's utility for the object and for every $i \neq j$, e_{ij} is the externality incurred by agent j if agent i gets the object. If all bids are 0, the seller keeps the object; agent i's utility is normalized to 0 in this case. Let $a = (a_k)_{k\in N}$; the utility of player i is

$$\begin{aligned} u_i(a) &= t_i - a_i \quad \text{if } a_i > \left[\max_{j\neq i} a_j\right]^+ \\ &= e_{ji} \quad \text{if } a_j > \left[\max_{k\neq j} a_k\right]^+ \text{ for some } j \neq i \\ &= 0 \quad \text{if } a = 0 \end{aligned}$$

To complete this description, we assume that if several players make the highest bid, they all get the object with the same probability.

Recalling (2), we have here

$$v(N) = \max_{a\in A}\left[\sum_{i\in N} u_i(a)\right] = \left[\max_{i\in N}\left\{t_i + \sum_{j\neq i} e_{ij}\right\} - \epsilon\right]^+$$

Since Γ is a game with complete information, the $\alpha-$characteristic function v_α is defined by (9).

JM mostly consider negative externalities, i.e., $e_{ij} \leq 0$ for every $i \neq j$. We shall keep that assumption throughout the section, except in example 3. If externalities are negative, given any strategy profile $(a_i)_{i\in S}$ of S, $N \setminus S$ can inflict a negative payoff on S by bidding over $\max_{i\in S} a_i$; hence $v_\alpha(S) \leq 0$ for $S \subsetneq N$; since $v(N) \geq 0$, the $\alpha-$core $C(v_\alpha)$ is not empty.

JM consider the following strategy profile $(b_j)_{j\in N}$: if $t_i - \min_j e_{ji} \leq 0$ for every $i = 1, ..., n$, then $b_i = 0$ for every i.[9] Otherwise, let (i, k) be a pair of bidders $i \neq k$ such that $t_i - e_{ki}$ is maximal over all $t_j - e_{lj}$, $j \neq l$ (that is, bidder i is willing to pay the highest price for the object, given

[8] As in JM, we assume that there is a smallest money unit $\epsilon > 0$, in order to guarantee the existence of equilibria.

[9] This particular case can be discarded by assumption in the original game Γ ($t_i > 0$ for every i) but can occur in the game $\Gamma(P)$ between cartels.

his valuation and the externalities he might suffer); take $b_i = t_i - e_{ki} - \epsilon$, $b_k = t_i - e_{ki} - 2\epsilon$ and $b_j < b_k$, $j \neq i, k$. JM's proposition 1 states that, under appropriate genericity conditions, these strategies form an equilibrium in Γ. We refer to it as to the JM-equilibrium (see Biran (2009), Appendix A, for a full characterization of equilibria)[10]. At the JM-equilibrium, bidder i's payoff is $e_{ki} + \epsilon \leq 0$ and all other bidders $j \neq i$ get $e_{ij} \leq 0$. This enables us to proceed as above for the $\alpha-$core to show that the grand coalition is core-stable w.r.t. the JM-equilibrium.

Proposition 4 *In a first price auction with complete information and direct negative externalities, the grand coalition N is core-stable w.r.t. the coalitional equilibrium mapping σ defined by the JM-equilibrium, namely, $C(w_\sigma^N) \neq \emptyset$.*

Proof: $w_\sigma^N(N) = v(N) \geq 0$. We shall check that for every $S \subsetneq N$, $w_\sigma^N(S) \leq 0$. Let $P \in \mathcal{P}(N)$ be a nontrivial partition of N. Let $S \in P$ and let $r_S \in S$ be an efficient agent in S, i.e., $r_S = \arg\max_{i \in S}(t_i + \sum_{j \in S, j \neq i} e_{ij})$. Given strategies of the other cartels in P, S cannot do better than choosing a bid profile $(a_i)_{i \in S}$ with $a_i = 0$ for $i \neq r_S$ (r_S represents S at the auction). Hence, $\Gamma(P)$, the auction between the cartels, has the same structure as the original game Γ, with valuations $t_S = \max_{i \in S}(t_i + \sum_{j \in S, j \neq i} e_{ij})$ and externalities $e_{SK} = \sum_{j \in K} e_{r_S j} \leq 0$. The JM-equilibrium applies to this game and yields negative payoffs to all players. It follows that $v_\sigma(S; P) \leq 0$ for every S, P and thus $w_\sigma^N(S) \leq 0$. ■

From the previous proposition, we recover that the $\alpha-$core $C(v_\alpha)$ is not empty. This result is quite different from JM's proposition 6, which establishes that the α-core of a flexible market game is in general empty. In JM's market game, all agreements are conceivable, including bribing the seller, for instance. Here, we stick to the rigid original format of the first price auction, so that we do not allow for any collusion between the bidders and the seller. The only possible form of collusion between the potential buyers is to coordinate their bids and to make side-payments to each other.

The proof of the previous proposition shows that, if, the coalitional equilibrium mapping σ is defined by the JM-equilibrium, *all* associated conceivable cores (e.g., the $s-$core and the $m-$core, see subsection 2.3) will be

[10]Let $k \in N$ and $a_k^* = t_k - \min_{l \neq k} e_{lk}$. As pointed out in JM, any bid $a_k \geq a_k^*$ is weakly dominated (by $a_k^* - \epsilon$) for player k. In the JM-equilibrium, the strategy b_k of the second highest bidder satisfies $b_k > a_k^* - 2\epsilon$ and is thus typically dominated.

nonempty. The example below illustrates that this property does not necessarily hold for coalitional equilibrium mappings which may lead to positive payoffs.[11]

Example 1: $n = 4$; the matrix of valuations/externalities is

$$E = \begin{pmatrix} t_1 & -2 & -2 & -2 \\ 0 & 1 & 0 & 0 \\ -8 & -8 & 1 & -8 \\ -7 & -7 & -7 & 1 \end{pmatrix}$$

Let us start with $t_1 = 8$. $v(N) = 2 - \epsilon$. Assume first that the bidders act individually. Then the following strategies form an equilibrium: $a_1 = a_4 = 8 - 2\epsilon$, $a_2 = 8$, $a_3 = 8 - \epsilon$. Bidder 2 wins the auction and the payoffs are $(0, -7, 0, 0)$. Hence,

$$v_\sigma\left(\{i\}\,; \{\{1\}, \{2\}, \{3\}, \{4\}\}\right) = 0,\ i = 3, 4 \tag{14}$$

Assume next that the first two bidders collude, while the two others remain singletons. The relevant matrix becomes

$$\begin{pmatrix} 6 & -2 & -2 \\ -16 & 1 & -8 \\ -14 & -7 & 1 \end{pmatrix}$$

The following strategies now form an equilibrium: $a_1 = 3$, $a_2 = 0$, $a_3 = 3 - \epsilon, a_4 = 3 - 2\epsilon$. Coalition $\{1, 2\}$ gets a payoff of 3 so that

$$v_\sigma\left(\{1, 2\}\,; \{\{1, 2\}, \{3\}, \{4\}\}\right) = 3 \tag{15}$$

(14) and (15) imply that the characteristic function f^s_σ is not grand coalition superadditive, hence that the $s-$core $C_s(v_\sigma)$ is empty in that example.

Let us take $t_1 = 4$. We now have $v(N) = 1 - \epsilon$. Let us assume that bidder 3 competes with the cartel $\{1, 2, 4\}$. The matrix is

$$\begin{pmatrix} 1 & 0 \\ -24 & 1 \end{pmatrix}$$

[11] The features of the next examples depend crucially on the direct externalities. In a first price auction with complete information and no externalities, there exists a coalitional equilibrium mapping σ in which the outcome (namely, the winner and the price) is as in the equilibrium in undominated strategies of the second price auction. For that σ, the $s-$core and the $m-$core of v_σ are not empty. Furthermore, every bidding ring is core-stable w.r.t. σ.

where the first row corresponds to the utilities in case the cartel obtains the object. The strategies $a_1 = 0$, $a_2 = 1$, $a_3 = 1 - \epsilon$, $a_4 = 0$ form an equilibrium. Hence

$$v_\sigma\left(\{3\}\,;\{\{3\}\,,\{1,2,4\}\}\right) = 0 \tag{16}$$

and similarly for bidder 4. Let us assume again that the first two bidders collude, but facing the opposite ring $\{3,4\}$. The relevant matrix is now

$$\begin{pmatrix} 2 & -4 \\ -14 & -6 \end{pmatrix}$$

The strategies $a_1 = \epsilon$, $a_2 = a_3 = a_4 = 0$ are in equilibrium, so that

$$v_\sigma\left(\{1,2\}\,;\{\{1,2\}\,,\{3,4\}\}\right) = 2 - \epsilon \tag{17}$$

(16), the analog of (16) for bidder 4 and (17) imply that the characteristic function f_σ^m is not grand coalition superadditive, hence that the $m-$core $C_m(v_\sigma)$ is empty in that example. ■

One of the motivations of JM for studying cooperative agreements was to understand why two European firms did not cooperate in a procurement auction opposing them to an Asian competitor. JM suggest that negative externalities might explain the failure of the natural partners' association. However, as explained above, the emptiness of the $\alpha-$core that they consider only shows that no stable agreement can be found between the three potential buyers and the seller. In this particular example, cooperation between the European firms and the Asian one looked unlikely, but the stability of the European coalition could be considered. This kind of stability is captured by our concept of core-stability, which applies to any cartel. We illustrate below that, in the presence of externalities, a two firm cartel may not be stable.

Example 2: $n = 3$; the matrix of valuations/externalities is

$$E = \begin{pmatrix} 5 & -4 & -3 \\ -4 & 6 & -9 \\ -10 & -1 & 3 \end{pmatrix}$$

If a first price auction takes place between the 3 agents, in every equilibrium, agent 1 wins and agent 3 is the second highest bidder; in undominated strategies, $10 \leq p \leq 12$; at the lowest price $p = 10$, the utilities are $(-5, -4, -3)$. Provided that $p < 11$, bidders 1 and 2 get a total utility

> -10. If they form a joint venture, in every equilibrium, agent 2 represents $R = \{1, 2\}$ at the auction and wins; in undominated strategies, $p = 12$: the price *raises* when agent 1 and agent 2 do not compete. The total utility of $\{1, 2\}$ is -10, which is less than the sum of agents 1 and 2's individual payoffs (in our previous notation, $w^R_\sigma(\{1\}) = -5$, $w^R_\sigma(\{2\}) = -4$, $w^R_\sigma(\{1, 2\}) = -10$). The interpretation is the following: if agents 1 and 2 get together, they cannot expect more than -10; if agent 3 plays a dominated strategy, they will even get less. If agent 1 breaks the agreement, he does not expect that agents 2 and 3 (like a European firm and the Asian firm above) will collude, but considers a noncooperative equilibrium between the three competitors. At an equilibrium leading to the lowest price, he can expect -5. Similarly, agent 2 can expect -4. ■

In this example, we did not rely on the JM-equilibrium. However, it can be shown that the grand coalition is core-stable[12]. More generally, we show in the appendix that if $n \leq 3$, the grand coalition is core-stable w.r.t. every coalitional equilibrium mapping, even if externalities can be positive. In the previous example, with negative externalities, the grand coalition can decide not to participate in the auction so as to guarantee 0 to its members, a strategy that is not feasible for small coalitions. Of course, that the grand coalition would be stable if it could form does not mean that it is viable. In the competition between two European firms and an Asian one, the grand coalition is not to be expected.

We conclude this section by illustrating that, if sufficiently many players face possibly positive externalities, the grand coalition may not be stable. In the next example, with five players, the $\alpha-$core, $C(v_\alpha)$, is empty.

Example 3: $n = 5$; every agent i has two neighbors ($i-1 \bmod 5$, $i+1 \bmod 5$); $t_i = 3$, $e_{ji} = 2$ if agent j is a neighbor of agent i, $e_{ji} = -2$ otherwise.

One computes that $v(N) = 3 - \epsilon$. By symmetry, if $C(v_\alpha) \neq \emptyset$, the payoff vector in which every agent gets $\frac{3-\epsilon}{5}$ must be in $C(v_\alpha)$. Let us consider a coalition of the form $S = \{i, i+1, i+3\}$ where $+$ is $\bmod 5$, i.e., S contains agent i, a neighbor of agent i and a non-neighbor of agent i. S guarantees $\max\{3 - \epsilon, 2\}$ if agent i bids ϵ and the other members of S bid 0; hence,

[12] Focusing on undominated strategies, $\{1, 3\}$ gets the payoff -8 against 2 and $\{2, 3\}$ gets the payoff -7 against 1. All noncooperative equilibria of the original three player auction give negative payoffs to all. Hence, even with a coalitional equilibrium mapping σ involving dominated strategies, the s-core $C_s(v_\sigma)$ is nonempty.

$v_\alpha(S) \geq 2 \geq 3\times \frac{3-\epsilon}{5}$, contradicting $C(v_\alpha) \neq \emptyset$.[13] Hence the grand coalition is not stable in this example. ■

A Appendix

In this section, we focus on three players; the model is the same as in section 3.3. In particular, information is complete; externalities e_{ji}, $j \neq i$, can be positive or negative, but every agent gets more utility if he himself possesses the object than if another agent (including the seller) possesses it. We assume generic parameters. Recall that $f^+ = \max\{f, 0\}$.

Proposition 5 *In every 3-player first price auction with direct externalities such that $t_i > e_{ji}^+$ for every $i, j \neq i$, the grand coalition is core-stable w.r.t. every coalitional mapping σ.*

Proof: Let us fix an arbitrary coalitional mapping σ, namely, for every partition P of $N = \{1, 2, 3\}$, a Nash equilibrium $\sigma(P)$ of the auction game in which the players are the coalitions in P. We will show that the core with singleton expectations $C_s(v_\sigma)$ is not empty, i.e., that $C(f_\sigma^s) \neq \emptyset$, where the characteristic function f_σ^s is defined by (7).

Let us assume w.l.o.g. that player 1 is efficient in N, namely that

$$t_1 + e_{12} + e_{13} \geq \max\{t_2 + e_{21} + e_{23}, t_3 + e_{31} + e_{32}\} \tag{18}$$

Then

$$f_\sigma^s(N) = [t_1 + e_{12} + e_{13} - \epsilon]^+$$

We will consider the modified characteristic function g_σ defined by

$$\begin{aligned} g_\sigma(N) &= t_1 + e_{12} + e_{13} - \epsilon \\ g_\sigma(S) &= f_\sigma^s(S) \quad \text{for every } S \subsetneq N \end{aligned}$$

and show that $C(g_\sigma) \neq \emptyset$. Let us set $x_i = g_\sigma(\{i\})$, $i = 1, 2, 3$. x_i is player i's payoff at the equilibrium $\tau \equiv \sigma(\{\{1\}, \{2\}, \{3\}\})$ induced by σ in the 3-person original auction game. Since $t_i > 0$ for every i, the seller cannot keep the object at τ. If player i gets the object at a positive price p at τ,

[13] Equivalently: $\mathcal{S} = \{\{i, i+1, i+3\}, i = 1, ..., 5\}$ is balanced (with weights $\lambda_S = \frac{1}{3}$) and $\sum_{S\in\mathcal{S}} \lambda_S v_\alpha(S) \geq \frac{10}{3} > 3 - \epsilon$.

$x_i = t_i - p < t_i$; if player $j \neq i$ wins the object at τ, $x_i = e_{ji} < t_i$ by assumption. Hence $x_i < t_i$. Furthermore, $x_2 + x_3 \leq e_{12} + e_{13}$. Indeed, if player 1 wins the object at τ, $x_2 + x_3 = e_{12} + e_{13}$. If, say, player 2 wins the object at τ, the price p must exceed $t_1 - e_{21}$, otherwise player 1 would deviate from τ: $x_2 + x_3 = t_2 - p + e_{23} \leq t_2 - t_1 + e_{21} + e_{23} \leq e_{12} + e_{13}$, where the last inequality follows from (18).

Let us set[14]

$$y = (t_1 - \epsilon, qx_2 + (1-q)t_2, qx_3 + (1-q)t_3)$$

where q is computed so that

$$y_1 + y_2 + y_3 = g_\sigma(N), \text{ i.e., } y_2 + y_3 = e_{12} + e_{13}$$

namely

$$q = \frac{(t_2 + t_3) - (e_{12} + e_{13})}{(t_2 + t_3) - (x_2 + x_3)}$$

From the properties of x_2 and x_3, q is well-defined and $0 < q \leq 1$. We will show that $y \in C(g_\sigma)$. By construction, y is efficient and individually rational. Let S be a 2-player coalition. $g_\sigma(S)$ is the payoff of S at the equilibrium $\zeta_S \equiv \sigma(\{S, N \setminus S\})$ of the 2-player auction game in which S competes against the singleton $N \setminus S$. It is easily checked that, at every equilibrium of an auction game with 2 players, the most efficient one wins the object (see, e.g., JM's proposition 2). Let $S = \{2,3\}$; by (18), player 1 wins the object at $\zeta_{\{2,3\}}$, so that $g_\sigma(\{2,3\}) = e_{12} + e_{13} = y_2 + y_3$. Let $S = \{1,2\}$; if player 3 wins the object at $\zeta_{\{1,2\}}$, $g_\sigma(\{1,2\}) = e_{31} + e_{32} \leq t_1 + e_{12} + e_{13} - t_3 \leq y_1 + y_2$, where the first inequality follows from (18) and the second one from $t_3 \geq y_3 + \epsilon$. If $\{1,2\}$ wins the object at $\zeta_{\{1,2\}}$, let $k = 1$ or 2 be the most efficient player in $\{1,2\}$, i.e., $\max\{t_1 + e_{12}, t_2 + e_{21}\} = t_k + e_{k,k+1}$, where $k+1$ is $\bmod 2$. The price p to be paid by $\{1,2\}$ at $\zeta_{\{1,2\}}$ must exceed $t_3 - e_{k3}$, otherwise player 3 would deviate from $\zeta_{\{1,2\}}$. Hence, $g_\sigma(\{1,2\}) \leq t_k + e_{k,k+1} - t_3 + e_{k3}$ so that $g_\sigma(\{1,2\}) \leq t_1 + e_{12} + e_{13} - t_3$ by (18); the proof is completed as above. $S = \{1,3\}$ is similar. ■

[14] The idea is that the grand coalition, if it forms, allocates the object to the efficient player 1. Then player 2 and player 3 must share $e_{12} + e_{13}$. Transfers are organized between these two players so that they get at least their individually rational level.

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