# Walrasian Analysis via
# Two-Player Games

## Carlos Hervés-Beloso

RGEA. Facultad de Económicas. Universidad de Vigo.

e-mail: cherves@uvigo.es

## Emma Moreno-García

Facultad de Economía y Empresa. Universidad de Salamanca.

e-mail: emmam@usal.es

Corresponding author: Carlos Hervés-Beloso. Facultad de Económicas. Universidad de Vigo. Campus Universitario, E-36310 Vigo, Spain. Phone +34 986812446, Fax +34 986812401, e-mail: cherves@uvigo.es

**Abstract.** We associate to any pure exchange economy a game with only two players, regardless of the number of consumers. In this two-player game, each player represents a different role of the society, which is formed by all the individuals in the economy. Player 1 selects feasible allocations trying to make Pareto improvements. Player 2 chooses an alternative from the wider range of allocations that are feasible in the sense of Aubin. The set of Nash equilibria of our game is non-empty and our main result provides a characterization of Walrasian equilibria allocations as strong Nash equilibria of the associated society game.

2

# 1 Introduction

Game theoretic approaches to economic equilibrium and, in particular, to Walrasian or competitive equilibrium, provide insights into the market mechanism through which trade is conducted.

The Walrasian equilibrium reflects the spirit of "the invisible hand" and of decentralization. However, the power and appeal of this equilibrium concept appears to be far greater than that of mere decentralization. This is reflected in the finding that under the appropriate conditions the Walrasian equilibrium may be regarded either as the solution or as the limit solution for several cooperative and non-cooperative game-theoretic notions of equilibria.

Regarding cooperative notions, a great deal of work has been done by showing the connection of Walrasian equilibria with the core of the economy. Edgeworth (1881) proved, in the case of two agents and two commodities, that the core shrinks to the set of Walrasian equilibria and claimed that his result applies for an arbitrary number of commodities and an arbitrary number of agents. Nearly eighty years after, Shubik (1959) recognized the importance of Edgeworth's contribution and pointed out the relationship between the core and Edgeworth's idea of the contract curve. Debreu and Scarf (1963) provided a rigorous formulation of the Edgeworth's conjecture and showed that the intersection of the cores of the sequence of the replications coincides with the set of Walrasian equilibrium allocations. Further, Aumann (1964) introduced a model of an economy with a continuum of agents and showed the core-Walras equivalence; that is, in the Aumann's model the core exactly coincides with the set of competitive equilibrium allocations. Following these pioneering contributions, the relations between the core and the set of equilibrum allocations have been a major focus of the literature in mathematical economics during the 70s and 80s. Notable contributions on this direction include Arrow-Hahn (1971), Bewley (1973), Hildenbrand (1974), Dierker (1975), Khan (1976), Trockel (1976) Anderson (1978, 1981, 1985) Shubik and Wooders (1982), Hammond, Kaneko and Wooders (1989) and Kaneko and Wooders (1989). These works establish the existence of an equilibrium price system as a result of a theory whose prime concern is with the power of coalitions and makes no mention of prices.

The search for non-cooperative game theoretic foundations of Walrasian equilibrium rests on the Nash equilibrium solution. The seminal paper of Nash

(1950), on the existence of equilibrium points in non cooperative games, was historically critical for Walrasian analysis and founded the genesis for a rapidly growing series of papers on strategic approaches to economic equilibrium. In order to prove existence of Walrasian equilibrium, Debreu (1952), Arrow and Debreu (1954) and Debreu (1962) extended Nash's model to "generalized games" by adding a fictitious price player whose payoff was the value of the excess demand. Walras equilibrium was then obtained as Nash equilibrium of a "pseudo-game" that included this additional price player. Hence, the game theoretical interpretation of the Walrasian equilibrium in the proof of this existence result is not based on a game but merely in a pseudo-game. One would, therefore, hope for an economic or game theoretical model that formulates an exchange process and/or a price-setting process in addition to the consumer behavior in the market. Actually, this is the aim of all the papers on strategic market games, which was initiated by Shubik (1973), Shapley (1976) and Shapley and Shubik (1977) and constitutes now a well known alternative to the Walrasian model. Other attempts to provide strategic foundations of competitive equilibria make use of cooperative game theory. In this direction, Shapley and Shubik (1969) showed that the class of market games and the class of totally balanced games are the same and Wooders (1994) proved an equivalence between large games with effective small groups of players and games generated by markets.

Most of the literature on market games deals with the basic problem of redistribution of goods in the framework of an exchange economy and models this procedure by describing explicitly the behavior of the agents and the corresponding process of formation of prices and exchange of goods. There is a first mechanism (game form), where the agents are the players and their strategies are signals (in terms of money and/or commodities to buy or sell on each trading post), which specify as outcome a new allocation of the quantities announced. Prices appear at this stage as an interim technical device. Once endowments and utilities are added to the model, one can describe the set of feasible strategies for each player and evaluate the outcome in terms of utilities. In this way, one faces a strategic game.

Strategic market games may be classified into different categories depending basically on the underlying strategy sets for players and on the way in which every agent's signal is used to determine market prices. In any case, these market games can be viewed as extensions of the single market analysis of Cournot (1838) and Bertrand (1883) to multiple markets within a general equilibrium framework.

The extension of the Cournot tradition to general equilibrium was pioneered by the already cited works by Shubik (1973), Shapley (1976) and Shapley and Shubik (1977). In order to overcome the difficulty that an agent might want to sell in one market and buy in another, Shapley and Shubik explicitly introduced money as the stipulated medium of exchange. Their model was carried forward by several other authors, who showed that the Cournot-Nash equilibria converge to Walrasian equilibria (see, for instance, Dubey and Shapley 1994 and Dubey and Geanakoplos 2003).

Hurwicz (1979), Schmeidler (1980) and Dubey (1982) followed the Bertrand tradition, which naturally led to discontinuous payoff functions, and established the exact coincidence of Nash and Walrasian equilibria, relying on the existence of Walrasian equilibrium of the economy to show the existence of the Nash equilibrium of the game.

This paper adds to the broad range of literature on strategic approaches to Walrasian equilibrium. Our aim is to show a characterization of Walrasian equilibria within a strategic game approach which differs from those contemplated in the previous papers. Actually, the game we consider is neither a generalized game nor a game in the tradition of Cournot or Bertrand, but a two-player game played by the society, representing all the agents in the economy. Furthermore, money is not considered in our game and prices are not involved either in the strategy sets or the payoff functions defining the game. Another important difference to be noted is that we do not need to define outcome functions from the strategy profiles. In fact, in our society game the outcomes are given by the strategies.

Given any pure exchange economy with a finite number of agents, we define an associated game with only two players. We refer to this game as the society game because it can be interpreted as a game in which the society plays two different roles. The first one consists in acting as a Paretian player in pursuit of efficiency. The second role involves an impartial and fair behavior against the Paretian player.

The Paretian player, player 1, selects feasible allocations and tries to make Pareto improvements. On the other hand, the society, acting as player 2, chooses strategies among the wider range of the feasible allocations in the sense of Aubin, that is, allocations that are feasible by considering strictly positive participation

or weights of each member of the society.

A strategy profile in this society game is given by a feasible allocation chosen by player 1, and by the weights and the Aubin allocation chosen by player 2. The payoff function for the Paretian player depends on her strategy and on the Aubin allocation proposed by player 2, but it is not directly affected by the selected weights, which only appear explicitly in the second player's payoff function. By definition, the payoffs for both players can not be strictly positive simultaneously. Moreover, the player 2 can always get a null payoff by choosing the same allocation as player 1 and her payoff can be strictly positive only in the case in which the Paretian player's strategy is not a Walrasian allocation.

The assumption stated for our exchange economy leads to the existence of Walrasian equlibria. It is not difficult to show that the strategy in which both players play the same Walrasian allocation is a Nash equilibrium for the society game. As the Paretian player can increase her payoff when her strategy is not an efficient allocation, at any Nash equilibrium the strategy selected by player 1 is required to be Pareto optimal. Hence, the first player represents efficiency whereas the second one is a weighting player who tries to give balance. Actually, we will show that at any Nash outcome both payoffs are zero, Paretian player achieves efficiency and the Aubin player restores Walrasian equilibrium allocations.

We recall that all the previous works on market games model the economy as a game where each consumer is a player that announces quantities or both quantities and prices and, therefore, the economy is recast as a game with at least as many players as consumers. However, in this paper the game we associate to a pure exchange economy is just a two-player game, regardless of the number of consumers in the economy.

Our main result provides an equivalence between the Walrasian allocations and the set of Nash equilibria of the associated society game. We obtain a characterization of Walrasian equilibria of the n-agent economy in terms of Nash equilibria of a game with only two players. In other words, we show that the Walrasian mechanism is implementable as a Nash equilibrium of a two-player game.

In order to obtain our results, the key idea is to exploit the veto power of the society. In Hervés-Beloso et al. (2005a, 2005b), we provide characterizations of Walrasian allocations in terms of the blocking power of the "society"

6

called there the "grand coalition". Precisely, in Hervés-Beloso et al. (2005b), it is shown that the set of Walrasian allocations coincides with the set of allocations which are not blocked, in the sense of Aubin, by the society. Therefore, in order to obtain the equilibria it suffices to consider the Aubin blocking power of just one coalition, namely, the society formed by all the individuals in the economy. This equivalence between the set of Walrasian allocations and the set of Aubin non dominated allocations by the society, which is already stated for the more general case of differential information economies, will be used in order to show our main result. The proof of that characterization rests on a Core-Walras equivalence theorem and on the blocking power of large coalitions in continuum economies (Vind 1972). That is, we apply results which connect Walrasian equlibria to subtle cooperative solutions. Then, although in this article we follow a non-cooperative approach to Walrasian analysis, the underlying arguments are somehow related to a cooperative approach.

To sum up, this paper is related to the literature on non-cooperative market games and the main result shows a coincidence between Walrasian equilibria and Nash equilibria of this society game. On the other hand, it is also related to a cooperative approach to equilibria because we are using, indirectly, core equivalence results. Moreover, it is easy to show that Nash equilibria of the society game exist, and that they are strong Nash equilibria. Therefore, we can conclude that the Walrasian mechanism is implementable as a strong Nash equilibrium of a game with two players which represent the society. Finally, we remark that, for simplicity, the model we consider in this paper addresses a complete information pure exchange economy with a finite number of consumers and a finite number of commodities. However, the same results hold for more general settings. More precisely, the equivalence between Walrasian equilibria and Nash equilibria of this society game does still hold for economies with infinitely many commodities and for differential information economies. This is so because the key result is the theorem of Hervés-Beloso et al. (2005b), which was actually proved for those more general models.

The remainder of the paper is organized as follows. Section 2 states the model of a finite exchange economy and recalls the solution concepts and the already mentioned results that will be used in the proof of our theorem. In Section 3, we define the society game associated to a pure exchange economy and discuss the definitions of strategy sets and payoff functions that describe the game. Section

4 includes our main result and some previous lemmas. In Section 5 we modify the game in order to include extreme situations not contemplated in the previous section. Finally, Section 6 is the conclusion.

## 2   The Economy

Consider a pure exchange economy $\mathcal{E}$ with $n$ consumers and $\ell$ commodities. The commodity space is $\mathbb{R}^\ell$. Each consumer $i$ is characterized by the consumption set $\mathbb{R}^\ell_+$, a preference relation $\succeq_i$ on $\mathbb{R}^\ell_+$ and an initial endowment $\omega_i = (\omega_{ij})_{j=1}^\ell \in \mathbb{R}^\ell_+$.

We state the following assumptions on endowments and preference relations for every consumer $i$:

(H.1) The initial endowment, $\omega_i$, is strictly positive, i.e., $\omega_i \gg 0$.

(H.2) The preference relation, $\succeq_i$, is continuous, strictly monotone and convex.

Note that the assumption (H.2) means that each preference relation, $\succeq_i$, is represented by a continuous, quasi-concave and strictly increasing utility function $U_i : \mathbb{R}^\ell_+ \to \mathbb{R}_+$, with $U_i(0) = 0$. So, the economy is defined by $\mathcal{E} \equiv (\mathbb{R}^\ell_+, (U_i, \omega_i)_{i=1}^n)$.

An allocation $x$ is a consumption bundle $x_i \in \mathbb{R}^\ell_+$ for every agent $i = 1, \ldots, n$. The allocation $x$ is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i$. A price system is an element of $\Delta$, where $\Delta$ denotes the $(\ell - 1)$-dimensional simplex of $\mathbb{R}^\ell_+$, that is, $\Delta = \{p \in \mathbb{R}^\ell_+ \text{ such that } \sum_{h=1}^\ell p_h = 1\}$. A Walrasian equilibrium for the economy $\mathcal{E}$ is a pair $(p, x) \in \Delta \times \mathbb{R}^{\ell n}_+$, where $p$ is a price system and $x$ is a feasible allocation such that, for every agent $i$, the bundle $x_i$ maximizes the preference relation $\succeq_i$ (or equivalently, the utility function $U_i$) in the budget set $B_i(p) = \{y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leq p \cdot \omega_i\}$.

A feasible allocation $x$ is blocked by a coalition $S$ if there exists $y_i, i \in S$, such that $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$ and $U_i(y_i) > U_i(x_i)$ for every member $i$ in the coalition $S$. The core of the economy is the set of feasible allocations which are not blocked by any coalition of agents. It is well known that under the hypothesis H.1 and H.2 the economy $\mathcal{E}$ has Walrasian equilibrium, and that if $x$ is a Walrasian allocation for the economy $\mathcal{E}$, then $x$ belongs to the core of $\mathcal{E}$.

Aubin (1979), addressing pure exchange economies with a finite number of agents and commodities, introduced the pondered veto concept. The veto system proposed by Aubin extends the notion of ordinary veto in the sense that allows the agents to participate with a fraction of their endowments when forming a coalition. This veto mechanism is known in the literature as *fuzzy veto*. The term *fuzzy* is usually used in relation to the elements that belong to a set with certain probability. In the veto system introduced by Aubin, agents actually participate in a coalition with a fraction of their endowments (which, under standard assumptions, is equivalent to the classical Debreu-Scarf participation of a coalition in a replicated economy - see Florenzano (1990)). Therefore, we prefer to designate this veto system as *Aubin veto* or *veto in the sense of Aubin*.

Following Aubin (1979), we define the Aubin veto as follows: an allocation $x$ is blocked in the sense of Aubin by the coalition $S$ via the allocation $y$ if there exist $\alpha_i \in (0,1]$, for each $i \in S$, such that (i) $\sum_{i \in S} \alpha_i y_i \leq \sum_{i \in S} \alpha_i \omega_i$, and (ii) $U_i(y_i) > U_i(x_i)$, for every $i \in S$.

The Aubin core of the economy $\mathcal{E}$ is the set of all feasible allocations which cannot be blocked in the sense of Aubin.

This definition of Aubin core is equivalent to the original one, Aubin (1979). However, it is important to remark that we require the coefficients $\alpha_i$ to be strictly positive for every agent forming the coalition. If we consider (as in the original definition by Aubin) the possibility of null weights or contributions, the coalition formed by all the agents (the society) contains, implicitly, any other coalition (see Hervés-Beloso and Moreno-García (2001) for more details).

Aubin (1979) showed that, under standard assumptions, any Walrasian allocation is in the Aubin core, and, reciprocally, any non-Walrasian allocation is blocked in the sense of Aubin (see Florenzano (1990) for economies with an infinite-dimensional commodity space and without ordered preferences).

Hervés-Beloso et al. (2005b), provide a new characterization of Walrasian equilibrium allocations in terms of the blocking power of the "society" called there the "grand coalition". Precisely, under assumptions (H.1) and (H.2), a feasible allocation is a Walrasian equilibrium allocation in $\mathcal{E}$, if and only if $x$ is not blocked by the society in the sense of Aubin. It should be remarked that in the characterization above, the society is able to block, in the sense of Aubin, any non-walrasian allocation with a contribution of each member close to the total participation.

To be more precise, we can write the above result in the following way:

(*) *Let $\mathcal{E}$ be a pure exchange economy under assumptions (H.1) and (H.2). The next statements hold:*

*If $x$ is a feasible allocation which is Aubin blocked by the society, then $x$ is not a Walrasian allocation.*

*Reciprocally, if $x$ is feasible but not a Walrasian allocation then, for any positive $\alpha < 1$ there exist coefficients $\alpha_i \in [\alpha, 1]$, and consumption bundles $y_i$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \alpha_i y_i \leq \sum_{i=1}^{n} \alpha_i \omega_i$, and $U_i(y_i) > U_i(x_i)$, for every agent $i$.*

This equivalence between the set of Walrasian allocations and the set of allocations that the society cannot block in the sense of Aubin with participations of every member arbitrarily close to the total participation will be used in the rest of the paper and we will refer to this characterization by the symbol (*).

# 3    The Associated Game

Consider the pure exchange economy $\mathcal{E} \equiv (X = \mathbb{R}^{\ell}, (U_i, \omega_i)_{i=1}^{n})$ defined in the previous Section.

We define a game $\mathcal{G}$ associated to the economy $\mathcal{E}$ in order to analyze the relation between the non-cooperative solution of Nash equilibrium and the decentralized solution of Walrasian equilibrium.

There are two players. The strategy set for the player 1 is denoted by $S_1$ and is given by

$$S_1 = \{ \ x = (x_1, ..., x_n) \in \mathbb{R}^{\ell n}_+ \ \text{ such that } \ x_i \neq 0 \text{ and } \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \omega_i \}.$$

That is, the strategy set for player 1 is the set of feasible allocations that assign a non-zero consumption for each agent. Observe that $\omega = (\omega_1, \ldots, \omega_n) \in S_1$.

Let $\alpha$ be a real number such that $0 < \alpha < 1$. The strategy set for the player 2 is denoted by $S_2$ and is defined as follows:

$$S_2 = \{ (a, y) \in [\alpha, 1]^n \times \mathbb{R}^{\ell n}_+ \ \text{ such that } \ \sum_{i=1}^{n} a_i y_i \leq \sum_{i=1}^{n} a_i \omega_i \}.$$

That is, the strategy set for player 2 is the set of allocations that are feasible in the sense of Aubin with a participation greater or equal to $\alpha$ for every member of the society. Observe that $S_2$ is a non empty set ( $(\mathbf{1}, \omega) \in S_2$, where $\mathbf{1}$ is the vector in $[\alpha, 1]^n$ whose coordinates are constant and equal to 1).

Let $S$ denote the product set $S_1 \times S_2$. A strategy profile is any $s = (x, a, y) \in S$, that is, a strategy profile is a strategy $x \in S_1$ for player 1 and a strategy $(a, y) \in S_2$ for player 2.

Given a strategy profile $s = (x, a, y) \in S$, the payoff functions $\Pi_1$ and $\Pi_2$, for player 1 and 2, respectively, are defined as follows:

$$\Pi_1(x, a, y) = \min_i \{U_i(x_i) - U_i(y_i)\}$$

$$\Pi_2(x, a, y) = \min_i \{a_i (U_i(y_i) - U_i(x_i))\}$$

In short, the game $\mathcal{G}$ is defined by the strategy sets and the payoff functions for each player. We write

$$\mathcal{G} \equiv \{S_1, S_2, \Pi_1, \Pi_2\}.$$

From the definition of the game $\mathcal{G}$ we can obtain some immediate consequences.

Let $s$ be a strategy profile. If $\Pi_1(s) > 0$, then $\Pi_2(s) < 0$. Reciprocally, if $\Pi_2(s) > 0$, then $\Pi_1(s) < 0$. That is, $\Pi_1(s)$ and $\Pi_2(s)$ can not be strictly positive for any $s$, although both $\Pi_1(s)$ and $\Pi_2(s)$ can be strictly negative for some strategy profile $s$.

Given the strategy profile $s = (x, a, y) \in S$, note that if $x \in S_1$ is not an efficient allocation, then there exists a feasible allocation $z$ such that $U_i(z_i) > U_i(x_i)$ for every $i = 1, \ldots, n$; and then $U_i(z_i) - U_i(y_i) > U_i(x_i) - U_i(y_i)$ for every $i = 1, \ldots, n$, and for any $(a, y) \in S_2$. In other words, if $x$ is not a Pareto optimum, there exists an allocation $z \in S_1$ such that $\Pi_1(z, a, y) > \Pi_1(x, a, y)$, for any $(a, y) \in S_2$. That is, if $x$ is not a Pareto optimum, player 1 can improve upon her payoff.

On the other hand, if player 2 selects $(a, x)$, where $x$ is a feasible and efficient allocation, then the best response of player 1 is also the Pareto optimum $x$. To be precise, if $x$ is a Pareto optimum, we have $\Pi_1(x, a, x) \geq \Pi_1(z, a, x)$, for any

$z \in S_1$. To see this, note that $\Pi_1(x, a, x) = 0$ and if there exists $z \in S_1$ such that $\Pi_1(z, a, x) > \Pi_1(x, a, x) = 0$, then $U_i(z_i) > U_i(x_i)$ for every individual in the society, which is in contradiction with the efficiency of $x$.

Moreover, if $\Pi_2(x, a, y) > 0$, then $x$ is blocked by the society in the sense of Aubin. Reciprocally, if $x$ is an allocation blocked by the society in the sense of Aubin, then there exists $(a, y) \in S_2$ such that $\Pi_2(x, a, y) > 0$. Furthermore, according to the characterization (*) if $x$ is a Walrasian allocation then $\Pi_2(x, a, y) \leq 0$ for any $(a, y) \in S_2$.

**Definition** 3.1 *A Nash equilibrium for the game $\mathcal{G}$ is a strategy profile $s^* = (x^*, a^*, y^*) \in S$ such that*

$$\Pi_1(s^*) \geq \Pi_1(x, a^*, y^*), \quad \textit{for every } x \in S_1 \quad \textit{and}$$

$$\Pi_2(s^*) \geq \Pi_2(x^*, a, y), \quad \textit{for every } (a, y) \in S_2.$$

**Proposition** 3.1 *The set of Nash equilibria in pure strategies for the game $\mathcal{G}$ is not empty.*

*Proof.* This is a consequence of the existence of Walrasian equilibrium of the economy $\mathcal{E}$. In fact, if $x$ is a Walrasian allocation, then $(x, \mathbf{1}, x)$ a Nash equilibrium of the society game $\mathcal{G}$. To see this, note that $\Pi_1(x, \mathbf{1}, x) \geq \Pi_1(z, \mathbf{1}, x)$, for all $z \in S_1$, because $x$ is a Pareto-optimum. On the other hand, if there existed $(a, y) \in S_2$ such that $\Pi_2(x, a, y) \geq \Pi_2(x, \mathbf{1}, x)$, it would imply that $x$ could be blocked in the sense of Aubin, which is a contradiction with the fact that $x$ is Walrasian.

<div align="right">Q.E.D.</div>

**Remark.** Note that given any $x \in S_1$, the strategy $(\mathbf{1}, x)$ belongs to $S_2$. Therefore the payoff for agent 2 at any Nash equilibrium can not be negative. That is, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium, then $\Pi_2(s^*) \geq 0$.

As was already observed, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium of the game $\mathcal{G}$ then the allocation $x^*$ is necessarily Pareto-efficient.

Finally, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium of the game $\mathcal{G}$ and $\Pi_2(s^*) = 0$, then the allocation $x$ is a non dominated allocation in the sense of Aubin and, therefore, applying the characterization (*), $x^*$ is a Walrasian allocation of the economy $\mathcal{E}$.

# 4 The Main Result

In this Section, we state our main result which shows the equivalence between the Walrasian equilibria of the economy $\mathcal{E}$ and the set of Nash equilibria of the associated game $\mathcal{G}$. That is, we obtain a characterization of Walrasian equilibria in terms of Nash equilibria of a game with only two players, independently of the number of consumers in the economy. Thus, we show that the Walrasian mechanism is implementable as a Nash equilibrium of a two-player game.

As we have remarked in the Introduction, the game $\mathcal{G}$ can be interpreted as a society game where the society plays two different roles: in the first role, the society, as player 1, selects feasible allocations and tries to make Pareto improvements, while as player 2, society comes up with alternative allocations that are feasible in the sense of Aubin.

As we have already observed, the Paretian player has an incentive to deviate whenever the strategy she chooses is not an efficent allocation. Hence, at any Nash equilibrium, the strategy for player 1 is required to be Pareto-optimal. Then, we may argue that the society, as player 1, seeks efficiency.

The society, as player 2, acts as an adviser who recommends different assignments trying to dominate the allocation proposed by the Paretian player. Player 2 has incentives to deviate whenever the strategy she selects is a dominated allocation in the sense of Aubin. On the other hand, these "Aubin" player can always get a non-negative payoff (by choosing the same allocation as player 1) and can reach a strictly positive payoff only in the case that the allocation proposed by player 1 is not Walrasian.

We will show that the values of the payoff functions which come from any Nash equilibrium coincide for both players and are equal to zero. As we will see, this fact avoids the allocation proposed by player 1 to be Aubin dominated at any Nash equilibrium. Thus, at any Nash equilibrium $s^* = (x^*, a^*, y^*)$, the outcome $x^*$ is feasible and efficient while player 2 ensures that it is Walrasian.

In order to show our main result we need some previous lemmas. Given a strategy profile $s = (x, a, y)$ let us define the sets:

$$B(s) = \left\{ k \in \{1, \ldots, n\}, \text{ s.t. } U_k(x_k) - U_k(y_k) = \min_i \{U_i(x_i) - U_i(y_i)\} \right\};$$

$$B'(s) = \left\{ k \in \{1, \ldots, n\}, \text{ s.t. } a_k(U_k(x_k) - U_k(y_k)) = \min_i \{a_i(U_i(x_i) - U_i(y_i))\} \right\}.$$

Given a real number $a \in [0, 1]$ we denote $\mathbf{a} = (a, \ldots, a) \in [0, 1]^n$, i.e., $\mathbf{a}$ is the

vector in $[0,1]^n$ whose coordinates are identical and equal to $a$.

The next lemmas show that, in a Nash equilibrium, the minima which define the payoff functions $\Pi_1$ and $\Pi_2$ are attained by every consumer. That is, if $s^*$ is a Nash equilibrium of the game $\mathcal{G}$, then $B(s^*) = B'(s^*) = \{1, \ldots, n\}$.

**Lemma** 4.1 *If $x^*$ is player 1's best response to the strategy $(a^*, y^*)$ selected by player 2, in particular, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium of the game $\mathcal{G}$, then $U_i(x_i^*) - U_i(y_i^*) = U_j(x_j^*) - U_j(y_j^*)$ for every $i, j \in \{1, \ldots, n\}$.*

*Proof.* Let $x^*$ be player 1's best response to the strategy $(a^*, y^*)$ selected by player 2. Assume that the statement of the Lemma does not hold. Then, $B(s^*) \neq \{1, \ldots, n\}$. This implies that there exists a consumer $j$ such that $U_j(x_j^*) - U_j(y_j^*) > U_i(x_i^*) - U_i(y_i^*)$ for every $i \in B(s^*)$. By continuity of the utility functions, there exists some $\delta > 0$ such that player 1 can deviate to $x'$, where $x_j' = (1 - \delta)x_j^*$ and $x_i' = x_i^* + \frac{\delta}{n-1}x_j^*, \forall i \neq j$, and still have $U_j(x_j') - U_j(y_j^*) > U_i(x_i') - U_i(y_i^*)$ for every $i \in B(s^*)$. By monotonicity of preferences, $U_i(x_i') > U_i(x_i^*)$ for every $i \neq j$, which implies that $\Pi_1(x', a^*, y^*) > \Pi_1(s^*)$. This contradicts the fact that $x^*$ is player 1's best response to $(a^*, y^*)$.

<div align="right">Q.E.D.</div>

**Lemma** 4.2 *If $(a^*, y^*)$ is player 2's best response to the strategy $x^*$ selected by player 1, in particular, if $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium of the game $\mathcal{G}$, then $a_i(U_i(y_i^*) - U_i(x_i^*)) = a_j(U_j(y_j^*) - U_j(x_j^*))$ for every $i, j \in \{1, \ldots, n\}$.*

*Proof.* Let $(a^*, y^*)$ be player 2's best response to the strategy $x^*$ selected by player 1. Then, since $(\mathbf{1}, x^*) \in S_2$, one has that $\Pi_2(s^*) \geq 0$. Assume that the statement of the Lemma does not hold. Then, $B'(s^*) \neq \{1, \ldots, n\}$. This implies that there exists a consumer $j$ such that $a_j(U_j(y_j^*) - U_j(x_j^*)) > a_i(U_i(y_i^*) - U_i(x_i^*)) \geq 0$ for every $i \in B'(s^*)$. This implies that $U_j(y_j^*) > U_j(x_j^*)$, so, by strict monotonicity of preferences, $y_j^* > 0$. By continuity of the utility functions, we can take a non null commodity bundle $\varepsilon \in \mathbb{R}_+^\ell$ such that $a_j(U_j(y_j^* - \varepsilon) - U_j(x_j^*)) > a_i(U_i(y_i^*) - U_i(x_i^*))$ for every $i \in B'(s^*)$. Let $\delta = \dfrac{a_j^*}{\displaystyle\sum_{i \in B'(s^*)} a_i^*}\, \varepsilon$. Consider the allocation $y = (y_1, \ldots, y_n)$ defined as follows:

<div align="center">14</div>

$$
y_i = \begin{cases} y_i^* - \varepsilon & \text{if } i = j \\[2ex] y_i^* + \delta & \text{if } i \in B'(s^*) \\[2ex] y_i^* & \text{otherwise} \end{cases}
$$

By construction, we obtain:

$$
\sum_{i=1}^{n} a_i^* y_i = a_j^*(y_j^* - \varepsilon) + \sum_{i \in B'(s^*)} a_i^*(y_i^* + \delta) + \sum_{\substack{i \notin B'(s^*) \\ i \neq j}} a_i^* y_i^* =
$$

$$
= a_j^* y_j^* - a_j^* \varepsilon + \sum_{i \in B'(s^*)} a_i^* y_i^* + \sum_{i \in B'(s^*)} a_i^* \delta + \sum_{\substack{i \notin B'(s^*) \\ i \neq j}} a_i^* y_i^* =
$$

$$
= \sum_{i=1}^{n} a_i^* y_i^* \ \leq \ \sum_{i=1}^{n} a_i^* \omega_i.
$$

Then we have that $(a^*, y) \in S_2$. On the other hand, by monotonicity of preferences, $U_i(y_i) > U_i(y_i^*)$ for every $i \in B'(s^*)$. Therefore, we conclude that $\Pi_2(x^*, a^*, y) > \Pi_2(s^*)$, which is a contradiction with the fact that $(a^*, y^*)$ is player 2's best response to $x^*$.

Q.E.D.

As an immediate consequence of the previous lemmas we obtain the following proposition.

**Proposition** 4.1 *If $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium for the game $\mathcal{G}$, then $U_i(y_i^*) = U_i(x_i^*)$ for every $i = 1, \ldots, n$, and $\Pi_1(s^*) = \Pi_2(s^*) = 0$.*

*Proof.* Let $s^* = (x^*, a^*, y^*)$ be a Nash equilibrium for the game $\mathcal{G}$. Since $(\mathbf{1}, x^*) \in S_2$, one has that $\Pi_2(s^*) \geq 0$. Assume that the statement of the proposition does not hold. Then, $\Pi_2(s^*) > 0$, which implies that $\Pi_1(s^*) < 0$. By the two previous lemmas, $a_i^* = a > 0$ for every $i = 1, \ldots, n$. This implies that $\sum_{i=1}^{n} y_i^* \leq \sum_{i=1}^{n} \omega_i$, so $y^*$ belongs to $S_1$. Hence, $\Pi_1(y^*, a^*, y^*) = 0 > \Pi_1(s^*)$, a contradiction.

Q.E.D.

Before stating our characterization result, we show as an easy consequence of the previous lemmas that the Nash equilibria of the society game are actually strong Nash equilibria.

**Proposition** 4.2 *Any Nash equilibrium of the associated game $\mathcal{G}$ is a strong Nash equilibrium.*

*Proof.* Let $s^* = (x^*, a^*, y^*)$ be a Nash equilibrium for the game $\mathcal{G}$. Since there are only two players in the game, it is enough to show that the coalition formed by both players has no incentive to deviate. Otherwise, there is a strategy profile $s = (x, a, y) \in S$ such that player 1 and player 2 get better. Then, by Proposition 4.1, one has that $\Pi_1(s) > \Pi_1(s^*) = 0$ and $\Pi_2(s) > \Pi_2(s^*) = 0$. But, by the definition of the payoff functions, this is impossible, that is, both inequalities above can not hold together because if $\Pi_1(s) > 0$, then necessarily $\Pi_2(s) < 0$.

$$\text{Q.E.D.}$$

The next Theorem is our main result in this paper. It shows the relation between the set of Walrasian equilibria of the economy $\mathcal{E}$ and the set of Nash equilibria of the associated game $\mathcal{G}$. This characterization of Walrasian equilibria allows us to conclude that the Walrasian mechanism is implementable as a Nash equilibrium of the society game.

**Theorem** 4.1 *Let $\mathcal{E}$ be a pure exchange economy under assumptions (H.1) and (H.2).*

*If $s^* = (x^*, a^*, y^*)$ is a Nash equilibrium for the game $\mathcal{G}$, then $x^*$ is a Walrasian equilibrium allocation for the economy $\mathcal{E}$.*

*Reciprocally, if $x^*$ is a Walrasian equilibrium allocation for the economy $\mathcal{E}$, then any $s^* = (x^*, a^*, y^*) \in S$, with $U_i(y_i^*) = U_i(x_i^*)$ for every $i = 1, \ldots, n$, is a Nash equilibrium for the game $\mathcal{G}$.*

*In particular, $x^*$ is a Walrasian equilibrium allocation for the economy $\mathcal{E}$, if and only if $(x^*, \mathbf{b}, x^*)$ with $\mathbf{b}_i = b$, for every $i = 1, \ldots, n$, (for instance $(x^*, \mathbf{1}, x^*)$) is a Nash equilibrium for the game $\mathcal{G}$.*

*Proof.* Let $s^* = (x^*, a^*, y^*)$ be a Nash equilibrium for the game $\mathcal{G}$. Assume that $x^*$ is not a Walrasian equilibrium allocation. Then, by (*), we can take $a = (a_1, \ldots, a_n) \in [\alpha, 1]^n$ and $y_i$, for each $i \in \{1, \ldots n\}$, such that

(a) $\sum_{i=1}^{n} a_i y_i \leq \sum_{i=1}^{n} a_i \omega_i$ and

*(b)* $U_i(y_i) > U_i(x_i^*)$ for every $i = 1, \dots, n$.

Then, there exists $(a, y) \in S_2$ such that $\Pi_2(x^*, a, y) > \Pi_2(s^*)$, which contradicts the fact that $s^*$ is a Nash equilibrium.

Reciprocally, let $x^*$ be a Walrasian equilibrium allocation. Assume that $(x^*, a, y)$ is not a Nash equilibrium and $(a, y) \in S_2$ is such that $U_i(x_i^*) = U_i(y_i)$ for every $i = 1, \dots, n$. Then, we have: (i) there exists $x \in S_1$ such that $\Pi_1(x, a, y) > \Pi_1(x^*, a, y) = 0$; or (ii) there exists $(b, z) \in S_2$ such that $\Pi_2(x^*, b, z) > \Pi_2(x^*, a, y) = 0$.

If *(i)* is the case, we obtain that $U_i(x_i) > U_i(y_i) = U_i(x_i^*)$ for every consumer $i$. Since $x$ is a feasible allocation in the economy $\mathcal{E}$, we conclude that $x^*$ is not an efficient allocation. By the first Welfare Theorem, it is a contradiction with the fact that $x^*$ is a Walrasian allocation.

Assume that *(ii)* holds. Then, $U_i(z_i) > U_i(x_i^*)$ for every $i = 1, \dots, n$, and $\sum_{i=1}^{n} b_i z_i \leq \sum_{i=1}^{n} b_i \omega_i$, with $b_i > 0$ for all $i$. This implies that $x^*$ is a dominated allocation in the sense of Aubin, which is a contradiction with the fact that $x^*$ is a Walrasian equilibrium allocation.

<div align="right">Q.E.D.</div>

Observe that in spite of the fact that we may obtain a continuum of Nash equilibria for the game $\mathcal{G}$ with the same strategy for player 1, we can select a canonical representation. If $(x^*, a^*, y^*)$ is a Nash equilibrium then $(x^*, \mathbf{1}, x^*)$ is also a Nash equilibrium for the game $\mathcal{G}$.

# 5 A Remark

The reader may observe that we have excluded null consumption from the strategy set of player 1. In particular, Pareto-optimal allocations that assign all the initial endowments to one of the agents are excluded as strategies for player 1.

Note that $x_i \neq 0$ was used only in the proof of Lemma 4.1. Without this technical device, there could be an equilibrium of the game that did not correspond to a Walrasian allocation (see the example below).

On the other hand, assuming that the utility functions are concave, we may allow null consumptions and still obtain the same result if we define a new game

$\mathcal{G}'$ where the strategy set and the payoff function for player are changed to $S_1'$ and $\Pi_1'$.

To be precise, $S_1'$ is the set of feasible allocations:

$$S_1' = \{ x = (x_1, ..., x_n) \in \mathbb{R}_+^{\ell n} \text{ such that } \sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i \},$$

Let $f(x)$ be a positive differentiable function defined in $[\alpha, 1]$, such that $f'(x) \, x > f(x)$. For example, $f(x) = e^{bx}$, with $b > \frac{1}{\alpha}$. Observe that $\frac{f(x)}{x}$ is a positive and strictly increasing function, therefore: $\max \{f(x)/x\} = f(1)$.

Given a strategy profile $s = (x, a, y) \in S'$, the new payoff function of player 1 is:

$$\Pi_1'(x, a, y) = \min_i \{f(a_i) \, (U_i(x_i) - U_i(y_i))\}.$$

Both strategy sets, $S_1'$ and $S_2$, are compact. Obviously, $S_1$ is convex, but $S_2$ is not (we owe this observation to an anonymous referee). Thus, we cannot conclude from the strategies and payoff functions that the game has a Nash equilibrium. Observe that the considerations made for the game $\mathcal{G}$ also apply to $\mathcal{G}'$. In particular, Proposition 3.1 also establishes existence of Nash equilibrium of the game $\mathcal{G}'$.

In order to prove Proposition 4.1, assume that $\Pi_2(s^*) = C > 0$.

Start by seeing that if $\Pi_2(s^*) > 0$, then $a_{max}^* = \max\{a_i^*\} = 1$. Or else, by deviating to $s = (x^*, a, y^*)$, where $a_i = \frac{a_i^*}{a_{max}^*}$, player 2 improves its payoff. Since player 2 cannot improve its payoff, $\max\{a_i^*\} = 1$.

If all $a_i^* = 1$, then $y^*$ is feasible and player 1 can obtain a null payoff by selecting $x^* = y^*$. This would give $\Pi_2 = 0$.

Lemma 4.2 is still valid in this modified game. In a Nash equilibrium: $\Pi_1 = \min\{f(a_i)(U_i(x_i) - U_i(y_i))\} = \min\{f(a_i)\frac{-C}{a_i}\} = -Cf(1)$. The properties of the function $f(\cdot)$ would guarantee that the payoff of player 1 is determined in the $i$ such that $a_i = 1$.

Now observe that if $a_i^* < 1$, then $x_i^* \neq 0$. Otherwise, player 2 could deviate, in the $i^{th}$ coordinate, to $(a', y') = [(1 + \varepsilon)a_i^*, y_i^*/(1 + \varepsilon)]$, obtaining at least the same payoff:

$$a_i'[U_i(y_i') - U_i(x_i^*)] = a_i'U_i(y_i') = (1 + \varepsilon)a_i^* \, U_i(y_i^*/(1 + \varepsilon)) \geq a_i^*U_i(y_i^*) = a_i^*[U_i(y_i^*) - U_i(x_i^*)].$$

This is an interior Aubin allocation:

$$\sum a'_j y'_j = \sum a^*_j y^*_j \leq \sum a^*_j \omega_j < \sum a^*_j \omega_j + \varepsilon a^*_i \omega_i = \sum a'_j \omega_j.$$

Therefore, we can redistribute the remaining resources $(\varepsilon a^*_i \omega_i)$ and obtain a higher payoff. This contradiction implies that if $a^*_i < 1$, then $x^*_i \neq 0$.

The coefficients aren't all equal, so there is some $i$ such that $a^*_i < 1$ (and $x^*_i \neq 0$) for which $f(a^*_i) \left[ U_i(x^*_i) - U_i(y^*_i) \right] > \Pi_1(s^*)$. Then, by continuity, player 1 can select a strategy $x'$ in which $x'_i = (1-\delta)x^*_i$ and $x'_j = x^*_j + \frac{\delta}{n-1}x^*_i$, obtaining a higher payoff. This contradiction implies that $\Pi_2 = 0$, proving Proposition 4.1 and, as a consequence, Theorem 4.1.

**An example:**

We will show that if null consumptions are allowed as strategies for player one in game $\mathcal{G}$, then the main result is no longer true.

Consider an economy with two agents and one commodity. Both agents have the same preference relation represented by the utility function $U(x) = x$. Let $\omega_1 = \omega_2 = \omega > 0$. We will see that the non-Walrasian allocation that assigns all the resources to one consumer can be a Nash equilibrium strategy for player one. Let $s^* = (s^*_1, s^*_2) \in S_1 \times S_2$ with $s^*_1 = (2\omega, 0)$ and $s^*_2 = ((a, b), (x, y))$, such that $\Pi_2(s^*) > 0$. As we have seen, if $s^*$ is a Nash equilibrium, then $a < 1$, $b = 1$ and, by Lemma 4.2, $a(x - 2\omega) = y$. On the other hand, $s^*_2 \in S_2$ implies $ax + y = (a+1)\omega$. This implies $x = 2\omega + \dfrac{1-a}{2a}\omega$ and $y = \dfrac{1-a}{2}\omega$. Then $\Pi_2(s^*) = \dfrac{1-a}{2}\omega$ and $a = \alpha$ guarantees that player two has no incentive to deviate. Observe that in the game $\mathcal{G}$, given $s^*_2$, when player one chooses the strategy $(z, t)$ her payoff $\Pi_1((z,t), s^*_2) = \min \left\{ z - \left( 2\omega + \dfrac{1-\alpha}{2\alpha}\omega \right), \ t - \dfrac{1-\alpha}{2}\omega \right\}$ is attained in the first term and then the best response for player 1 is $(2\omega, 0)$. It is now clear that the profile $s^* = ((2\omega, 0), (\alpha, 1), (2\omega + \dfrac{1-\alpha}{2\alpha}\omega, \dfrac{1-\alpha}{2}\omega))$ is a Nash equilibrium.

However, in the modified game $\mathcal{G}'$ the profile $s^*$ can not be a Nash equilibrium because $s^*_1$ is not a Walrasian allocation. Observe that the payoff for player one is $\Pi'_1((z,t), s^*_2) = \min \left\{ f(\alpha) \left( z - 2\omega - \dfrac{1-\alpha}{2\alpha}\omega \right), \ f(1) \left( t - \dfrac{1-\alpha}{2}\omega \right) \right\}$ where each term is increasing in $z$ and $t$ respectively. We have $\Pi'_1(s^*) = f(1) \left( \dfrac{1-\alpha}{2}\omega \right)$. Then player 1 increases her payoff by choosing the strategy $(2\omega - \varepsilon, \varepsilon)$ for $\varepsilon$ small.

# 6 Conclusion

In this paper, we have provided a characterization of Walrasian equilibria allocations in terms of Nash equilibria of an associated two-player game that we have referred to as the society game. Moreover we have established that, independently of the number of consumers and commodities, Walrasian equilibrium is implementable as a strong Nash equilibrium of a two-player game.

This equivalence result adds to the great deal of works on strategic approaches to economic equilibrium. However, our society game differs substantially from those games considered in the literature on strategic market games with respect to several points: our society game involves only two players (it makes no difference the number of consumers in the economy); each player represents a role of the society formed by all the consumers in the economy and not an individual; the outcomes are given by the strategies themselves and prices appear neither in the strategy sets nor in the payoff functions.

The parameter $\alpha$ and the utility functions representing preferences are used in the definition of the game. However, as a consequence of our main result, we conclude that the allocations underlying Nash equilibria are the same, independently of the value of $\alpha$ and only depend on preferences.

Finally, as we have pointed out in the Introduction, our results do still hold for economies with infinitely many commodities and also for differential information economies. For this, it suffices to apply the characterization result (*) which is proved in Hervés-Beloso et al. (2005b) for those more general settings.

# References

Anderson, R.M., 1978. An Elementary Core Equivalence Theorem. Econometrica, 46, 1483-1487.

Anderson, R.M., 1981. Core Theory with Strongly Convex Preferences. Econometrica, 49, 1457-1468.

Anderson, R.M., 1985. Strong Core Theorems with Nonconvex Preferences. Econometrica, 53, 1283-1294.

Arrow, K.J., Debreu G., 1954. Existence of an Equilibrium for a Competitive Economy. Econometrica 22, 265-290.

Arrow, K., Hahn, F., 1971. General Competitive Analysis. Holden-Day, San Francisco.

Aubin, J.P., 1979. *Mathematical Methods of Game Economic Theory.* North-Holland, Amsterdam, New York, Oxford.

Aumann, R. J.,1964. Markets with a continuum of traders. Econometrica 32, 39-50.

Bertrand, J., 1883. Théorie Mathématique de la Richesse Sociale. Journal de Savants, 499-508.

Bewley, T.F., 1973. Edgeworth's Conjecture. Econometrica, 41, 425-454.

Cournot, A., 1838. Recherches sur les Principes Mathématiques de la Théorie des Rechesses. In: Bacon, N. (Ed.), English ed., Researches into the Mathematical Principles of Theory of Wealth. Macmillan, New York, 1897.

Debreu, G., 1952. A Social Equilibrium Existence Theorem. Proceedings of the National Academy of Sciences, 38, 886-893.

Debreu, G., 1962. New Concepts and Technics for Equilibrium Analysis. International Economic Review 3, 257-273.

Debreu, G., Scarf, H., 1963. A Limit theorem on the Core of an Economy. International Economic Review, 4, 235-246.

Dierker, E., 1975. Gains and Losses at Core Allocations. Journal of Mathematical Economics, 2, 119-128.

Dubey, P., 1982. Price-Quantity Strategic Market Games. Econometrica 50, 111-126.

Dubey, P., Geanokoplos, J., 2003. From Nash to Walras via Shapley-Shubik. Journal of Mathematical Economics, 39, 391-400.

Dubey, P., Shapley, L.S., 1994. Non-Cooperative General Exchange with a Continuum of traders: Two Models. Journal of Mathematical Economics 23, 253-293.

Edgeworth, F.Y., 1881. Mathematical Psychics. Paul Kegan, London.

Florenzano, M., 1990. Edgeworth Equilibria, Fuzzy Core, and Equilibria of a Production Economy without ordered Preferences. Journal of Mathematical Analysis and Applications , 153, 18-36.

Hammond, P.J, Kaneko, M., Wooders, M. 1989. Continuum economies with finite coalitions; core, equilibrium and widespread externalities. Journal of Economic Theory 49, 113-134.

Hervés-Beloso, C., Moreno-García, E., 2001. The Veto Mechanism Revisited. In *Approximation, Optimization and Mathematical Economics*; p. 147-159, 2001. Marc Lassonde (Ed.). Heidelberg, New York. Physica-Verlag.

Hervés-Beloso, C., Moreno-García, E., Yannelis, N.C., 2005a. An Equivalence Theorem for a Differential Information Economy. Journal of Mathematical Economics 41, 844-856.

Hervés-Beloso, C., Moreno-García, E., Yannelis, N.C., 2005b. Characterization and Incentive Compatibility of Walrasian Expectations Equilibrium in Infinite Dimensional Commodity Spaces. Economic Theory 26, 361-381.

Hildenbrand, W., 1974. Core and Equilibria of a Large Economy. Princeton University Press, Princeton.

Hurwicz, L., 1979. Outcome Functions Yielding Walrasian and Lindhal Allocations at Nash Equlibrium Points. Review of Economic Studies, 46, 217-227.

Kaneko, M., Wooders, M., 1989. The Core of a Continuum Economy with Widespread Externalities and Finite Coalitions; from Finite to Continuum Economies. Journal of Economic Theory 49, 135-168.

Khan, M.A. 1976. Oligopoly on Markets with a Continuum of Traders: An Asymptotic Interpretation. Journal of Economic Theory, 12, 273-297.

Nash, J.F. 1950. Equilibrium Points in N-Person Games. Proceedings of the National Academy of Sciences, 36, 48-49.

Schmeidler, D. 1980 Walrasian Analysis via Strategic Outcome Functions. Econometrica 48, 1585-1593.

Shapley, L.S. 1976 Non-Cooperative General Exchange. In Theory of Measure of Economic Externalities. Academic Press, New York.

Shapley, L.S. Shubik, M., 1969. On market games. Journal of Economic Theory 1, 9-25.

Shapley, L.S., Shubik, M., (1977): Trade Using one Commodity as a Mean of Payment. Journal of Political Economy, 85, 937-968.

Shubik, M. (1959): Edgeworth Market Games. Vol. 4 in Harold W. Kuhn and A.W. Tucker (editors). Contributions to the Theory of Games. Princeton University Press.

Shubik, M. (1973): Commodity Money, Oligopoly, Credit and Bankruptcy in a General Equilibrium Model. Western Economic Journal, 11, 24-38.

Shubik, M., Wooders, M.H., 1982. Near markets and market games. Cowles Foundation Discussion Paper No. 657. Published as Clubs, near markets and market games. Topics in Mathematical Economics and Game Theory; Essays in Honor of Robert J. Aumann, eds. M. H. Wooders American Mathematical Society Fields Communication Volume 23.

Trockel, W. (1976): A Limit Theorem on the Core. Journal of Mathematical Economics, 3, 247-264.

Vind, K. (1972): A Third Remark on the Core of an Atomless Economy. Econometrica, 40, 585-586.

Wooders, M.H., 1994. Equivalence of games and markets. Econometrica 62, 1141-1160.

# Large Economies and Two-Player Games

Carlos Hervés-Beloso

RGEA. Facultad de Económicas. Universidad de Vigo.

e-mail: cherves@uvigo.es

Emma Moreno-García

Facultad de Economía y Empresa. Universidad de Salamanca.

e-mail: emmam@usal.es

**Abstract.** We characterize the core and the competitive allocations of a continuum economy as the strong Nash equilibria of an associated game with only two players.

# 1 Introduction

In this paper we consider a pure exchange economy with a continuum of agents and finitely many commodities (Aumann 1964, 1966). We associate a game with only two players to each Aumann's economy. Our aim is to characterize the core and the competitive allocations of the economy as Nash equilibria of the associated two-player game.

In the associated two players game, the strategies of the first payer are feasible allocations of the economy. Each strategy of the second player consists in a coalition of agents and a feasible allocation for this coalition. Given a strategy profile, if all the agents in the coalition proposed by player 2 are better off with the allocation proposed by player 1, then player 1's payoff depends on the difference of utilities that the agents in the coalition obtain with the allocations proposed by the two players. Otherwise, her payoff is the essential infimum of the difference of utilities in that coalition. The payoff of the second player is defined symmetrically.

Given any strategy selected by player 1, player 2 can get zero payoff by choosing the strategy given by the coalition of all agents and the same allocation as player 1. Furthermore, she could obtain a positive payoff if and only if the allocation proposed by player 1 can be blocked by a coalition. Moreover, by showing that in any Nash equilibrium both payoffs are zero, we prove that in this two-plater game any Nash equilibrium is a strong Nash equilibrium.

Our main result (Theorem 3.1) proves that any allocation in the core of the economy is a strong Nash equilibrium of the two-player game and, reciprocally, any Nash equilibrium can be identified with a core allocation in the economy. Therefore, the assumptions that guarantee the core-Walras equivalence and the non-emptiness of the core (Aumann 1964, 1966) lead us to conclude (Corollary 3.1) that the competitive allocations of the continuum economy are characterized as Nash equilibria of the associated two-player game.

In Hervés-Beloso and Moreno-García, (2008), we have characterized the walrasian allocations of an $n$-agents economy as the Nash equilibria of an associated two-player game. Then Corollary 3.1 is an extension to the continuum case of our previous result. Moreover, Theorem 3.1 holds not only for continuum (atomless) economies but also for economies with a finite or countable number of agents or mixed economies. Thus, in the case of a finite pure exchange economy, we have

two different two-player games, associated to the economy, which characterize the core and the Walrasian allocations, respectively.

In order to emphasize the power of the veto mechanism in atomless economies, we address the particular case of a continuum economy with $n$ types of agents. We then consider a finite pure exchange economy with $n$ consumers which we identify with this $n$-types continuum economy. Assuming convexity of preferences, for each competitive allocation in the $n$ types economy, we can construct a competitive allocation which is constant on types, i.e., a competitive allocation with the equal treatment property. This step function corresponds to a Walrasian allocation in the finite economy (see García-Cutrín and Hervés-Beloso, 1993). The attempt to apply our main result to the finite economy, via an $n$-types continuum economy, does no longer allow us to exploit the veto power of the coalitions as we do in the continuum case. This is due to the fact that the measure of the set of agents of each type forming the coalition proposed by player 2 is the only coalitional data collected in the payoff functions. For this reason we cannot ensure that any Nash equilibrium underlies a Walrasian allocation, as we show in an example. Actually, in our example there is a Nash equilibrium of the game associated to the finite economy with a positive payoff for player 2 and, therefore, the allocation proposed by player one cannot be Walrasian.

In our study we have only considered a finite number of commodities. This assumption is not essential. Our main result is actually a characterization of the core of the economy. Therefore, if we consider a continuum economy with infinitely many commodities, in which the core-Walras equivalence holds (Bewley, 1973), we would also obtain the characterization of the competitive allocations as Nash equilibria of the associated two-player game.

The remainder of this paper is organized as follows. In Section 2 we define the continuum economy and the associated two-players game. In Section 3 we present the properties of the game, we prove our main results and we state some remarks regarding the size of the coalition selected by player 2 in which the main results still hold. In Section 4, we consider the particular case of a continuum economy with $n$ types of agents in order to recast the associated game for an Arrow-Debreu pure exchange economy. A final example points out that the discrete version of the game does not allow us to characterize Nash equilibria as Walrasian allocations.

## 2 The economy and the game

Consider a pure exchange economy $\mathcal{E}$ with $\ell$ commodities. The space of consumers is represented by an atomless finite measure space $(I, \mathcal{A}, \mu)$.

Each agent $t \in I$ is characterized by her consumption set $\mathbb{R}_+^\ell$, her initial endowment $\omega(t) \in \mathbb{R}_+^\ell$ and her preference relation $\succeq_t$ on consumption bundles, which is represented by the continuous utility function $U_t : \mathbb{R}_+^\ell \to [0, 1]$. The mapping $\omega : I \to \mathbb{R}_+^\ell$, which assigns to each agent her recourses, is $\mu$-integrable and the mapping $U$ that assigns to each consumer her utility function is measurable.

An allocation is a $\mu$-integrable function $f : I \to \mathbb{R}_+^\ell$. An allocation $f$ is feasible in the economy $\mathcal{E}$ if $\int_I f(t) d\mu(t) \leq \int_I \omega(t) d\mu(t)$.

A price system is an element of $\Delta$, where $\Delta$ denotes the $(\ell - 1)$-dimensional simplex of $\mathbb{R}_+^\ell$, that is, $\Delta = \{p \in \mathbb{R}_+^\ell \text{ such that } \sum_{h=1}^\ell p_h = 1\}$.

A competitive equilibrium for $\mathcal{E}$ is a pair $(p, f)$, where $p$ is a price system and $f$ is a feasible allocation such that, for almost every agent $t$, the bundle $f(t)$ maximizes the utility function $U_t$ in the budget set $B_t(p) = \{y \in \mathbb{R}_+^\ell \text{ such that } p \cdot y \leq p \cdot \omega(t)\}$.

A coalition is any measurable set $S$ with $\mu(S) > 0$. A coalition $S$ blocks an allocation $f$ via another allocation $g$ in the economy $\mathcal{E}$ if:

(i) $\int_S g(t) d\mu(t) \leq \int_S \omega(t) d\mu(t)$  and

(ii) $U_t(g(t)) > U_t(f(t))$ for almost all $t \in S$.

A feasible allocation belongs to the core of the economy if it is not blocked by any coalition of agents.

We define a game $\mathcal{G}$ associated to the economy $\mathcal{E}$ in order to analyze the relation between the non-cooperative solution of Nash equilibrium and the decentralized solution of competitive equilibrium.

There are two players. The strategy set for the player 1 is denoted by $\Theta_1$ and is given by

$$\Theta_1 = \{f : I \to \mathbb{R}_+^\ell : \int_I f(t) d\mu(t) = \int_I \omega(t) d\mu(t)\}$$

That is, a strategy for player 1 is a feasible allocation $f$. Observe that $\omega \in \Theta_1$.

The strategy set for the player 2 is denoted by $\Theta_2$ and is defined as follows:

$$\Theta_2 = \{(S, g) : \int_S g(t)d\mu(t) = \int_S \omega(t)d\mu(t)\}$$

That is, the strategy set for player 2 is the set of pairs which specify a coalition of agents and a feasible trade for such a coalition. Observe that if $f$ is a feasible allocation then $(I, f) \in \Theta_2$. Furthermore, $(S, \omega) \in \Theta_2$ whatever coalition $S$ may be.

Let $\Theta$ denote the product set $\Theta_1 \times \Theta_2$. A strategy profile is any $\theta = (f, S, h) \in \Theta$, that is, a strategy profile is a strategy $\theta_1 = f \in \Theta_1$ for player 1 and a strategy $\theta_2 = (S, g) \in \Theta_2$ for player 2.

In order to define the payoff functions, given a function $F : I \to \mathbb{R}$ and a coalition of agents $S \subset I$, let be

$$ess\inf\{F(t), t \in S\} = \sup\{c \in \mathbb{R} \mid F(t) \geq c \text{ for almost all } t \in S\}.$$

Given a strategy profile $(f, S, g)$, we define the following real valued functions

$$\alpha(f, S, g) = ess\inf\{U_t(f(t)) - U_t(g(t)), t \in S\}$$
$$\beta(f, S, g) = ess\inf\{U_t(g(t)) - U_t(f(t)), t \in S\}$$

Now for every $(f, S, g) \in \Theta$, the payoff functions $\Pi_1$ and $\Pi_2$ for player 1 and 2, respectively, are defined as follows

$$\Pi_1(f, S, g) = \begin{cases} \int_S (U_t(f(t)) - U_t(g(t)))\, d\mu(t) & \text{if } \alpha(f, S, g) \geq 0 \\ \alpha(f, S, g) & \text{otherwise} \end{cases}$$

$$\Pi_2(f, S, g) = \begin{cases} \int_S (U_t(g(t)) - U_t(f(t)))\, d\mu(t) & \text{if } \beta(f, S, g) \geq 0 \\ \beta(f, S, g) & \text{otherwise} \end{cases}$$

A strategy profile is a Nash equilibrium if no player has an incentive to deviate. That is, $\theta = (f, S, g)$ is not a Nash equilibrium of $\mathcal{G}$ if either there exists $\hat{f} \in \Theta_1$, such that $\Pi_1(\theta) < \Pi_1(\hat{f}, S, g)$ or there exists $(\hat{S}, \hat{g}) \in \Theta_2$ such that $\Pi_2(\theta) < \Pi_2(f, \hat{S}, \hat{g})$.

A strategy profile is a strong Nash equilibrium if no coalition has an incentive to deviate. Therefore, in our game $\mathcal{G}$ a profile is a strong Nash equilibrium if and only if it is an efficient Nash equilibrium.

# 3 Main Results

In this section we analyze some properties of the game $\mathcal{G}$, which we have previously associated to the economy $\mathcal{E}$. The aim is to present our main results which characterize the core and the competitive equilibrium allocations of the economy as strong Nash equilibria of $\mathcal{G}$.

Given the associated game $\mathcal{G}$, note that if $\Pi_1(f, S, g) > 0$ (resp. $\Pi_2(f, S, g) > 0$) then $\Pi_2(f, S, g) < 0$ (resp. $\Pi_1(f, S, g) < 0$). That is, both payoffs can be negative for some strategy profiles but cannot be strictly positive at the same time. Observe also that $\Pi_1(f, S, g) = 0$ if and only if $\Pi_2(f, S, g) = 0$.

Note that given any strategy $f \in \Theta_1$, player 2 can always get null payoff by selecting $(I, f) \in \Theta_2$. We also show that for each $(S, g) \in \Theta_2$, there exists $f \in \Theta_1$ such that $\Pi_1(f, S, g) = 0$ (see the proof of the Lemma 3.1 below).

**Lemma 3.1** *If $\theta^*$ is a Nash equilibrium then $\Pi_1(\theta^*) = \Pi_2(\theta^*) = 0$.*

*Proof.* Let $\theta^* = (f^*, S^*, g^*)$ be a Nash equilibrium. Since $(I, f^*)$ is a possible strategy for player 2, we have $\Pi_2(\theta^*) \geq 0$. Assume $\Pi_2(\theta^*) > 0$. Then $\Pi_1(\theta^*) < 0$. Consider the allocation $f$ given by

$$f(t) = \begin{cases} g^*(t) \text{ if } t \in S^* \\ \omega(t) \text{ otherwise} \end{cases}$$

Note that $f \in \Theta_1$ and $\Pi_1(f, S^*, g^*) = 0$, which is a contradiction.

Q.E.D.

**Remark 1.**[1]  Consider now a strategy $(S, g) \in \Theta_2$, with $\mu(S) > 0$ and $\int_{I \setminus S} \omega(t) d\mu(t) > 0$. Define the following feasible allocation

$$f(t) = \begin{cases} g(t) + \dfrac{1}{\mu(S)} \displaystyle\int_{I \setminus S} \omega(t) d\mu(t) \ \text{ if } t \in S \\[2em] 0 \ \text{ otherwise} \end{cases}$$

---

[1]We thank Andrés Carvajal for pointing out this remark.

If preferences are monotone we have $\Pi_1(f, S, g) > 0$. Therefore, under monotonicity of preferences and requiring that $\int_A \omega(t) d\mu(t) > 0$ for every coalition $A$, we can conclude that if $(f, S, g)$ is a Nash equilibrium then $\mu(S) = \mu(I)$.

**Proposition** 3.1 *Any Nash equilibrium of the game $\mathcal{G}$ is a strong Nash equilibrium.*

*Proof.* Let $\theta^*$ be a Nash equilibrium. By Lemma 3.1., $\Pi_1(\theta^*) = \Pi_2(\theta^*) = 0$. By definition of the payoff functions, both $\Pi_1$ and $\Pi_2$ cannot be strictly positive at the same time and $\Pi_1(\theta) = 0$ if and only if $\Pi_2(\theta) = 0$. This implies that the coalition formed by the two players has no incentive to deviate from the profile $\theta^*$.

Q.E.D.

**Theorem** 3.1 *If $\theta^* = (f^*, S^*, h^*)$ is a Nash equilibrium for the game $\mathcal{G}$, then $f^*$ belongs to the core of the economy $\mathcal{E}$.*

*Reciprocally, if $f^*$ is core allocation for the economy $\mathcal{E}$, then any strategy profile $(f^*, I, h^*) \in \Theta$, with $U_t(f^*(t)) = U_t(h^*(t))$, for almost all $t \in I$, is a Nash equilibrium for the game $\mathcal{G}$.*

*In particular, $f^*$ belongs to the core of the economy $\mathcal{E}$ if and only $(f^*, I, f^*)$ is a strong Nash equilibrium for the game $\mathcal{G}$.*

*Proof.* Let $\theta^* = (f^*, S^*, h^*)$ be a Nash equilibrium. By Lemma 3.1 $\Pi_1(\theta^*) = \Pi_2(\theta^*) = 0$. Assume that $f^*$ does not belong to the core of the economy $\mathcal{E}$. Then there exists $(S, g) \in \Theta_2$ such that $\Pi_2(f^*, S, g) > 0 = \Pi_2(\theta^*)$.

Reciprocally, let $f^*$ be a core allocation and $(I, h^*) \in \Theta_2$ such that $U_t(f^*(t)) = U_t(h^*(t))$, for almost all $t \in I$. Assume that $(f^*, I, h^*) \in \Theta$ is not a Nash equilibrium. Then, either player 1 or player 2 has an incentive to modify her strategy. If player 1 has an incentive to deviate then the allocation $f^*$ is not efficient, that is, it is blocked by the coalition $I$. If there exists $(S, g) \in \Theta_2$ such that $\Pi_2(f^*, S, g) > 0 = \Pi_2(f^*, I, h^*)$ then $f^*$ is blocked by the coalition $S$, via $g$ which is a contradiction with the fact that $f$ belongs to the core of $\mathcal{E}$.

Q.E.D.

We remark that the previous theorem provides a characterization of the core of a general economy. In fact, the above equivalence result holds not only for

atomless economies but also for economies with a finite or countable number of agents or mixed economies. However, this equivalence result could be empty in the sense that, without any assumption on the model, core allocations of the economy or, equivalently, Nash equilibria of the game $\mathcal{G}$, could not exist. Then, thereafter, we suppose that our economy $\mathcal{E}$ is a continuum economy that fulfills the assumptions that guarantee the core-Walras equivalence and therefore, the core and the set of the competitive allocation are non-empty (see Aumman 1964, 1966).

**Proposition** 3.2 *The set of Nash equilibria for the game $\mathcal{G}$ is nonempty.*

*Proof.* The non-emptyness of the set of Nash equilibria is a consequence of the existence of core allocations (or competitive allocations) for the continuum economy $\mathcal{E}$. In fact, if $f$ is a core allocation for the economy $\mathcal{E}$ then $(f, I, f)$ is a Nash equilibrium for the game $\mathcal{G}$. To see this, note that $\Pi_1(f, I, f) = \Pi_2(f, I, f) = 0$. If player 1 has an incentive to deviate then there exists $g \in \Theta_1$ such that $\Pi_1(g, I, f) > 0$ and therefore $f$ is not efficient. If $\Pi_2(f, S, g) > 0$ for some $(S, g) \in \Theta_2$, then $f$ is blocked by the coalition $S$, which is a contradiction with the fact that $f$ belongs to the core.

<div align="right">Q.E.D.</div>

We remark that the assumptions on the continuum economy $\mathcal{E}$ which guarantee the core-Walras equivalence allow us to obtain as an immediate consequence of Theorem 3.1 the corresponding characterization of competitive allocations.

**Corollary** 3.1 *If $\theta^* = (f^*, S^*, h^*)$ is a Nash equilibrium for the game $\mathcal{G}$, then $f^*$ is a competitive equilibrium allocation for the economy $\mathcal{E}$.*

*Reciprocally, if $f^*$ is a competitive equilibrium allocation for the continuum economy $\mathcal{E}$, then any strategy profile $(f^*, I, h^*) \in \Theta$, with $U_t(f^*(t)) = U_t(h^*(t))$, for almost all $t \in I$, is a Nash equilibrium for the game $\mathcal{G}$.*

*In particular, $f^*$ is a competitive equilibrium allocation for the economy $\mathcal{E}$, if and only $(f^*, I, f^*)$ is a strong Nash equilibrium for the game $\mathcal{G}$.*

**Remark 2.** We highlight that from the proof of the previous results we can deduce that the requirement of finitely many commodities is not essential. In particular, Corollary 3.1 is still true for atomless economies with infinitely

many commodities whenever the core-Walras equivalence holds (see for instance Bewley, 1973).

**Remark 3.** Let us consider a number $\varepsilon \in (0, \mu(I))$. Let $\mathcal{G}(\varepsilon)$ the game which coincides with $\mathcal{G}$ except for the strategy set of player 2 that is restricted to those coalitions $S$, with $\mu(S) \geq \mu(I) - \varepsilon$, and feasible allocations for such coalitions. For the case of finitely many commodities, Vind (1972) showed that in atomless economies it is enough to consider the blocking power of coalitions with a fixed measure $\varepsilon$ in order to get the core or, alternatively, the competitive equilibria (see Hervés-Beloso et al. 2000, 2005 and Evren-Husseinov, 2008). Then, we can conclude that for atomless economies our results hold for any game $\mathcal{G}(\varepsilon)$.

# 4   Economies with $n$ types of consumers

Let us consider the particular case of a continuum economy $\mathcal{E}_c$ with only $n$ types of agents. The set of agents is represented by the real interval $[0, 1]$, with the Lebesgue measure $\mu$. We write $I = [0, 1] = \bigcup_{i=1}^{n} I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$, if $i \neq n$, and $I_n = \left[\frac{n-1}{n}, 1\right]$. Every consumer $t \in I_i$ is characterized by her consumption set $\mathbb{R}_+^\ell$, her utility function $U_t = U_i$ and her initial endowment $\omega(t) = \omega_i \in \mathbb{R}_+^\ell$. We will refer to $I_i$ as the set of agents of type $i$ in the atomless economy $\mathcal{E}_c$.

This particular economy $\mathcal{E}_c$ can be considered as a representation of a finite economy $\mathcal{E}_n$, with $n$ consumers and $\ell$ commodities, where each consumer $i$ is characterized by the utility function $U_i$ and the initial endowments $\omega_i \in \mathbb{R}_+^\ell$.

In this Section, we assume convexity of preferences and the hypotheses that guarantee the core-Walras equivalence for the $n$-types continuum economy $\mathcal{E}_c$.

Observe that an allocation $x$ in $\mathcal{E}_n$ can be interpreted as an allocation $f_x$ in $\mathcal{E}_c$, where $f_x$ is the step function given by $f_x(t) = x_i$, if $t \in I_i$. Reciprocally, an allocation $f$ in $\mathcal{E}_c$ can be interpreted as an allocation $x^f = (x_1^f, \ldots, x_n^f)$ in $\mathcal{E}_n$, where $x_i^f = \dfrac{1}{\mu(I_i)} \displaystyle\int_{I_i} f(t) d\mu(t)$. Observe also that $(x, p)$ is an equilibrium for the economy $\mathcal{E}_n$ if and only if $(f, p)$ is an equilibrium for the continuum economy $\mathcal{E}_c$, where $f(t) = x_i$ if $t \in I_i$.

Let $\mathcal{G}_c$ denote the two-player game associated to the $n$-types continuum economy $\mathcal{E}_c$. A discrete approach of the game $\mathcal{G}_c$ to an associated game $\mathcal{G}_n$ for the finite economy $\mathcal{E}_n$ is related with the equal treatment property of allocations.

Note that if the strategy profiles in $\mathcal{G}_c$ are required to satisfy the equal treatment property, then each player selects the same bundle for agents of the same type and, therefore, the payoff functions depend on the weight of the types in the coalitions selected by player 2 and on the corresponding step functions but do not reflect the possibility of any other different distribution of resources among members of the same type. On the other hand, any Nash equilibrium for the $\mathcal{G}_c$ underlies a core or competitive allocation. Therefore, under convexity of preferences, we can deduce any Nash equilibrium for the game $\mathcal{G}_c$, defines a Nash equilibrium for which the equal treatment property holds.

Next we show how the game $\mathcal{G}_c$ provides a two-player game $\mathcal{G}_n$ associated to the economy with $n$ consumers. For this, observe that, without loss of generality, we can assume that the strategy set of player 2 can be restricted to those strategies $(S, g)$ such that $g$ is feasible for the coalition $S$ and $\mu(S) > 1 - \frac{1}{n}$ (see Remark 3 in the previous Section). This guarantees that all types are represented in the coalition selected by player 2.

Thus, in the game $\mathcal{G}_c$ the strategy set for the player 1 is is given by

$$\{x \in \mathbb{R}_+^{\ell n} : \sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i\}$$

The strategy set for the player 2 is as follows:

$$\{(a, y) \in [\delta, 1]^n \times \mathbb{R}_+^{\ell n} : \sum_{i=1}^n a_i y_i = \sum_{i=1}^n a_i \omega_i\},$$

where $\delta$ is any real number in the interval $(0, 1)$.

Given $a \in [\delta, 1]^n$ let us denote by $\mathcal{S}(a)$ the set of coalitions $S_a$ in the continuum economy $\mathcal{E}_c$ such that $n\mu(S_a \bigcap I_i) = a_i$ for every $i = 1, \ldots, n$. Observe that the game $\mathcal{G}_n$ does not distinguish among the coalitions in $\mathcal{S}(a)$.

Then, in practice, the strategy set for player 2 is the set of pairs which specify a parameter (a rate of participation) and a commodity bundle for each agent such that the resulting allocation is feasible in the sense of Aubin (1979).

Observe that if $x$ is a feasible allocation then $(\mathbf{1}, x)$ is a possible strategy for player 2, where $\mathbf{1}$ denotes the vector $(1, \ldots, 1) \in [\delta, 1]^n$. Furthermore, $(a, \omega)$ is also a strategy that player 2 can choose whatever $a \in [\delta, 1]^n$ may be.

Now, the payoff functions $\Phi_1$ and $\Phi_2$ for player 1 and 2, respectively, are defined as follows

$$\Phi_1(x, a, y) = \begin{cases} \sum_{i=1}^{n} a_i(U_i(x_i) - U_i(y_i)) & \text{if } U_i(x_i) \geq U_i(y_i) \text{ for every } i \\ \min\{U_i(x_i) - U_i(y_i), \ i = 1, \ldots, n\} & \text{otherwise} \end{cases}$$

$$\Phi_2(x, a, y) = \begin{cases} \sum_{i=1}^{n} a_i(U_i(y_i) - U_i(x_i)) & \text{if } U_i(y_i) \geq U_i(x_i) \text{ for every } i \\ \min\{U_i(y_i) - U_i(x_i), \ i = 1, \ldots, n\} & \text{otherwise} \end{cases}$$

**Theorem** 4.1 *If $(x^*, 1, x^*)$ is a Nash equilibrium for the game $\mathcal{G}_n$, then $x^*$ is a Walrasian equilibrium allocation for the economy $\mathcal{E}_n$.*

*Reciprocally, if $x^*$ is a Walrasian equilibrium allocation for the economy $\mathcal{E}_n$, then any strategic profile $(x^*, a^*, y^*)$ with $U_i(y_i^*) = U_i(x_i^*)$ for every $i = 1, \ldots, n$, is a Nash equilibrium for the game $\mathcal{G}_n$.*

*In particular, $x^*$ is a Walrasian equilibrium allocation for the economy $\mathcal{E}_n$, if and only if $(x^*, \mathbf{b}, x^*)$ with $\mathbf{b}_i = b$, for every $i = 1, \ldots, n$, (for instance $(x^*, \mathbf{1}, x^*)$) is a Nash equilibrium for the game $\mathcal{G}_n$.*

*Proof.* Let $s^* = (x^*, 1, x^*)$ be a Nash equilibrium for the game $\mathcal{G}_n$. If $x^*$ is not a Walrasian allocation, then $x^*$ is blocked in the sense of Aubin with weights $a_i$ as closed to one as one wants, for every $i = 1, \ldots, n$ (see Hervés-Beloso and Moreno-García, 2001, 2005, for details). That is, a strategy $(a, y)$ for player 2 exists, such that $\Phi_2(x^*, a, y) > 0 = \Phi_2(s^*)$.

Reciprocally, let $x^*$ be a Walrasian allocation and let $(x^*, a^*, y^*)$ be a strategy profile, with $U_i(y_i^*) = U_i(x_i^*)$ for every $i = 1, \ldots, n$, a strategy profile. If player 1 has an incentive to deviate, then $x^*$ is not efficient. If there is a strategy $(a, y)$ for player 2 such that $\Phi_2(x^*, a, y) > 0$ then $x^*$ is blocked by the grand coalition in the sense of Aubin which is in contradiction to the fact that $x^*$ is Walrasian (see again Hervés-Beloso and Moreno-García, 2001, 2005, for details in the infinite dimensional case).

Q.E.D.

# 5 Some Remarks

Let us consider the finite economy $\mathcal{E}_n$ and the continuum $n$-types economy $\mathcal{E}_c$ with their associated games $\mathcal{G}_n$ and $\mathcal{G}_c$, respectively.

Let $(x, a, y)$ be a strategy profile in the game $\mathcal{G}_n$. Note that if $a_i < 1$ for every $i$, then player 2 has an incentive to deviate by selecting the strategy $(b, y)$ where $b_i = \dfrac{a_i}{\max_i a_i}$. Therefore if $(x, a, y)$ is a Nash equilibrium in the game $\mathcal{G}_n$ then $a_i = 1$ for some $i$.

Let $x$ be a feasible allocation in the economy $\mathcal{E}_n$. Recall that if $(f_x, S, g)$ is a Nash equilibrium for the game $\mathcal{G}_c$ then both players get a null payoff. This is due to the fact that if player 2 obtains a strictly positive payoff, then player 1 can select the strategy which assigns $g$ to the individuals in the coalition $S$ and $\omega$ to the individuals outside $S$. However, this kind of strategy is not possible for player 1 in the game $\mathcal{G}_n$. The reason is that the game $\mathcal{G}_n$ only takes into account the size of the members of a coalition belonging to each type and does not reflect differences in the distribution of commodities among agents with the same type as the $\mathcal{G}_c$, associated to the continuum economy, does.

Furthermore, as we have already remarked, in the continuum case, the strategy set for player 2 can be restricted to coalitions with any size and, therefore, to arbitrarily big coalitions. This implies that, in the particular case of an atomless $n$-types economy we can consider, without loss of generality, that player 2 only selects coalitions where all types are actually represented. Therefore, when one goes from the continuum to the finite economy, in the associated game $\mathcal{G}_n$ player 1 would be restricted to select equal treatment allocations. This implies that the distribution properties among agents of the same type are not contemplated as strategies. That is, when we recast the game $\mathcal{G}_c$ as the game $\mathcal{G}_n$ we lose possibilities of distribution among agents of the same type leading to a reduction of strategies (basically for player 1) which can result in the existence of Nash equilibria where player 2 obtains a strictly positive payoff and, therefore, the allocation proposed by player 1 is not Walrasian. The next example shows our claim:

**An Example.** Consider an economy with two agents and one commodity. Both agents have the same preference relation represented by the utility function $U(x) = x$. Let $\omega_1 = \omega_2 = \omega > 0$ be the initial endowments. Let us consider the associated game $\mathcal{G}_2$ where, without loss of generality, the parameter $a \in [1/2, 1]^2$.

Consider that player 1 chooses feasible allocation $x^* = (2\omega, 0)$ which is efficient but it is not a Walrasian allocation. The best response for player 2 is obtained by maximizing $\alpha(y_1 - 2\omega) + \beta y_2$ subject to $\alpha y_1 + \beta y_2 = (\alpha + \beta)\omega$, $y_1 \geq 2\omega$ and $y_2 \geq 0$. Then $\beta = 1$ and $\alpha < 1$ (see remarks above). Furthermore, taking into account the restrictions, the payoff function for player 2 takes the value $(1-\alpha)\omega$. Therefore, the player 2's best response is given by the weights $a^* = (1/2, 1)$ and the allocation $y^* = (5\omega/2, \omega/4)$. Observe that $\Phi_2(x^*, a^*, y^*) = \omega/2$ whereas $\Phi_1(x^*, a^*, y^*) = -\omega/2$. Also note that when player 2 selects $(a^*, y^*)$ player 1 is not able to get a positive payoff and then $\Phi_1(x, a^*, y^*) = \min\{x_1 - 5\omega/2, x_2 - \omega/4\}$. It is easy to conclude that $(x^*, a^*, y^*)$ is a Nash equilibrium.

Consider now the associated continuum economy with two types of agents. Let the strategy profile $(f_{x^*}, S_{a^*}, f_{y^*})$, where $S_{a^*}$ is any coalition $S$ such that $\mu(S \cap I_1) = 1/4$ and $\mu(S \cap I_2) = \mu(I_2) = 1/2$. Note that $\Pi_2(f_{x^*}, S_{a^*}, f_{y^*}) = \Phi_2(x^*, a^*, y^*) > 0$ which allows us to conclude that $(f_{x^*}, S_{a^*}, f_{y^*})$ is not a Nash equilibrium for the two player game associated to the continuum economy (see Lemma 3.1). Actually, player 1 has an incentive to deviate by selecting the feasible allocation $f$ given by $f(t) = f_{y^*}(t)$ if $t \in S_{a^*}$ and $f(t) = \omega(t)$ otherwise.

We refer the reader to Hervés-Beloso and Moreno-García (2008) where the Walrasian allocations of a finite economy are characterized as the Nash equilibria of a two-player game. In that game, the player 1's payoff are affected by coefficients $f(a_i)$ that depend on the parameters $a_i, i = 1, \ldots, n$. It suffices to consider $f(a_i) = a_i^2$ for each $i$ in order to avoid the situation presented in the example above.

# References

Aubin, J.P., 1979. *Mathematical Methods of Game Economic Theory.* North-Holland, Amsterdam, New York, Oxford.

Aumann, R. J.,1964. Markets with a continuum of traders. Econometrica 32, 39-50.

Aumann, R.J., 1966. Existence of competitive Equilibria in markets with a continuum of traders. Econometrica 34, 1-17.

Bewley, T. (1973): The equality of the core and the set of equilibria in economies with infinitely many commodities and a continuum of agents. International Economic Review 14, 383-393.

Evren, Ö, Hüsseinov, F.: Theorems on the core of a large square economy. Journal of Mathematical Economics, forthcoming.

García-Cutrín, J., Hervés-Beloso, C. (1993): A Discrete Approach to Continuum Economies. Economic Theory 3, 577-584.

Hervés-Beloso, C., Moreno-García, E., Núñez-Sanz, C. and M. Páscoa (2000): Blocking Efficacy of Small Coalitions in Myopic Economies. Journal of Economic Theory, 93, 72-86.

Hervés-Beloso, C., Moreno-García, E., 2001. The Veto Mechanism Revisited. In *Approximation, Optimization and Mathematical Economics*; p. 147-159, 2001. Marc Lassonde (Ed.). Heidelberg, New York. Physica-Verlag.

Hervés-Beloso, C., Moreno-García, E., Yannelis, N.C., 2005. Characterization and Incentive Compatibility of Walrasian Expectations Equilibrium in Infinite Dimensional Commodity Spaces. Economic Theory 26, 361-381.

Hervés-Beloso, C., Moreno-García, E., 2008. Walrasian analysis via two-player games. Games Econ. Behav. doi:10.1016/j.geb.2007.12.001.

Vind, K. (1972): A Third Remark on the Core of an Atomless Economy. Econometrica, 40, 585-586.