REGULARITY AND STABILITY OF EQUILIBRIA IN AN OVERLAPPING GENERATIONS MODEL WITH EXOGENOUS GROWTH

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Abstract. In an exogenous-growth economy with overlapping generations (OG) we analyse local stability of the balanced growth equilibria with respect to perturbations of consumption endowments, thought of as the "monetised" value of a government policy to individuals. We show that perturbed economies have a unique equilibium in the neighbourhood, that the equilibrium allocation expressed in terms of efficient labour units is Fréchet differentiable in L_∞ with derivatives given by kernels, and that the equilibrium is stable in the sense that if perturbations converge to 0 at $\pm\infty$, the corresponding equilibria converge back to the unperturbed equilibrium at $\pm\infty$.

As a corollary this implies a proof of non-vacuity of the main result in Mertens and Rubinchik (2006).

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Contents

1. Introduction

1.1. **Motivation.** Our main objective is to demonstrate the non-vacuity of the main result in Mertens and Rubinchik (2006), namely, an example where the relative utilitarian welfare function is differentiable at a (competitive) equilibrium of an exogenously growing economy with overlapping generations (OG), when viewed as a map from individual (consumption) endowments at birth. The non-vacuity result holds for a generic set of parameters of the economy.

The example provides a template for extending Debreu's regularity result to such economies, and in addition, a stability result of the following form: if perturbations have a bounded support, the corresponding equilibria converge *exponentially* back to the unperturbed equilibrium at $\pm\infty$.

App. D contains a bird's-eye view, suggestive of the generality of our approach.

1.2. **Related Literature.** Gale (1973), who analysed an exchange economy with overlapping generations (OG), demonstrated it has two types of equilibria: balanced ones, with zero net savings; and the golden rule, in which the economy as a whole can hold a debt. Further, Diamond (1965) showed that a Pareto efficient equilibrium in a production economy with overlapping generations should typically involve some debt. Introducing an arbitrary life-time productivity of individuals and exogenous growth we show that Gale's insight is still true: in a golden rule equilibrium net savings almost always differ from the value of accumulated capital, while in any other balanced growth equilibrium the two are equal; see appendix B for the explicit derivation of this dichotomy. The number of equilibria of the latter sort is not necessarily odd as in Kehoe and Levine (1985); their parity varies with the specification of individual life-cycle productivity.

It is well known that OG models are prone to indeterminacy (Kehoe and Levine, 1985; Geanakoplos and Polemarchakis, 1991), even in the presence of capital accumulation (Muller and Woodford, 1988); the reason we avoid this might be that we use for time the more natural real line.

Analysis of regularity of infinite economies with a finite number of consumers (Chichilnisky and Zhou, 1998; Shannon and Zame, 2002)[1] is based on extensions of Sard's theorem, that are not applicable here. We use instead Wiener's theorem on the spectrum of convolution operators to assure the generic invertibility of the the derivative of the equilibrium map required by the implicit function theorem. Although we only demonstrate this approach with an example, it should help to identify a way to verify regularity for a wide class of infinite economies.

1.3. **The Roadmap.** Section 2 contains the specification of the economy, whose equilibria are characterised in section 3. Section 7 is devoted to the regularity result, local uniqueness and differentiability of the balanced growth equilibria (thm. 1), which is followed by establishing stability of those equilibria (cor. 4 and 8), and the description of the properties of the derivative. Finally, section 8 contains the non-vacuity result, differentiability of the relative utilitarian welfare function with respect to perturbations of (normalised) endowments.

2. The setup

2.1. **Individuals.** $N_0 e^{\nu x} dx$ ($N_0 > 0$) individuals get born in $[x, x + dx]$, $\forall x \in \mathbb{R}$. Individual preferences over consumption, a non-negative Lebesgue-measurable function of time $c \geq 0$, are represented as a discounted sum of homogeneous instantaneous utility functions with intertemporal substitution $\sigma > 0$: with $u(x) = \frac{x^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$ for $\sigma \neq 1$ and $u(x) = \mathsf{ln}(x)$ for $\sigma = 1$ (extended by continuity to $[0, +\infty]$),

$$U(c) = \int_0^1 e^{-\beta s} u(c(s)) ds$$

We will ignore the case $\sigma = 1$ till section 7. Cardinal properties of U will play no role till section 8; there we will assume as in Mertens and Rubinchik (2006) U homogeneous of degree $1 - \rho$, but with same ordinal preferences as here.

[1]The latter also contains a detailed overview of the literature

An individual can rent his time endowment (1 at each instant, =100%) out as labour; its efficiency varies according to some integrable function $\varepsilon_s \geq 0$ with age $s \in [0,1]$. Besides, labour productivity grows with time at rate γ, as in classical exogenous growth models. So, aggregate (productive) labour available equals:

$$L_t = N_0 e^{\gamma t} \int_{t-1}^{t} \varepsilon_{t-x} e^{\nu x} dx = N_0 e^{(\gamma+\nu)t} \int_0^1 \varepsilon_s e^{-\nu s} ds$$

His time sells for $\int_0^1 w_{x+s}\varepsilon_s ds$, where x is his birth-date and w_t the per efficiency-unit wage rate at time t. In addition, his null consumption endowment may be perturbed by $\omega_{x,s}$ at age s, so his lifetime wealth is $\int_0^1 p_{x+s}\omega_{x,s} ds + \int_0^1 w_{x+s}\varepsilon_s ds$.

2.2. **Endowments.** Endowments are 0 on the baseline, but else are given by a locally integrable aggregate endowment Ω_t, distributed across age-groups according to some time-invariant (integrable) distribution ϑ_s, $\int_0^1 \vartheta_s ds = 1$, such that $\omega_{x,s} = \vartheta_s \frac{\Omega_{x+s}}{N_0 e^{\nu x}}$, (so "pure redistribution" is excluded, i.e., $\Omega = 0 \Rightarrow \omega = 0$).[2]

2.3. **Production.** All firms are finitely lived, so profits are well-defined.

2.3.1. *Instantaneous production set* is a subset of $\mathbb{R}^5$ describing feasible transformations of effective labour L_t capital K_t, investment I_t, consumption C_t and an intermediate good called "output" Y_t, produced using a Cobb-Douglas technology

$$Y_t = A K_t^{\alpha} L_t^{1-\alpha}, \quad 0 < \alpha < 1, A > 0$$

The instantaneous production cone is any closed cone satisfying free-disposal, containing the graph of the production function and the activities of transforming output into consumption or investment, and contained in the closed convex cone spanned by the production function, free-disposal, and 2-way transformations of output into consumption and investment.

2.3.2. *Capital* K_t accumulates as $K'_t = I_t - \delta K_t$, with $R \stackrel{\text{def}}{=} \gamma+\nu+\delta > 0$; formally:

$$K_t = e^{-\delta(t-t_0)} K_{t_0} + \int_{t_0}^{t} e^{-\delta(t-s)} I_s ds \ , \quad \text{as a (wide) Denjoy integral,}^3$$

(e.g., Čelidze and Džvaršeĭšvili, 1989, p. 27), with as initial condition:

Assumption 1 (Weak Initial Condition). $e^{\delta t} K_t$ converges to 0 at $-\infty$.[4]

2.3.3. *Production and Merchandising Firms. Production firms* use the Cobb-Douglas technology to manufacture undifferentiated output Y_t from labour L_t purchased from individuals at a price of w_t, and capital K_t rented from investment firms at rate r_t. The output is sold to merchandising firms.

Merchandising firms transform Y_t in a one-to-one way into either the consumption good C_t or the investment good I_t. This transformation may or may not be partially reversible depending on the instantaneous production set. C_t is sold to individuals and I_t to investment firms.

[2]See sect. 4.2 in Mertens and Rubinchik (2006) for the discussion.

[3]As argued in Mertens and Rubinchik (2008), this is the right interpretation of the capital-accumulation equation; Denjoy rather than Lebesgue integration is needed in order not to exclude classical solutions of this differential equation a priori; on the other hand, the possibility to use arbitrary integrable I_t rather than exact derivatives is important in this paper, allowing to use the more natural L_p spaces. The interpretation of the equation is as direct as that of the differential equation: K_t is what remains after depreciation from K_{t_0} and the intervening investments.

[4]This is the initial condition of Mertens and Rubinchik (2008) in its weakest form, as in App. B *loc. cit.* It would indeed have been unsatisfactory to require an exponentially fast convergence at $-\infty$ when one of the purposes of the paper is to establish such a stability property. Nevertheless the equilibria we find do satisfy the strongest forms of that initial condition (cf. fn. 5), so in this respect too there is no ambiguity as to what are equilibria.

2.3.4. *Investment Firms* buy some capital K_{t_0} at time t_0, incur flows of outlays for investment $p_t I_t$ and of rents $r_t K_t$, and sell K_{t_1} at time $t_1 > t_0$.

Recall our "standing assumption" (Mertens and Rubinchik, 2006, fn. 16), that investment firms can disvest as well as invest: all restrictions on disvestment are written in the production set of the manufacturing firm, i.e., if disvestment is not possible for some capital good, any sale of that investment good by an investment firm can only be to another investment firm, and can be interpreted as being the transfer of the corresponding capital.

We allow for a measure space $(F, \mathcal{F}, \mu)$ of investment firms, with $F_t \in \mathcal{F}$ denoting the subset of firms alive at time t. Let $[t_0^f, t_1^f]$ with $t_0^f < t_1^f$ be the lifetime of firm f, K_t^f the capital holding of, and I_t^f the investment by firm f at time t. All those functions of f are measurable. The measure space of firms allows a.o. to include the case where the consumers would individually do all the investing. The need for assumption 2.(v) is illustrated in app. A.1.

Assumption 2. (i) $\forall t, \mu\{f \mid t_0^f \le t \le t_1^f\} > 0$;
(ii) I_t^f and K_t^f are locally in t jointly integrable in (t, f);[5]
(iii) $\forall t \notin [t_0^f, t_1^f], K_t^f = I_t^f = 0$; $K_t^f \ge 0$;
(iv) $\int I_t^f \mu(df) = I_t$ a.e.; $\int K_t^f \mu(df) = K_t$ $\forall t$;
(v) $K_t > 0 \Rightarrow \exists F_{t_+}, F_{t_-} \in \mathcal{F}\colon \mu(F_{t_+}) > 0, \mu(F_{t_-}) > 0$, $\exists \varepsilon > 0\colon K_s^f \ge \varepsilon$ on $(F_{t_-} \times [t-\varepsilon, t[) \cup (F_{t_+} \times]t, t+\varepsilon])$.

2.3.5. *Variants.* The *constrained model* satisfies, in addition, irreversibility: neither consumption, nor investment can be transformed back into output. It is a particular case of the model described in Mertens and Rubinchik (2008). So this is the variant that will provide the "proof of non-vacuity" for that paper.

Another variant is where both of the above assumptions are dropped, so that consumption and investment are freely transformable into each other, thus, effectively defining a 1 good model; this variant will be referred to as the *basic model*, which will be used to establish results for the constrained model.

3. Characterisation of Equilibria

We allow as price-systems all Lebesgue-measurable functions p_t with values in $[0, +\infty]$, and similarly for individual consumption streams. Note individual utility functions are well-defined over all Lebesgue-measurable consumption streams c_t with values in $[0, +\infty]$. Following the usual convention in measure theory, define for any product of prices and quantities $p \cdot c$ as 0 in case of a product $0 \times \infty$ or $\infty \times 0$ — thus allowing to think of either prices or quantities as measures. So the cost of any consumption bundle is well defined.

The evaluation of profits of the investment firms is discussed in sect. A.

3.1. **Individual Demand.** Observe, that for any function c in the demand correspondence any equivalent function (coinciding with c a.e.) has the same utility and the same budget, therefore we can think of the demand correspondence as a set of equivalence classes. Similar observation applies to prices.

[5]This assumption implies I_t is locally Lebesgue-integrable, apparently contradicting fn. 3. Lemma 7 in Mertens and Rubinchik (2008), plus prop. 4 (*ibidem*) to deal with our weak form of the initial condition, does not imply that Lebesgue integrability holds nevertheless, because those lemmas rely on irreversibility, which does not hold for the basic variant. But for our purposes here the restriction to local Lebesgue integrability does not matter, since the only equilibria that can appear in our statements in sect. 7 and 8 must satisfy, with $i_t = \frac{I_t}{L_t}$, $\|i_t\| \overset{\text{def}}{=} \mathsf{sup}_x \int_x^{x+1} |i_t| dt < \infty$, hence local Lebesgue-integrability of I_t, and proper Lebesgue integrability in lemma 3, and, e.g. by the inequality in fn. 8, exponential convergence to zero (at rate R) of $e^{\delta t} K_t$ when $t \to -\infty$: thus also the strongest form of Initial Condition (*loc. cit.*). So our results are not affected by this restriction.

Remark 1. Individual demand is derived using the Lagrange technique, thus allowing, a.o. the optimal utility to attain any values including $\pm\infty$. The latter solutions (with marginal utility of income being undefined, and therefore, 'Euler equations' unapplicable) can be consistent, as is shown in lemma 2, with prices and income being positive everywhere, so they are not a-priori 'pathological'.

The budget set is left undefined when both income and prices are infinite, so this is the only case in which the indirect utility is undefined and individual demand is unrestricted. Such case is ruled out by equilibrium restrictions (cf. prop. 1), so the conclusion that the budget set is well-defined in an equilibrium is 'convention-free'.

Lemma 1. *(i)* $\forall a > 0, \forall p \in \overline{\mathbb{R}}_+$, $\max_{0\le c\le\infty}[au(c) - pc] = \frac{1}{\sigma-1}a^\sigma p^{1-\sigma}$, *where the left hand member is defined by continuity in* c *at* ∞.

(ii) $c = (\frac{a}{p})^\sigma$ *is a maximiser, and the only one iff either* $p < \infty$ *or* $\sigma > 1$.

Proof. Note that the bracket is concave and u.s.c. on $\mathbb{R}_+$ (lack of continuity if $p = \infty$ and $\sigma > 1$). Therefore the extension by continuity at ∞ is well-defined, and a maximum always exists in $\overline{\mathbb{R}}_+$. For $p = \infty$, $c = (\frac{a}{p})^\sigma = 0$ is a maximum, and the only one iff $\sigma > 1$. So the maximal value equals $au(0)$, i.e. 0 if $\sigma > 1$ and $-\infty$ else, as given by the right hand member. And the case $p < \infty$ is obvious. ∎

Notation. λ denotes Lebesgue measure on $\mathbb{R}$.

Lemma 2. *For any budget* $M \in \overline{\mathbb{R}}_+$ *and price-system* p_s ($s \in [0, 1]$), *let*

$$z_s = \frac{p_s}{M}\ , \qquad \chi_s = e^{-\beta\sigma s} z_s^{1-\sigma}\ , \qquad c_s^* = \frac{(e^{\beta s} z_s)^{-\sigma}}{\int_0^1 \chi_t dt} \tag{1}$$

where $\frac{0}{0}$ *is defined as* 0, *a negative power of* 0 *as* $+\infty$, *and* $\frac{\infty}{\infty}$ *is left undefined* ≥ 0.

Let also $J = \int_0^1 z_s^{1-\sigma} ds$ *and* $U^* \stackrel{\text{def}}{=} \frac{\sigma}{\sigma-1}\left[\int_0^1 \chi_s ds\right]^{\frac{1}{\sigma}}$.

Note those integrals may be well-defined even when z_s *is not a.e. well-defined, e.g., if the integral over the set where* z_s *is well-defined is already infinite.*

Then:

(i) Indirect utility is unspecified, even as a sup *(the budget set itself being unspecified), iff* $M = \infty$, $\lambda\{p_s = \infty\} > 0$, *and* ($\sigma > 1 \Rightarrow p_s = \infty$ *a.e.*).

This is also the case where U^* *is not defined.*

Else indirect utility equals U^* *and is achieved.*

(ii) Demand is unique (as an equivalence class) iff both (1) U^* *is well defined and (2) either* $U^* \in \mathbb{R}$ *or* ($\sigma < 1$ *and)* $z_s = \infty$ *a.e.*

Demand is also unique ($= 0$) *for all* s *such that* $z_s = \infty$.

(iii) Whenever demand is unique, z_s *and* c_s^* *are well-defined a.e., and demand is given by the equivalence class of* c_s^*.

(iv) U^* *is well defined iff* J *is so, and then* $U^* \in \mathbb{R}$ *iff* $J < \infty$.

Proof. The last point (iv) is obvious.

If $M = 0$ and $p_t > 0$ a.e., the result is obvious: $c_t = 0$, so if $\sigma > 1$, then $U^* = 0$, if $\sigma < 1$, then $U^* = -\infty$.

When $M = 0$, $\sigma > 1$, and $\lambda\{p_t = 0\} > 0$, many feasible bundles achieve $U^* = \infty$, so demand is not unique, hence the lemma is established in this case.

When $M = 0$, $\sigma < 1$, and $0 < \lambda\{p_t = 0\} < 1$, the agent's instantaneous optimal consumption is clearly $c_t = \infty$ when $p_t = 0$, $c_t = 0$ otherwise; but since $\lambda\{p_t = 0\} > 0$ this gives him utility $-\infty$, so any point in his budget set is optimal, and the lemma is established in this case too. And if $p_t = 0$ a.e., $c_t = \infty$ a.e., so $z_t = 0$ a.e., $U^* = 0$, and this case is covered too.

Thus the lemma is established when $M = 0$. So, henceforth $M > 0$.

Assume now $M < \infty$. To calculate the indirect utility, consider, after Lagrange, for $\mu > 0$ the maximum of $\mathcal{L}(c) \stackrel{\text{def}}{=} \int_0^1 \left[\mu e^{-\beta t} u(c_t) - p_t c_t\right] dt$. By lemma 1, it equals $\frac{1}{\sigma-1}\mu^\sigma\int_0^1 e^{-\beta\sigma t}p_t^{1-\sigma}dt$, and the set of maximisers is the equivalence class of $\tilde{c}_t=(\frac{\mu e^{-\beta t}}{p_t})^\sigma$, which is unique iff the maximum of $\mathcal{L}$ is finite and either $p_t<\infty$ a.e. or $\sigma>1$. Clearly the maximum is finite iff $J<\infty$. Since for $\sigma<1$, $J<\infty$ implies $p_t<\infty$ a.e., uniqueness too holds iff $J<\infty$.

For $J<\infty$, the budget $M=\int_0^1 p_t c_t dt=\mu^\sigma\int_0^1 e^{-\beta\sigma t}p_t^{1-\sigma}dt$ is finite.

In particular, if $0<J<\infty$, by varying μ we can obtain any $0<M<\infty$; so for any such M, and the corresponding $\mu(M)$, we obtain $\tilde{c}(\mu(M))=c^*$ and $U(c^*)=U^*$ as in the statement.

And c^* is the agent's unique optimal choice given his budget M: for any $c'\neq c^*$ s.t. $\langle p,c'\rangle\stackrel{\text{def}}{=}\int p_t c'_t dt\leq M$, the integrability of pc' implies $\mu U(c')-\langle p,c'\rangle=\mathcal{L}(c')<\mathcal{L}(c^*)=\mu U(c^*)-\langle p,c^*\rangle=\mu U^*-M$, where the strict inequality is by the uniqueness property of the maximiser $\tilde{c}$. So $\langle p,c'\rangle\leq M$ and $c'\neq c^*$ implies $U(c')<U^*$.

Thus the statement is proved for $0<J<\infty$ and $M<\infty$.

$J=0$ means, when $0<M<\infty$, that, if $\sigma>1$, $p_t=\infty=z_t$ a.e., so $c=0=c^*$, and if $\sigma<1$, $p_t=0=z_t$ a.e., so $c=\infty=c^*$, and in both cases the utility $U^*=0$ is attained, thus the statement is established in that case too.

To summarize, the lemma is proved when $M<\infty$ and either $M=0$ or $J<\infty$.

If $J=\infty$ (and, recall, $0<M<\infty$), then, for $\sigma<1$, $\mathcal{L}(c)=-\infty$ $\forall c$. So, whenever $p_t c_t$ is integrable, the indirect utility is $\int_0^1\mu e^{-\beta t}u(c_t)dt=-\infty$. If $p_t=\infty$ a.e. then the demand is unique, $c=0$; otherwise all points in the budget set are utility maximisers. Thus this case is solved too.

So, in case $0<M<\infty$ it remains to prove the lemma for $J=\infty$ and $\sigma>1$, which then is assumed to hold for the next two paragraphs.

Consider the indirect utility function $V(M)$ (for fixed price system p): by homogeneity, it must be of the form $vu(M)$ for some $v\geq 0$. Assume now $v<\infty$. Then by lemma 1 for any $\mu>0$, $\max_{0<M<\infty}(\mu V(M)-M)=\frac{1}{\sigma-1}(\mu v)^\sigma$. So for any c such that $p_t c_t$ is integrable we get $\mathcal{L}(c)=\int_0^1\left[\mu e^{-\beta t}u(c_t)-p_t c_t\right]dt\leq\frac{1}{\sigma-1}(\mu v)^\sigma$. As was shown above, the unique maximiser of $\mathcal{L}$ is $\tilde{c}(\mu)$. Let then $c_t^N=\min(\frac{N}{p_t},\tilde{c}_t(\mu))$. $p_t c_t^N$ being integrable, c_t^N satisfies our bound above. If $p_t=\infty$ then $\tilde{c}_t(\mu)=0$ and so is $\frac{N}{p_t}$ for any N. And $p_t\neq 0$ a.e., as $J=\infty$. Since then c_t^N increases to $\tilde{c}_t(\mu)$, the corresponding integrands in $\mathcal{L}(c_t^N)$ are non-negative and increase to that for $\tilde{c}_t(\mu)$: by the monotone convergence theorem, $\tilde{c}_t(\mu)$ still satisfies the same inequality, i.e., as seen above, $\frac{1}{\sigma-1}\mu^\sigma\int_0^1 e^{-\beta\sigma t}p_t^{1-\sigma}dt\leq\frac{1}{\sigma-1}(\mu v)^\sigma<\infty$, contradicting $J=\infty$.

Thus $v=\infty$, i.e., $V(M)=+\infty$. We claim next that therefore, $\forall M\colon 0<M<\infty$, there exist (many) c in the budget set with $U(c)=\infty$. Indeed, note first that there exists a partition of $[0,1]$ in 2 borel subsets of equal Lebesgue measure such that $J=\infty$ on each (e.g., consider the distribution of the integrand of J, and on each atom use non-atomicity of Lebesgue measure). Next re-use this on one of the subsets, etc., to obtain a borel partition into a sequence B_n with $\lambda(B_n)=2^{-n}$ s.t. $J=\infty$ on each B_n. Hence for each B_n the supremum of utility derived on that subset of time with a strictly positive finite budget should be infinite by the argument above. Therefore one can choose for each n a consumption plan on B_n costing $\leq 2^{-n}M$ and with "utility on B_n" ≥ 1: the resulting total consumption plan costs $\leq M$ and has infinite utility. Thus, $U^*=+\infty$ and demand is multivalued.

Remains thus only to establish the lemma when $M=\infty$. Then, for $\sigma>1$, if $p=\infty$ a.e., demand is unspecified, and if $\lambda\{p_t<\infty\}>0$, $U^*=\infty$ and demand is multivalued. While for $\sigma<1$, if $p<\infty$ a.e., $U^*=0$ and $c^*=\infty$, and if $\lambda\{p_t=\infty\}>0$, demand is unspecified. ■

3.2. **Equilibrium restrictions.** The price system p appearing above is the price p_t^C of consumption. The prices p_t^Y, p_t^I and p_t of output, investment, and capital resp., can a-priori be different. We want to prove all four are equal.

Let also w_t be the wage rate and r_t the rental rate of capital. p_t prices a stock, so is an — a priori arbitrary — function of t; but all others price flows, so are naturally thought of as equivalence classes of Lebesgue-measurable functions.

In deriving equilibrium restrictions, we will use equilibrium conditions only when completely non-anbiguous. E.g., for consumer maximisition, we will use only for consumers for whom the integral defining defining their wealth is a well-defined Lebesgue integral, and even then only when in addition their budget set is well-defined, and their utility attains a maximum on it. Similar precautions concerning the profits of investment firms are discussed in app. A.2. At the end, we will show in prop. 1, that nevertheless the equilibria thus characterised are fully satisfactory (i.e., wealth is always well-defined, utility always attains a maximum on the budget set, etc.)

Note that if $p_t^Y = \infty$, profits of any production plan with positive output are either infinite or, if also w_t or r_t are ∞, undefined. This is incompatible with any equilibrium concept, so we exclude it formally, as part of the definition of equilibrium:

Definition 1. In equilibrium, $p_t^Y < \infty$ a.e.

Lemma 3. *$K_t = e^{-\delta t}\int_{-\infty}^{t} I_s e^{\delta s} ds$ as an improper Lebesgue integral.*

Proof. Let $t_0 \to -\infty$ in the capital accumulation equation (initial condition). ■

Lemma 4. *In equilibrium, $r_t \geq 0$, $w_t \geq 0$, and $\left(\frac{r_t}{\alpha}\right)^{\alpha}\left(\frac{e^{-\gamma t}w_t}{1-\alpha}\right)^{1-\alpha} \geq Ap_t^Y$ whenever the left hand side is well-defined.*

Proof. Profits equal $p_t^Y AK_t^{\alpha}L_t^{1-\alpha} - r_tK_t - e^{-\gamma t}w_tL_t$. Thus $K_t = L_t = 0$ shows that maximal profits are ≥ 0. So we have to show that the condition is necessary and sufficient for profits to be ≤ 0. The maximal profit is the maximum over the 2 cases $L_t = 0$ and $L_t > 0$. The maximum with $L_t = 0$ being ≤ 0 is equivalent to $r_t \geq 0$. For $L_t > 0$, dividing by L_t, it means that $p_t^Y Ak_t^{\alpha} - r_tk_t - e^{-\gamma t}w_t \leq 0\ \forall k_t \stackrel{\text{def}}{=} \frac{K_t}{L_t} \geq 0$. Now, since $p^Y < \infty$, $p_t^Y Ak_t^{\alpha} - r_tk_t$ is well-defined $\forall k_t \geq 0$, so the condition is equivalent to $e^{-\gamma t}w_t \geq \mathsf{sup}_{k_t\geq 0}(p_t^Y Ak_t^{\alpha} - r_tk_t)$, which equals 0 if $p_t^Y = 0$, and else, by lemma 1 (using $\frac{1}{\sigma_t} = 1-\alpha$, $a = \alpha Ap_t^Y$, $p = r_t$), $\frac{1-\alpha}{\alpha}(\alpha Ap_t^Y)^{\frac{1}{1-\alpha}} r_t^{\frac{-\alpha}{1-\alpha}} \geq 0$; so $w_t \geq 0$ anyway, and $(\frac{\alpha e^{-\gamma t}w_t}{1-\alpha})^{1-\alpha} \geq \alpha Ap_t^Y r_t{}^{-\alpha}$ needs to hold if $p_t^Y > 0$. Multiplying by r_t^{α} yields an equivalent inequality, given the "whenever" part of the statement: the equivalence is obvious if $0 < r_t < \infty$; if $r_t = 0$, it is because then both inequalities mean $w_t = \infty$; and if $r_t = \infty$, it is because then both inequalities mean $w_t \geq 0$. Hence the statement, since the inequality there holds obviously also when $p_t^Y = 0$. ■

Lemma 5. *(i) $p_t^C \leq p_t^Y \geq p_t^I = p_t$ a.e., and $\forall t$ $p_t < \infty$. Further,*

$$(1)\quad \varsigma(t) \stackrel{\text{def}}{=} e^{-\delta t}p_t - \int_t^{\infty} r_s e^{-\delta s} ds \geq 0 \textit{ is decreasing, and constant wherever } K_t > 0.$$

(ii) Wherever the constraint that consumption can not be transformed into output is not binding $p^C = p^Y$ a.e. Wherever the constraint that investment can not be transformed into output is not binding, $p^I = p^Y$ a.e.

Proof. The zero profit condition for the merchandising firm implies that $p_t^I \leq p_t^Y$ and $p_t^C \leq p_t^Y$ a.e. If the constraint that consumption can not be transformed into output is not binding, it also implies $p^C = p^Y$ a.e. If, in addition, the constraint that investment can not be transformed into output is not binding, then $p^Y = p^I$ a.e.

To show $p_t < \infty\ \forall t$, assume else $p_{t_0} = \infty$. But then the firms alive just before t_0 can make infinite profits. Indeed, consider F_{t_0-} and the corresponding ε; since $p_t^I \leq p_t^Y < \infty$ a.e., $\exists M < \infty$: $\lambda\{t \in [t_0 - \varepsilon, t_0] \mid p_t^I \leq M\} > 0$. So if those firms invest at unit rate during this set they get a positive amount of capital at finite cost, that can be re-sold for ∞ at t_0; contradiction.

Next, r_t is locally integrable: if it was not integrable on $[t_0-\varepsilon, t_0]$, let the firms in F_{t_0-} buy some capital at $t_0-\varepsilon$, cash its returns until t_0, and sell it then, yielding infinite profit, since $p_t<\infty$. Similarly with F_{t_0+} if r_t is not integrable on $[t_0, t_0+\varepsilon]$.

Consider a policy variation (satisfying the requirements sub A.2 above for completely arbitrary p_t) where each firm f s.t. $K_t^f \geq \varepsilon$ for $a<t<b$ buys, with $\delta K_t^f = \xi e^{-\delta t}\mathbf{1}_{]a,b[}$, δK_{a_+} additional capital at time a, and sells δK_{b_-} at time b, while cashing the returns in between. Then $\delta\pi^f = \xi(g(b)-g(a))$, with $g(t) = e^{-\delta t}p_t + \int_0^t r_s e^{-\delta s}ds$. Since r_s is locally integrable and $p_t<\infty$, $g(t)<\infty$.

Fix now t, and assume either $K_t>0$ or $\xi>0$. By assumption 2.v, the above deviation is feasible $\forall f\in F_{t_+}$, $\forall \xi$: $|\xi|\leq \varepsilon e^{\delta t}$, $\forall a,b$: $t\leq a<b\leq t+\varepsilon$. So, since $\mu(F_{t_+})>0$, absence of profitable deviations implies g is decreasing on $[t,t+\varepsilon]$ and is constant there if $K_t>0$. Similarly on $[t-\varepsilon,t]$, thus, t being arbitrary, g is decreasing, and is constant wherever $K_t>0$.

So $\forall t\geq 0$ $g(0)\geq \int_0^t r_s e^{-\delta s}ds$, and $g(0)<\infty$, hence $\int_0^\infty r_s e^{-\delta s}ds<\infty$; subtracting this quantity from $g(t)$ we get that $\varsigma(t) = e^{-\delta t}p_t - \int_t^\infty r_s e^{-\delta s}ds$ is decreasing and (letting $t\to\infty$) ≥ 0, and is constant wherever $K_t>0$.

Next we show, following A.2, that $p_t^I = p_t$ a.e.

Else, p being borel by the previous conclusion, there would be, by Lusin's theorem, a non-empty compact set K to which p_t^I, p_t and I_t have a continuous restriction, with either (1) $p_t>p_t^I$ $\forall t\in K$ or (2) $p_t<p_t^I$ $\forall t\in K$ and which equals the support of the restriction of Lebesgue measure to itself. By the joint local-integrability of I_t^f (assumption 2.ii), remove from F the set where I_t^f is not integrable over f's lifetime, this set is negligible by Fubini's theorem. We now construct a policy variation. Fix some $T\in K$ and let $K_n = K\cap[T-n^{-1}, T+n^{-1}]$, F_n is the set of firms alive during a non-negligible subset of K_n ($F_n = \{f \mid \lambda(K_n\cap[t_0^f, t_1^f])>0\}$) and let $\tau_{n,0}^f = \tau_{n,1}^f = t_1^f$ $\forall f\notin F_n$, and $\forall f\in F_n$, $\tau_{n,0}^f = \mathsf{min}\{K_n\cap[t_0^f,t_1^f]\}$, $\tau_{n,1}^f = \mathsf{max}\{K_n\cap[t_0^f,t_1^f]\}$. Further, $\mu(F_n)>0$ because T is in the support of the Lebesgue measure on K_n and by assumption 2.i. Let the firm buy/sell additional investment $\delta I_t^{n,f} = \xi\mathbf{1}_{K_n\cap[t_0^f,t_1^f]}(t)$, where $\xi \stackrel{\text{def}}{=} \mathsf{sign}(p_T - p_T^I)$ and sell the additional accumulated capital at time $\tau_{n,1}^f$, resp., buy additional capital at time $\tau_{n,0}^f$ such that it will be exactly offset by $\delta I^{n,f}$.

So, if $\xi=1$, then $\delta K_t^{n,f} = e^{-\delta t}\mathbf{1}_{\tau_{n,0}^f\leq t\leq\tau_{n,1}^f}\int_{-\infty}^t e^{\delta s}\delta I_s^{n,f}ds$ (sold at $t=\tau_{n,1}^f$). And for $\xi=-1$, $\delta K_t^{n,f} = e^{-\delta t}\mathbf{1}_{\tau_{n,0}^f\leq t\leq\tau_{n,1}^f}\int_t^{\tau_{n,1}^f} e^{\delta s}|\delta I_s^{n,f}|ds$ (bought at $t=\tau_{n,0}^f$). Observe that $\delta K_t^{n,f}$ is clearly of bounded variation and ≥ 0, and is jointly measurable (by the same property of $\delta I_t^{n,f}$), and vanishes outside $[\tau_{n,0}^f, \tau_{n,1}^f]$.

We can finally compute the induced variation in profit, denoting the transaction date (resp. $\tau_{n,1}^f$ and $\tau_{n,0}^f$) by $t_n(f)$

$$\delta\pi^{n,f} = \xi p_{t_n(f)}\delta K_{t_n(f)}^{n,f} + \int_{\tau_{n,0}^f}^{\tau_{n,1}^f}\big(r_t\delta K_t^{n,f} - p_t^I\delta I_t^{n,f}\big)dt \tag{2}$$

The last term in the integrand is jointly integrable in (t,f), by the same property of $\delta I_t^{n,f}$ and the continuity of p^I on the compact set K, $\delta I_t^{n,f}$ being 0 outside of K.

And the first term of the integrand is $r_t\delta K_t^{n,f} = r_t e^{-\delta t}\mathbf{1}_{\tau_{n,0}^f\leq t\leq\tau_{n,1}^f}\int_{-\infty}^t e^{\delta s}\delta I_s^{n,f}ds$, where all terms are clearly non-negative and jointly measurable. This is majorised by $\int_{\tau_{n,0}^f}^{\tau_{n,1}^f} r_t e^{-\delta t}\mathbf{1}_{\tau_{n,0}^f\leq t\leq\tau_{n,1}^f}e^{\delta s}\delta I_s^{n,f}ds$, where the integrand is clearly jointly integrable in (s,t,f): $r_t e^{-\delta t}\mathbf{1}_{\tau_{n,0}^f\leq t\leq\tau_{n,1}^f} \leq r_t e^{-\delta t}\mathbf{1}_{\mathsf{min}(K)\leq t\leq\mathsf{max}(K)}$, which is an integrable function of t alone by the local integrability of r_t and the compactness of K, while $e^{\delta s}\delta I_s^{n,f}$ is jointly integrable in (s,f) by the joint integrability of δI^n and the boundedness of $e^{\delta s}$ on the compact set K. This joint integrability in (s,t,f) ensures then in particular that the first term in our integrand, $r_t\delta K_t^{n,f}$, is also jointly integrable in (t,f).

Both terms in the integrand being jointly integrable, we can use linearity of the integral and integrate them separately. And for the first term, since it comes by integration from this jointly integrable expression in (s,t,f), we can permute the order of integration there between s and t. We get thus $\int_{\tau^f_{n,0}}^{\tau^f_{n,1}} r_t \delta K^{n,f}_t dt = \int_{\tau^f_{n,0}}^{\tau^f_{n,1}} \big(\int_s^{\tau^f_{n,1}} r_t e^{-\delta t} dt\big) e^{\delta s} \delta I^{n,f}_s ds$, and hence, replacing also $\delta K^{n,f}_{t_{n,f}}$ by its value, and re-using linearity of the integral: $\delta\pi^{n,f} = \int_{\tau^f_{n,0}}^{\tau^f_{n,1}} \big[p_{\tau^f_{n,1}} e^{-\delta\tau^f_{n,1}} + \int_t^{\tau^f_{n,1}} r_s e^{-\delta s} ds - p^I_t e^{-\delta t}\big] e^{\delta t} \delta I^{n,f}_t dt, = \int_{K_n \cap [t^f_0, t^f_1]} \big[p_{t_n(f)} e^{-\delta t_n(f)} + \int_t^{t_n(f)} r_s e^{-\delta s} ds - p^I_t e^{-\delta t}\big] e^{\delta t} dt$.

$\delta\pi^{n,f}$ must be a.e. non-positive (equivalently—Fubini again—$\int_S \delta\pi^{n,f}\mu(df) \le 0 \ \forall S \in \mathcal{F}$); since F_n is non-negligible there exists thus $f_n \in F_n$ s.t. $\delta\pi^{n,f_n} \le 0$. Since by definition of F_n, $\lambda(K_n \cap [t^f_0, t^f_1])dt > 0$ for $f \in F_n$, $\exists s_n \in K_n$ s.t. $\xi\big[p_{t_n(f_n)} e^{-\delta t_n(f_n)} + \int_{s_n}^{t_n(f_n)} r_s e^{-\delta s} ds - p^I_{s_n} e^{-\delta s_n}\big] \le 0$. Since K_n shrinks to $\{T\}$, $t_n(f_n)$ and s_n converge to T. So by the continuity of p^I and p on K, we get in the limit $\xi\big[p_T e^{-\delta T} - p^I_T e^{-\delta T}\big] \le 0$, contradiction. ■

Remark 2. As the "hot potato" example (app. A.1) shows, the assumption 2.v is clearly needed to derive the lemma. Without the assumption, one cannot deduce the constancy of ς_t in equation 1, even where $K_t > 0$ (though one can obtain that there $\varsigma(t)$ is the sum of countably many jumps, i.e., its continuous part is 0). And one gets then similarly in prop. 1 the analog of (6) for such ς_t. So the example presents really the pure form of the difficulty.

Assumption 3. $\omega_{x,s}$ is jointly locally Lebesgue-integrable.

Lemma 6. *(i) Aggregate consumption C_t is locally integrable, p_t is locally bounded, and $\lambda\{t \in [x, x+1] \mid p^C_t > 0\} > 0$ for all but countably many x's.*

(ii) If either p^C is locally bounded or $\omega \ge 0$, p_t is locally bounded away from 0, $M_x \stackrel{\text{def}}{=} \int_0^1 (p^C_{x+s}\omega_{x,s} + w_{x+s}\varepsilon_s)ds$ and U^*_x are well-defined a.e., Lebesgue measurable, and a.e. $M_x < \infty$ and $U^*_x < u(\infty)$. For $\sigma > 1$, one has further $p^C_t > 0$ a.e. and, for a.e. x, $M_x > 0 \Rightarrow \int_x^{x+1} (p^C_t)^{1-\sigma} dt < \infty$.

Proof. (i): C_t is bounded by the sum of $N_0 \int_0^1 e^{\nu(t-s)}\omega_{t-s,s} ds$, and of Y_t; the first is locally integrable by assumption 3 on ω, the latter is locally bounded: L_t is so by definition, the initial condition (assumption 1) implies that $\exists t$: $K_t < \infty$, local Denjoy-integrability of I_t implies then, by lemma 3, $K_t < \infty \ \forall t$, and K_t is continuous by lemma 3, so it is locally bounded. Thus C_t is locally integrable. That p_t is locally bounded follows from the two statements in lemma 5.(i).

(ii): As $w_t \ge 0$ by lemma 4 and $\varepsilon_t \ge 0$, $M_x = \int_0^1 (p^C_{x+s}\omega_{x,s} + w_{x+s}\varepsilon_s)ds$ is a.e. well-defined: if $\omega \ge 0$, by non-negativity of the integrand, and if p^C is locally bounded, the negative part of the integrand is a.e. integrable, by the assumption above and by Fubini's theorem. M_x is Lebesgue measurable by the joint measurability of ω.

Since $p^C_t < \infty$ by lemma 5.i and def. 1, z_t is well-defined in lemma 2, and so are U^*_x and J_x (resp. by lemma 2.i and 2.iv).

If $\sigma < 1$ assume, contrary to the statement, that, in the notation of lemma 2, $z_t = 0$ a.e. (so that $p^C_t = 0$ a.e. or $M_x = \infty$). Then by the same lemma, $U^* = 0$ and the unique optimal individual consumption is infinite a.e. (achieving $U^* = 0$), thus contradicting $C_t < \infty$ a.e. that follows from the previous point (i).

If $\sigma > 1$, let us show that the aggregate utility of individuals born between a and $b > a$, $\int_a^b U_x(c_x)dx$ is bounded over all feasible reallocations $c_{x,s}$ of C_t, or equivalently, since $e^{-\beta s}$ is bounded and bounded away from 0, that $\int_a^b \int_x^{x+1} u(c_{x,t})dtdx$ is bounded given that $\int_{t-1}^t c_{x,t}e^{\nu x}dx \le C_t \ \forall t \in [a, b+1]$, or again equivalently, since $e^{\nu x}$ is bounded and bounded away from 0 on that interval, and since the maximisation of $\int_a^b U_x(c_x)dx$ clearly implies distributing nothing to agents x not born in $[a,b]$, given that $\int_{\max\{a,t-1\}}^{\min\{b,t\}} c_{x,t}dx \le C'_t$, where $C'_t \stackrel{\text{def}}{=} \frac{C_t}{\min\{e^{\nu a}, e^{\nu(b+1)}\}}$.

By Fubini, our objective function equals thus $\int_a^{b+1}\int_{\max\{a,t-1\}}^{\min\{b,t\}} u(c_{x,t})dxdt$, so by concavity of u the maximiser is $c_{x,t} = \frac{C'_t}{\min\{b,t\}-\max\{a,t-1\}}$, hence $\int_a^b U_x(c_x)dx \leq \frac{\sigma}{\sigma-1}\int_a^{b+1}(C'_t)^{1-\frac{1}{\sigma}}[\min\{b,t\}-\max\{a,t-1\}]^{\frac{1}{\sigma}}dt < \infty$, since the bracket is bounded and $\sigma > 1$. Thus $U_x^* < \infty$ a.e., i.e., $J_x < \infty$ (by lemma 2.iv), hence $z_t > 0$ a.e., so $M_x < \infty$ and $p_t^C > 0$ a.e. Then $J_x < \infty$ and $M_x > 0$ imply $\int_x^{x+1}(p_t^C)^{1-\sigma}dt < \infty$.

As to $\lambda\{t \in [x, x+1] \mid p_t^C > 0\} > 0$, we just proved, under the assumptions of (ii), that it was > 0 a.e. So this holds anyway: else there would exist $a < b$ s.t., for $x \in [a, b]$, $p_t^C = 0$ a.e. on $[x, x+1]$ — so bounded! Thus the proof of (ii) applies to those x, contradiction. It is then clear that for $x \neq y$ in the exceptional set, $|x - y| > 1$.

If p_t is not locally bounded away from 0, there exists by lemma 5(i) some t s.t. $p_{t+} = 0$, ς being decreasing. The same equation implies then $\varsigma_s = r_s = 0$ a.e. on $]t, \infty[$. Note that, by (i), $\forall x\ \lambda\{t > x \mid p_t^C > 0\} > 0$. By lemma 4, $r_s = 0$ implies $w_s = \infty$ wherever $p_t^C > 0$, since $p^C \leq p^Y$ a.e. Since M_x is a.e. well-defined, $w_s = \infty$ on a set of positive measure implies the same for M_x, contradiction again. ■

Remark 3. In the following, we also select w.l.o.g. canonical representatives within equivalence classes, so as to make maximisation hold everywhere instead of just a.e.

Lemma 7. *In any equilibrium where $p^Y = p^I$ there is full-employment, i.e., we can assume L_t is given $\forall t$ by our formula in sect. 2.1, and no free-disposal, i.e., $\forall t\ Y_t = AK_t^\alpha L_t^{1-\alpha}$. Also $0 \leq p_t^C \leq p_t\ \forall t$, so p^C is locally bounded, and:*[6]

(1) $K_t > 0$ a.e.
(2) $g_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha}$ *is locally integrable, and* $r_t = g_t p_t$
(3) $w_t = (1-\alpha)e^{\gamma t}\frac{Y_t}{L_t}p_t$
(4) $p_t = e^{-\pi_t+\int_0^t(\delta-g_s)ds}$, *with π_t real-valued, increasing, and constant on $K_t > 0$.*
(5) $M_x \stackrel{\text{def}}{=} \int_0^1 [p_{x+s}^C\omega_{x,s} + w_{x+s}\varepsilon_s]ds \geq 0$ *a.e., and is locally integrable.*

Proof. $0 \leq p_t^C \leq p_t\ \forall t$ by lemma 5.i; so p^C is locally bounded and lemma 6.ii applies.

Next, note first that any increase in budget increases the utility for a.e. agent. For $0 < M < \infty$, let $V_x(M) = u(M)\left(\int_0^1 e^{-\beta\sigma s}p_{x+s}^{1-\sigma}ds\right)^{\frac{1}{\sigma}}$. It is the indirect utility of consumer born at x with income M by lemma 2.i. For $\sigma < 1$, the integral is positive (lemma 6.i), and finite, p^C being locally bounded, so V_x is well-defined even on $[0, \infty]$ and is strictly increasing in M. For $\sigma > 1$, $(p_t^C)^{1-\sigma} > 0$, p^C being finite, so the integral is positive, and assume first it is finite. Then V_x is strictly increasing in M as above. While if the integral is infinite, lemma 6.ii implies that $M_x = 0$. Since $p_t^C > 0$ a.e. (lemma 6), this implies in lemma 2 that $z_t = \infty$ a.e., so $J_x = 0$ and hence U_x^* is well-defined and $= 0$. On the other hand for $M > 0$, the integral being infinite implies that $V_x(M)$ is so. Thus in this case too any increase in budget increases utility — from 0 to $+\infty$.

Now, since $M_x < \infty$ a.e. (lemma 6.ii), any additional amount of money earned increases the budget, and hence the utility.

Thus, if $Y_t = 0$, we must have $w_t = 0$. Indeed, else there would be positive labour supply, since agents have no disutility for labour and gain additional utility from any increase in budget. However this positive amount of labour, at positive cost, would imply that the production firm makes negative profits, Y_t being 0, which

[6]Condition 4 is equivalent to the differential equation $H'_t = g_t$, with $H_t \stackrel{\text{def}}{=} \delta t - \ln p_t$, where the equality holds everywhere, and H' may have $\overline{\mathbb{R}}$ values (but must be well-defined, so $H_y - H_x$ must be well-defined for y sufficiently close to x, so that H must be $\mathbb{R}$-valued to be differentiable). Indeed, π_t is increasing only where $K_t = 0$, i.e., $g_t = \infty$, so for those t the equation $H'_t = g_t$ is unaffected and p_t is a solution of the initial differential equation. Conversely, any solution H of that differential equation implies a p_t as specified, using that a monotone function H on $[0, 1]$ is a.e. differentiable and $H_1 - H_0 \geq \int_0^1 H'_t dt$. This is in turn closely related to Perron's approach to the Denjoy integral, and to our argument at the end of the proof, and in fn. 7.

contradicts optimality of the production plan. Now, $w_t = 0$ and $p_t > 0$ implies by lemma 4 that $r_t = \infty$; by (1) in lemma 5 this can happen only on a negligible set of t, since $p < \infty$. So $Y_t > 0$ a.e.

$Y_t > 0$ implies both $K_t > 0$ and $L_t > 0$; then profit maximisation by the production firm implies (2) (in the form $r_t = \alpha\frac{Y_t}{K_t}p_t$) and (3). Since this is on a set of full measure, and r_t and w_t play a role only as equivalence classes, we can assume the equations hold everywhere the right hand member is well-defined. In particular, $w_t > 0$ a.e., and hence, by the above argument, all agents work full-time, so L_t is indeed given by the formula in sect. 2.1. Similarly, $p_t > 0$ implies no free-disposal, i.e. $Y_t = AK_t^\alpha L_t^{1-\alpha}$, and thus the equation for r_t becomes $r_t = g_t p_t$. Here the right-hand member is always well defined, since $L_t > 0$, so we assume those equations for L_t, Y_t, r_t and w_t to hold everywhere. In particular, wherever $K_t > 0$, p_t is continuous by (1) in lemma 5, so w_t is continuous real-valued and a.e. > 0, and r_t is > 0, and continuous and locally integrable as an $\overline{\mathbb{R}}$-valued function. Because of this, equation (1) in lemma 5 can be differentiated term by term on any interval where $K_t > 0$. Doing this with $z_t = e^{-\delta t}p_t$ and substituting r_s by its value (2) we get $z_t' = -g_t z_t$, where $z_t > 0$ since $p_t > 0$, so $z_t = z_0 e^{-\int_0^t g_s ds}$ (we can integrate from 0 on because local (Lebesgue-)integrability of g follows from that of r, p being locally bounded away from 0 by lemma 6.ii). (4) always holds for an appropriate choice of π_t, since $p_t > 0$ and g is locally integrable; the above argument shows π_t is constant wherever $K_t > 0$. Let $H_t \stackrel{\text{def}}{=} \pi_t + \int_0^t g_s ds$: since $z_t = e^{-H_t}$ is decreasing by (1) in lemma 5, H_t is increasing. But π_t equals H_t minus an absolutely continuous function, and since π_t is constant on $K_t > 0$, the whole variation of π_t happens on a negligible set, by (1), so the variation of the absolutely continuous part is null. Thus π itself is increasing.[7]

(5) is an obvious feasibility condition, M being well-defined (lemma 6.ii). Local integrability follows from that of ω and ε, p^C and w being locally bounded. ∎

Corollary 1. *If* $p^Y = p^I$, $\omega_{x,\cdot} \geq 0 \Rightarrow M_x > 0$.

Proof. $w_t > 0$ a.e., by lemma 7. ∎

3.3. **Aggregate Demand.** Following-up on the conventions at the start of this section, note that for aggregate consumption the classic integration of correspondences (Aumann, 1965; Debreu, 1967) doesn't apply, consumption bundles being (equivalence classes of) arbitrary $\overline{\mathbb{R}}_+$-valued Lebesgue-measurable functions, so do not lie in any vector space. Use the following very close analog: let $\mathcal{M}$ (or $\mathcal{M}^{\mathbb{R}}$ to denote the domain) be the set of all such equivalence classes with the topology of convergence in measure on all compact sets, for any fixed distance on $\overline{\mathbb{R}}_+$. The topology is independent of the distance, and is Polish, so the usual measurable selection theorems hold. Define thus the integral of a measurable $\mathcal{M}$-valued correspondence with a.e. well-defined and non-empty values as the set of integrals of all its measurable selections, and the integral of a measurable function $x \mapsto F_x$ with values $t \mapsto F_x(t)$ in $\mathcal{M}$ as the unique point G in $\mathcal{M}$ s.t. $\forall p \in \mathcal{M}$, $\int p(t)G(t)dt = \iint p(t)F_x(t)dtdx$, with the usual measure-theoretic convention that $0 \times \infty = 0$.

To prove the above is well-defined (and to show how it is used), observe that by Doob's (1953) classical martingale argument, there exists for any such F a jointly measurable function $f(x,t)$ s.t. $f(x,\cdot) \in F_x$ $\forall x$ (use first a homeomorphism of $\overline{\mathbb{R}}_+$ with $[0,1]$ to reduce to the case where $\mathsf{sup}_x \|F_x\|_\infty \leq 1$). Fubini's theorem implies then that $\int f(x,t)dx$ satisfies the requirements for G. Uniqueness is obvious.

Conversely, given any jointly measurable $\overline{\mathbb{R}}$-valued function $f(x,t)$, $F\colon x \mapsto f(x,\cdot)$ is a measurable $\mathcal{M}$-valued map. Indeed, assume first f is bounded; then

[7]The argument can be reversed, to show that for any π_t as in (4), the corresponding p_t will also satisfy (1) in lemma 5 for an appropriate ς_t.

F is measurable to L_∞ with the weak*-topology, since bounded subsets there are compact metric. But those bounded subsets are Polish for the topology of convergence in measure on compact sets, so the borel structure is the same. For general f, approximate it by the sequence $f \wedge n$.

Note finally that, G being well-defined, it suffices to check the definition with indicator functions of compact sets for p (intervals do not suffice!).

Lemma 8. *Assume an equilibrium with p^C locally bounded, and let*

$$C_t = N_0\big(e^{\beta t}p_t^C\big)^{-\sigma}\int_{t-1}^{t} e^{(\nu+\beta\sigma)x}\frac{M_x}{\int_0^1 (p_{x+s}^C)^{1-\sigma}e^{-\beta\sigma s}ds}dx \tag{1}$$

The integral in the right hand is everywhere well-defined and finite, hence a continuous function of t, If the right hand member is undefined, involving thus $\infty\times 0$, let $C_t = \overline{\mathbb{R}}_+$. Then aggregate demand (the integral of individual demand) is the set of equivalence classes of all measurable selections from C_t. Further, the integrand is null iff M_x is so, thus $C_t > 0$ when M_x is not negligible on $[t-1,t]$.

Proof. Neglect all negligible sets of birthdates x of lemmas 6 and 7.(5), and take as domain D the remaining part of $\mathbb{R}$. In particular, M_x is everywhere well-defined on D and $\in \mathbb{R}_+$, so lemma 2 is applicable, with M_x for M and $s \mapsto p_{x+s}^C$ for p, and demand is everywhere well-defined and non-empty, by lemma 2.i.

The demand correspondence from D to $\mathcal{M}^{\mathbb{R}}$, $x \mapsto \Gamma(x)$, has a measurable graph, as the intersection of the following 3 measurable graphs: (1) $\{(x,c)\in D\times\mathcal{M}^{\mathbb{R}} \mid c_t = 0 \text{ a.e. for } t\notin[x,x{+}1]\}$, (2) $\{(x,c)\in D\times\mathcal{M}^{\mathbb{R}} \mid \int_{-\infty}^{\infty} p_t^C c_t dt \le M_x\}$. (3) $\{(x,c)\in D\times\mathcal{M}^{\mathbb{R}} \mid U(s\mapsto c_{x+s}) \ge U_x^*\}$. Indeed, (1) is closed, measurability of (2) follows from that of M_x (lemma 6.ii) and the lower semi-continuity of $c\mapsto \int_{-\infty}^{\infty} p_t^C c_t dt$ (Fatou), and of (3) from that of U^* (lemma 6.ii), of U on $\mathcal{M}^{[0,1]}$ (being by Fatou lower semi-continuous if $\sigma > 1$ and else upper semi-continuous), and from the continuity of $(x,c)\mapsto(s\mapsto c_{x+s})\colon \mathbb{R}\times\mathcal{M}^{\mathbb{R}}\to\mathcal{M}^{[0,1]}$, which follows from the continuity of $(\mu,f)\mapsto \mu\star f$ sub 'Notation' in sect. 4.3, using point masses at $-x$ for μ and $h\circ c$ for f, with h a homeomorphism from $\overline{\mathbb{R}}$ to $[0,1]$, and of the projection from $\mathcal{M}^{\mathbb{R}}$ to $\mathcal{M}^{[0,1]}$.

Thus, its integral is well-defined — recall we allow for correspondences to be defined only a.e., so equivalently, define, for $x\notin D$ where $\Gamma(x)$ is not defined (M_x being not defined, or $\notin\mathbb{R}_+$), $\Gamma(x) \stackrel{\text{def}}{=} \mathcal{M}$ —, and is the set of integrals over $x\in\mathbb{R}$ of all jointly measurable functions $c(x,t)$ s.t. $s\mapsto c(x,x+s)\in\Gamma(x)\ \forall x$.

Observe that requirement (1) was not part of our assumptions, nor did we prove that in equilibrium no agent would buy any goods dated outside his life-span. But the same proof obviously shows that without this the demand-correspondence is also measurable; we claim the integrals are the same, so our result is independent of any such assumption. Indeed, take a selection $c(x,t)$ as above from the larger correspondence, and define $\tilde c(x,t) = \mathbf{1}_{t\le x\le t+1}c(x,t) + \frac{1}{\Phi(-\nu)}\int_{y\notin[t-1,t]} e^{\nu(y-t)}c(y,t)dy$. Then clearly $\tilde c$ is measurable, has the same intergral as c, and is a selection from the smaller correspondence: indeed agents would have bought something at times t outside their life-span only if $p_t^C = 0$, since as seen in the proof of lemma 7 any increase in budget would increase their utility, so nobody's budget is affected by the change.

Lemma 2.ii and 2.iii imply then that the selection $c(x,t)$ must equal $c_x^*(t-x)$ given there when either $U_x^*\in\mathbb{R}$ or $p_t^C > 0$ and $M_x = 0$ (indeed, this is a measurable region, and $c_x^*(t-x)$ is jointly measurable on this region, so we can assume equality up to a (joint) null set, which does not affect the equivalence class of the integral).

Note that (1) follows then by integration at all t s.t. either $p_t^C > 0$ or, a.e. on $[t{-}1,t]$, $M_x > 0 \Rightarrow U_x^*\in\mathbb{R}$. And for the right-hand side to be well-defined, note that the denominator in the integrand is a.e. > 0, by lemma 6.i if $\sigma < 1$ and by local boundedness of p^C if $\sigma > 1$, so, since $M_x < \infty$ a.e. (lemma 7.(5)), the outer integral is everywhere well defined. Thus, again since p^C is locally bounded, the right hand member is well-defined except possibly where both the integral and p_t^C are 0. This is the case where it involves $\infty \times 0$, and cannot occur under our assumptions: if $p_t^C > 0$, clearly, if $\sigma > 1$, because (lemma 6.ii) $p_t^C > 0$ a.e., and if $\sigma < 1$ and M_x is not negligible on $[t-1,t]$, because then the outer integral is > 0, the integrand being non-negligible since $M_x > 0 \Rightarrow \int_x^{x+1}(p_t^C)^{1-\sigma}dt < \infty$ (lemma 6.ii).

Next point is to show equality in (1), under our conditions. Since it is already established whenever $p_t^C > 0$ or, a.e. on $[t-1,t]$, $M_x > 0 \Rightarrow U_x^* \in \mathbb{R}$, remains to take care of the case where $p_t^C = 0$ and, for a non-negligible set of x in $[t-1,t]$, both $M_x > 0$ and $U_x^* \notin \mathbb{R}$. Since by lemma 6.ii, $U_x^* < \infty$, this means $U_x^* = -\infty$ and hence $\sigma < 1$. But this, with $M_x < \infty$ (lemma 7.(5)), implies $J_x < \infty$ in lemma 2, since $p^C \le p$ is locally bounded, thus contradicting $U_x^* = -\infty$: the remaining case is vacuous, and (1) is established under our conditions.

As to the demand correspondence in the "else" case, if $\sigma < 1$ and $M_x = 0$ a.e. on $(t-1,t)$, (almost) all living agents have a null lifetime wealth; since they (almost) all face some non-negligible period in their lifetime where $p^C > 0$ by lemma 6.i, $U_x^* = -\infty$, so any consumption at times where $p_t^C = 0$ is both feasible and optimal for them. Recall the integral in the denominator is > 0 a..e., so the integrand is well-defined and null a.e. where $M_x = 0$: the right hand integral is 0 a.e., i.e., the right hand side is undefined, just as demand.

For the "further" clause, since by lemma 6.i aggregate demand is locally integrable, it is a.e. finite, and thus so is our integral in the right hand side. This implies in turn the integrand is locally integrable everywhere, and so the integral is everywhere finite, and is continuous in t.

Finally, if $M_x > 0$, the denominator is finite, by local boundedness of p^C if $\sigma < 1$, and by lemma 6.ii if $\sigma > 1$. So the integrand is null iff M_x is so. ■

3.4. **The equilibrium equations.** We need for the moment the following assumption just for the end of next proof; it should be dispensible...

Assumption 4. For some $\delta > 0$, $\varepsilon_t > \delta$ a.e. on some non-empty open set.

Proposition 1. *Let $\Omega_t \overset{\text{def}}{=} N_0 \int_0^1 e^{\nu(t-s)}\omega_{t-s,s}ds$ be the aggregate endowment at date t. The equilibria where $0 < I_t < Y_t$ a.e. are the solutions (satisfying this condition) of the following:*

(1) $I_t = \Omega_t + Y_t - C_t$

(2) $Y_t = AK_t^\alpha L_t^{1-\alpha}$

(3) $K_t = e^{-\delta t}\displaystyle\int_{-\infty}^t I_s e^{\delta s} ds$

(4) $C_t = \left(e^{\beta t}p_t\right)^{-\sigma}\displaystyle\int_{t-1}^t \frac{N_0 e^{(\nu+\beta\sigma)x}M_x}{\int_0^1 p_{x+s}^{1-\sigma}e^{-\beta\sigma s}ds}dx,\ M_x \overset{\text{def}}{=} \int_0^1 (p_{x+s}\omega_{x,s} + w_{x+s}\varepsilon_s)ds \ge 0$ a.e.

(5) $g_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha}$

(6) $p_t = p_0 e^{\int_0^t(\delta - g_s)ds}$, *with $0 < p_t < \infty\ \forall t$ (so g is locally integrable)*

(7) $w_t = (1-\alpha)e^{\gamma t}\frac{Y_t}{L_t}p_t$

The same holds for all equilibria of the unperturbed economy ($\omega = 0$) where constraints on disvestment are not binding, and for all its balanced-growth equilibria. The same holds for all equilibria of the basic model, replacing (6) *by* (4) *of lemma 7.*

Proof. (3) comes from lemma 3, and (1) is market clearing.

$0 < I_t$ a.e. implies that constraints on disvestment are not binding, so our previous results are applicable. It also implies $K_t > 0$ by (3), hence (6) by (4) in lemma 7. Since $I_t < Y_t$ a.e., equilibrium requires $p_t^C \ge p_t^Y$ (for the merchandising activity transforming output into consumption not to imply a loss). So $p^C = p$ by lemma

5.ii (using equality everywhere as explained in remark 3). Finally, equation (1) of lemma 8 holds everywhere because $p_t > 0$ (lemma 7.4).

As to the unperturbed economy, cor. 1 implies that equation (1) of lemma 8 holds everywhere, and with $C_t > 0$ since $M_x > 0$ a.e.. Since $p^C \leq p$ is locally bounded (lemma 6.i), we first conclude that the integral is > 0 everywhere, thus locally bounded away from 0, and next that C_t itself is so. Thus $K_t = 0$ is impossible: immediately after such a time, it is impossible to have C_t bounded away from 0. The rest of the proof is as in the previous case.

Finally, in the basic model, by no-arbitrage, $p^C = p$, so the positivity of p (lemma 6.(ii)) implies that the condition $p_t^C > 0$ for (1) in lemma 8 is satisfied everywhere.

Remains thus only to deal with the BGE of the unperturbed constrained model. By definition of balanced growth, $K_t = K_0 e^{(\gamma+\nu)t}$, and $K_0 \geq 0$, so $I_t = RK_0 e^{(\gamma+\nu)t}$, with $R > 0$ by 2.3.2. Thus, if $K_0 > 0$, $I_t > 0$ $\forall t$, so constraints on disvestment are not binding, and the result is established. Else, $K_t = 0$ $\forall t$, and hence also $I_t = Y_t = C_t = 0$ $\forall t$. Then there is no loss to increase p_t^C such as to achieve equality with p_t^Y in lemma 5.i: a fortiori demand will still be 0.

For demand to be identically 0, we need according to lemma 2, since $p^C < \infty$ a.e. by def. 1, that for a.e. x either $\sigma > 1$ and $\frac{p^C_{x+s}}{M_x} = \infty$ a.e. for $0 < s < 1$, or $\sigma < 1$ and $J_x = \infty$. Thus, if $\sigma > 1$, $p_t^C > 0$ a.e. and $M_x = 0$ a.e., and if $\sigma < 1$, for a.e. x, $M_x < \infty$, p_t^C is not a.e. 0 on $[x, x+1]$, and either $M_x = 0$ or $\int_x^{x+1} (p_t^C)^{1-\sigma} dt = \infty$.

By lemma 5, changing prices except p on a null set we can assume $p \leq p^C = p^Y < \infty$ everywhere, and p is locally bounded and locally bounded away from 0 (the latter by lemma 6.ii).

Lemma 4 implies now, since r_t is locally integrable, and hence a.e. finite, and since $p_t^Y > 0$ a.e., that $w_t > 0$ a.e. Thus every agent can achieve $M_x > 0$, by putting in some labour. If $\sigma > 1$, such positive M_x would, by lemma 2, guarantee him $U_x^* > 0$, since $p^C < \infty$ a.e.: he is not optimising.

Remains thus to deal with the case $\sigma < 1$, where for a.e. x, either $M_x = 0$, or $M_x < \infty$ and $\int_x^{x+1} (p_t^C)^{1-\sigma} dt = \infty$. But by the argument in last paragraph, the agent can achieve $M > 0$, even if $M_x = 0$, and hence could achieve $U_x^* > -\infty$ by lemma 4 if $J_x < \infty$, which would contradict his optimising behaviour. Thus our case reduces to $M_x < \infty$ and $\int_x^{x+1} (p_t^C)^{1-\sigma} dt = \infty$ a.e.

Consider now one such x: a fortiori $\int_x^{x+1} p_t^C dt = \infty$, so Hölder's inequality applied to lemma 4 yields $\int_x^{x+1} w_t dt = \infty$, r being locally integrable. By the assumption, this implies for a non-empty open set of x that those agents can achieve $M_x = \infty$, hence $U_x^* = 0 > -\infty$, contradiction again.

Finally, to prove those equations really define equilibria, suffices to observe that (6) implies p_t is locally bounded and locally bounded away from 0; this implies first that M_x is well-defined and finite a.e., next, given $M_x \geq 0$, that all those agents have, by lemma 2, ii and iii, $c^*_{x,s}$ as unique maximiser in their budget set, and that those indeed aggregate to C_t. The rest is obvious (cf. e.g. fn. 20 in case of $I \geq 0$). ∎

Remark 4. The "intellectual reason" why the "0-equilibrium" (where $K_t = 0$ $\forall t$) doesn't exist is individual rationality: a single Robinson Crusoe with no starting capital can produce output and capital and consumption goods in his lifetime—cf. the differential equation where he works full-time and all output is converted into investment. The problem with this "argument" is that if ε is identically 0 in some initial part of his lifetime, capital (and hence consumption possibilities) will start to build up only after that initial segment, i.e., if $\sigma < 1$, his lifetime utility is still $-\infty$: that's why trading is needed with other Robinson's born at different dates, and hence the whole apparatus of equilibrium analysis...

Notation. Express key variables in efficient labour units: $k_t = \frac{K_t}{L_t}$, $y_t = \frac{Y_t}{L_t}$, $i_t = \frac{I_t}{L_t}$, $E_t = \frac{\Omega_t}{L_t}$, $c_t = \frac{C_t}{L_t}$; and let $\eta = (\gamma + \nu)\big(1 - \sigma\big) + \beta\sigma$.

Remark 5. *No Bubbles:* To get rid of the solutions of (1) in lemma 5 with (even constant) $\varsigma > 0$ ("bubbles", or: indeterminacy) one might expect to need a transversality condition, e.g., $\lim_{t\to\infty} e^{-\delta t} p_t = 0$, or infinitely-lived investment firms making arbitrage operations like buying some capital now and renting it out forever after. But (6) does imply $\varsigma = 0$: (3) and feasibility (plus the initial condition and $R > 0$) imply k_t is bounded,[8] so $\inf g_t > 0$, thus $\int_0^\infty g_t dt = \infty$, and hence $\varsigma = \lim_{t\to\infty} e^{-\delta t} p_t = 0$.

Proposition 2. *Given an endowment perturbation E_t, a distribution of endowments ϑ_s, and $\varphi_s \overset{\text{def}}{=} \frac{e^{-\nu s}\varepsilon_s \mathbf{1}_{0\le s\le 1}}{\int_0^1 e^{-\nu u}\varepsilon_u du}$, define $\Upsilon\colon (i, E) \mapsto \tilde{\imath}$ as the composition of:*

(i) $i \mapsto k\colon k_t = e^{-Rt}\int_{-\infty}^t e^{Rs} i_s ds > 0$ a.e.
(ii) $k \mapsto y\colon y_t = Ak_t^\alpha$
(iii) $k \mapsto f\colon f_t = R - \alpha A k_t^{\alpha-1}$ $(= R - \frac{\alpha y_t}{k_t}, = \gamma + \nu + \frac{p_t'}{p_t})$ *is locally integrable*
(iv) $(y, f, E) \mapsto \mathcal{N}\colon \mathcal{N}_x = \int_0^1 e^{\int_x^{x+s} f_t dt}\big(\vartheta_s E_{x+s} + (1-\alpha)\varphi_s y_{x+s}\big) ds \ge 0$
(v) $f \mapsto \mathcal{D}\colon \mathcal{D}_x = \int_0^1 e^{-\eta s + (1-\sigma)\int_x^{x+s} f_t dt} ds$
(vi) $(\mathcal{N}, \mathcal{D}) \mapsto \mathcal{B}\colon \mathcal{B}_x = \frac{\mathcal{N}_x}{\mathcal{D}_x}$
(vii) $(f, \mathcal{B}) \mapsto c\colon c_t = \int_0^1 e^{-\eta u - \sigma\int_{t-u}^t f_s ds}\mathcal{B}_{t-u} du$
(viii) $(y, E, c) \mapsto \tilde{\imath}\colon \tilde{\imath}_t = y_t + E_t - c_t$

The equilibrium i_t's with $0 < i_t < y_t$ a.e. are the zeros (s.t. $0 < i_t < y_t$ a.e.) of $F(i, E) \overset{\text{def}}{=} \Upsilon(i, E) - i$, i.e., the fixed points of Υ.

The same holds for all equilibria of the unperturbed economy ($E = 0$) where $i_t > 0$ a.e., and for all its balanced-growth equilibria.

The same holds also for all equilibria of the basic model where $K_t > 0$ $\forall t$.

Proof. $K_t > 0$ for the basic model ensures that (6) of prop. 1 holds. Rewrite the conditions of prop. 1 in the new notation:

- $y_t = Ak_t^\alpha$
- $k_t = e^{-Rt}\int_{-\infty}^t e^{Rs} i_s ds > 0$
- $p_t = p_0 e^{\int_0^t \left(\delta - \alpha\frac{Y_s}{K_s}\right) ds} > 0$
- $i_t = y_t + E_t - c_t$

Next eliminate the price equation; only aggregate consumption depends on it. First, using $p_{x+s} = p_x \psi(x, s)$ with $\psi(x, s) = \exp(\delta s - \alpha A\int_x^{x+s} k_v^{\alpha-1} dv)$,

$$c_t = \frac{\big(e^{\beta t} p_t\big)^{-\sigma}}{e^{(\gamma+\nu)t}\int_0^1 \varepsilon_s e^{-\nu s} ds}\int_{t-1}^t e^{(\nu+\beta\sigma)x}\frac{\int_0^1 (p_{x+s}\omega_{x,s} + w_{x+s}\varepsilon_s)\, ds}{\int_0^1 p_{x+s}^{1-\sigma} e^{-\beta\sigma s} ds} dx$$
$$= \int_{t-1}^t \frac{e^{(\nu+\beta\sigma)(x-t)-\gamma t}}{\int_0^1 \varepsilon_s e^{-\nu s} ds}\frac{\int_0^1 \psi(x, s)\big(\omega_{x,s} + (1-\alpha)\, e^{\gamma(x+s)} y_{x+s}\varepsilon_s\big)\, ds}{(\psi(x, t-x))^\sigma\int_0^1 (\psi(x, s))^{1-\sigma} e^{-\beta\sigma s} ds} dx$$

Use now $\omega_{x,s} = \vartheta_s E_{x+s} e^{\gamma(x+s)+\nu s}\int_0^1 \varepsilon_u e^{-\nu u} du$, from the definitions at the beginning of this section to re-write the numerator of the second ratio:

$$\omega_{x,s} + (1-\alpha)\varepsilon_s e^{\gamma(x+s)} y_{x+s}$$
$$= e^{\gamma(x+s)+\nu s}\Big(\vartheta_s E_{x+s}\int_0^1 \varepsilon_u e^{-\nu u} du + (1-\alpha) e^{-\nu s}\varepsilon_s y_{x+s}\Big)$$
$$= e^{\gamma(x+s)+\nu s}\Big(\int_0^1 \varepsilon_u e^{-\nu u} du\Big)\Big(\vartheta_s E_{x+s} + (1-\alpha)\varphi_s y_{x+s}\Big)\,; \quad \text{so:}$$

[8]Cf. Mertens and Rubinchik (2006, lemma 3) for $\omega = 0$. For any E s.t. $\|E\| \overset{\text{def}}{=} \sup_x \int_x^{x+1} |E_t| dt < \infty$ one gets similarly $\sup_t k_t \le B_{\|E\|} + \|E\|$ with B_x the root of $AB_x^\alpha - RB_x + x = 0$.

$$c_t = \int_0^1 e^{A\alpha\sigma \int_{t-u}^t k_s^{\alpha-1}ds-(\nu+\gamma+\sigma(\beta+\delta))u}\mathcal{B}_{t-u}du\,, \quad \text{where } \mathcal{B}_x \stackrel{\text{def}}{=} \frac{\mathcal{N}_x}{\mathcal{D}_x} \quad \text{and:}$$

$$\mathcal{N}_x \stackrel{\text{def}}{=} \int_0^1 e^{Rs-A\alpha\int_x^{x+s} k_v^{\alpha-1}dv}\Big(\vartheta_s E_{x+s} + (1-\alpha)\varphi_s y_{x+s}\Big)ds$$

$$\mathcal{D}_x \stackrel{\text{def}}{=} \int_0^1 e^{s(\delta-\sigma(\beta+\delta))-A\alpha(1-\sigma)\int_x^{x+s} k_v^{\alpha-1}dv}ds$$

Using the definition of η we obtain now the equilibrium conditions stated. ■

Remark 6. We will assume E is bounded. There is a specific advantage to L_∞: if E is small in L_∞, we know (or: prove) that all quantities remain positive, in particular investment. So everything is independent of the presence or not of non-negativity constraints on some types of investment (Mertens and Rubinchik, 2006, fn. 16).

3.5. **Balanced growth equilibria.**

Definition 2. A *balanced growth equilibrium* (BGE) is an equilibrium with $E_t = 0$ and k_t constant (and hence i, y,...). It is a *golden rule equilibrium* (GRE) if $\frac{i}{y} = \alpha$.

Corollary 2. *The* BGE *are the same for all variants; they are characterized by:*

(i) $k = \frac{1}{R}i$
(ii) $y = Ak^\alpha$
(iii) $f = R - \alpha Ak^{\alpha-1}$
(iv) $\mathcal{N} = (1-\alpha)y\int_0^1 e^{sf}\varphi_s ds$
(v) $\mathcal{D} = \Phi\big(f(1-\sigma)-\eta\big)$, *where* $\Phi(x) \stackrel{\text{def}}{=} \frac{e^x-1}{x}$.
(vi) $\mathcal{B} = \frac{\mathcal{N}}{\mathcal{D}}$
(vii) $\frac{R-\frac{f}{1-\alpha}}{R-f} = F(f)\int_0^1 \varphi_s e^{sf}ds$, *where* $F(f) = \frac{\Phi\big(-f\sigma-\eta\big)}{\Phi\big(f(1-\sigma)-\eta\big)}$

Proof. Condition viii in prop. 2 becomes $i = y - \mathcal{B}\,\Phi(-\eta-f\sigma)$. Dividing that equation by y and re-arranging we get (vii), since $\frac{i}{y} = R\frac{k}{y} = \frac{R\alpha}{R-f}$. Given any solution of this equation in f the rest of the BGE can be re-computed from the above formulae and $p_t = p_0 e^{(\delta-\alpha\frac{y}{k})t}$, $r_t = \alpha\frac{y}{k}p_t$, $w_t = (1-\alpha)ye^{\gamma t}p_t$. ■

Remark 7.
- $F(f)$ decreases from ∞ to $(1-\sigma^{-1})^+$; $F(0) = 1$.
- In any BGE $R - f > 0$ by condition iii, so, since $F > 0$, $R(1-\alpha) > f$.
- $\int_0^1 \varphi_s e^{sf}ds$ increases in f, and $= 1$ at 0.
- Equation vii has $f = 0$, the GRE, as solution; cf. App. B for explanation.

Remark 8. We plot (cf. also App. B) BGE making in (vii) $\frac{\alpha}{1-\alpha}$ explicit as a function $\mathcal{F}$ of $x = 1 - f/R$ $(= \alpha Y_t/I_t$ by i–iii).[9,10] Figures 1–4 show the BGE of economies with $\varphi(s) = \frac{1}{b-a}\mathbf{1}_{[a,b]}(s)$ and with reasonable parameters (recall time unit is 1 lifetime).

Corollary 3. *At the golden rule equilibrium :*

(i) $k^* = \big(\frac{A\alpha}{R}\big)^{\frac{1}{1-\alpha}}$
(ii) $y^* = A^{\frac{1}{1-\alpha}}\big(\frac{\alpha}{R}\big)^{\frac{\alpha}{1-\alpha}}$
(iii) $i^* = \big(\frac{A\alpha}{R^\alpha}\big)^{(1-\alpha)^{-1}}$
(iv) $p_t^* = p_0^* e^{-(\gamma+\nu)t}$
(v) $w_t^* = p_0^*(1-\alpha)y^*e^{-\nu t}$
(vi) *and* $f^* = 0$, $i^* = Rk^* = \alpha y^*$, $\mathcal{N}^* = (1-\alpha)y^*$, $\mathcal{D}^* = \Phi(-\eta)$

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[9]With those coordinates, 1) the relevant region becomes the positive orthant, 2) the units are dimensionless, thus easier to interpret, and 3) the function is analytic, so the graph, more reliable.

[10]With a the "minimal working age" (minimum of the support of the distribution $\varphi(s)ds$), since the curve passes through the origin: 1) if $a \geq \min(\sigma, 1)$, the function converges to $\frac{-1}{\max(1,\sigma)} < 0$, so the number of equilibria on the curve is even for generic α; 2) Else the function converges to $+\infty$, so the number is odd; (contrast with Gale (1973); Kehoe and Levine (1984)). Figures 3 and 4 are right at the edge. Anyway, the number is finite (analyticity).

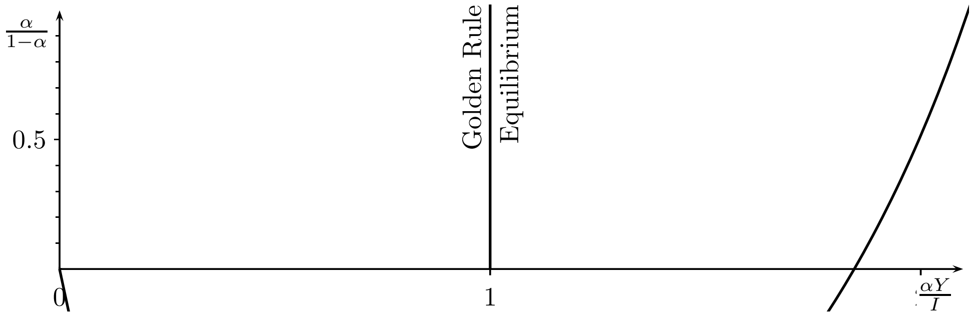

Figure 1: $R = 11$, $\sigma = .5$, $\eta = 2$, $a = .2$, $b = .75$. Two equilibria $\forall \alpha$.

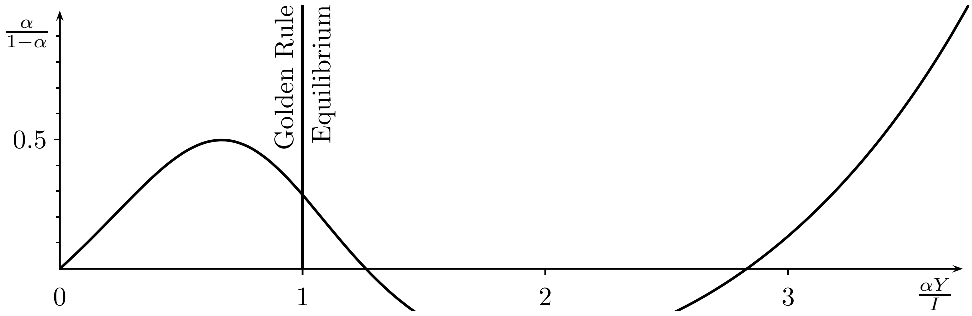

Figure 2: $R = 11$, $\sigma = .25$, $\eta = 2$, $a = .135$, $b = .5$. Two to four equilibria.

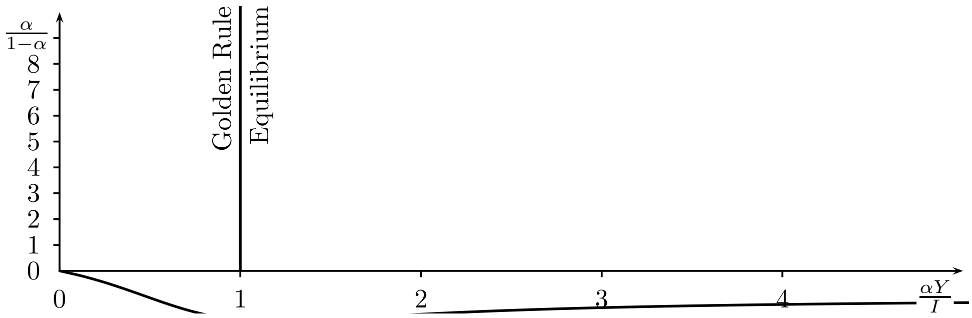

Figure 3: $R = 10$, $\sigma = .25$, $\eta = 2.5$, $a = .25$, $b = .75$. 1 equilibrium $\forall \alpha$.

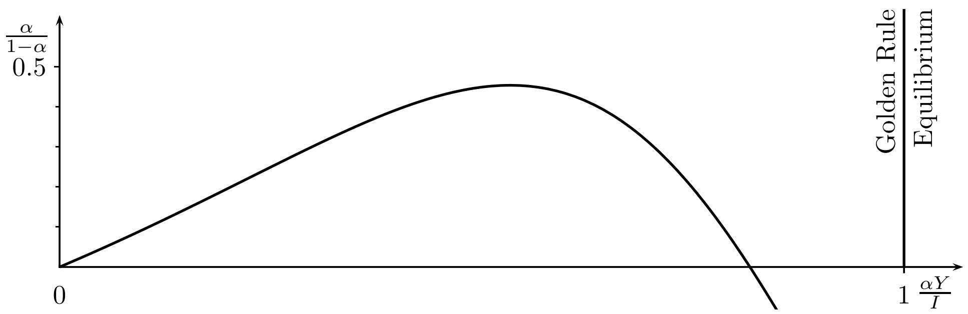

Figure 4: $R = 15$, $\sigma = .24$, $\eta = 1.9$, $a = .24$, $b = .55$. 1 or 3 equilibria.

4. Tools

4.1. **Banach Pairs and the Implicit Function Theorem.**

Notation. For Banach spaces X and Y, $\mathbb{L}(X,Y)$ is the Banach space of continuous linear maps from X to Y; so $\mathbb{L}(X,X)$ is the Banach algebra of *operators* on X.

Definition 3. A *Banach pair* is a pair (B,B') of Banach spaces with $B'\subseteq B$ and s.t. $\|\cdot\|'\geq\|\cdot\|$. By definition, for any Banach space B, $\|x\|_B \overset{\text{def}}{=} \infty$ if $x\notin B$; hence also, for any map φ, define its operator norm in $\mathbb{L}(B,B')$ as ∞ if B is not in the domain of φ (i.e., interpret "$\varphi(x)$ undefined" as implying $\varphi(x)\notin B'$) (as a notational convention; we won't involve the set of all sets…).

For Banach pairs (X,X') and (Y,Y'),

(i) $\mathbb{L}(X,X';Y,Y')$ is the Banach space $\{\varphi\in\mathbb{L}(X,Y)\mid\varphi'\in\mathbb{L}(X',Y')\}$, where $\varphi'=\varphi_{|X'}$, and $\|\varphi\|=\max\{\|\varphi\|_{\mathbb{L}(X,Y)},\|\varphi'\|_{\mathbb{L}(X',Y')}\}$.

(ii) For O open in X, a map $F\colon O\to Y$ is *Fréchet differentiable* at $x\in O$ if $\exists\varphi\in\mathbb{L}(X,X';Y,Y')$ s.t. both for $p(\cdot)=\|\cdot\|$ and $p(\cdot)=\|\cdot\|'$ one has that $\forall\varepsilon>0\exists\delta>0$: $p[F(x+\delta x)-F(x)-\varphi(\delta x)]\leq\varepsilon p(\delta x)$ when $p(\delta x)\leq\delta$.

It is S^1 if it is Fréchet differentiable at each $x\in O$, with differential φ_x, and $x\mapsto\varphi_x$ is continuous on O.

Remark 9. For $X'=\{0\}$, $\mathbb{L}(X,X';Y,Y')=\mathbb{L}(X,Y)$ and the definitions reduce to the usual ones for 'non-pairs'.

Lemma 9. *Equivalently, $F\colon O\to Y$ is S^1 iff:*

(i) F is C^1. Let φ'_x be the restriction to X' of its derivative φ_x at x.
(ii) $\forall x\in O,\forall\varepsilon>0,\exists\delta>0$: $\|F(x+\delta x)-F(x)-\varphi_x(\delta x)\|'\leq\varepsilon\|\delta x\|'$ for $\|\delta x\|'\leq\delta$.
(iii) $\varphi'\colon x\mapsto\varphi'_x$ is continuous from O to $\mathbb{L}(X',Y')$.

Proposition 3. *For Banach pairs (X,X') and (Y,Y'), and $F\colon O\to Y$ S^1 with O open in X, $F\colon O\to Y$ is C^1. For $x\in O$ denote by V the connected component of* 0 *in $(O-x)\cap X'$, with the X' topology. Then V and its complement in $(O-x)\cap X'$ have disjoint closures in $(O-x,\|\cdot\|)$, and $\delta x\mapsto F(x+\delta x)-F(x)$ is C^1 from V to Y', with $\varphi'_{x+\delta x}$ as derivative at δx.*

Proof. F being by (i) Fréchet differentiable for $\|\cdot\|$ at each $x\in O$, (ii) ensures that $F(x+\delta x)-F(x)\in Y'$ for δx sufficiently small in X', since by (iii) $\varphi_x(\delta x)\in Y'$, and is Fréchet differentiable at 0 with φ'_x as derivative. Those 2 together imply there is an open neighbourhood V_x of 0 in X' s.t. $\delta x\mapsto F(x+\delta x)-F(x)$ is Fréchet differentiable from V_x to Y', with $\varphi'_{x+\delta x}$ as derivative at δx. This being continuous by (iii), the map is C^1 on V_x.

$V'=\{z\in V\mid\exists m,\exists x_i$ with $i=1\dots 2m+1$: $x_1=x$, $x_{2m+1}=x+z$, $x_{2i\pm1}-x_{2i}\in V_{x_{2i}}$ for $i=1\dots m\}$ is trivially open and closed in V, so $V'=V$. Since $F(x_{2i\pm1})-F(x_{2i})\in Y'$, $F(x+z)-F(x)\in Y'$ $\forall z\in V$, so $\delta z\mapsto F(x+z+\delta z)-F(x)$ is C^1 on V_{x+z} $\forall z\in V$, hence the second statement. For the first, let else z belong to both closures: an $\|\cdot\|$-ball around z is contained in $O-x$ and intersects V and its complement, say in z_1 and z_2. Then the segment from z_1 to z_2 lies in the ball, hence in $O-x$, and also in X': z_2 is connected to V, hence $\in V$: contradiction. ■

Corollary 4. *If $F\colon O\to Y$ is S^1 and O is convex, then $z\mapsto F(x+z)-F(x)$ is, $\forall x\in O$, C^1 from $(O-x)\cap X'\subseteq X'$ to Y'.*

Lemma 10. *A composition of S^1 maps is S^1.*

Probably a proper version of cor. 5 should come after this.

Proof. Use the same result for C^1 (Schwartz, 1957-59, vol. 1, thm 11) for points i and ii (cf. prop. 3), and the continuity of composition for point iii. ■

Proposition 4 (IFT). *For Banach pairs (X, X') and (Y, Y'), let $F\colon X \times Y \to X$ be S^1 in a neighbourhood of (x_0, y_0), with $F(x_0, y_0) = 0$. If $\frac{\partial F}{\partial x}$ is invertible in $\mathbb{L}(X, X'; X, X')$ at (x_0, y_0), then $\exists \delta, \delta' > 0$ and an S^1 map ϖ from $\{y \mid \|y - y_0\| < \delta\}$ to X s.t. $x = \varpi(y)$ is the unique solution of $F(x, y) = 0$ with $\|x - x_0\| \leq \delta'$.*

Proof. The theorem without pairs (i.e., with $X' = Y' = 0$) is classical (e.g. Schwartz, 1957-59, theorems 25, 26, vol. 1). Use it first for that case, to obtain just the C^1 aspect of ϖ, i.e., (i). Next use (iii) for F to conclude that $\frac{\partial F}{\partial x}$ is still invertible at all $(\varpi(y), y)$ with $\|y - y_0\| < \delta$, reducing δ if needed, invertible maps forming an open set. Re-using thus the theorem, and prop. 3 for F, at each such (x, y) for the spaces X' and Y', translating (x, y) back to $(0, 0)$, yields now (ii) for ϖ. As to (iii) for ϖ, it follows now straight from $\frac{d\varpi}{dy} = -\left(\frac{\partial F}{\partial x}\right)^{-1} \frac{\partial F}{\partial y}$, from (iii) for F, and from the continuity of the composition and the inverse. ■

4.2. **Kernels.**

Notation. M is the space of bounded measures on $\mathbb{R}$, and $C_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$, with the uniform topology.

Definition 4. A *kernel operator* is a continuous linear map A from L_∞ to $C_b(\mathbb{R})$, s.t. $A(f_n)$ converges pointwise to 0 whenever $f_n \to 0$ a.e. and is uniformly bounded.

Proposition 5. *Let A be a kernel operator. Then $\exists k$ jointly borel from $\mathbb{R}^2$ to $\mathbb{R}$ s.t. $\forall f \in L_\infty, \forall s \in \mathbb{R}, [A(f)](s) = \int k(s, t) f(t) dt$.*

Also $\sup_s \int |k(s, t)| dt = \|A\| < \infty$, and A is continuous under the Mackey topologies $\tau(L_\infty, L_1)$ and $\tau(C_b(\mathbb{R}), M)$.

Proof. Let $k_s\colon L_\infty \to \mathbb{R}\colon f \mapsto [A(f)](s)$: $k_s \in L_\infty^*$, and the pointwise convergence condition ensures then $k_s \in L_1$. Doob's classical martingale argument yields then a jointly borel version $k(s, t)$. The first point in the 'also' clause is obvious; it allows to use Fubini's theorem to obtain $\int [A(f)](s) \varsigma(ds) = \int k(\varsigma, t) f(t) dt \, \forall f \in L_\infty, \varsigma \in M$, where $k(\varsigma, t) = \int k(s, t) \varsigma(ds)$. This implies that $A^t\colon \varsigma \mapsto k(\varsigma, \cdot)$ is $\sigma(M, C_b(\mathbb{R}))$-$\sigma(L_1, L_\infty)$ continuous, and thus A, by duality, Mackey continuous. ■

4.3. **The spaces L_p^λ and Wiener's theorem.**

Notation. L_1 is identified with a subspace of M. The convolution $\mu \star f$ of $f \in L_p$ ($p \geq 1$) with $\mu \in M$ is $t \mapsto \int f(t - s) \mu(ds)$, and $\|\mu \star f\|_p \leq \|\mu\| \|f\|_p$, and similarly for $\mu \star \nu$. This way, M is a commutative Banach subalgebra (of *convolution operators*) of $\mathbb{L}(L_p, L_p)$ $\forall p \geq 1$. For $1 \leq p < \infty$, $(\mu, f) \mapsto \mu \star f$ is (weak*,$\|\cdot\|_p$)-$\|\cdot\|_p$ continuous when restricted to bounded subsets of M.[11] The Banach algebra (*Wiener algebra*) W is the subspace of M spanned by L_1 and δ_0, the unit mass at 0.

For $\mu \in M$, its Fourier transform (FT) $\widehat{\mu}(\omega) = \int e^{\mathrm{i}\omega t} \mu(dt)$ ($\widehat{g}$ for $g \in L_1$). $\widehat{\mu \star \nu} = \widehat{\mu}\widehat{\nu}$, so the FT is an injective algebra homomorphism of norm 1 from M to $C_b(\mathbb{R})$.

For $\lambda \in \mathbb{R}$, let ϕ_λ be the multiplication operator by $e^{\lambda t}$ on the space of functions of a real variable into a vector space; i.e., $\phi_\lambda(f) = [t \mapsto e^{\lambda t} f(t)]$ — so $\lambda \to \phi_\lambda$ is a group isomorphism. For $1 \leq p \leq \infty$, let $L_p^\lambda \stackrel{\text{def}}{=} \phi_{-\lambda}(L_p)$, with $\|f\|_p^\lambda \stackrel{\text{def}}{=} \|\phi_\lambda(f)\|_p$.

Lemma 11. *Let $h \in L_1^\lambda$, and denote by $h^\star$ the convolution with h. Then we have the commutative diagram — so, L_1^λ is a Banach algebra of operators on all L_p^λ, extended by δ_0 to a Wiener algebra W^λ, and ϕ_λ an algebra-isomorphism on W^λ:*

$$
\begin{array}{ccc}
L_p^\lambda & \xrightarrow{\;h^\star\;} & L_p^\lambda \\
\Big\downarrow \phi_\lambda & & \Big\downarrow \phi_\lambda \\
L_p & \xrightarrow{\;\phi_\lambda(h)^\star\;} & L_p
\end{array}
$$

[11] Exercise! Consider first f fixed, and continuous with compact support.

In particular, from the formula for $\lambda = 0$ we get $\|h \star f\|_p^\lambda \le \|h\|_1^\lambda \|f\|_p^\lambda$, and hence W^λ is (isometrically) a subalgebra of the operator algebra on L_p^λ ($1 \le p \le \infty$).

Proposition 6. *Let $f\colon \mathbb{R} \to \overline{\mathbb{R}}$ be Lebesgue-measurable and $J = \{\lambda \mid f \in L_1^\lambda\}$. Then J is connected; denote its interior by J°. Let, for $\Re(z) \in J$, $h(z) = \int e^{zt} f(t) dt$, so h is analytic for $\Re(z) \in J^\circ$, and let $D = \{\Re(z) \in J \mid h(z) = 1\}$. D is closed in J, and $D \cap J^\circ$ discrete. For any connected component Λ of $J \backslash D$, $f - \mathbf{1}$ has a convolution inverse $g - \mathbf{1}$ in $\bigcap_\Lambda W^\lambda$, and $\int e^{zt} g(t) dt = \bar{h}(z) \overset{\text{def}}{=} \frac{1}{1 - \frac{1}{h(z)}}$ for $\Re(z) \in \Lambda$.*

Am I missing something? Could it be that whenever the FT of an integrable function is analytic in a band around the real axis, that function belongs to all corresponding L_1^λ?

Remark 10. There can be several distinct sets Λ with different convolution inverse $g - 1$ in each, as illustrated in app. C, though, as assured by the proposition, within any given set Λ the inverse is independent of λ.

Proof. J is connected: for any $\lambda \in [\lambda_1, \lambda_2]$ $e^\lambda \le e^{\lambda_1} + e^{\lambda_2}$, so with $\lambda_i \in J$, $\int e^{\lambda t} |f(t)| dt \le \int (e^{\lambda_1 t} + e^{\lambda_2 t}) |f(t)| dt < \infty$, so $f \in L_1^\lambda$.

Observe that $h(z)$ is analytic on $\Re(z) \in J^\circ$, since the integral under $|e^{zt} f(t)| dt$ of the power series of e^{at} converges absolutely for sufficiently small $|a|$.

Next we show that $h(\lambda + \mathrm{i}\omega)$ converges when $\omega \to \pm\infty$ uniformly to 0 for λ in compact subsets of J. Indeed, this compact subset can be taken as an interval $[\lambda_1, \lambda_2]$; with $\varphi(t) = e^{\lambda_1 t} + e^{\lambda_2 t}$, approximate now φf up to ε in L_1 by $\varphi\psi$, where ψ is a linear combination of indicator functions of intervals: so it suffices to prove the claim when f is such an indicator function, where it results e.g. by direct calculation.

The same proves also continuity of h, so h is bounded on compact subsets of J.

By continuity, $R = \{z \mid \Re(z) \in J,\ h(z) = 1\}$ is closed in $\{z \mid \Re(z) \in J\}$, and, by the above uniform convergence on compact subsets, the projection to J is proper (compact sets have compact inverse images), so D is closed in J. The analyticity of h implies that $\{z \in R \mid \Re(z) \in J^\circ\}$ is discrete, thus so is $D \cap J^\circ$, again by properness.

By lemma 11, to compute the inverse of $f - \mathbf{1}$ in W^λ, map everything to W ($= W^0$), and use there Wiener's theorem (Jörgens, 1982, thm. 32 p. 340),[12] that, for $\phi_\lambda(f) = r \in L_1$, $\mathbf{1} - r$ is invertible in W iff $\widehat{r}$ does not take the value 1 (i.e., $1 \notin$ the closure of $\{\widehat{r}(\omega)\}$, since $\widehat{r} \in C_0$). Then the inverse must be of the form $\mathbf{1} - r'$, with FT's $\widehat{r}$ and $\widehat{r}'$ satisfying $\widehat{r}' = \frac{1}{1 - 1/\widehat{r}}$; the inverse of $f - \mathbf{1}$ in W^λ is then $g_\lambda - \mathbf{1}$ with $g_\lambda = \phi_{-\lambda}(r')$.

By definition, $h(\lambda + \mathrm{i}\omega) = \widehat{\phi_\lambda(f)}(\omega) = \widehat{r}(\omega)$. So, $\mathbf{1} - r$ is invertible iff $h(z) \ne 1$ for $\Re(z) = \lambda$, with as inverse $\mathbf{1} - r'$ where $\widehat{r}'(\omega) = \bar{h}(\lambda + \mathrm{i}\omega)$. Now, since $g_\lambda = \phi_{-\lambda}(r')$, Parseval's formula (Gel'fand and Shilov, 1959, II.2) yields, for $\varphi \in K$, $\int \varphi(t) g_\lambda(t) dt = \int \varphi(t) e^{-\lambda t} r'(t) dt = \frac{1}{2\pi} \int \bar{h}(\lambda + \mathrm{i}\omega) \left[\int \varphi(t) e^{-(\lambda + \mathrm{i}\omega)t} dt\right] d\omega$. The bracket is integrable in ω and $\bar{h}$ bounded. Let now $r'_T(t) = \frac{1}{2\pi} \int_{-T}^{T} e^{-(\lambda + \mathrm{i}\omega)t} \bar{h}(\lambda + \mathrm{i}\omega) d\omega$; then we get $\int \varphi(t) r'_T(t) dt = \frac{1}{2\pi} \int_{-T}^{T} \bar{h}(\lambda + \mathrm{i}\omega) \left[\int \varphi(t) e^{-(\lambda + \mathrm{i}\omega)t} dt\right] d\omega$; by the integrability of the integrand this converges for $T \to \infty$ to our previous formula for $\int \varphi(t) g_\lambda(t) dt$: so $r'_T \to g_\lambda$ in K^*.

For $\lambda_1, \lambda_2 \in \Lambda \cap J^\circ$, the integrand in r'_T, $e^{-zt} \bar{h}(z)$, is analytic for $\lambda_1 \le \Re(z) \le \lambda_2$, so r'_T at λ_1 and λ_2 differs by $\int_{\lambda_1}^{\lambda_2} e^{-(x - \mathrm{i}T)t} \bar{h}(x - \mathrm{i}T) dx - \int_{\lambda_1}^{\lambda_2} e^{-(x + \mathrm{i}T)t} \bar{h}(x + \mathrm{i}T) dx$. $\bar{h}$ converges, as h, uniformly for $\lambda \in [\lambda_1, \lambda_1]$ to 0 when $\omega \to \infty$, so each of those 2 integrals is bounded in norm by $|\lambda_1 - \lambda_2| e^{\max |\lambda_1 t|, |\lambda_2 t|} o(T)$. By the dominated convergence theorem, this bound for the difference of the r'_T remains valid $\forall \lambda_1, \lambda_2 \in \Lambda$. It tends to 0 in K^* as $T \to \infty$. So g_λ, the limit in K^*, is independent of $\lambda \in \Lambda$; call it g: $g \in \bigcap_\Lambda L_1^\lambda$, and $r' = \phi_\lambda(g), \widehat{r}'(\omega) = \bar{h}(\lambda + \mathrm{i}\omega) \Rightarrow \int e^{zt} g(t) dt = \bar{h}(z)$ for $\Re(z) \in \Lambda$. ∎

Definition 5. The Banach space $L_p^\lambda \cap L_\infty$ has $\max\{\|\cdot\|_p^\lambda, \|\cdot\|_\infty\}$ as norm.

A S_λ^p*-map* is an S^1 map of pairs where the pairs are of the form $(L_\infty, L_p^\lambda \cap L_\infty)^n$.

[12] The theorem also states that W is inverse-closed in $\mathbb{L}(L_p, L_p)$.

We could use this proposition just for kernels, which would somewhat lighten the proof.

Proposition 7. *(i) The operator norm of a measurable function k as a convolution operator on $L^\lambda_\infty \cap L_\infty$ equals $\max\{\|k\|^\lambda_1, \|k\|_1\}$.*
(ii) The operator norm on $(L_\infty, L^\lambda_\infty \cap L_\infty)$ of any operator is convex in λ.

Proof. (ii): Let ψ be the operator. Let $\|\psi\|^\lambda$ be the operator norm of ψ on $L^\lambda_\infty \cap L_\infty$. Since the norm equals $\max\{\|\psi\|^0, \|\psi\|^\lambda\}$, suffices to prove convexity of $\|\psi\|^\lambda$ on $\mathbb{R}_+$ (hence dually on $\mathbb{R}_-$), and we can assume $\|\psi\|^0 < \infty$. Thus ψ is an operator on L_∞, and since $\|f\|_{L^\lambda_\infty \cap L_\infty} = \|\max(1, e^{\lambda t}) f(t)\|_\infty$, we get, with $h^\lambda(t) = \min(1, e^{-\lambda t})$, $\|\psi\|^\lambda = \sup_{\|g\|_\infty \le 1} \operatorname{ess\,sup}_s \max(1, e^{\lambda s}) |\psi_s(h^\lambda g)|$.

So $\|\psi\|^\lambda = \sup_{\|g\|_\infty \le 1} \sup_n \operatorname{ess\,sup}_s \max(1, \min(n, e^{\lambda s})) |\psi_s(h^\lambda g)|$, hence, using a strong (i.e., that is the identity on bounded continuous functions, for the max term to factor out) lifting $\frac{1}{\sigma}$, we can replace the ess sup by a sup: $\|\psi\|^\lambda = \sup_{\|g\|_\infty \le 1} \sup_n \sup_s \max(1, \min(n, e^{\lambda s})) |(\frac{1}{\sigma}{}_s \circ \psi)(h^\lambda g)| = \sup_s \max(1, e^{\lambda s}) \sup_{\|g\|_\infty \le 1} (\frac{1}{\sigma}{}_s \circ \psi)(h^\lambda g)$. Now, $\frac{1}{\sigma}{}_s \circ \psi \in L^*_\infty$ (i.e., is a "finitely additive measure"), say μ_s, and $h^\lambda \ge 0$. Thus $\sup_{\|g\|_\infty \le 1} (\frac{1}{\sigma}{}_s \circ \psi)(h^\lambda g) = \sup_{\|g\|_\infty \le 1} \mu_s(h^\lambda g) = \nu_s(h^\lambda)$, where $\nu_s = |\mu_s|$ is the absolute value of μ_s, i.e., $\nu_s \in (L^*_\infty)_+$.

Hence $\|\psi\|^\lambda = \sup_s \max(1, e^{\lambda s}) \nu_s(h^\lambda)$. Now, since we consider $\lambda \in \mathbb{R}_+$, $\nu_s(h^\lambda) = \nu_s(\mathbf{1}_{t \le 0}) + \nu_s(\mathbf{1}_{t>0} e^{-\lambda t})$, so $e^{\lambda s} \nu_s(h^\lambda) = \nu_s(\mathbf{1}_{t \le 0}) e^{\lambda s} + \nu_s(\mathbf{1}_{t>0} e^{\lambda(s-t)})$, and both are clearly convex in λ, as positive linear combinations of exponentials. Thus so is $\|\psi\|^\lambda$, as a sup of convex functions.

(i): By the same argument (without liftings) as above for $\|\psi\|^\lambda$, we get for the operator norm $\|k\| = \sup_s \max(1, e^{\lambda s}) (|k| \star h_\lambda)_s, = \sup_s \int |k(t)| \max(1, e^{\lambda s}) \min(1, e^{-\lambda(s-t)}) dt = \sup_s \int |k(t)| e^{[\lambda s]^+ - [\lambda(s-t)]^+} dt$. Now, the exponent is increasing to λt when $\lambda s \ge 0, s \to \infty$, and to 0 when $\lambda s \le 0, s \to \infty$, so by the monotone convergence theorem $\|k\| = \max\{\int |k(t)| e^{\lambda t} dt, \int |k(t)| dt\}$. ■

5. F is S^p_λ for $\lambda < R$ and $p = 1, \infty$

Lemma 12. *For O open in $\mathbb{R}^n$, define $\tilde{O}$ open in $L^{\mathbb{R}^n}_\infty$ as $\tilde{O} = \{g \in L^{\mathbb{R}^n}_\infty \mid \exists \varepsilon > 0 \colon d(g_t, \complement O) \ge \varepsilon \text{ a.e.}\} = \{g \in L^{\mathbb{R}^n}_\infty \mid \exists \tilde{g} \in g \colon \{\tilde{g}_t, t \in \mathbb{R}\} \text{ is relatively compact in } O\}$.*
Then for $f \colon O \mapsto \mathbb{R}$ continuous, $\tilde{f} \colon \tilde{O} \to L_\infty \colon g \mapsto f \circ g$ is continuous.

Proof. For $g \in \tilde{O}$ and $\tilde{g} \in g$ let $K \subseteq O$ be compact such that $\tilde{g}_t \in K$ $\forall t$. K has a compact neighbourhood $K_1 \subseteq O$. By Stone-Weierstrass, approximate f uniformly on K_1 by polynomials: this reduces the proof of the continuity at g to the case where f is a polynomial. That follows in turn since L_∞ is a Banach algebra. ■

Lemma 13. *Assume f C^1 in lemma 12. Then $\tilde{f}$ is S^p_λ on $\tilde{O}$, with $A_g \colon L^{\mathbb{R}^n}_\infty \to L_\infty \colon \delta\tilde{g} \mapsto \big(\sum_i \big(\frac{\partial f}{\partial x^i}\big)_{\tilde{g}_t} \delta\tilde{g}^i_t\big)_{t \in \mathbb{R}}$ as derivative at g.*

Proof. Define $K_1 \subseteq O$ as in the proof of lemma 12. We deal first with point (i).

- $A_{\tilde{g}}(\delta\tilde{g}) \in \mathcal{L}_\infty$ since, for all i, $\delta\tilde{g}^i_t \in L_\infty$, and by lemma 12 $\big(\frac{\partial f}{\partial x^i}\big)_{\tilde{g}_t} \in L_\infty$.
- $A_{\tilde{g}}(\delta\tilde{g})$ depends only on the equivalence classes δg and g, so $A_g \colon L^{\mathbb{R}^n}_\infty \to L_\infty$.
- A_g is linear by construction.
- A_g is continuous since $\|A_g\| \le \sum_i \big\|\big(\frac{\partial f}{\partial x^i}\big)_{\tilde{g}_t}\big\|_\infty$ and $\frac{\partial f}{\partial x^i}$ is bounded on K_1.

So $A_g \in \mathbb{L}(L^{\mathbb{R}^n}_\infty, L_\infty)$, and to show it is the Fréchet differential of $\tilde{f}$, it suffices to prove that $\forall \varepsilon > 0$ $\exists \delta > 0$ such that, for any $t \in \mathbb{R}$ and $z \in \mathbb{R}^n$, $\sum_i |z^i| \le \delta \Rightarrow$

$$(*) \qquad \Big| f(g_t + z) - f(g_t) - \sum_i \Big(\frac{\partial f}{\partial x^i}\Big)_{g_t} z^i \Big| \le \varepsilon \sum_i |z^i|$$

For that, following the continuous path $h(s)$ indexed by $s \in [0,1]$ from g_t to $g_t + z$ where only the i-th coordinate is varied during the interval $\big[\frac{i-1}{n}, \frac{i}{n}\big]$:

$$f(g_t + z) - f(g_t) - \sum_i \Big(\frac{\partial f}{\partial x^i}\Big)_{g_t} z^i = \int_0^1 \sum_i \Big[\Big(\frac{\partial f}{\partial x^i}\Big)_{h(s)} - \Big(\frac{\partial f}{\partial x^i}\Big)_{h(0)}\Big] dh^i_s$$

So: $$\Big|f(g_t+z)-f(g_t)-\sum_i\Big(\frac{\partial f}{\partial x^i}\Big)_{g_t}z^i\Big|\le\int_0^1\sum_i\Big|\Big(\frac{\partial f}{\partial x^i}\Big)_{h(s)}-\Big(\frac{\partial f}{\partial x^i}\Big)_{h(0)}\Big|dh_s^i$$
$$\le\sum_i\sup_s\Big|\Big(\frac{\partial f}{\partial x^i}\Big)_{h(s)}-\Big(\frac{\partial f}{\partial x^i}\Big)_{h(0)}\Big|\,|z^i|$$

Recall that $\frac{\partial f}{\partial x^i}$ is continuous on the compact set K_1. Hence for any $\varepsilon>0$ $\exists\delta'>0$ such that whenever $h,h_0\in K_1$, and $\|h-h_0\|\le\delta'$, then $\forall i$, $\big|\big(\frac{\partial f}{\partial x^i}\big)_h-\big(\frac{\partial f}{\partial x^i}\big)_{h_0}\big|<\varepsilon$. Thus choose $\delta\le\delta'$ such that the δ-neighbourhood of K is included in K_1. Therefore if $\|z\|=\sum_i|z^i|\le\delta$, then $\|h(s)-h(0)\|\le\delta$ and hence $h(s),h(0)\in K_1$ for all s.

Thus A_g is indeed the Fréchet derivative of $\tilde f$. Remains to prove that $g\mapsto A_g$ is continuous. Since the sum of continuous maps is continuous, it suffices to prove that $g\mapsto\big(\frac{\partial f}{\partial x^i}\big)_{g_t}$ is continuous for all i when the right-hand member is viewed as the corresponding (multiplication) operator from L_∞ to L_∞. But then, clearly, the operator norm coincides with the L_∞ norm; so we need that $g\mapsto\big(\frac{\partial f}{\partial x^i}\big)_{g_t}$ is continuous from $L_\infty^{\mathbb{R}^n}$ to L_∞. This follows from lemma 12; so point (i) is established.

(iii) follows too, the norm of the operator on $L_p^\lambda\cap L_\infty$ being also its L_∞-norm.

For point (ii), it suffices, all norms on $\mathbb{R}^n$ being equivalent, to replace in (*) z by δg_t, and take the L_p^λ-norm on both sides, assuming $\|\delta g\|_\infty\le\delta$. ■

Corollary 5. *Sums, products, etc. of S_λ^p maps are S_λ^p.*

Proof. Apply lemma 13 with f the sum function, and lemma 10. ■

Lemma 14. *For $a\in\mathbb{R}$ and ϑ integrable between 0 and a, the map $(g,E)\mapsto T\colon L_\infty\times L_\infty\to L_\infty\colon T_x=\int_0^a\exp\{\int_x^{x+s}g(u)\,du\}E_{x+s}\vartheta_s ds$ is S_λ^1 and S_λ^∞, with derivative at (g,E) $A_{g,E}\colon(\delta g,\delta E)\curvearrowright\big[x\mapsto\int_x^{x+a}\delta E_t h_1(x,t)dt+\int_x^{x+a}\delta g_t h_2(x,t)dt\big]$, where $h_1(x,t)=\exp\{\int_x^t g_s ds\}\vartheta_{t-x}$, $h_2(x,t)=\int_t^{x+a}h_1(x,s)E_s ds$*

For further streamlining, should extend to arbitrary p, instead of dealing separately with 2 cases.

Proof. We first prove a couple of inequalities. Assume $h(x,t)$ Lebesgue-measurable from $\mathbb{R}^2$ to $\mathbb{R}$, well-defined for $t-x$ between 0 and a. Let then

$$\|h\|=\Big\|\int_x^{x+a}|h(x,t)|\,dt\Big\|_\infty$$
$$\|h\|_*=\Big\|\int_{t-a}^{t}|h(x,t)|\,dx\Big\|_\infty$$

Then: (i.e., if right-hand term is finite, left-hand term is well-defined and)

(1) $$\Big\|x\mapsto\int_x^{x+a}f(t)h(x,t)\,dt\Big\|_\infty^\lambda\le e^{|\lambda a|}\|f\|_\infty^\lambda\,\|h\|$$

(2) $$\Big\|x\mapsto\int_x^{x+a}f(t)h(x,t)\,dt\Big\|_1^\lambda\le e^{|\lambda a|}\|f\|_1^\lambda\,\|h\|_*$$

So the operator norm of h on L_∞ is $\le\|h\|$, on $L_1^\lambda\cap L_\infty$, $\le e^{|\lambda a|}\max\{\|h\|,\|h\|_*\}$, and on $L_\infty^\lambda\cap L_\infty$, $\le e^{|\lambda a|}\|h\|$.

Let $\|\vartheta\|_1=\big|\int_0^a|\vartheta_s|ds\big|$ and $\|f\|_a=\sup_x\big|\int_x^{x+a}|f_t|dt\big|$, $\le|a|\,\|f\|_\infty$:

(3) $$\|h_1\|\le e^{\|g\|_a}\|\vartheta\|_1\,,\quad\text{(so } h_2 \text{ is well-defined)}$$

(4) $$\|h_1\|_*\le e^{\|g\|_a}\,\|\vartheta\|_1$$

and $\|h_2\|\le\|E\|_\infty\sup_x\int_x^{x+a}\int_t^{x+a}|h_1(x,s)|ds\,dt$, so, by Fubini:

(5) $$\|h_2\|\le|a|\|E\|_\infty\|h_1\|$$

and $\|h_2\|_*\le\sup_t\int_{t-a}^t\int_t^{x+a}|h_1(x,s)||E_s|ds\,dx=\sup_t\int_t^{t+a}|E_s|\int_{s-a}^t|h_1(x,s)|dx\,ds$, so:

(6) $$\|h_2\|_*\le\|E\|_a\,\|h_1\|_*\quad\text{(alternatively: } \|h_2\|_*\le|a|\|E\|_\infty\|h_1\|\text{)}$$

Thus $\|h_i\|, \|h_i\|_* < \infty$, and to show that $A_{g,E} \in \mathbb{L}(L_\infty \times L_\infty, L_\infty)$, suffices to do this for each of the two terms, so, linearity being obvious, this follows from $\|h_i\| < \infty$. Similarly for $L_\infty^\lambda \cap L_\infty$, and, by $\|h_i\|_* < \infty$, for $L_1^\lambda \cap L_\infty$.

Since $T_x = h_2(x,x)$, $\text{DIFF}_x \stackrel{\text{def}}{=} |T_x(g+\delta g, E+\delta E) - T_x(g,E) - A_x(\delta g, \delta E)| =$

$$= \Big| \int_x^{x+a} h_1^{g+\delta g}(x,t)(E_t + \delta E_t)dt - \int_x^{x+a} h_1^g(x,t)E_t dt - \int_x^{x+a} h_1^g(x,t)\delta E_t dt - \int_x^{x+a} h_2^{g,E}(x,t)\delta g_t dt \Big|$$

$$\leq \Big| \int_x^{x+a} |h_1^{g+\delta g}(x,t) - h_1^g(x,t)|\,|\delta E_t|\,dt \Big| + \Big| \int_x^{x+a} \big[h_1^{g+\delta g}(x,t) - h_1^g(x,t) - h_2^{g,E}(x,t)\delta g_t\big] dt \Big|$$

With a change of order of integration (by Fubini), and the definition of h_2, the term in h_2 becomes $\int_x^{x+a} \big(\int_x^s \delta g_t dt\big) h_1^g(x,s) E_s ds$; so, by definition of h_1, suffices to bound: $\Big| \int_x^{x+a} |h_1^g(x,t)| \big| e^{\int_x^t \delta g_s ds} - 1 \big|\,|\delta E_t|\,dt \Big| + \Big| \int_x^{x+a} |h_1^g(x,t)| \big[e^{\int_x^t \delta g_s ds} - 1 - \int_x^t \delta g_s ds \big]\,|E_t|\,dt \Big|$
Since, with $M(x) = e^x - 1$, $e^x - 1 - x \leq x M(x)$, and since $\int_x^t \delta g_s ds$ is bounded by $\|\delta g\|_a$, the bracket is bounded by $M(\|\delta g\|_a) \big| \int_x^t \delta g_s ds \big|$, thus:

$$\text{DIFF}_x \leq M(\|\delta g\|_a) \Big\{ \Big| \int_x^{x+a} |h_1^g(x,t)|\,|\delta E_t|\,dt \Big| + \|E\|_\infty \int_x^{x+a} |h_1^g(x,t)| \int_x^t |\delta g_s| ds\, dt \Big\}$$

The second integral equals $\int_x^{x+a} |\delta g_s| \int_s^{x+a} |h_1^g(x,t)| dt\, ds$, $\leq \|h_1^g\| \big| \int_x^{x+a} |\delta g_s| ds \big|$. So:
$\|\text{DIFF}\|_\infty^\lambda \leq e^{|\lambda a|} M(\|\delta g\|_a) \big\{ \|\delta E\|_\infty^\lambda \|h_1^g\| + |a| \|\delta g\|_\infty^\lambda \|E\|_\infty \|h_1^g\| \big\}$, using (1),
$\|\text{DIFF}\|_1^\lambda \leq M(\|\delta g\|_a) \big\{ e^{|\lambda a|} \|\delta E\|_1^\lambda \|h_1^g\|_* + \big| \frac{1 - e^{-\lambda a}}{\lambda} \big| \|\delta g\|_1^\lambda \|E\|_\infty \|h_1^g\| \big\}$, by (2).

Using the first also for $\lambda = 0$, those imply the Fréchet differentiability conditions in 9.i and 9.ii; remains thus only to prove continuity of A for 9.i and A' for 9.iii.

For those 3 continuity properties, suffices, by our bounds on the operator norms of the h_i, to show $g \mapsto h_1$ and $(g,E) \mapsto h_2$ are locally Lipschitz for $\|h_i\|$ and $\|h_i\|_*$.

For h_1, this stems from the following for $\|\cdot\|$, and an identical argument for $\|\cdot\|_*$:[13]

$$\big\| h_1^{g+\delta g} - h_1^g \big\| = \sup_x \Big| \int_x^{x+a} \big| e^{\int_x^t \delta g_s ds} - 1 \big| \big| h_1^g(x,t) \big| dt \Big| \leq M(\|\delta g\|_a) \|h_1^g\|$$

For h_2, from (5), in $\big\| h_2^{g_1,E_1} - h_2^{g_2,E_2} \big\| \leq \big\| h_2^{g_1,E_1} - h_2^{g_1,E_2} \big\| + \big\| h_2^{g_1,E_2} - h_2^{g_2,E_2} \big\|$, $\big\| h_2^{g_1,E_1 - E2} \big\| \leq |a| \|E_1 - E_2\|_\infty \|h_1^{g_1}\|$, and $\|h_2^{g_1,E_2} - h_2^{g_2,E_2}\| \leq |a| \|E_2\|_\infty \|h_1^{g_1} - h_1^{g_2}\|$, so the result follows from that for h_1. Same argument, with (6), for $\|\cdot\|_*$. ■

Lemma 15. *(i) If $\lambda < R$, $i \mapsto k$ is S_λ^p with derivative $\delta k_t = e^{-Rt} \int_{-\infty}^t e^{Rs} \delta i_s ds$.*
(ii) $k \mapsto y$ is S_λ^p with derivative $\delta y_t = \alpha A k_t^{\alpha - 1} \delta k_t$, if $\inf k_t > 0$.
(iii) $k \mapsto f$ is S_λ^p with $\delta f_t = \alpha(1-\alpha) A k_t^{\alpha - 2} \delta k_t$, if $\inf k_t > 0$. So $\delta f = (1-\alpha) \frac{\delta y}{k}$.
(iv) $(f, E, y) \mapsto \mathcal{N}$ from L_∞^3 to L_∞ is S_λ^1 and S_λ^∞ with derivative

$$\delta \mathcal{N}_x = \int_x^{x+1} [H(x,t)\delta f_t + h(x,t,\delta y_t, \delta E_t)]\, dt \quad \textit{where}$$

$$h(x,t,u,v) = \big((1-\alpha) u \varphi_{t-x} + v \vartheta_{t-x} \big) e^{\int_x^t f_s ds}; \; H(x,t) = \int_t^{x+1} h(x,z,y_z,E_z) dz$$

(v) $f \mapsto \mathcal{D}$ is S_λ^1 and S_λ^∞ with derivative

$$\delta \mathcal{D}_x = (1-\sigma) \int_x^{x+1} \delta f_t \zeta(x,t) dt \quad \textit{where}$$

$$\zeta(x,t) \stackrel{\text{def}}{=} \int_t^{x+1} \exp\big\{ \textstyle\int_x^s [(1-\sigma) f_v - \eta] dv \big\} ds$$

(vi) $(\mathcal{N}, \mathcal{D}) \mapsto \mathcal{B}$ is S_λ^p with derivative $\delta \mathcal{B}_t = \frac{\mathcal{D}_t \delta \mathcal{N}_t - \mathcal{N}_t \delta \mathcal{D}_t}{\mathcal{D}_t^2}$, if $\inf \mathcal{D}_t > 0$.

[13] Using the formula for h_1 in small steps along the segment joining g_1 and g_2, since $g \mapsto \|h_1^g\|$ is convex, we get: $\|h_1^{g_1} - h_1^{g_2}\| \leq \|g_1 - g_2\|_a \max_i \|h_1^{g_i}\|$—and recall (3).

(vii) $(f, \mathcal{B}) \mapsto c$ *is* S^1_λ *and* S^∞_λ *with derivative*

$$\delta c_t = \int_{t-1}^{t} \delta \mathcal{B}_x G(x,t)dx - \sigma \int_{t-1}^{t} \delta f_x \int_{t-1}^{x} \mathcal{B}_s G(s,t)\, ds\, dx \quad \textit{where}$$
$$G(x,t) \stackrel{\text{def}}{=} \exp\left\{-\int_x^t (\eta + \sigma f(u))du\right\}$$

(viii) $(y, E, c) \mapsto \tilde{\imath}$ *is* S^p_λ *with derivative* $\delta \tilde{\imath}_t = \delta y_t + \delta E_t - \delta c_t$

Proof. i: the map equals $g \star i$ with $g(x) = \mathbf{1}_{x \geq 0} e^{-Rx} \in L^\lambda_1\ \forall \lambda < R$ ($\|g\|^\lambda_1 = \frac{1}{R-\lambda}$), so the inequality in lemma 11 implies its continuity as an operator, both on L_∞ and on L^λ_p. Being linear, it is its own derivative, hence is S^p_λ.

ii and iii: by lemma 13.

iv: recall that $\mathcal{N}_x = \int_0^1 e^{\int_x^{x+s} f_t dt}\big(\vartheta_s E_{x+s} + (1-\alpha)\,\varphi_s y_{x+s}\big)ds$. By lemma 14 the derivative of the first term is

$$\int_x^{x+1} \delta E_t h_1(x,t)dt + \int_x^{x+1} \delta f_t h_2(x,t)dt \quad \text{where}$$
$$h_1(x,t) \stackrel{\text{def}}{=} \exp\left\{\int_x^t f_s ds\right\}\vartheta_{t-x}\ ; \quad h_2(x,t) \equiv \int_t^{x+1} h_1(x,s)E_s ds$$

Similarly, the derivative of the second term is

$$\int_x^{x+1} \delta y_t h_3(x,t)dt + \int_x^{x+1} \delta f_t h_4(x,t)dt \quad \text{where}$$
$$h_3(x,t) \stackrel{\text{def}}{=} (1-\alpha) \exp\left\{\int_x^t f_s ds\right\}\varphi_{t-x}\ ; \quad h_4(x,t) \stackrel{\text{def}}{=} \int_t^{x+1} h_3(x,s)y_s ds$$

Combining the two (cor. 5), the derivative of $\mathcal{N}$ maps $(\delta f, \delta y, \delta E)$ to

$$x \mapsto \int_x^{x+1} \delta E_t h_1(x,t)dt + \int_x^{x+1} \delta y_t h_3(x,t)dt + \int_x^{x+1} \delta f_t \left[h_2 + h_4\right](x,t)dt$$

Hence the answer, by regrouping terms.

v: by lemma 14, setting $a = 1$, $g_t = \big(1-\sigma\big)f_t - \eta$, $\vartheta_s = 1$ and $E_t = 1$.

vi: by lemma 13.

vii: by lemma 14, setting $a = -1$, $E_t = \mathcal{B}_t$, $\vartheta_s = 1$ and $g_t = \eta + \sigma f_t$.

viii: by cor. 5. ∎

Proposition 8. $F\colon L^2_\infty \to L_\infty$ *is* S^1_λ *and* S^∞_λ *on* $\{i \mid \inf k_t > 0\}$ *for* $\lambda < R$.

Proof. By lemmas 10 and 15, since $\inf k_t > 0$ implies the same for $\mathcal{D}$. ∎

6. Generic invertibility of $\frac{\partial F}{\partial i}$ at BGE

Notation. For a function X on a (subset of a) group define $\bar{X}(x) = X(-x)$.

Lemma 16. *For* $O \subseteq \mathbb{C} \times \mathbb{C}^n$ *open and* $F\colon O \to \mathbb{C}$ *analytic,* $\frac{F(x,z)-F(y,z)}{x-y}$ *is so too on* $\{(x,y,z) \mid (x,z) \in O, (y,z) \in O\}$.

Proof. Suffices to prove analyticity at points of the form (x_0, x_0, z_0). Replace F by its power series around $(x_0, z_0) \in O$, getting $a_n(z)\frac{(x-x_0)^n-(y-x_0)^n}{(x-x_0)-(y-x_0)}$ as a typical term, and then verify that after division the resulting power series still has positive (e.g., the same) radius of convergence. ∎

6.1. **Parameterisation of the equilibrium graph.**

Definition 6. The *parameter space*, or the space of economies, is $\wp = \{(R, \alpha, \eta, \sigma, \varphi(ds)) \mid (R,\sigma) \in \mathbb{R}^2_{++}, \alpha \in\,]0,1[,\ \varphi(ds) \in \Delta([0,1])\}$, with the weak*-topology on $\Delta([0,1])$, the probabilities on $[0,1]$.

Please, include price equations in that BGE corollary as well, individual consumption formula should be there too, explain why f is a "price."

Definition 7. Let $\mathfrak{G}$ be the cross product of $\wp$ and the set containing all allocations and prices in the economy. The equilibrium graph (restricting attention to BGE) is the subset G^* of $\mathfrak{G}$ composed of all points satisfying conditions (i)-(vii) of cor. 2.

Definition 8. A real-valued function defined on a subset of $\mathbb{R}^n \times \Delta([0,1])$, is *JE* (or *JA*) if its complex extension by analytic continuation (to a subset of $\mathbb{C}^n \times \Delta([0,1])$) is jointly continuous in all variables and for each fixed $\varphi(ds) \in \Delta([0,1])$ jointly entire (or analytic) in all variables but $\varphi(ds)$.

There should be a good place to list the equilibrium variables and where they live.

Lemma 17. *(i) Let* $\mathcal{H}(x) = \frac{1-X}{1-x}$, $X = \digamma(R(1-x)) \int e^{sR(1-x)}\varphi(ds)$, $\mathcal{T}(x) = \frac{\mathcal{H}(x)}{1+x\mathcal{H}(x)}$, $\bar{\bar{\wp}} = \{(R, x, \eta, \sigma, \varphi(ds)) \mid R > 0, 1+x\mathcal{H}(x) \neq 0, \mathcal{T}(x) \geq 0, \varphi(ds) \in \Delta([0,1])\}$, $\bar{\wp} = \{(R, \alpha, \eta, \sigma, \varphi(ds)) \mid R > 0, \varphi(ds) \in \Delta([0,1])\}$. *Let the map*[14] $\Omega_b : \bar{\bar{\wp}} \to \bar{\wp} \times \mathbb{R}^2$ *be such that all the parameters but* α *are mapped into themselves, and* $\alpha = x\mathcal{T}(x)$, $f = R(1-x)$, $y = A(A\mathcal{T}(x)/R)^{x\mathcal{H}(x)}$. *Then* Ω_b *is one-to-one and is jointly continuous, in addition it is JA where* $\mathcal{T}(x) \neq 0$ *except for poles at* $(1-\sigma)R(1-x) - \eta = 2n\pi\mathrm{i}$ *with* $n \neq 0$. *The inverse defined on* $\bar{\wp} \times \mathbb{R}^2$ *(by* $x = 1 - f/R$*) is also JA.*

(ii) Let $\tilde{\wp} = \{(R, x, \eta, \sigma, \varphi(ds)) \mid (R, \sigma, x, \mathcal{H}(x)) \in \mathbb{R}^4_{++}, \varphi(ds) \in \Delta([0,1])\}$, *and let* $\Omega_g : \wp \to \wp \times \mathbb{R}^2$ *map all the parameters into themselves, return* 0 *as* f *and* $A^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{R}\right)^{\frac{\alpha}{1-\alpha}}$ *as* y; *and define* $G_g \stackrel{\text{def}}{=} \Omega_g(\wp)$, $G_b \stackrel{\text{def}}{=} \Omega_b(\tilde{\wp})$, $G \stackrel{\text{def}}{=} G_g \cup G_b$. *Then* G *is consistent with conditions* (vii), (ii) *and* (iii) *of cor. 2. Let* $\Gamma : G \mapsto G^*$, *be an identity on* G, *and for the rest of the coordinates return all the BGE quantities and prices according to conditions* (i), (iii), (iv), (vi) *of cor. 2. Then* Γ *is one-to-one and is JA except for poles at* $(1-\sigma)R(1-x) - \eta = 2n\pi\mathrm{i}$ *with* $n \neq 0$. *The inverse defined on* $\mathfrak{G}$ *is also JA.*

Again need all the conditions!!

Proof. We prove (i) in two steps: (a) $\mathcal{H}$ is JA except for poles at $(1-\sigma)R(1-x)-\eta = 2n\pi\mathrm{i}$ with $n \neq 0$ and (b) the rest of the statement.

For (a), let us first prove that X is JA except for those poles. Suffices to do this for each of the 2 terms in the product. Since $\Phi(z)$ is entire by lemma 16 and has as only zeros $2n\pi\mathrm{i}$ with $n \neq 0$, the conclusion follows immediately for $\digamma$, and for the integral it follows from the fact that the φ have bounded support. Remains thus only to prove that $\mathcal{H}$ is JA at any point with $x = 1$.[15] This is easier in terms of the variable f; letting then $Z = 1 - \digamma$, $I = 1 - \int e^{sf}\varphi(ds)$, we have $1 - X = Z + I - ZI$, so it suffices to prove that both $\frac{Z}{f}$ and $\frac{I}{f}$ are JA whenever $f = 0$.

$\frac{Z}{f} = \frac{\Phi(a)-\Phi(b)}{a-b}\frac{1}{\Phi(a)}$, with $a = (1-\sigma)f - \eta$, $b = -\sigma f - \eta$. The first factor is entire by lemma 16 and the second factor has poles at $2n\pi\mathrm{i}$, $n \neq 0$ as mentioned above.

$\frac{I}{f} = -\int \Phi(sf)s\varphi(ds)$, hence again the result since Φ is entire by lemma 16.

For (b), we start with the continuity claim. For f it is obvious, and for α note $1 + x\mathcal{H}(x) \neq 0$ guarantees the continuity of $\mathcal{T}(x)$. So, as $(A/R)^{x\mathcal{H}(x)}$ is continuous by (a) and $A/R > 0$, there only remains to prove continuity of $(\frac{\mathcal{H}(x)}{1+x\mathcal{H}(x)})^{x\mathcal{H}(x)}$. If $1 + x\mathcal{H}(x) > 0$, then $\mathcal{H}(x) \geq 0$ by $\mathcal{T}(x) \geq 0$. The continuity of the function $(1+u)^{-u}$ allows to reduce the problem to the continuity of $[(\mathcal{H}(x))^{\mathcal{H}(x)}]^x$, which follows from first applying the continuity of u^u for $u \geq 0$ to the bracket, then that of a^b for $a > 0$ to the whole expression. And if $1 + x\mathcal{H}(x) < 0$, $\mathcal{T}(x) \geq 0$ implies $\mathcal{H}(x) \leq 0$, and hence $\mathcal{H}(x) < 0$ and $x > 0$, so $\mathcal{T}(x) > 0$, thus continuity is trivial. The JA property follows from (a).

As for the inverse map, it is a projection, apart from the x coordinate, which is obtained from f: $x = 1 - f/R$. Thus the inverse map is linear and therefore is JE.

[14] The image of the map is a reduced equilibrium graph with one quantity y and one 'price' f.

[15] The GRE corresponding to the intersection of BGE and GRE graphs, i.e., $\Gamma(G_b \cap G_g)$.

To show (ii) we start by claiming that G is consistent with a definition of BGE: the formula for α is the solution of condition vii in cor. 2, y is determined by eliminating k from conditions (ii), (iii) of the same corollary. Next, we claim that $\Omega_g(\tilde{\wp}) \subset \wp \times \mathbb{R}^2$ with the last coordinates being (f, y), such that $y > 0$, and conversely, the inverse map from the image of $\Omega_g(\tilde{\wp})$ maps into a subset of $\tilde{\wp}$. Indeed, given any $x > 0$ s.t. $\mathcal{H}(x) > 0$, the corresponding y, f, and α clearly satisfy $0 < \alpha < 1$, $y > 0$, and (ii), (iii) and (vii) in cor. 2. Conversely, $x > 0$ follows then by the remark 8 and then $\mathcal{H}(x) > 0$ from $0 < \frac{\alpha}{1-\alpha} = x\mathcal{H}(x)$. hence the conclusion.

Next, observe that BGE's are completely described just by the variables y, f, as related by (ii), (iii) and (vii) in cor. 2: all other equilibrium quantities are JE functions of those and of the parameters. Indeed, from cor. 2 we get then $k = y\mathcal{T}(x)/R$, $i = Rk$, and thus $c = y - i$; all other relations just serve to define additional quantities; next, since $\alpha\frac{y}{k} = R - f$, all prices become JE in the parameters and t; and by lemma 2 $c_{t,s} = (1-\alpha)ye^{\sigma(\gamma+\nu)t+(\nu-\eta)s+(\gamma-\sigma f)(t+s)}\frac{\int_0^1 e^{(f-\nu)u}\varepsilon_u du}{\Phi((1-\sigma)f-\eta)}$, which is thus also JE in the parameters and t, s, except for poles at $(1-\sigma)f - \eta = 2n\pi\mathrm{i}$ with $n \neq 0$.

For the GRE use cor. 3. Again, the inverse map, being a projection, is JE. ■

Corollary 6. *The maps* $\tilde{\wp} \mapsto \Gamma(\Omega_b(\tilde{\wp}))$ *and* $\wp \mapsto \Gamma(\Omega_g(\wp))$ *are both JA.*

6.2. The derivative of the fixed point map.

Lemma 18. *(i) The derivative* $T = \frac{\partial \Upsilon}{\partial i}(i^*, 0)$ *at a BGE is a convolution operator, with kernel* $\tau \in L_1^\lambda\ \forall \lambda < R$ *having as FT* $\widehat{\tau} = \frac{R-f}{R-\mathrm{i}\omega}(1 - \widehat{H})$*, where*

$$\widehat{H}(\omega) = \Phi(-\varkappa + \mathrm{i}\omega)\widehat{Q}(-\omega) - C\sigma v(f)\widehat{\chi}^{-\varkappa}(\omega) \tag{1}$$

$$\widehat{Q}(\omega) = \frac{C}{\Phi(-\varkappa)}\Big[\widehat{\psi}(\omega) + Bv(f + \mathrm{i}\omega) - (1-\sigma)v(f)\widehat{\chi}^{f-\varkappa}(\omega)\Big] \tag{2}$$

$$\widehat{\psi}(\omega) = \frac{1}{\mathrm{i}\omega}\big(v(f + \mathrm{i}\omega) - v(f)\big), \quad \psi_t = \mathbf{1}_{0 \le t \le 1}\int_t^1 e^{fs}\varphi_s ds \tag{3}$$

$$\widehat{\chi}^x(\omega) = \frac{1}{\mathrm{i}\omega}\big(\Psi(-x, \omega) - 1\big), \qquad \chi_t^x = \mathbf{1}_{0 \le t \le 1}\Big(1 - \frac{t\Phi(xt)}{\Phi(x)}\Big) \tag{4}$$

with $\Phi(z) = \dfrac{e^z - 1}{z}$ $\quad \Psi(x, y) = \dfrac{\Phi(-x + \mathrm{i}y)}{\Phi(-x)}$ $\quad v(z) = \displaystyle\int_0^1 e^{zt}\varphi(dt)$

$\varkappa = f\sigma + \eta$ $\quad B = \dfrac{\alpha}{(1-\alpha)(R-f)}$ $\quad C = \dfrac{(1-\alpha)\Phi(-\varkappa)}{B\Phi(f-\varkappa)}$

or, equivalently,

$$\widehat{H}(\omega) = \frac{C}{\mathrm{i}\omega}\Big(\Psi(\varkappa, \omega)\Big[(B\mathrm{i}\omega - 1)v(f - \mathrm{i}\omega) + (1-\sigma)v(f)\Psi(\varkappa - f, -\omega)\Big] + v(f)\sigma\Big) \tag{5}$$

(ii) $H \in L_p([-1, 1])$*,* $p < \infty$*, is norm-continuous function on the BGE graph* G*.*

Proof. By lemma 15 and cor. 2, $\frac{\partial \Upsilon}{\partial i}$ is given by the following at a BGE, if $k > 0$:

(i) $i \mapsto k$ has derivative $\delta k_t = e^{-Rt}\int_{-\infty}^t e^{Rs}\delta i_s ds$, i.e., with $g(t) = \mathbf{1}_{t \ge 0}e^{-Rt}$, $\delta k = g \star \delta i$.

(ii) $k \mapsto y$ has derivative $\delta y_t = \alpha A k^{\alpha-1}\delta k_t$, so: $\delta y = (R-f)\delta k = (R-f)g \star \delta i$

(iii) $y \mapsto f$ has derivative $\delta f = (1-\alpha)\frac{\delta y}{k}$

(iv) $(f, y) \mapsto \mathcal{N}$ has derivative, with $\varrho_s^f \stackrel{\text{def}}{=} \mathbf{1}_{0 \le s \le 1}e^{fs}\varphi_s$, $(\varphi_s = \varphi_s(ds))$ $\psi_s \stackrel{\text{def}}{=} \mathbf{1}_{0 \le s}\int_s^\infty \varrho_t^f(dt)$:

$$\delta\mathcal{N}_x = (1-\alpha)\int\Big[\varrho_s^f + \frac{\psi_s}{B}\Big]\delta y_{x+s} ds$$

(v) $(f, y) \mapsto \mathcal{D}$ has derivative, with $\chi_s^z \stackrel{\text{def}}{=} \mathbf{1}_{0\le s\le 1}\big(1 - \frac{s\Phi(zs)}{\Phi(z)}\big)$, $\varkappa = \eta + f\sigma$:

$$\delta\mathcal{D}_x = (1-\sigma)\frac{1-\alpha}{k}\Phi(f-\varkappa)\int \chi_s^{f-\varkappa}\delta y_{x+s} ds$$

(vi) $(f, y) \mapsto \mathcal{B}$ has derivative, with $Q_s \stackrel{\text{def}}{=} \frac{C}{\Phi(-\varkappa)}\big[\psi_s + B\varrho_s^f - (1-\sigma)\psi_0\chi_s^{f-\varkappa}\big]$:

$$\delta\mathcal{B}_x = \int Q_s \delta y_{x+s} ds = \int \bar{Q}_s \delta y_{x-s} ds, \quad \text{so: } \delta\mathcal{B} = \bar{Q} \star \delta y$$

(vii) $(f, y) \mapsto c$ has derivative, with $h_s = \mathbf{1}_{0\le s\le 1} e^{-\varkappa s}$, $Z_s = C\sigma\psi_0\chi_s^{-\varkappa}$:

$$\delta c_t = \int (h_s \delta\mathcal{B}_{t-s} - Z_s \delta y_{t-s}) ds, \quad \text{so: } \delta c = h \star \delta\mathcal{B} - Z \star \delta y$$

and thus, with $H = h \star \bar{Q} - Z$, $\delta c = H \star \delta y$.

(viii) $(y, c) \mapsto \tilde{\imath}$ has derivative $\delta\tilde{\imath} = \delta y - \delta c$.

So, with δ_0 the unit mass at 0, $\delta\tilde{\imath} = (\delta_0 - H) \star ((R-f)g) \star \delta i$, i.e., $\frac{\partial\Upsilon}{\partial i}$ is indeed a convolution operator with kernel $\tau = (R-f)g \star (\delta_0 - H)$. $\tau \in L_1^\lambda\ \forall\lambda < R$ since g is so and $\delta_0 - H$ has compact support. Finally, taking FT's, $\widehat{\tau} = \frac{R-f}{R-\mathrm{i}\omega}(1 - \widehat{H})$.

Observe that, for any Q, $\widehat{\bar{Q}} = \bar{\widehat{Q}}$, so $\widehat{H} = \widehat{h}\bar{\widehat{Q}} - \widehat{Z}$.

Now $\widehat{h}(\omega) = \Phi(-\varkappa + \mathrm{i}\omega)$, $\widehat{\varrho}^f(-\omega) = v(f - \mathrm{i}\omega)$, and $\psi_0 = v(f)$, Hence representation (1), and, by direct computation, formula (5).

Point ii. We first show that h, Q and Z are jointly continuous, using the $\|\|p$ topology for h and Z and weak* topology for Q. For h note that for any converging sequence in $\wp \times \mathbb{R}^2$, with limit $\varkappa_0$, h_s converges uniformly to its limit $\mathbf{1}_{0\le s\le 1}e^{-\varkappa_0 s}$. The coefficients in the definitions of Q and Z, i.e., B, σ, $\Phi(-\varkappa)$, C, ψ_0 are clearly continuous in the parameters and f, as for any point in G we have $\Phi(-\varkappa) > 0$, $0 < \alpha < 1$ and $R > f$ (see remark 7), so $B > 0$. The conclusion then follows by the joint weak*-$\|\cdot\|_p$ continuity of the maps $(\varphi(ds), f) \mapsto \psi$, $z \mapsto \chi^z$; and the weak*-weak* continuity of $(\varphi(ds), f) \mapsto \varrho^f(ds)$ on G. Next note that the map $h, Q \mapsto h \star \bar{Q}$ is $\|\cdot\|_p$-continuous using weak* topology on Q and $\|\cdot\|_p$ topology on h (cf. notation section in 4.3). [16] ■

6.3. **Generic invertibility.**

Definition 9. A subset of $\wp$ or of G is *negligible* if its section for any fixed probability distribution $\varphi(ds)$ in $\Delta([0,1])$ has Lebesgue measure 0.

A subset is *generic* if its complement is contained in a countable union of closed negligible sets.

Lemma 19. *Let $f\colon O \to \mathbb{R}$ be analytic and non-null, where O is open and connected in $\mathbb{R}^n$. Then the set of zeros of f is closed and negligible.*

Remark 11. The same statement holds with the same proof replacing $\mathbb{R}$ by $\mathbb{C}$.

Remark 12. The conclusion can obviously be strengthened to 0 measure for any measure whose conditionals on any factor given the other factors are non-atomic.

Proof. For $n = 0$ the statement is trivial. Proceeding by induction, let the statement hold for $n-1$. Assume first O is a product of two open connected sets $X \times Y$, $X \in \mathbb{R}^{n-1}$,$Y \in \mathbb{R}$. By assumptions there is a point, $(x^0, y^0) \in X \times Y$, at which f is non-null. Then by the induction hypothesis the set of zeros in X of $f(x, y^0)$, $Z_{y_0} \subset X$, is closed and has measure zero. For any fixed $x \in X\backslash Z_{y_0}$, $f(x, y)$ is an analytic function defined on Y, non-zero (at y_0), thus the set of its zeros, Z_x, is discrete. The set of zeros of f on $X \times Y$ then is a union of $\{(x, y_0) : x \in Z_{y_0}\}$ and $\{(x, y) : y \in Z_x\}$, both of measure zero. For general O, cover then O with countably

[16] Note that for continuous distributions φ the convergence of H is uniform.

many products of the form $X \times Y$; since we know the set of zeros is closed in O, it suffices to show that its intersection with each of those product sets has measure 0. This follows from our previous argument provided f does not vanish identically on any of those product sets. But if it did, connexity of O would imply by analytic continuation that f vanishes everywhere on O. ■

Proposition 9. *Generically on $\wp$, 1 is not a value of $\widehat{\tau}$ for any BGE.*

Proof. Multiplying $1-\widehat{\tau}=0$ by the non-null factor $\frac{R-\mathrm{i}\omega}{C(R-f)}$ yields $(f-\mathrm{i}\omega)\big(D(\omega)+\frac{1}{C(R-f)}\big)=0$, where $D(\omega)\stackrel{\text{def}}{=}\frac{\widehat{H}(\omega)}{C(f-\mathrm{i}\omega)}$; and hence the exceptional set $N\stackrel{\text{def}}{=}N_0\cup\tilde{N}$, where $N_0=\{g\mid f=0,\widehat{\tau}(0)=1\}$, and $\tilde{N}=\{g\in G:\exists\omega\colon \frac{1}{C(f-R)}=D(\omega)\}$.

Claim 1. *If $f=0$, the coefficient of σ in D equals*

$$\Xi(\omega,\eta)=\frac{1}{\omega^2+\eta^2}\left[1-\left(\frac{\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}}{\frac{\sinh\frac{\eta}{2}}{\frac{\eta}{2}}}\right)^2\right]<\infty$$

$0<\Xi(\omega,\eta)\le\frac{1}{\omega^2+\eta^2}$ *everywhere, and* $\Xi(\omega,\eta)\sim\frac{1}{\omega^2+\eta^2}$ *for* $(\omega,\eta)\to\infty$

Proof. $\Xi(\omega,\eta)\xrightarrow[\omega,\eta\to 0]{}\frac{1}{12}>0$. Now $\frac{\sin x}{x}$ (resp., $\frac{\sinh x}{x}$) is, for $x\neq 0$, in absolute value <1 (resp., >1), so for $(\omega,\eta)\neq(0,0)$, we also get $\Xi(\omega,\eta)>0$; so $\Xi>0$ everywhere. Remains thus only to show that $\Xi(\omega,\eta)\sim\frac{1}{\omega^2+\eta^2}$ for $(\omega,\eta)\to\infty$, which follows from $\frac{\sin\omega}{\omega}\xrightarrow[\omega\to\infty]{}0$ and $\frac{\sinh\eta}{\eta}\xrightarrow[\eta\to\infty]{}\infty$. ■

$N_0=\{g\mid f=0,\frac{1}{C}\widehat{H}(0)=0\}$, and since by claim 1 the coefficient of σ is $(f-\mathrm{i}\omega)\Xi=0$, and since $v(0)=\Psi(\eta,0)=1$, suffices to express that, at $\omega=0$,

$$B+\frac{1}{\mathrm{i}\omega}\big(\Psi(\eta,-\omega)-v(-\mathrm{i}\omega)\big)=0$$

By (3) and (4) in lemma 18 and the definition of B, this equation is equivalent to

$$\widehat{\chi}^{-\eta}(0)-\widehat{\psi}(0)-\frac{\alpha}{(1-\alpha)R}=0$$

Now $\widehat{\psi}(0)=\int_0^1 s\varphi_s ds$ and $\widehat{\chi}^{-\eta}(0)=\frac{1}{\eta}-\frac{1}{e^\eta-1}$; thus N_0 is the set of zeros of

$$\frac{1}{\eta}-\frac{1}{e^\eta-1}-\int_0^1 s\varphi_s ds-\frac{\alpha}{1-\alpha}\frac{1}{R}$$

Since $\frac{1}{\eta}-\frac{1}{e^\eta-1}$ decreases from 1 to 0, there is at most 1 value of any the 4 parameters $R,\alpha,\eta,\int s\varphi_s ds$ that fits, given values of the other 3.[17] So N_0 is closed and negligible.

To show that $\tilde{N}$ is negligible we establish, first, that the imaginary part of $D(\omega)$ has only a discrete set of zeros as a function of ω on $G\backslash G_g$ (G_g), this set depends on all parameters but R (σ), and second that, for those ω, $\frac{1}{C(f-R)}=\Re D(\omega)$ holds only for a discrete set of values R (σ). Finally we show N is closed.

Definition 10. For $g\in G$, $Z(g)\stackrel{\text{def}}{=}\{\omega\in\mathbb{R}\mid\Im D(\omega)=0\}$.

Step 1. *The set of g where $Z(g)$ is not discrete is negligible. On G_g, $Z(g)$ depends only on (η,α,R,φ), and on $G\backslash G_g$, D (and hence $Z(g)$) only on (η,f,σ,φ).* double-check the list

Proof of step 1. On G_g, since $f=0$, formula 5 of lemma 18 implies $\widehat{H}$ is purely imaginary (and so $\widehat{H}/\mathrm{i}\omega$ is real) iff $(B\mathrm{i}\omega-1)v(-\mathrm{i}\omega)$ is real, i.e. iff $\frac{v(-\mathrm{i}\omega)}{1+B\mathrm{i}\omega}$ is real. But given $v(-\mathrm{i}\omega)=\widehat{\varphi}(-\omega)$, so the ratio is a Fourier transform of the convolution of $\bar{\varphi}$ with $B^{-1}\mathbf{1}_{t\le 0}e^{B^{-1}t}$ (recall $B>0$). As $\varphi\gneqq 0$ has compact support, the support of the convolution is bounded on one side and unbounded on the other, so the convolution can not be even, hence its FT is not real.

[17]The last 2 equations of App. B show this is the condition for $M=0$ (autarchy) in the GRE.

Also, by formula 5 of lemma 18, the imaginary part of $D(\omega) = \frac{\widehat{H}(\omega)}{-C\mathrm{i}\omega}$ is independent of σ, hence its set of real zeros is so too, and it is discrete by lemma 19.

Remains to prove the statement on $G\backslash G_g$.

Claim 2. *(i) On $G\backslash G_g$, $B = \frac{\Phi(f-\varkappa)-\Phi(-\varkappa)v(f)}{f\Phi(f-\varkappa)}$ and $C = \frac{f/B}{R\frac{\Phi(f-\varkappa)}{\Phi(-\varkappa)}-(f-R)v(f)}$, so D only depends on $(\eta, f, \sigma, \varphi)$.*

(ii) D is JA on $(G\backslash G_g)\times\mathbb{R}$, where the last coordinate is ω, and is the FT of the bounded measure $\frac{1}{C}H\star\ell_f$ on $[-1,1]$, where $\ell_f(x) = \mathsf{sign}(f)\mathbf{1}_{fx>0}e^{-fx}$.

Proof. Expressing α as a function of f by lemma 17, (i) follows by definition of B and C in lemma 18. Thus the last clause, using also lemma 18.i.

Point ii. To show that D is entire in ω, note that H is a measure with bounded support, $[-1,1]$, so its Fourier transform is an entire function, i.e., limit of a power series (converging everywhere) with infinite radius of convergence.[18] As $\widehat{H}$ is entire, the only possible pole of D is at $-\mathrm{i}f$, but a direct computation, using the formula for B from point (i), shows that $\widehat{H}(-\mathrm{i}f)$ is identically zero, so, using lemma 16 with $x=\omega$ and $y=-\mathrm{i}f$, D is entire.

Since it is the FT of $\frac{1}{C}H\star\ell_f$ with $\ell_f(x) = \mathsf{sign}(f)\mathbf{1}_{fx>0}e^{-fx}$, and since this convolution must be proportional to ℓ_f outside $[-1,1]$, it follows that the proportionality factor must be 0, else the FT would have a pole at $-\mathrm{i}f$. Thus this convolution is carried by $[-1,1]$.

Since $D(\omega)$ is the FT of this convolution, the joint continuity follows from the same property for H (point (ii) of lemma 18) and ℓ_f.

To establish joint analyticity, note that for any point in $G\backslash G_g$, $f\neq 0$, so $f-\mathrm{i}\omega\neq 0$. Given equation (5) for $\frac{\widehat{H}}{C}$ in lemma 18, possible poles are at $\omega=0$, $\varkappa = 2k\pi\mathrm{i}$ for $k\neq 0$ and $\varkappa - f = 2k\pi\mathrm{i}$ also for $k\neq 0$. The latter two are far away from $G\backslash G_g$, where $\varkappa$ and f are real, so remains to prove joint analyticity of $\frac{\widehat{H}}{C}$ at $\omega=0$.

$$\widehat{H}(\omega)/C = \frac{1}{\mathrm{i}\omega}(\Psi(\varkappa,\omega)((B\mathrm{i}\omega-1)v(f-\mathrm{i}\omega)+(1-\sigma)v(f)\Psi(\varkappa-f,-\omega))+v(f)\sigma)$$

Since B and its coefficient are clearly analytic, it suffices to concentrate on $\frac{1}{\mathrm{i}\omega}(\Psi(\varkappa,\omega)((1-\sigma)v(f)\Psi(\varkappa-f,-\omega)-v(f-\mathrm{i}\omega))+v(f)\sigma)$, which equals $-\Psi(\varkappa,\omega)(1-\sigma)\tilde{\Psi}(\varkappa-f,-\omega)v(f)-\Psi(\varkappa,\omega)(1-\sigma)V-\sigma v(f-\mathrm{i}\omega)\tilde{\Psi}(\varkappa,\omega)-\sigma V$, with $\tilde{\Psi}(x,y) = \frac{\Psi(x,y)-1}{\mathrm{i}y}$, $V = \frac{v(f-\mathrm{i}\omega)-v(f)}{\mathrm{i}\omega}$.

So we need that V and $\tilde{\Psi}(\varkappa,\omega)$ are JA at $\omega=0$. $V = -\int_0^1 e^{ft}t\Phi(-\mathrm{i}\omega t)\varphi(dt)$, and since the integrand is JE, the integral is so too. And $\tilde{\Psi}(\varkappa,\omega)$ is analytic by lemma 16 except for poles at $\varkappa = 2k\pi\mathrm{i}$, $k\neq 0$ (i.e., the poles of $\Psi(-\varkappa,y)$). ■

Claim 3. *$\{g\in G\backslash G_g \mid \Im D(\omega) = 0\ \forall\omega\}$ is negligible.*

Proof. By claim 2.ii, $D(\omega)$ is the FT of a bounded measure. To show that $\Im D\neq 0$ it is sufficient to show that the derivative at zero is distinct from zero. Indeed, for a FT of a positive measure the real part is maximized at $\omega=0$, so the derivative at zero has zero real part. This conclusion is preserved for sums and differences of any positive measures, and thus for an arbitrary measure.

Then to prove the claim it is sufficient to show that $(\frac{d}{d\omega}D)(0)$ is distinct from zero for all but a negligible set of parameters. Given representation 1 of $\widehat{H}$ in lemma 18, it is affine in σ when expressed in terms of $\varphi,\varkappa,f,\sigma$ and so $(\frac{d}{d\omega}D)(0)$ is so too.

It remains to show then that the coefficient of σ in $(\frac{d}{d\omega}D)(0)$ is zero for a negligible set of $(\varphi,\varkappa,f)$. Let $A \stackrel{\text{def}}{=} \frac{f^2}{v(f)\mathrm{i}}(\frac{d^2}{d\omega d\sigma}D)_{|\omega=0}$: since $f^2/v(f)>0$ on $G\backslash G_g$, it suffices by lemma 19 to show that A is JA and is not identically zero.

[18] Indeed, as the exponential function is entire, the series $\sum_n \frac{(zt)^n}{n!}$ converges everywhere, so by the Lebesgue's dominated convergence theorem $\widehat{H}$ is $\int e^{zt}H(t)dt = \sum_n \frac{z^n}{n!}\int t^n H(t)dt$, i.e., a power series in z with infinite radius of convergence, $z\in\mathbb{C}$.

Given D is JA on $G\backslash G_g$ by claim 2, $(\frac{d}{d\omega}D)(0)$ is so too. Hence it is so for $\sigma = 0$, then the JA property of the coefficient of σ, and therefore that of A follows. Using again representation 1 of $\widehat{H}$ in lemma 18, $A = f^2 \frac{d}{d\omega}(\frac{\Psi(\varkappa,\omega)\widehat{\chi}^{f-\varkappa}(-\omega)-\widehat{\chi}^{-\varkappa}(\omega)}{\mathrm{i}(f-\mathrm{i}\omega)})|_{\omega=0}$. So $A(f,\varkappa) = f\int\big((\frac{\Phi(\varkappa)-1}{\varkappa\Phi(\varkappa)} - t)\chi^{f-\varkappa}(t) - t\chi^{-\varkappa}(t)\big)dt + \int(\chi^{f-\varkappa}(t) - \chi^{-\varkappa}(t))dt$. It is not identically zero since, given the identities, $\int\chi^z(t)dt = \frac{\Phi(z)-1}{z\Phi(z)}$ and $\int t\chi^z(t)dt = \frac{z-2+2\Phi(-z)}{2z^2\Phi(-z)}$, $A(1,0) = \frac{e-4}{3(e-1)} \neq 0$. ■

Claim 4. *The subset of G_b where $Z(g)$ is not discrete is negligible, in addition, on $G\backslash G_g$ $Z(g)$ is independent of R.*

Proof. Given the representation of B in claim 2.i, $D(\omega)$ is independent of R. In view of lemma 19, given $\Im D$ is real-analytic for real arguments, this implies that the set of zeros of $\Im D(\omega)$ is discrete and is independent of R. ■

This finishes the proof of step 1. ■

Step 2. *N is negligible in G.*

Proof. Since N_0 is negligible, suffices to prove this for $\tilde{N}$. Partition $\tilde{N}$ into two sets: $\tilde{N}^g = \tilde{N} \cap G_g$ and its complement, $\tilde{N}^b$.

For $\tilde{N}^g$, given the definition of the exceptional set and the previous step, it suffices to verify that for any ω in the countable set $Z(g)$ there exists at most one value of σ for which the real part of D is equal to $-CR$. This is because $C = \frac{(1-\alpha)^2 R}{\alpha}$ and $D = \Xi\sigma + const$ with $\Xi > 0$ by claim 1.

Note $\tilde{N}^b = \tilde{N}_1 \cup \tilde{N}_2$, where $\tilde{N}_1 = \{g \in G\backslash G_g \mid Z(g) \text{ is not discrete}\}$ and $\tilde{N}_2 = \{g \in G\backslash G_g \mid Z(g) \text{ is discrete}, \exists\omega \in Z(g)\colon \Re D(\omega) = \frac{1}{C(f-R)}\}$. By step 1, $\tilde{N}_1$ is negligible. By claim 2.i, $\frac{1}{C(f-R)} = \frac{B}{f}\big(\frac{R}{f-R}\frac{\Phi(f-\varkappa)}{\Phi(-\varkappa)} - v(f)\big)$, where B is independent of R, and, recall, B and $f \neq 0$. By step 1, $\Re D(\omega)$ does not change with R, so there is at most one value of R that satisfies the equality for every $\omega \in Z(g)$. Since $Z(g)$ is discrete, there are at most countably many values of R that satisfy the equality, so $\tilde{N}_2$ is negligible. ■

Step 3. *N is closed in G.*

Proof. Given the previous steps, it remains to show that N is closed. Consider a sequence $g_n \in N$ with $g_n \to g_0$. Choose corresponding ω_n with $\widehat{\tau}(\omega_n, g_n) = 1$. Since $\|\widehat{H}\|$ is bounded on the sequence g_n by lemma 18.ii used with $p = 1$, and since $R-f$ is obviously bounded on the sequence, $\exists K\colon 1 = \|\widehat{\tau}(\omega_n, g_n)\| \leq \frac{K}{\|R_n - \mathrm{i}\omega_n\|}$, so ω_n is bounded. Thus, extracting a convergent subsequence, one can assume $\omega_n \to \omega_0$.

By lemma 18.ii, the map $H\colon G \to L_1$ is continuous, so, composing with FT: $L_1 \to C_b(\mathbb{R})$ (see notation in section 4.3), the composite map $\widehat{H}\colon G \to C_b(\mathbb{R})$ is also continuous; hence the joint continuity of $\widehat{H}$ in (ω, g). Given $R > 0$, $R - \mathrm{i}\omega \neq 0$, so $\widehat{\tau}$ is also jointly continuous in (ω, g). This implies then $\widehat{\tau}(\omega_0, g_0) = 1$, so $g_0 \in N$. ■

To complete the proof of the proposition observe that G is a countable union of compact sets, the intersection of N with each of those is compact and negligible by the previous steps and its projection onto $\wp$, i.e., the set of exceptional parameters, is compact. Remains to show this projection is negligible. This is obvious for $N \cap G_g$, since there the projection is basically the identity map. And on the complement, Fubini's theorem ensures that, outside a negligible set of $(R, \eta, \sigma, \varphi)$, the set of exceptional values of f is negligible. For fixed $(R, \eta, \sigma, \varphi)$, our projection basically maps f to α, as in the figures above, and this map is C^1, thus preserves negligible sets. ■

Remark 13. By example-specific tricks, we reduced the problem to show that negligibility is preserved when going from the equilibrium graph to the parameter space to the (trivial) 1-dimensional version of a statement that a C^1 map from $\mathbb{R}^n$ to $\mathbb{R}^n$ preserves negligibility (or, more generally, replacing $\mathbb{R}^n$ above by a n-dimensional manifolds with boundary, the first one being a K_σ). Such a statement seems easily provable from Sard's theorem and the implicit function theorem (and still doesn't seem "the right form": why should e.g. locally Lipschitz not suffice?); we just didn't find the right reference yet.

It is such a statement that would be the right tool to handle the above problem in general. It is also the one (even its 1-dimensional version) that shows that neglicting above the difference between the equilibrium graph including the y coordinate (as defined) and the graph without it (as used) is immaterial.

Remark 14. On the other hand, our technique above to prove genericity, relying on the fact that Z_g is independent of one the parameters, seems very specialised, and would probably need to be replaced by something else for a generalisation.

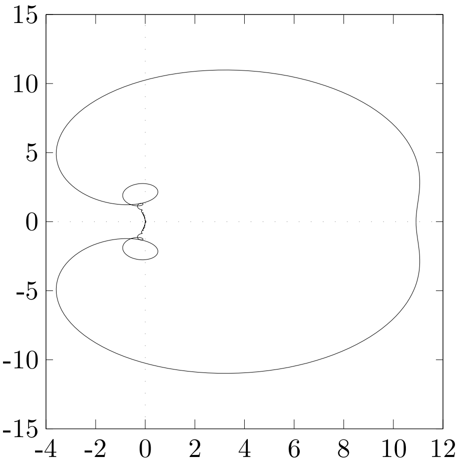

Figure 5: BGE of fig. 1, $\frac{\alpha Y}{I} = 2$

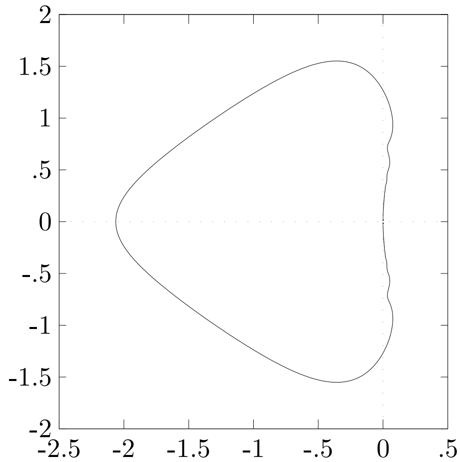

Figure 6: GRE of fig. 1, $\alpha = .3$

Corollary 7. *For $z \in \mathbb{C}$ let $G(z) = \int_{-1}^{1} e^{zt} H(t)dt$, with H from lemma 18.ii. The set $D = \{\Re(z) \mid G(z) = \frac{z-f}{R-f}\}$ is closed in $\mathbb{R}$ and discrete. Generically $0 \notin D$. Let then Λ be the connected component of 0 in $\mathbb{R}\backslash(D \cup \{R\})$. Then $\frac{\partial F}{\partial i}$ has as inverse in $\bigcap_\Lambda W^\lambda$ a convolution operator $g - \mathbf{1}$, and $\int e^{zt} g(t)dt = \frac{1-G(z)}{\frac{z-f}{R-f} - G(z)}$ for $\Re(z) \in \Lambda$.*

In particular, $g - \mathbf{1}$ is also the inverse on all $L_p^\lambda \cap L_\infty$ with $\lambda \in \Lambda$ and $1 \le p \le \infty$.

If $R \notin D$, $z = R$ is not a singularity of $\frac{1-G(z)}{\frac{z-f}{R-f} - G(z)}$. What happens beyond R, till the first point in D?

Proof. Since H has support in $[-1, 1]$, G is entire and $|G(z)| \le \|H\|_1 e^{|\Re(z)|}$, so for $\Re(z)$ bounded $\{z \mid G(z) = \frac{z-f}{R-f}\}$ must be compact. By analyticity, it is discrete, hence finite: D is finite on every bounded set, thus closed in $\mathbb{R}$ and discrete.

By prop. 9, generically $0 \notin D$.

The rest follows now by applying prop. 6 with $f = \tau$ $(= \frac{\partial \Upsilon}{\partial i})$, since $\frac{\partial F}{\partial i} = \tau - \mathbf{1}$ (prop. 2)—and thus (lemma 18.i) $J =]-\infty, R[$, and $h(z) = \frac{R-f}{R-z}(1 - G(z))$, by analytic continuation, since by lemma 18.i $h(\mathrm{i}\omega)$ is given by this formula, and since h is analytic by prop. 6 and G entire as seen above. ■

7. Local properties of equilibrium selections

7.1. Local Uniqueness and S^p_λ.

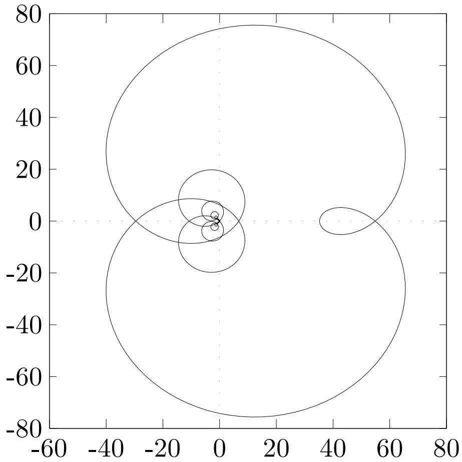

Figure 7: BGE of fig. 2, $\frac{\alpha Y}{I} = 3$

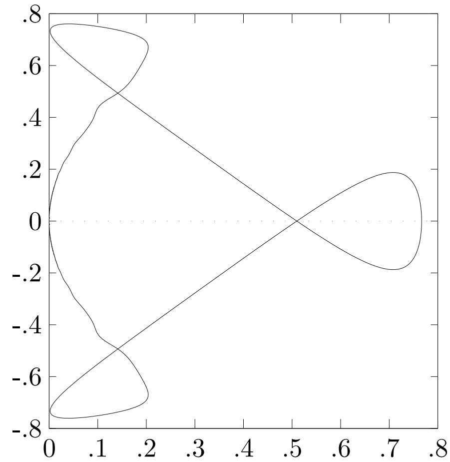

Figure 8: GRE of fig. 2, $\alpha = .3$

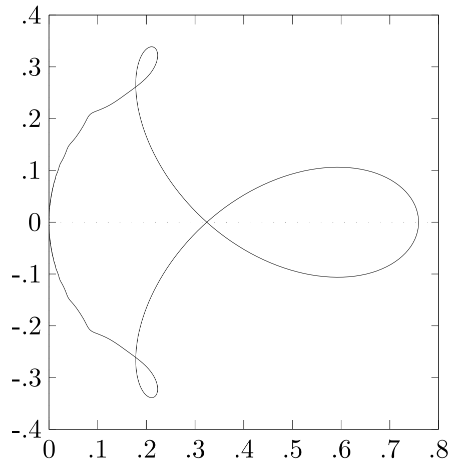

Figure 9: BGE of fig. 2, $\frac{\alpha Y}{I} = \frac{1}{2}$

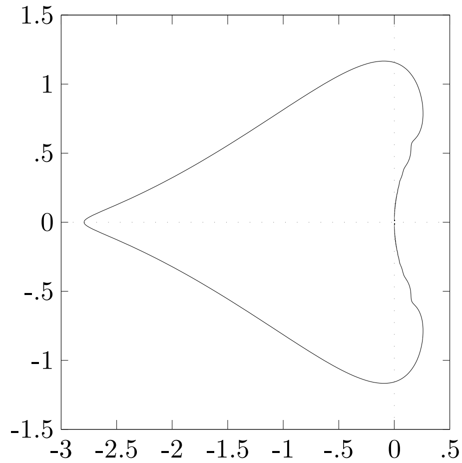

Figure 10: GRE of fig. 3, $\alpha = .3$

Theorem 1. *Generically, $\exists \delta' > 0$, and for any BGE $\varpi(0)$, $\forall \Lambda^0 \subseteq \Lambda$ compact and $\forall \Lambda^1 \subseteq \Lambda$ finite, $\exists \delta > 0$ s.t., for $\|E\|_\infty \leq \delta$, the E-perturbed economy has a unique equilibrium $(i, k, y, f, c, \ldots)$, say $\varpi(E)$, with $\|\varpi(E) - \varpi(0)\|_\infty \leq \delta'$ and s.t. $E \mapsto \varpi(E)$ is $S^1_\lambda \, \forall \lambda \in \Lambda^1$ and $S^\infty_\lambda \, \forall \lambda \in \Lambda^0$ on $\{E \mid \|E\|_\infty < \delta\}$.*

Remark 15. Conditions for regularity of the BGE's w.r.t. variations is the parameters are trivial: it suffices that when restricting all functions in Υ in prop. 2 to be constants, at each BGE $\frac{d\tilde{\imath}}{di} \neq 1$, i.e., equivalently $\widehat{\tau}(0) \neq 1$. In particular, on our generic set, regularity w.r.t. variations in the parameters also holds.

Proof. Suffices to do the proof for a fixed BGE, then to replace δ' by its minimum over all (finitely many, recall fn. 10) BGE, then to decrease accordingly the corresponding δ's. Note that the set Λ depends on the chosen BGE.

By lemma 15 (and 10) it suffices to show that the normalised investment i_t is S^1_λ and S^∞_λ with respect to E around the BGE. By prop. 8 and cor. 7, this follows

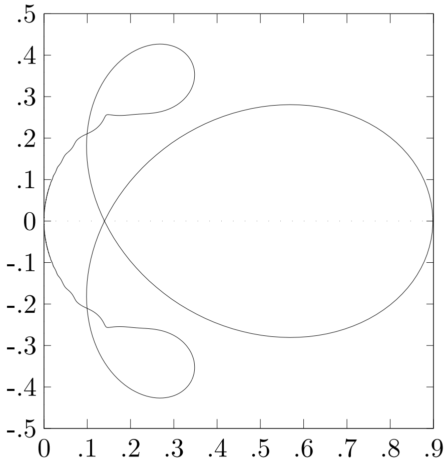

Figure 11: BGE of fig. 4, $\frac{\alpha Y}{I} = \frac{1}{2}$

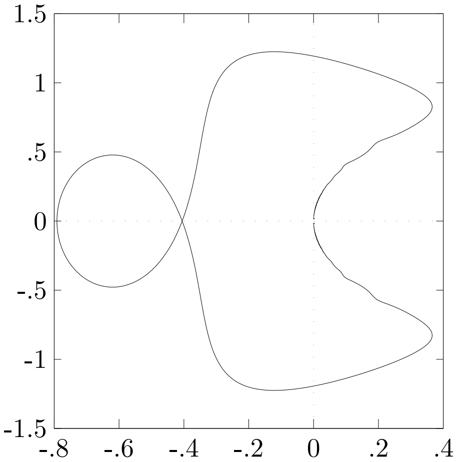

Figure 12: GRE of fig. 4, $\alpha = .3$

from applying prop. 4, first for $p = 1$, for each $\lambda \in \Lambda^1$, to $F\colon L_\infty^2 \to L_\infty\colon (i, E) \mapsto \Upsilon(i, E) - i$ at the GRE $x_0 = i^*$, $y_0 = E^* = 0$. Doing this first with $\Lambda^1 = 0$ fixes δ and δ'. To ensure the fixed points are really equilibria, reduce δ further if needed to ensure that $\|E\|_\infty < \delta$ implies $\mathcal{N}$ in prop. 2 is bounded away from 0. Repeating now with the other λ's, reduce δ as needed—no need to change δ'.

Next, for $p = \infty$, repeat the above for the 2 values $\max \Lambda^0$ and $\min \Lambda^0$ of λ. Use the continuity of ϖ and prop. 8 to reduce δ so that for those 2 λ's, and $\forall E\colon \|E\|_\infty < \delta$, the norm in $\mathbb{L}(L_\infty, L_\infty \cap L_\infty^\lambda; L_\infty, L_\infty \cap L_\infty^\lambda)$ of $X \overset{\text{def}}{=} \mathbf{1} - (g - \mathbf{1}) \star \big(\frac{\partial F}{\partial i}\big)(\varpi(E), E)$ is < 1, where $g - \mathbf{1}$ is as in cor. 7 — i.e., choose, with that operator norm throughout, $\|\big(\frac{\partial F}{\partial i}\big)(\varpi(E), E) - \big(\frac{\partial F}{\partial i}\big)(\varpi(0), 0)\| < 1/\|g - \mathbf{1}\|$. Prop. 7.ii implies then that for $\|E\|_\infty < \delta$, X has norm < 1 in $\mathbb{L}(L_\infty, L_\infty \cap L_\infty^\lambda; L_\infty, L_\infty \cap L_\infty^\lambda)$ for all $\lambda \in \Lambda^0$. So $\big(\frac{\partial F}{\partial i}\big)(\varpi(E), E)$ is invertible in all those spaces, with as inverse $(\sum_0^\infty X^n) \star (g - \mathbf{1})$. Thus, for each $\lambda \in \Lambda^0$, prop. 4 is applicable at any such E, implying that ϖ is S_λ^∞ in the neighbourhood of E, hence is so on $\{E \mid \|E\|_\infty < \delta\}$. ■

7.2. **Smoothness of equilibrium paths.**

Theorem 2. *In $\varpi\{E \mid \|E\|_\infty < \delta\}$, the functions k, y, f are uniformly Lipshitz, and the c, C^1 with uniformly bounded uniformly equicontinuous derivatives.*

Proof. The $\varpi(E)$ are uniformly bounded by thm. 1. Thus the k are uniformly Lipshitz by prop. 2.i. Next so are the y, f, by prop. 2.ii and iii, since the k are uniformly bounded away from 0. Then $\mathcal{N}$ (and $\mathcal{D}$) are uniformly equicontinuous (e.g., for the first term, approximate ϑ in L_1 by a continuous function on $\mathbb{R}$ with support in $[0, 1]$), hence so is $\mathcal{B}$, and thus the conclusion for c by prop. 2.vii. ■

7.3. **Continuity of the equilibrium selection.**

We obtain here continuity for some more reasonable topologies.

Theorem 3. *ϖ is continuous on $\{E \mid \|E\|_\infty < \delta\}$ with the weak*-topology, and with the topology of uniform convergence on, for i, compact sets in L_1, and, for k, y, f, c, and its time-derivative c', tight sets of measures.*

Proof. Suffices to establish sequential continuity, the domain being metrisable. Let thus $E_n \to E$ weak*. Extracting a subsequence, we can assume the $\varpi(E_n)$ converge weak*, say to ϖ_∞, and it suffices to prove weak*-convergence for i and pointwise convergence—this implies uniform convergence on tight sets for uniformly bounded equicontinuous (cf. thm. 2) sequences—for the others, and that $\varpi_\infty = \varpi(E)$.

Prop. 2.i shows that $k(i_n)$ converge pointwise to $k(i_\infty)$, The other equations show then the same for the other variables, and the last equation shows then that $\tilde{i}_n$ $(= i_n)$ converges weakly to $\tilde{i}_\infty$ $(= i_\infty)$: so i_∞, i.e., ϖ_∞, is a solution for E, and hence, by uniqueness, it equals $\varpi(E)$. ∎

7.4. **Stability of equilibrium.**

The next results show a (strong) form of stability, or, "no hysteresis":

Theorem 4. *If $\|E^i\|_\infty < \delta$ then for $p = 1$, $\lambda \in \Lambda^1$ and $p = \infty$, $\lambda \in \Lambda^0$, $E^1 - E^0 \in L_p^\lambda$ implies $\varpi(E^1) - \varpi(E^0) \in L_p^\lambda$.*

Proof. By cor. 4, using convexity of the δ-ball O. ∎

Remark 16. Just for $p = \infty$, since Λ^0 can be taken as a compact interval approximating Λ as close as desired from inside (so with 0 interior), the theorem implies a very strong form of stability, towards both $-\infty$ and $+\infty$, "at any exponential rate in Λ".

Corollary 8. *For $\|E^i\|_\infty < \delta$ and $\lambda \in \Lambda^1$, $E^1 - E^0 \in L_1^\lambda$ implies all but the i coordinate of $\varpi(E^1) - \varpi(E^0)$ belong to $C_0^\lambda \cap L_1^\lambda$.*

Proof. Let $\triangle x = x^{E_1} - x^{E_0}$ for any variable x. By thm. 4, $\triangle\varpi \in L_1^\lambda$, and in particular $\triangle i$. In prop. 2, (i) implies then $\triangle k \in C_0^\lambda$ (i.e., is continuous and $\phi_\lambda(\triangle k)$ converges to 0 at ∞). The other equations imply then successively the same for all $\triangle$'s. ∎

7.5. **The derivatives of the equilibrium selection.**

Theorem 5. *For $\|E\|_\infty < \delta$, the derivative of the i-component of ϖ w.r.t. E is the identity plus a kernel operator, and is a kernel operator for the other components.*

At 0, those kernel operators are convolution operators, with kernels $k \in \bigcap_\Lambda L_1^\lambda$.

Is still missing the analog of $k \in \bigcap_\Lambda L_1^\lambda$ at $E \neq 0$.

Proof. We first show that, if $\sup\|f_n\|_\infty < \infty$, $f_n \to 0$ a.e. implies $\frac{di}{dE}(f_n) \to 0$ a.e.

Since the partials of F clearly preserve a.e.-convergence of uniformly bounded sequences, this follows from $\frac{di}{dE} = -\left(\frac{\partial F}{\partial i}\right)^{-1}\frac{\partial F}{\partial E}$, if, with $A \stackrel{\text{def}}{=} \frac{\partial F}{\partial i}$, A^{-1} preserves this convergence. Let A_0 be the value of A at $(i(0), 0)$; by cor. 7, A_0^{-1} preserves this convergence. So, since also A does, $X \stackrel{\text{def}}{=} I - A_0^{-1}A$ does too; and since $\|X\| < 1$ for the δ chosen in the second part of the proof of thm. 1, $(I - X)^{-1}$ is the norm limit of $\sum_0^n X^i$, hence preserves also the convergence; thus $A^{-1} = (I - X)^{-1}A_0^{-1}$ does too.

For the first part, this together with prop. 5 and lemma 15.i implies the result for k. The other points of lemma 15 (paying attention to the occurrence of δE in iv and viii) imply then the result for the other components.

In the second part, the convolution aspect follows then by shift-invariance (or from the formulas). Then, for $\lambda \in \Lambda$, we can use thm. 1 adding λ to Λ^0, and getting the same statement with some $\delta_\lambda \leq \delta$ instead of δ. So $E \mapsto \varpi(E)$ is S_λ^1 on $\{E \mid \|E\|_\infty < \delta_\lambda\}$. In particular, its derivative at 0, i.e. our convolution operator, has finite norm as an operator on $L_\infty^\lambda \cap L_\infty$. So, by prop. 7.i, $\|k\|_1^\lambda < \infty$. ∎

8. Welfare

8.1. **Utility functions.**

The utility function was used till now only in its ordinal aspect; here the cardinal aspect will play a role, so we first characterise the cardinal utility functions V, concave and homogeneous as required by Mertens and Rubinchik (2006), which induce the same ordinal preferences. This allows in particular to separate risk-aversion (denoted ρ) from intertemporal substitution σ. Then

$\rho \geq 0$, $\rho \neq 1$, and up to additive and multiplicative constants:

$$\text{if } \sigma \neq 1 \qquad V(c) = \frac{1}{1-\rho}\Big(\int_0^1 e^{-\beta s} c_s^{1-\frac{1}{\sigma}} ds\Big)^{\frac{1-\rho}{1-\frac{1}{\sigma}}}$$

$$\text{if } \sigma = 1 \qquad V(c) = \frac{1}{1-\rho}\exp\Big(\frac{1-\rho}{\Phi(-\beta)}\int_0^1 e^{-\beta s}\ln c_s ds\Big)$$

Multiplying the integral in the first equation by $\frac{1}{\Phi(-\beta)}$ yields a representation continuous in σ, while $\rho = 1$ must be excluded for homogeneity. We'll continue as in previous sections to avoid the limiting case $\sigma = 1$.

8.2. **Normalising utility functions.** By Mertens and Rubinchik (2006, p. 25), the normalised utility of an individual born at date x is then $V_x^\star(c) = e^{(\rho-1)\gamma x} V(c)$.

8.3. **Equilibrium utility.** Substituting in U^* of lemma 2, p and w using prop. 1, and $\omega_{x,s}$ from sect. 2.1, and using the notation from after prop. 1, we obtain the equilibrium utility $U_x^* =$

$$\frac{\big(\int_0^1 e^{-\nu s}\varepsilon_s ds\big)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}\Big[e^{\gamma x}\int_0^1 [\vartheta_s E_{x+s} + (1-\alpha)\varphi_s y_{x+s}] e^{\int_x^{x+s} f_u du} ds\Big]^{1-\frac{1}{\sigma}}\Big[\int_0^1 e^{-\eta s + (1-\sigma)\int_x^{x+s} f_u du} ds\Big]^{\frac{1}{\sigma}}$$

Then the normalised equilibrium utility $\tilde{V}_x = \frac{e^{(\rho-1)\gamma x}}{1-\rho}\big[(1-\frac{1}{\sigma})U_x^*\big]^{\frac{1-\rho}{1-\frac{1}{\sigma}}}$ equals

$$\frac{\big(\int_0^1 \varepsilon_s e^{-\nu s} ds\big)^{1-\rho}}{1-\rho}\Big[\Big(\int_0^1 e^{-\eta s + (1-\sigma)\int_x^{x+s} f_u du} ds\Big)^{\frac{1}{\sigma-1}}\int_0^1 e^{\int_x^{x+s} f_u du}\big(E_{x+s}\vartheta_s + (1-\alpha) y_{x+s}\varphi_s\big) ds\Big]^{1-\rho}$$

8.4. **Welfare diffs.** What we have to sum are the differences w_x of those utilities $\tilde{V}_x$ with those on the baseline, the BGE. Those are obtained by replacing, in $\tilde{V}_x$, E by 0, and y_s and f_s by y and f There is no harm then to divide troughout by $\big((\Phi(f-\varkappa))^{\frac{1}{\sigma-1}}\int_0^1 \varepsilon_s e^{-\nu s} ds\big)^{1-\rho}$. We obtain thus $w_x =$

$$\frac{\Big[\Big(\int_0^1 \frac{e^{-\eta s}}{\Phi(f-\varkappa)} e^{(1-\sigma)\int_x^{x+s} f_u du} ds\Big)^{\frac{1}{\sigma-1}}\int_0^1 e^{\int_x^{x+s} f_u du}\big(\vartheta_s E_{x+s} + (1-\alpha) y_{x+s}\varphi_s\big) ds\Big]^{1-\rho} - \big[(1-\alpha) y v(f)\big]^{1-\rho}}{1-\rho}$$

Here $\|E\|_\infty \leq \delta$ is assumed, as in thm. 1, and $y_\cdot, f_\cdot$ are given by $\varpi(E)$.

So our SWF equals $W = \int_{-\infty}^{\infty} e^{\lambda x} w_x dx$, where $\lambda = \nu$ in principle, but is left arbitrary for greater generality.

8.5. **The derivative of welfare.**

Lemma 20. *The map $(E, y, f) \mapsto w$ is, $\forall \lambda < R$, S_1^λ and S_∞^λ from an open subset of L_∞^3 containing $\varpi\{E \mid \|E\|_\infty < \delta\}$ (notation of thm. 1) to L_∞.*

Proof. Using lemmas 10 and 13, as well as cor. 5, it suffices to prove that each of the 2 integrals in our expression for w_x is S_p^λ, since the second integral is bounded away from 0 by our choice of δ in thm. 1.

For the first integral, this follows from lemma 14, with $g = (1-\sigma)f$, $\vartheta(s) = \frac{e^{-\eta s}}{\Phi(f-\varkappa)}$ and $E = 1$, while for the second ($\mathcal{N}$ in prop. 2.iv) this was shown in lemma 15.iv. ■

Theorem 6. *The map $E \mapsto w$ (the composite with ϖ) can be added as an additional coordinate of ϖ, leaving all our previous statements valid. I.e., or further:*

(i) *In thm. 2, the w are uniformly equicontinuous.*
(ii) *In thm. 3, the topology on w's is the same as for k, y, f.*

Proof. Thm. 1 and lemmas 10 and 20 imply the map is S_p^λ, so can be added to ϖ.

For (i), the proof of lemma 20 shows that the w have the same smoothness as the $\mathcal{N}$, which were shown to be equicontinuous in the proof of thm. 2. (ii) follows from (i) as in the proof of thm. 3.

For the rest, in the proof of cor. 8, the verification for the "other equations" included that for $\mathcal{N}$, which is the essential point for w as seen. And similarly the proof of thm. 5 refers explicitly to the equation for $\mathcal{N}$. ■

Corollary 9. *$\forall\lambda \in \Lambda^1$, $E \mapsto w$ is C^1 from the open subset $\{E \mid \|E\|_\infty < \delta\}$ of $L_1^\lambda \cap L_\infty$ to $L_1^\lambda \cap L_\infty$; further $E \mapsto W = \int e^{\lambda x} w_x dx$ is C^1 on this open set.*

Proof. The first part follows from thm. 6 by cor. 4, and the second part follows then since $w \mapsto \int e^{\lambda x} w_x dx$ is a continuous linear functional on $L_1^\lambda \cap L_\infty$. ■

Theorem 7. *For any BGE, and $\forall\lambda \in \Lambda$, W is differentiable on $L_1^\lambda \cap L_\infty$ at the BGE with as derivative $\delta W(E) = (\int e^{\lambda x} k(x)dx) \int e^{\lambda t} E_t dt$, for some $k \in \bigcap_\Lambda L_1^\lambda$.*

Proof. By thm. 5 (and thm. 6), $w'_x(E) = \int k(x-y)E_y dy$ with $k \in L_1^\lambda$; i.e., cf. lemma 11, $(\phi_\lambda(w'(E)))_x = \int (\phi_\lambda k)(x-y)(\phi_\lambda E)_y\, dy$; and so, since $E \in L_1^\lambda$, Fubini's theorem is applicable and $\delta W = \int (\phi_\lambda(w'(E)))_x dx = (\int e^{\lambda x} k(x)dx) \int e^{\lambda y} E_y dy$. ■

Remark 17. The "constant term" may seem of no interest, being just a normalisation, but this is not so in any extension of this to multidimensional policy variations (Mertens and Rubinchik, 2008), where it determines the evaluation over the policy space. We see here that it is very easy to evaluate: as a Laplace transform, it is Do it!
constructed from the Laplace transforms of the elementary building blocks by just replacing convolution products by usual products, and using the final formula (with $z = \lambda$) of prop. 7 for $(\frac{\partial F}{\partial i})^{-1}$.

Remark 18. It is trivial how to evaluate the effect of constant perturbations, since they lead again to balanced growth solutions. E.g., at the GRE a constant E is simply added to consumption, leading thus to $w_x = \frac{1}{1-\rho}\big[\big((1-\alpha)y^* + E\big)^{1-\rho} - \big((1-\alpha)y^*\big)^{1-\rho}\big]$, and hence $w'_x(1) = \big((1-\alpha)y^*\big)^{-\rho}$. Since $w'_x(1) = \int k(x)dx$ (cf. proof), this gives the coefficient in case $\lambda = 0$.

A bit more generally, at a given BGE, let $x \stackrel{\text{def}}{=} (1-\sigma)f - \eta$ and $\mu \stackrel{\text{def}}{=} f + \lambda$, and To be re-checked.
consider the solutions λ of $\Phi(x) = \Phi(x-\mu)\int_0^1 e^{\mu t}\vartheta_t dt$ ($\mu = 0$, i.e., $\lambda = -f$, is always Any significance?
one, but generically there is 1 other, the RHM being convex in μ and converging to $+\infty$ when $\mu \to \pm\infty$ — write it as $\int_0^1\int_0^1 e^{xy+(t-y)\mu}\vartheta_t dy dt$). For such λ's, $E_t = Be^{\lambda t}$ leads to no change in k, y, etc.; E_t is just added straight to consumption.

Appendix A. The evaluation of profits

A.1. **The "hot potato" example.** To illustrate the need of assumption 2.(v) for the correct evaluation of profits, consider the following example: $(F, \mathcal{F}, \mu)$ equals $[-1, 1]$ with Lebesgue measure; the "proposed equilibrium" is the Golden Rule equilibrium of our model, except that p_t is doubled for $t < 0$. Let $t_n = \frac{-1}{n+1}$ (and $t_0 = -\infty$); for $t_{n-1} \le t < t_n$, all capital is held and investment is done by the firms with $t_{n-1} \le f < t_n$ (say uniformly spread), and for $t \ge 0$, by the firms with $f \ge 0$. Then all firms make 0 profits, although on the aggregate they make a big loss (at time 0). Further, the technological constraint $K_t^f \ge 0$ prevents a profitable deviation by any firm. (Recall K is plant and equipment; markets for short sales of those are a bit hard to imagine.)

The same example can be re-cast with finitely many firms: take 2 firms active before time 0, exchanging the capital between them at times t_n, and a third, active from time 0 on.

We see thus that we need a reliable way to evaluate profits, that aggregates properly. Further, cf. infra, there are at least 2 such ways, applicable to different classes of functions.

A.2. **The variation.** Let $V_{a,b}(f) = \mathsf{sup}_n \, \mathsf{sup}_{a \le t_{i-1} \le t_i \le b} \sum_{i=1}^n \| f(t_i) - f(t_{i-1}) \|$, the variation of f on $[a,b]$. If $X(f,t)$ is jointly measurable on $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$, then $V_{a,b}(X(f,\cdot))$ can be shown to be measurable. X has *locally bounded variation* if $\forall a < b$, $\mathsf{E}\, V_{a,b}(X(f,\cdot)) < \infty$.

Consider first the case of a single firm (i.e., F is a singleton, so we can drop the superscript). Let us first compute the cumulative volume H_t of capital transactions: $K_t = e^{-\delta t} \int_{-\infty}^t e^{\delta s} [I_s ds + dH(s)]$, so $dH(t) = e^{-\delta t} d(e^{\delta t} K_t) - I_t dt$. Thus profits equal $-\int p_t^K dH(t) - \int p_t^I I_t dt + \int r_t K_t dt = \int (r_t - p_t^K \delta) K_t dt - \int p_t^K dK_t + \int (p_t^K - p_t^I) I_t dt$.

So, as long as we don't know universal measurability of p_t^K, we can only use the first formula for profits, and only in the case where $H(t)$ is purely atomic; further, to aggregate, the location of the atoms must be independent of f. But as soon as we know p_t^K is universally measurable and bounded, we can use the second, and allow for any H_t, or equivalently (integrability of I_t), K_t, of bounded variation. Note that the integral w.r.t. dK_t is really an integral w.r.t. a measure: even if, at some t, K_{t_-}, K_t and K_{t_+} are all different (where say $K_t - K_{t_-}$ represents the buys at time t that are, on the transaction date, registered in the name of the buyer, and $K_{t_+} - K_t$ those still registered in the name of the seller), all those transactions occured on date t and are thus valued at p_t^K—i.e., the mass at t equals $K_{t_+} - K_{t_-}$.

To aggregate well, the condition is then clearly that K_t^f has locally bounded variation. However, to show that there is a profitable deviation, suffices to exhibit an individually profitable deviation by a non-null coalition of firms (i.e., with just K_t^f of finite variation for each f); the deviations can then always be scaled down differentially for different f such as to get locally bounded variation (assuming just μ has no atoms of infinite measure).

Observe that this approach is one of "transactions-based" accounting: it is the cash-flow stemming from transactions that is recorded when they occur, and summed.

A.3. **Marking to market.** Assume we know now further that p_t^K is of bounded variation. Then we can use integration by parts in the previous formula. To this effect, define the linear functional $\int_a^b K_t dp_t$ as, for $a < b$, $(p_{a_+} - p_a) K_{a_+} + (p_b - p_{b_-}) K_{b_-} + \int_{]a,b[} K_t dp_t$, where, at a jump t of p_t inside $]a,b[$, the contribution of the jump is counted as $(p_t - p_{t_-}) K_{t_-} + (p_{t_+} - p_t) K_{t_+}$. Then, $\forall a,b,c$, $\int_a^b + \int_b^c + \int_c^a = 0$ and $\int_a^b f(G(x)) dF(G(x)) = \int_{G(a)}^{G(b)} f(x) dF(x)$ $\forall G$ continuous and monotone $[(x-y)(G(x)-G(y))(y-z)(G(y)-G(z)) \ge 0]$—using those formulas to define $\int_a^b$ for $a \ge b$. And, given our above interpretation of dK_t, the correct formula of integration by parts becomes: $\int_{[a,b]} p_t dK_t = p_b K_{b_+} - p_a K_{a_-} - \int_a^b K_t dp_t$.

Henceforth we'll think of (K,I) as a variation in policy over an interval $[a,b]$, so $K_{a_-} = K_{b_+} = 0$, hence $\int_{[a,b]} p_t dK_t = -\int_a^b K_t dp_t$, and we get as formula for the variation in profit: $\pi(K,I) = \int_a^b (r_t - p_t^K \delta) K_t dt + \int_a^b K_t dp_t^K + \int_a^b (p_t^K - p_t^I) I_t dt$.

This makes sense as soon as K_t is bounded and measurable, and such that $p_t > p_{t_+} \Rightarrow K_{t_+}$ exists and $p_{t_-} > p_t \Rightarrow K_{t_-}$ exists. In particular, any function K_t that has left- and right hand limits at every point satisfies this for all p.

To aggregate well over coalitions, one needs thus that $\forall S \in \mathcal{F}$, and for any monotone sequence t_n, $\mathsf{lim}_n \int_S K_{t_n}^f \mu(df) = \int_S \mathsf{lim}_n K_{t_n}^f \mu(df)$: K_t should be uniformly integrable, in addition to being $\mu \otimes \nu$-measurable for any measure ν on $\mathbb{R}$ and having, $\forall t$, a.e. left- and right hand limits. The aggregate K_t being continuous, and $K \ge 0$, uniform integrability is equivalent to the more intuitive market clearing: $\int K_{t_-}^f \mu(df) = \int K_t^f \mu(df) = \int K_{t_+}^f \mu(df)$.

Observe this is on the contrary a form of "marking to market" accounting: the integral $\int_a^b K_t dp_t$ shows that profits and losses are added daily to the account by adding to past profits the impact of today's price-variation on the value of the assets. Transactions at arbitrage-free prices don't alter the value of the portfolio, so are immaterial in this system.

Remark 19. The more is known about p, the more deviations can be evaluated this way. E.g., when one knows by prop. 1 that p is locally Lipshitz, any jointly-integrable K can be.

Because of this, there is no good reason to require anything more of K in the model than local joint integrability;[19,20] as a consequence however, this implies that as long as

[19]It is easily seen that the only thing more we required of K is equivalent to being minorised by some K' (with continuous, strictly positive aggregate) satisfying the aggregation conditions above.

[20]Still, any equilibrium is compatible with the strictest requirements: by assumption 2, $K_t > 0$ $\Rightarrow \mu\{f \mid t_0^f < t \le t_1^f\} > 0$ and $\mu\{f \mid t_0^f \le t < t_1^f\} > 0$. As soon as this holds, one can construct $I_t^f \ge 0$ and K_t^f of locally bounded variation, both $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$-measurable, satisfying all our requirements.

the Lipshitz character of p is not proved, the only arbitrage arguments we can use are that deviations from (K, I), satisfying the stronger assumptions above, would not be profitable.

Remark 20. In applying the above in lemma 5 to obtain the conditions for arbitrage-free prices, we will for further strength only use deviations of bounded variation.

Appendix B. Gale's dichotomy

Consider total net savings M_t of the economy at time t. We claim it equals $S_t - p_tK_t$, where S_t is total net savings of the consumers. Indeed, the 0-profit condition for investment firms ensures that p_tK_t is their total debt outstanding at time t. As to S_t, its derivative must be the flow of aggregate savings of the consumers, i.e., the difference $(1-\alpha)p_tY_t - p_tC_t$ between the wage bill and consumption. Since aggregate values like p_tY_t, p_tC_t or p_tK_t grow like e^{ft}, we deduce from S'_t that the primitive S_t equals $B + \frac{1}{f}\big((1-\alpha)p_tY_t - p_tC_t\big)$ for some constant B. And since savings or debt cannot exceed lifetime earnings, if $f \neq 0$, $B = 0$.

So,if $f \neq 0$, $S_t = (1-\alpha)p_tY_t\frac{1}{f}\big(1 - \frac{c}{(1-\alpha)y}\big)$, or, since $c = (1-\alpha)y\digamma(f)\int_0^1 \varphi_s e^{sf} ds$:

$$S_t = (1-\alpha)p_tY_t\frac{1 - \digamma(f)\int_0^1 \varphi_s e^{sf} ds}{f}$$

Observe that the fraction is continuous at $f = 0$ (in fact, everywhere jointly analytic in f, $\frac{1}{\rho}$ and η), with value $\frac{1}{\eta} - \frac{1}{e^\eta - 1} - \int_0^1 s\varphi_s ds$. A continuity argument yields then the same conclusion when $f = 0$ (the continuity argument is safe, since it only involves the dependence on f of the demand function of currently living consumers over their bounded lifetime; anyway, it is easy to confirm by direct computation). So, $\forall f$:

$$M_t = p_tK_t\Big((1-\alpha)\frac{y}{k}\frac{1 - \digamma(f)\int_0^1 \varphi_s e^{sf} ds}{f} - 1\Big)$$

Cor. 2, (ii) and (iii), imply $\frac{y}{k} = \frac{R-f}{\alpha}$, so this can be re-expressed using only the variable f.

Consider now market clearing: it implies (and, in a 1 good model like the present, is equivalent to) that at each instant t, the total net value of all transactions is 0—i.e., M_t should not change.[21] Formally, since $p_tK_t = p_0K_0e^{ft}$, $(p_tK_t)' = fp_tK_t$; so $M'_t = S'_t - fp_tK_t$, i.e., since by cor. 2, (i)–(iii), $fK_t = RK_t - \alpha Y_t = I_t - \alpha Y_t$:

$$M'_t = p_t\big((1-\alpha)Y_t - C_t - I_t + \alpha Y_t\big)$$

and thus market clearing implies $M'_t = 0$, i.e., $M_t = M$ is constant:

$$M = p_tK_t\Big((R-f)\frac{1-\alpha}{\alpha}\frac{1 - \digamma(f)\int_0^1 \varphi_s e^{sf} ds}{f} - 1\Big) \quad \text{is constant over time.}$$

Since $p_tK_t = p_0K_0e^{ft}$, it follows that

either: $M = 0$, i.e., $\frac{\alpha}{1-\alpha} = \Big(\frac{R}{f} - 1\Big)\Big(1 - \digamma(f)\int_0^1 \varphi_s e^{sf} ds\Big)$

or (GRE): $f = 0$, and then $M = p_tK_t\Big[R\frac{1-\alpha}{\alpha}\Big(\frac{1}{\eta} - \frac{1}{e^\eta - 1} - \int_0^1 s\varphi_s ds\Big) - 1\Big]$

The first equation is that plotted in our graphs, while the vertical there yields the GRE.

This is Gale's (1973) dichotomy between "balanced" equilbria and "golden rule" equilibria — and whether the bracket in the second alternative is positive or negative determines whether the model is "Samuelson" or "classical" in his terminology.[22]

[21] Think of all transactions being paid through individual- or firm-accounts at a single bank, in the numeraire underlying our price system p_t (so an interest-free money). Think of all those payments being made on the date of the corresponding physical transfer of goods, and of each account's balance as a function of time. Budget balance implies that only the accounts of currently living consumers or investment firms have a non-zero balance. So the total credit M_t extended by this bank at time t is the sum of the balances of all currently living consumers. But since any transaction credits one account by the same amount it debits another one, M_t is constant over time.

[22] I.e., in our graphs, values of α corresponding to a point on the GRE vertical lying above (below) the curve correspond to a "classical" ("Samuelson") model.

Appendix C. Speed of convergence

To see better the nature of the difficulty, why we obtain speeds of convergence only for $\lambda \in \Lambda$, and not $\forall \lambda \leq R$, consider the kernels $\varphi_\alpha(x) = \mathsf{sign}(\alpha)\mathbf{1}_{\alpha x > 0} e^{-\alpha x}$ for $\alpha \neq 0$. They are a simplified version of τ, with its main qualitative features. We get $\widehat{\varphi}_\alpha = y \mapsto \frac{1}{\alpha - \mathrm{i}y}$, hence $\widehat{\varphi}_\alpha - \widehat{\varphi}_\beta + (\alpha-\beta)\widehat{\varphi}_\alpha\widehat{\varphi}_\beta = 0$, so $\varphi_\alpha - \varphi_\beta + (\alpha-\beta)\varphi_\alpha \star \varphi_\beta = 0$, and thus, with $f = A\varphi_\alpha$, denoting by g_0 the solution in L_1 of the convolution equation $f+g = f\star g$ (i.e., $\mathbf{1}-g_0$ is the inverse in the Wiener algebra of $\mathbf{1}-f$, $\mathbf{1}$ denoting the identity): $g_0 = -A\varphi_{\alpha-A}$ for $A \neq \alpha$.

Observe that $\phi_\lambda(\varphi_\alpha) \in L_1$ — i.e., $\varphi_\alpha \in L_1^\lambda$ — iff $\alpha(\alpha-\lambda) > 0$, and then $\phi_\lambda(\varphi_\alpha) = \varphi_{\alpha-\lambda}$ and if $\alpha(\alpha-\lambda) \leq 0$ then $\phi_\lambda(\varphi_{-\alpha}) = \varphi_{\alpha-\lambda}$. Thus, by the inversion formula applied to $f = A\varphi_\lambda \in L_1^\lambda$, for such λ the solution in L_1^λ of our convolution equation equals g_0 if $(A-\alpha)(A-\alpha+\lambda) > 0$, and for $(A-\alpha)(A-\alpha+\lambda) < 0$ equals $g_* = -A\varphi_{A-\alpha}$, so $g_0 - g_* = A\,\mathsf{sign}(A-\alpha)e^{(A-\alpha)x}$. In particular for each λ at most 1 of g_0 and g_* belongs to L_1^λ, since an exponential belongs to no such space. Note there exists λ such that f and $g_\star \in L_1^\lambda$ (i.e., $\alpha(\alpha-\lambda) > 0$ and $(A-\alpha)(A-\alpha+\lambda) < 0$) iff $\alpha A > 0$ and $A \neq \alpha$. Assume this henceforth.

We have thus 2 solutions g_0 and g_* of our convolution equation, so the difference, hence $e^{(A-\alpha)x}$, belongs to $\mathsf{Ker}(\mathbf{1}-f)$ in $L_\infty^{\alpha-A}$. And $\mathbf{1}-f$ is trivially invertible on $L_1^\lambda \cap L_\infty$ if the inverse on L_1^λ equals g_0, i.e., for $(A-\alpha)(A-\alpha+\lambda) > 0$. Else, if $(A-\alpha)(A-\alpha+\lambda) \leq 0$, note that $e^{(A-\alpha)x}$ is a continuous linear functional on $L_1^\lambda \cap L_\infty$, and denote by K its kernel: $K = \{h \in L_1^\lambda \cap L_\infty \mid \int h(x)e^{(A-\alpha)x}dx = 0\}$. Clearly K is the set where g_0 and g_* coincide, hence the inverses of $\mathbf{1}-f$, $\mathbf{1}-g_0$ on L_∞ and $\mathbf{1}-g_*$ on L_1^λ, coincide on $h \in L_1^\lambda \cap L_\infty$ iff $h \in K$; thus the image of $L_1^\lambda \cap L_\infty$ by $\mathbf{1}-f$ equals K. And for $h \notin K$, $(\mathbf{1}-g_0)(h) \notin L_1^\lambda$ and $(\mathbf{1}-g_*)(h) \notin L_\infty$.

Appendix D. A cookbook description

The calculus developed here is basically quite general. Assuming the equilibrium conditions can be written as the set of zeros of a map $F\colon L_\infty^n \to L_\infty^n$, the derivative of this map will be given by kernels — if minimally reasonable, cf. e.g. prop. 5 —, and those will, at a BGE, have to be convolution operators by time-invariance (cf. e.g. thm. 5).

The spectrum of such an n by n matrix of convolution operators (i.e., elements of the Wiener algebra) should then be the union over all ω of the spectrum of the corresponding matrix of Fourier transforms at ω, plus their (singleton) limit at ∞, so the condition for invertibility becomes simply that the determinant of this matrix vanishes for no ω, ∞ included. A statement in this direction seems available as theorem 2 in Bochner and Phillips (1942) — however we still need to find a convenient reference or proof for the full statement.

The inverse matrix of convolution operators has then as Fourier transform the pointwise inverse of the above matrix of Fourier transforms, so all derivatives of equilibrium quantities w.r.t. variations in parameters can be obtained numerically applying a Fast Fourier Transform to this pointwise inverse. And for derivatives of welfare, this FFT is not even needed; they are obtained explicitly, staying in the realm of Fourier-Laplace transforms, welfare being the Laplace transform of the stream of individual lifetime utilities.

One obains then finally also as here the speeds at $-\infty$ and at $+\infty$ of convergence back to the original equilibrium (i.e., the interval Λ).

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