Satisficing and Salience

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Abstract

An earlier choice-theoretic characterization of satisficing is extended to allow the relative salience of different alternatives to influence the decision maker's behavior. Specifically, salience effects can break the agent's "pseudo-indifference" between alternatives that attain the relevant utility threshold and can therefore have welfare significance. Close parallels between the new (satisficing/salience), old (satisficing), and classical (utility maximization) theories are drawn using an analytical approach based on conditions of Richterian relational congruence. The theory is illustrated by application to pricing and advertising decisions in a differentiated-products duopoly with satisficing/salience consumers.

1 Introduction

The first task of revealed preference analysis, as envisioned by Samuelson [7, 8], was to axiomatize and thereby determine the "operational meaning" of the utility-maximization criterion

$$x \in \operatorname{Arg}\max_{y \in A} f(y) \tag{1}$$

for an alternative x to be choosable from a feasible set A. Here f is of course the decision maker's utility function, which can be (re-)constructed from observed choice data when the relevant axioms hold.

From the starting point of the representation in (1), many authors have proceeded to axiomatize criteria that either impose some useful structure on f (and are therefore logically stronger) or relax the stringent requirement of maximization (and are therefore logically weaker). Among the modifications of the second type, the author has in a recent paper [15] characterized (a special case of) the "satisficing" criterion

$$x \in \{y \in A : f(y) \geq \theta(A)\}, \tag{2}$$

where the function θ operates on menus and returns utility thresholds for choosability. Decisions made using this rule are seen to be optimal with respect to a constellation of

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"perceived preferences" having certain consistency properties, leading to an interpretation of satisficing behavior as the outcome of choice under cognitive constraints.

The present paper studies the "satisficing/salience" criterion

$$x \in \operatorname{Arg}\max_{y \in A} g(y) \text{ subject to } f(y) \geq \theta(A), \tag{3}$$

which is weaker still than that in (2). Here f and θ are the utility and threshold functions, as before, while g is a new function returning (an ordinal measure of) the "salience" of its argument. The alternatives deemed choosable are thus the maximally salient members of the set of options that attain the relevant utility threshold; that is to say, the decision maker maximizes salience subject to a (menu-dependent) utility constraint.

Why should we be interested in the representation in (3)? Visual salience — the relative success of different stimuli in attracting the attention of a viewer — is thought by neuroscientists to be a measurable variable that exerts a significant influence on animal behavior (see, e.g., Itti [4, 5]), and the auditory and other sensory systems exhibit similar effects. But within the framework of (1) such characteristics can have no welfare (and little behavioral) significance, since an agent who chooses the most salient alternative from among those that are utility maximal *a fortiori* maximizes utility. Moreover, the two postulated influences on decision making would in this framework be impossible to disentangle, since the components of a lexicographic (utility, then salience) ordering cannot be separated using choice data.

The situation is quite different in the case of satisficing. Here the decision maker is not (in general) genuinely indifferent between the alternatives that achieve the utility threshold, he or she simply cannot tell which option is best. This creates room for relative salience — imagined to break the agent's "pseudo-indifference" — to have consequences for welfare; good consequences if f (utility) and g (salience) are positively correlated, and bad if they are negatively correlated. In addition, the endogenous variation in the pseudo-indifference classes to which salience considerations are applied enables us to distinguish utility from mere attractiveness and thereby to construct f, θ, and g jointly from the decision maker's choices.

To appreciate the difference between welfare and salience, consider the matrix of money values shown in Table 1. Here a thorough inspection will reveal that B5 and D2 yield the highest available payoff, but these are not the cells that first attract the typical viewer's attention.[1] The more salient alternatives include cells with relatively low payoffs (such as B1 and E1), cells with somewhat larger payoffs (such as A4 and C6), and cells with high but not maximal payoffs (such as E5 and F3).[2]

Our theory does not by any means imagine that salience effects will lead a decision maker charged with choosing a cell from row B to select B1 rather than B5. On the contrary, higher salience will never trump higher utility *as long as the utility differential in question is perceived.* But in many choice problems of interest the determination of one's preferences is not as simple as comparing money values (and adding integers and converting pounds into dollars). In these situations it is plausible that the resolving power

[1] Here we apply the mid-market exchange rate £1.00 = \$1.99 as of this writing.

[2] The reader may be reminded of Schelling's [9] argument that relative salience can determine the "focal points" of a coordination game.

	1	2	3	4	5	6
A	\$30	\$9	\$0	\$33.33	\$0	\$42
B	−\$1000	\$24	\$44	\$22	\$56	\$51
C	\$15	\$0	\$47	\$4	\$55	£15
D	\$50	\$56	\$28	\$0	\$0	\$0
E	ten dollars	\$49	\$16	\$0	\$50	\$0
F	\$36	\$2	\$30 + \$25	\$0	\$0	\$0

Table 1: An assortment of money values. The largest available payoff appears in cells B5 and D2, but alternatives such as A4, B1, C6, E1, E5, and F3 may be more salient.

of the preference perception mechanism may be less than perfect — in any event this is the basic assumption behind the axiomatization of (2) in [15]. And as argued above, any satisficing practiced by the decision maker in the utility domain leaves room for salience considerations to have both behavioral and welfare significance.

Before proceeding, it may be helpful to examine a full-blown example of the type of choice procedure to be characterized. One such example appears in Figure 1, where each cell corresponds to a menu drawn from $X = wxyz$. (Here and throughout we employ a multiplicative notation for enumerated sets.) Within a particular cell are displayed the subset of chooseable alternatives, any (strict) preferences perceived and relevant salience comparisons, and the utility threshold assigned to the menu in question. Also shown are the utility and salience functions that together with the threshold mapping make up our representation.[3]

Observe that for each menu A in the example, the elements of the set $C(A)$ are maximal with respect to Q within the subset of alternatives that are not on the losing end of any *perceived* preference R. Or, equivalently, $C(A)$ is determined via (3) by the specified functions f, θ, and g. Our primary goal in this paper is to find conditions on the decision maker's behavior that must hold when it is generated in this fashion, and which conversely will allow us to deduce the underlying structure (in terms of either orderings or their numerical representations) in the revealed preference sense.

The remainder of the paper is organized as follows:

- Section 2 handles preliminary matters and proceeds to develop the characterization results that are central to our theory. We formulate first the classical analysis of preference maximization and then the analysis from [15] of cognitively-constrained choice in terms of "congruence" conditions of the sort introduced by Richter [6].[4]

[3]Here the system of perceived preferences and the utility and threshold functions obey the restrictions imposed in [15]; see Sections 2.3 and 3.2 below for a review.

[4]The reformulation in Theorem 2 of our characterization from [15, p. 56] may be less transparent but

<table>
<tr><td colspan="12">$C(wxyz) = x$
$x\mathrm{R}w$, $y\mathrm{R}w$, $z\mathrm{R}w$, $x\mathrm{Q}y$, $x\mathrm{Q}z$
$\theta(wxyz) = 1$</td></tr>
<tr><td colspan="3">$C(wxy) = x$
$x\mathrm{R}w$, $y\mathrm{R}w$, $x\mathrm{Q}y$
$\theta(wxy) = 1$</td><td colspan="3">$C(wxz) = z$
$x\mathrm{R}w$, $z\mathrm{R}w$, $z\mathrm{R}x$
$\theta(wxz) = 3$</td><td colspan="3">$C(wyz) = yz$
$y\mathrm{R}w$, $z\mathrm{R}w$
$\theta(wyz) = 2$</td><td colspan="3">$C(xyz) = z$
$y\mathrm{R}x$, $z\mathrm{R}x$, $z\mathrm{R}y$
$\theta(xyz) = 3$</td></tr>
<tr><td colspan="2">$C(wx) = x$
$x\mathrm{R}w$
$\theta(wx) = 1$</td><td colspan="2">$C(wy) = y$
$y\mathrm{R}w$
$\theta(wy) = 2$</td><td colspan="2">$C(wz) = z$
$z\mathrm{R}w$
$\theta(wz) = 3$</td><td colspan="2">$C(xy) = y$
$y\mathrm{R}x$
$\theta(xy) = 2$</td><td colspan="2">$C(xz) = z$
$z\mathrm{R}x$
$\theta(xz) = 3$</td><td colspan="2">$C(yz) = z$
$z\mathrm{R}y$
$\theta(yz) = 3$</td></tr>
<tr><td colspan="3">$C(w) = w$
$f(w) = 0 = \theta(w)$
$g(w) = 1$</td><td colspan="3">$C(x) = x$
$f(x) = 1 = \theta(x)$
$g(x) = 2$</td><td colspan="3">$C(y) = y$
$f(y) = 2 = \theta(y)$
$g(y) = 0$</td><td colspan="3">$C(z) = z$
$f(z) = 3 = \theta(z)$
$g(z) = 0$</td></tr>
</table>

Figure 1: An illustrative example. A menu is a nonempty subset of $X = wxyz$; R is the preference order, represented by f; Q is the salience order, represented by g; and θ assigns thresholds to menus. Each cell corresponds to a choice problem and displays the subset of alternatives deemed chooseable (e.g., $C(wxy) = x$), any preferences perceived and salience comparisons applied (e.g., $\{x\mathrm{R}w, y\mathrm{R}w, x\mathrm{Q}y\}$ in problem wxy), and the applicable threshold (e.g., $\theta(wxy) = 1$).

We then use the same analytical machinery to study the impact of salience on the cognitively-constrained case, obtaining our first main result (Theorem 3).

- Section 3 axiomatizes numerical representations corresponding to each of our three choice-theoretic characterizations. We treat both the classical criterion (1) and the satisficing criterion (2) before turning to the axiomatization of (3), our second main result (Theorem 6).
- Section 4 contains a detailed application to industrial organization, determining the pricing and advertising behavior of vertically-differentiated duopolists faced with a population of consumers who use the satisficing/salience criterion (3).
- Appendix A gives complete proofs of all results stated in the main text.

2 Characterization results

2.1 Preliminaries

Fix a nonempty set X and a domain $\mathcal{D} \subseteq \mathcal{A} := \{A \subseteq X : A \neq \emptyset\}$. A mapping $C : \mathcal{D} \to \mathcal{A}$ is a *choice function* if $\forall A \in \mathcal{D}$ we have $C(A) \subseteq A$. We shall refer to the argument A of C as the *menu* of available alternatives (drawn from the *universe* X), and to its value $C(A)$ at A as the associated *choice set*. Moreover, we shall adopt the simplifying assumption

constitutes a definite logical improvement, as we shall considerably weaken the required restrictions on the domain of C.

that $\mathcal{D} \supseteq \{xy : x, y \in X\}$; i.e., that the domain of C encompasses all menus containing either exactly one or exactly two alternatives.

A (*binary*) *relation* R is a subset of $X \times X$, with $\langle x, y\rangle \in \mathrm{R}$ more commonly written as $x\mathrm{R}y$. Given a relation R, its *complement* $\overline{\mathrm{R}}$ is defined by $x\overline{\mathrm{R}}y$ if and only if $\neg[x\mathrm{R}y]$, its *symmetric part* $\mathrm{R}^{\bowtie}$ by $x\mathrm{R}^{\bowtie}y$ if and only if $x\mathrm{R}y\mathrm{R}x$, its *asymmetric part* $\mathrm{R}^{\triangleright}$ by $x\mathrm{R}^{\triangleright}y$ if and only if $x\mathrm{R}y\overline{\mathrm{R}}x$, its *ancestral relation* (a.k.a. *transitive closure*) R^* by $x\mathrm{R}^*y$ if and only if $\exists n \geq 2$ and $\langle z_k\rangle_{k=1}^n$ such that $x = z_1\mathrm{R}z_2\mathrm{R}\cdots\mathrm{R}z_n = y$, and its *descendent relation* $\mathrm{R}^{\div}$ by $x\mathrm{R}^{\div}y$ if and only if $\neg[x\overline{\mathrm{R}}^*y]$.

A relation R is classified as a *partial order* if it is both irreflexive ($x\mathrm{R}y$ only if $x \neq y$) and transitive ($x\mathrm{R}y\mathrm{R}z$ only if $x\mathrm{R}z$), as a *weak order* if it is both asymmetric ($x\mathrm{R}y$ only if $y\overline{\mathrm{R}}x$) and negatively transitive ($x\overline{\mathrm{R}}y\overline{\mathrm{R}}z$ only if $x\overline{\mathrm{R}}z$), and as a *linear order* if it is both acyclic ($x_1\mathrm{R}x_2\mathrm{R}\cdots\mathrm{R}x_n$ only if $x_1 \neq x_n$) and weakly connected ($x \neq y$ only if either $x\mathrm{R}y$ or $y\mathrm{R}x$). Note that any linear order is a weak order and any weak order is a partial order.

Observe that for any relation R we have $\mathrm{R}^{\div} \subseteq \mathrm{R} \subseteq \mathrm{R}^*$, that R^* is a partial order if R is acyclic, and that $\mathrm{R}^{\div}$ is a weak order if R is asymmetric. Moreover, for any two relations R^1 and R^2 such that $\mathrm{R}^1 \subseteq \mathrm{R}^2$ we have both ${\mathrm{R}^1}^* \subseteq {\mathrm{R}^2}^*$ and ${\mathrm{R}^1}^{\div} \subseteq {\mathrm{R}^2}^{\div}$.

2.2 Preference

The classical theory of "rational" decision making (pioneered by Samuelson [7], Arrow [1], and Richter [6]) imagines the decision maker's choices being determined solely by his or her preferences among alternatives. Defining a relation R by $x\mathrm{R}y$ if and only if alternative x is strictly preferred to alternative y, this assumption can be expressed as the requirement that $\forall A \in \mathcal{D}$ we have

$$C(A) = \mathrm{R}\!\uparrow\!(A) := \{x \in A : \forall y \in A \text{ we have } y\overline{\mathrm{R}}x\}. \tag{4}$$

When this is the case we shall write $C = \mathrm{R}\!\uparrow$ and say that the relation R *generates* the choice function. And the latter equality can be usefully separated into the *upper bound* and *lower bound* inclusions $C \subseteq \mathrm{R}\!\uparrow$ and $\mathrm{R}\!\uparrow \subseteq C$.

Since C is presumed to be observable but R is not, notions of revealed preference are needed to characterize the classical (R-maximization) hypothesis.

Definition 1 A. (global relation) The *global relation* $\mathrm{P^g}$ is defined by $x\mathrm{P^g}y$ if and only if $\forall A \in \mathcal{D}$ such that $x, y \in A$ we have $y \notin C(A)$. **B. (separation relation)** The *separation relation* $\mathrm{P^s}$ is defined by $x\mathrm{P^s}y$ if and only if $\exists A \in \mathcal{D}$ such that $x, y \in A$ and $x \in C(A) \not\ni y$.

To paraphrase, the global relation records the *absence* of any choice problem in which the second alternative is deemed chooseable in the presence of the first, while the separation relation records the *presence* of at least one problem in which the first alternative is deemed chooseable and the second is not. Note that $\mathrm{P^g}$ is asymmetric (since $C(xy) \neq \emptyset$), included in $\mathrm{P^s}$ (since $x\mathrm{P^g}y$ implies that $C(xy) = x$ and so $x\mathrm{P^s}y$), and an upper bound for C (since $y \in A$ and $y\mathrm{P^g}x$ together imply that $x \notin C(A)$).

In addition to the hypothesis of preference maximization, the classical theory assumes that the relation being maximized is a weak order. Since $\mathrm{P^g}$ is asymmetric, its descendant relation will be in this class, and since $\mathrm{P}^{\mathrm{g}\div} \subseteq \mathrm{P}^{\mathrm{g}}$ we have also that $\mathrm{P}^{\mathrm{g}}\uparrow \subseteq \mathrm{P}^{\mathrm{g}\div}\uparrow$ and so $\mathrm{P}^{\mathrm{g}\div}$ is an upper bound for C.

Proposition 1 **A.** $\mathrm{P}^{\mathrm{g}\div}$ *is a weak order.* **B.** $C \subseteq \mathrm{P}^{\mathrm{g}\div}\uparrow$.

Now for an arbitrary choice function, the binary relations defined thus far need have no valid preference interpretation. When C is generated by a weak order, however, we can show both that $\mathrm{P}^{\mathrm{g}\div}$ *overestimates* and that P^{s} *underestimates* the preference relation.

Lemma 1 **A.** *If* $C \subseteq \mathrm{R}\uparrow$ *for some weak order* R*, then* $\mathrm{R} \subseteq \mathrm{P}^{\mathrm{g}\div}$. **B.** *If* $C = \mathrm{R}\uparrow$ *for some weak order* R*, then* $\mathrm{P}^{\mathrm{s}} \subseteq \mathrm{R}$.

It follows that in the classical case these two revealed preference constructs must be ordered by set inclusion, a condition referred to by Richter [6, p. 637] as "congruence" of the choice function.

Condition 1 (Global Congruence) $\mathrm{P}^{\mathrm{s}} \subseteq \mathrm{P}^{\mathrm{g}\div}$.

In view of Proposition 1B, it is clear that $\mathrm{P}^{\mathrm{g}\div}$ will generate the observed behavior if and only if it is a lower bound for C. This is easily seen to be guaranteed by Global Congruence.

Lemma 2 *If Global Congruence holds, then* $\mathrm{P}^{\mathrm{g}\div}\uparrow \subseteq C$.

And Richter's [6, p. 639] classical result on preference-maximizing choice behavior is then an immediate consequence of Proposition 1 and Lemmas 1–2.

Theorem 1 *A choice function is generated by a weak order if and only if it satisfies Global Congruence.*

2.3 Cognition

We proceed now to the characterization of behavior guided by preferences that may be imperfectly perceived. Our goal at this stage is to reformulate the relevant result from [15] in terms of a congruence condition — thereby demonstrating its parallels to Theorem 1 above and developing ideas that are essential for the treatment of salience to follow.

A *relation system* $\mathbb{R} = \langle \mathrm{R}_A \rangle_{A\in\mathcal{D}}$ is a vector of relations on the menus in the domain $\mathcal{D}$ of the choice function. Such an object can be used to encode imperfectly-perceived preferences: If R is the preference relation, the Ath component $\mathrm{R}_A \subseteq \mathrm{R} \cap [A \times A]$ of the "preference system" $\mathbb{R}$ may be imagined to contain those preferences among alternatives in A that the decision maker perceives when faced with this choice problem. The choice function is then generated by $\mathbb{R}$ when $\forall A \in \mathcal{D}$ we have

$$C(A) = \mathbb{R}\uparrow(A) := \{x \in A : \forall y \in A \text{ we have } y\overline{\mathrm{R}_A}x\}; \tag{5}$$

that is, when the upper and lower bound inclusions $C \subseteq \mathbb{R}\uparrow$ and $\mathbb{R}\uparrow \subseteq C$ both hold.

Whenever each component of a relation system exhibits a property such as irreflexivity, we shall say that the system itself exhibits the property. Similarly, a relation system whose components all belong to a class of relations will be described as, e.g., a "system of partial orders". And given two relation systems $\mathbb{R}^1$ and $\mathbb{R}^2$, we shall write $\mathbb{R}^1 \subseteq \mathbb{R}^2$ when $\forall A \in \mathcal{D}$ we have $\mathrm{R}^1_A \subseteq \mathrm{R}^2_A$.

Our basic cognitive assumption on the decision maker's preference system will be that of *nestedness*; the requirement that any preference perceived in a given choice problem remain perceived in any "smaller" problem in which it is relevant. In terms of $\mathbb{R}$, this means that $\forall x, y \in A \subseteq B$ we can have $x\mathrm{R}_B y$ only if $x\mathrm{R}_A y$ as well. (See [15, pp. 54–56] for extensive discussion of the rationale for and plausibility of this assumption.)

The following two notions of "revealed perceived preference" play the roles of the global and separation relations in the present setting.

Definition 2 A. (local relation system) The *local relation system* $\mathbb{P}^{\mathrm{l}}$ is defined, for each $B \in \mathcal{D}$, by $x\mathrm{P}^{\mathrm{l}}_B y$ if and only if $x, y \in B$ and $\forall A \in \mathcal{D}$ such that $x, y \in A \subseteq B$ we have $y \notin C(A)$. **B. (superseparation system)** The *superseparation (relation) system* $\mathbb{P}^{\mathrm{u}}$ is defined, for each $A \in \mathcal{D}$, by $x\mathrm{P}^{\mathrm{u}}_A y$ if and only if $x, y \in A$ and $\exists B \in \mathcal{D}$ such that $A \subseteq B$ and $x \in C(B) \not\ni y$.

Thus the Bth component of the local relation system is like a global relation that searches only the *subsets* of B, while the Ath component of the superseparation system is like a separation relation that searches only the *supersets* of A. Note that like P^{g}, the system $\mathbb{P}^{\mathrm{l}}$ is both asymmetric (since $C(xy) \neq \emptyset$) and an upper bound for C (since $y\mathrm{P}^{\mathrm{l}}_B x$ implies that $x \notin C(B)$). And note further that both $\mathbb{P}^{\mathrm{l}}$ and $\mathbb{P}^{\mathrm{u}}$ are by construction nested.

As before, we can manufacture an upper bound for C comprised of weak orders by forming from $\mathbb{P}^{\mathrm{l}}$ its descendant system $\mathbb{P}^{\mathrm{l}\,\div}$, an operation that *preserves nestedness.*[5]

Proposition 2 A. *$\mathbb{P}^{\mathrm{l}\,\div}$ is a nested system of weak orders.* **B.** $C \subseteq \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow$.

Continuing to mimic the classical analysis, we proceed by establishing that when the decision maker's perceived preferences have the weak order properties, the preference system is overestimated by $\mathbb{P}^{\mathrm{l}\,\div}$ and underestimated by $\mathbb{P}^{\mathrm{u}}$.

Lemma 3 A. *If $C \subseteq \mathbb{R}\!\uparrow$ for some nested system $\mathbb{R}$ of weak orders, then $\mathbb{R} \subseteq \mathbb{P}^{\mathrm{l}\,\div}$.* **B.** *If $C = \mathbb{R}\!\uparrow$ for some nested system $\mathbb{R}$ of weak orders, then $\mathbb{P}^{\mathrm{u}} \subseteq \mathbb{R}$.*

This suggests the appropriate congruence condition for the case of imperfect perception.

Condition 2 (Local Congruence) $\mathbb{P}^{\mathrm{u}} \subseteq \mathbb{P}^{\mathrm{l}\,\div}$.

It remains only to show that the relevant lower bound inclusion is guaranteed by the condition just stated.

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[5]Suppose that $\mathbb{R}$ is a nested relation system and let $x, y \in A \subseteq B$. If $x\overline{\mathrm{R}_A}^* y$, then $\exists n \geq 2$ and $\langle z_k \rangle_{k=1}^n$ such that $x = z_1 \overline{\mathrm{R}_A} z_2 \overline{\mathrm{R}_A} \cdots \overline{\mathrm{R}_A} z_n = y$. Since $\mathbb{R}$ is nested we have $x = z_1 \overline{\mathrm{R}_B} z_2 \overline{\mathrm{R}_B} \cdots \overline{\mathrm{R}_B} z_n = y$, and so $x\overline{\mathrm{R}_B}^* y$. But then $\overline{\mathrm{R}_A}^* \cap [A \times A] \subseteq \overline{\mathrm{R}_B}^*$, which is to say that $\mathrm{R}^{\div}_B \cap [A \times A] \subseteq \mathrm{R}^{\div}_A$ and $\mathbb{R}^{\div}$ is nested.

Lemma 4 *If Local Congruence holds, then* $\mathbb{P}^{\mathrm{l}\,\div}\!\uparrow \subseteq C$.

And the desired characterization now follows from Proposition 2 and Lemmas 3–4.

Theorem 2 *A choice function is generated by a nested system of weak orders if and only if it satisfies Local Congruence.*[6]

2.4 Salience

With the above characterizations in place, we are now in a position to pursue the main goal of this paper: allowing the relative salience of the available alternatives to influence our decision maker's choice behavior. In so doing we shall employ much of the same analytical machinery used already to establish Theorems 1–2, and the choice functions of interest will once again turn out to be those satisfying a Richterian congruence condition.

Continuing to write $\mathbb{R}$ for the decision maker's preference system, let us define a relation Q by $x\mathrm{Q}y$ if and only if alternative x is (strictly) more salient than alternative y. If the chooseable alternatives on a menu are those in the perceived-preference-maximal subset that are undominated in terms of salience, then it follows that $\forall A \in \mathcal{D}$ we have

$$C(A) = \mathrm{Q}\!\uparrow \circ\, \mathbb{R}\!\uparrow\!(A) = \{x \in \mathbb{R}\!\uparrow\!(A) : \forall y \in \mathbb{R}\!\uparrow\!(A) \text{ we have } y\overline{\mathrm{Q}}x\}. \tag{6}$$

When this is the case we shall write $C = \mathrm{Q}\!\uparrow \circ\, \mathbb{R}\!\uparrow$ and say that C is generated by the relation Q *composed with* the relation system $\mathbb{R}$ (once again the conjunction of upper and lower bound inclusions $C \subseteq \mathrm{Q}\!\uparrow \circ\, \mathbb{R}\!\uparrow$ and $\mathrm{Q}\!\uparrow \circ\, \mathbb{R}\!\uparrow \subseteq C$).

No less than the preference constructs R and $\mathbb{R}$, the salience relation Q is presumed to be unobservable and must therefore be deduced from the decision maker's choice behavior. Two "revealed salience" relations — again variants of the global and separation relations of the classical theory — will be instrumental for this task.

Definition 3 **A. (extended global relation)** The *extended global relation* $\mathrm{S^g}$ is defined by $x\mathrm{S^g}y$ if and only if $\forall A \in \mathcal{D}$ such that $x, y \in A$ and $y\, \overline{\mathrm{P}^{\mathrm{l}}_A}^{\,*}\, x$ we have $y \notin C(A)$. **B. (restricted separation relation)** The *restricted separation relation* $\mathrm{S^s}$ is defined by $x\mathrm{S^s}y$ if and only if $\exists A \in \mathcal{D}$ such that $x, y \in A$ and both $x\, \overline{\mathrm{P}^{\mathrm{l}}_A}^{\,*}\, y$ and $x \in C(A) \not\ni y$.

Thus the extended global relation records the absence of any choice problem in which the second alternative is deemed chooseable, the first is available, and the two stand in a particular further relationship (involving $\mathbb{P}^{\mathrm{l}}$). Similarly, the restricted separation relation records the presence of at least one problem in which the first alternative is chooseable, the second is not, and the two stand in the converse relationship. These definitions of course imply that $\mathrm{S^g} \supseteq \mathrm{P^g}$ and $\mathrm{S^s} \subseteq \mathrm{P^s}$, justifying the names of our new relations.

In our analysis of preference (resp., cognition), we first observed that $\mathrm{P^g}$ (resp., $\mathbb{P}^{\mathrm{l}}$) is asymmetric, then deduced that its descendant relation $\mathrm{P}^{\mathrm{g}\,\div}$ (resp., system $\mathbb{P}^{\mathrm{l}\,\div}$) has the

[6] Cf. [15, p. 56], where essentially this result is stated for the case of $\mathcal{D} = \mathcal{A}$. Under the latter restriction, Local Congruence amounts to an "expansion consistency" condition introduced in the context of social choice theory by Bordes [2, p. 452] and Sen [10, p. 66]. (For further details, see [14, pp. 44–47].)

weak order properties, and went on to show that the latter is also an upper bound for the choice function. But while there is no difficulty in verifying that $\mathrm{S}^{\mathrm{g}\,\div}$ composed with $\mathbb{P}^{\mathrm{l}\,\div}$ is an upper bound for C, the best that can be said for S^{g} itself is that it is irreflexive.[7]

Proposition 3 **A.** S^{g} *is irreflexive.* **B.** $C \subseteq \mathrm{S}^{\mathrm{g}\,\div}{\uparrow} \circ \mathbb{P}^{\mathrm{l}\,\div}{\uparrow}$.

Continuing to pursue the desired characterization, we argue now that when both the salience relation Q and the preference system $\mathbb{R}$ have the weak order properties, Q is overestimated by $\mathrm{S}^{\mathrm{g}\,\div}$ and underestimated by S^{s}.

Lemma 5 **A.** *If* $C \subseteq \mathrm{Q}{\uparrow} \circ \mathbb{R}{\uparrow}$ *for some weak order* Q *and nested system* $\mathbb{R}$ *of weak orders, then* $\mathrm{Q} \subseteq \mathrm{S}^{\mathrm{g}\,\div}$*.* **B.** *If* $C = \mathrm{Q}{\uparrow} \circ \mathbb{R}{\uparrow}$ *for some weak order* Q *and nested system* $\mathbb{R}$ *of weak orders, then* $\mathrm{S}^{\mathrm{s}} \subseteq \mathrm{Q}$*.*

This confirms the necessity of the appropriate congruence condition.

Condition 3 (Extended Global Congruence) $\mathrm{S}^{\mathrm{s}} \subseteq \mathrm{S}^{\mathrm{g}\,\div}$.

And we need then only show that this condition is sufficient for our composite hypothesis about the influences on our decision maker's choice behavior.

The difficulty that arises at this stage is that since $\mathrm{S}^{\mathrm{g}\,\div}$ is not certain to be a weak order, we cannot use it (composed with $\mathbb{P}^{\mathrm{l}\,\div}$) to show the sufficiency of Extended Global Congruence. We can, to be sure, remove from this relation any remaining symmetries by forming its asymmetric part $\mathrm{S}^{\mathrm{g}\,\div\triangleright}$, even while retaining our lower bound for C.

Lemma 6 *If Extended Global Congruence holds, then* $\mathrm{S}^{\mathrm{g}\,\div\triangleright}{\uparrow} \circ \mathbb{P}^{\mathrm{l}\,\div}{\uparrow} \subseteq C$*.*

But $\mathrm{S}^{\mathrm{g}\,\div\triangleright}$ may not be negatively transitive, and thus it too may fail to provide the weak order required.[8]

The key to demonstrating sufficiency can be found in Richter's proof [6, pp. 639–640] of Theorem 1 for an arbitrary domain $\mathcal{D}$. Here we state the relevant step as a general fact about binary relations, and our argument is essentially that in the original.

Proposition 4 *Given any irreflexive relation* R*, there exists a weak order* Q *such that* $\mathrm{R}^{\div\triangleright} \subseteq \mathrm{Q} \subseteq \mathrm{R}^{\div}$*.*

This will be true in particular of the revealed salience relation S^{g}, and a characterization of our composite hypothesis then follows from Propositions 2–3 and Lemmas 5–6.

Theorem 3 *A choice function is generated by a weak order composed with a nested system of weak orders if and only if it satisfies Extended Global Congruence.*

Observe that since $\mathrm{S}^{\mathrm{s}} \subseteq \mathrm{P}^{\mathrm{s}}$ and $\mathrm{P}^{\mathrm{g}\,\div} \subseteq \mathrm{S}^{\mathrm{g}\,\div}$, Extended Global Congruence is implied by Global Congruence (as we know also from Theorems 1 and 3). In Figure 2, panel A depicts the (classical) case in which both conditions hold, panel B that in which only Extended Global Congruence holds, and panel C that in which both conditions fail.

[7] In other words, strengthening P^{g} to S^{g} can lead to failures of asymmetry. For example, let $X = xyz$ and $\mathcal{D} = \mathcal{A}$ and consider the choice function with $C(xy) = x$, $C(xz) = x$, $C(yz) = y$, and $C(xyz) = y$. Here $\mathrm{P}^{\mathrm{g}} = \{\langle x, z\rangle, \langle y, z\rangle\}$ is asymmetric and so $\mathrm{P}^{\mathrm{g}\,\div}$ is a weak order (as guaranteed by Proposition 1A). But $\mathrm{S}^{\mathrm{g}} = \{\langle y, x\rangle, \langle x, z\rangle, \langle z, x\rangle, \langle y, z\rangle, \langle z, y\rangle\} \supsetneq \mathrm{P}^{\mathrm{g}}$ contains the symmetries $x\mathrm{S}^{\mathrm{g}}z\mathrm{S}^{\mathrm{g}}x$ and $y\mathrm{S}^{\mathrm{g}}z\mathrm{S}^{\mathrm{g}}y$, and $\mathrm{S}^{\mathrm{g}\,\div} = \mathrm{S}^{\mathrm{g}}$ is not a weak order.

[8] For example, the choice function in Footnote 7 has $\mathrm{S}^{\mathrm{g}\,\div\triangleright} = \{\langle y, x\rangle\}$, which is merely a partial order.

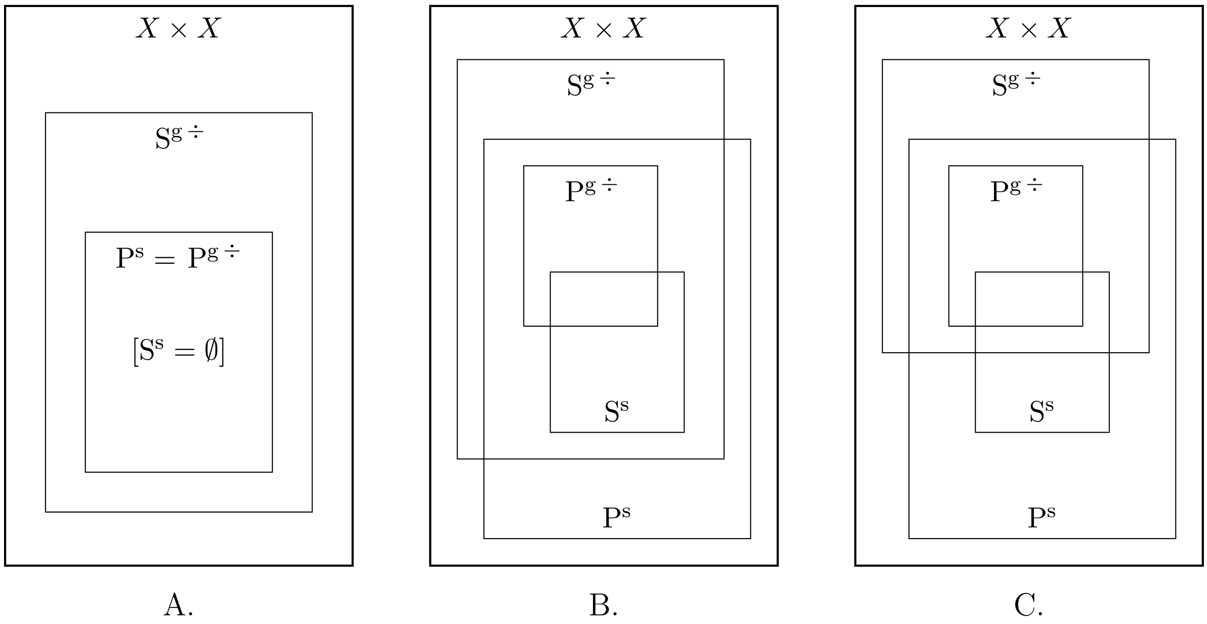

Figure 2: A. Schematic depiction of revealed preference and revealed salience relations for a choice function satisfying Global Congruence ($\mathrm{P^s} \subseteq \mathrm{P^{g\div}}$) and thus generated by a weak order. Note that here $\mathrm{S^s} = \emptyset$. B. A choice function satisfying Extended Global Congruence ($\mathrm{S^s} \subseteq \mathrm{S^{g\div}}$) but violating Global Congruence. Here the observed behavior is generated by a weak order composed with a nested system of weak orders. C. A choice function satisfying neither Global Congruence nor Extended Global Congruence.

3 Representation results

3.1 Preference

In the case of a finite universe X (to which, for simplicity, we restrict attention throughout our discussion of numerical representations), choice behavior is generated by a weak order if and only if it maximizes a real-valued "utility" function. Combining this fact with Theorem 1 above, we obtain a simple version of the classical representation result.

Theorem 4 *Let X be finite. Then a choice function C satisfies Global Congruence if and only if $\exists f : X \to \Re$ such that $\forall A \in \mathcal{D}$ we have $C(A) = \operatorname{Arg\,max}_{x \in A} f(x)$.*

3.2 Cognition

When preferences are not fully perceived, as under Theorem 2, utility maximization will of course also typically fail. Indeed, there is in general no coherent notion of utility to be maximized in this case, since a nested preference system consisting of weak orders does not rule out pathologies such as preference cycles.[9]

[9] For example, let $X = xyz$ and $\mathcal{D} = \mathcal{A}$ and consider the choice function with $C(xy) = x$, $C(xz) = z$, $C(yz) = y$, and $C(xyz) = xyz$. This function is generated by a nested system of weak orders but implies

Nestedness does, however, require that a preference perceived in any choice problem also be perceived in the relevant binary problem and thus affect the relation encoding pairwise choices.

Definition 4 (base relation) The *base relation* $\mathrm{P^b}$ is defined by $x\mathrm{P^b}y$ if and only if $C(xy) = x$ and $x \neq y$.

By imposing ordering properties on this relation strong enough to make it encodable in a real-valued function, we can therefore obtain utility assignments that are consistent with all of the choice data in C.

Condition 4 (Base Order) $\mathrm{P^b}$ is a weak order.[10]

But as shown in [15, pp. 58–59], this consistency falls short of utility maximization: When faced with a given choice problem A, our agent "satisfices", deeming as chooseable any alternative with utility greater than or equal to a (menu-dependent) threshold value $\theta(A)$.

In addition to the satisficing interpretation of the decision maker's choice behavior, two joint restrictions on the utility and threshold functions follow from the conjunction of Local Congruence and Base Order. Firstly, any menu A included in a second menu B and containing an alternative with utility at least $\theta(B)$ must be assigned a threshold no smaller than the latter value. (For discussion, see [15, p. 59].) And secondly, the threshold assigned to any menu containing no more than two alternatives must equal the maximum utility available (ensuring that the utility function is a genuine encoding of $\mathrm{P^b}$). With these provisos, we can state our representation result for cognitively-constrained choice.

Theorem 5 *Let X be finite. Then a choice function C satisfies both Local Congruence and Base Order if and only if $\exists f : X \to \Re$ and $\theta : \mathcal{D} \to \Re$ such that*

(i) $\forall A \in \mathcal{D}$ *we have* $C(A) = \{x \in A : f(x) \geq \theta(A)\}$,

(ii) $\forall A, B \in \mathcal{D}$ *such that* $A \subseteq B$ *and* $\max f[A] \geq \theta(B)$ *we have* $\theta(A) \geq \theta(B)$, *and*

(iii) $\forall x, y \in X$ *we have* $f(x) \geq f(y)$ *only if* $\theta(xy) = f(x)$.

3.3 Salience

We now develop a representation result to accompany Theorem 3, under which choices are generated by a weak (salience) order Q composed with a nested system $\mathbb{R}$ of weak (perceived preference) orders.

Imposing Base Order and appealing to Theorems 2 and 5, we can obtain utility and threshold functions f and θ such that $\forall A \in \mathcal{D}$ we have $\mathbb{R}{\uparrow}(A) = \{x \in A : f(x) \geq \theta(A)\}$. Similarly, appealing to Theorems 1 and 4, we can obtain a function g (to be interpreted as

the cycle $x\mathrm{R}y\mathrm{R}z\mathrm{R}x$ (as a consequence of the perceived preferences $x\mathrm{R}_{xy}y\mathrm{R}_{yz}z\mathrm{R}_{xz}x$).

[10] Note that in [15, p. 59] we demand only acyclicity of $\mathrm{P^b}$ and reach a correspondingly weaker conclusion.

an ordinal measure of salience) such that $\forall B \in \mathbb{R}\uparrow[\mathcal{D}]$ we have $\mathrm{Q}\uparrow(B) = \operatorname{Arg\,max}_{x\in B}\, g(x)$. Given $A \in \mathcal{D}$, the choice function then returns

$$C(A) = \mathrm{Q}\uparrow \circ \mathbb{R}\uparrow(A) = \mathrm{Q}\uparrow(\{x \in A : f(x) \geq \theta(A)\}) = \operatorname{Arg\,max}_{x\in A: f(x)\geq\theta(A)}\, g(x), \quad (7)$$

yielding the desired representation.

Theorem 6 *Let X be finite. Then a choice function C satisfies both Extended Global Congruence and Base Order if and only if $\exists f, g : X \to \Re$ and $\theta : \mathcal{D} \to \Re$ such that*

(i) $\forall A \in \mathcal{D}$ *we have* $C(A) = \operatorname{Arg\,max}_{x\in A: f(x)\geq\theta(A)}\, g(x)$,

(ii) $\forall A, B \in \mathcal{D}$ *such that* $A \subseteq B$ *and* $\max f[A] \geq \theta(B)$ *we have* $\theta(A) \geq \theta(B)$, *and*

(iii) $\forall x, y \in X$ *we have* $f(x) \geq f(y)$ *only if* $\theta(xy) = f(x)$.

4 Application: Vertically-differentiated duopoly

4.1 Setting

For a simple application of the above composite representation, we turn now to the model of duopoly with price competition and vertically-differentiated products analyzed, e.g., by Tirole [13, pp. 296–298].[11]

Each firm $m = 1, 2$ manufactures a product of quality $q_m \geq 0$ at constant marginal cost $c_m \geq 0$ and sells it at a price p_m. We write $\Delta q := q_2 - q_1 > 0$ and $\Delta c := c_2 - c_1$ for the interfirm quality and cost differentials, respectively, and $\Delta p := p_2 - p_1$ for the price differential. It is useful also to introduce a default, no-purchase "product" 0 with quality $q_0 = 0$ and price $p_0 = 0$.

Demand consists of a unit mass of consumers i with valuation characteristic $v_i \geq 0$ distributed uniformly over the interval $[\underline{v}, \overline{v}]$. We write $\Delta v := \overline{v} - \underline{v} > 0$ for the range of the value parameter across the population. Each consumer purchases a single product from the menu 012, and a buyer i purchasing product a enjoys utility

$$f_i(a) := v_i q_a - p_a \quad (8)$$

from its consumption.

The salience of product m to consumer i is determined stochastically by the level of advertising $\alpha_m \geq 0$ chosen by firm m. In particular, $g_i(m)$ has the Pareto distribution with scale and shape parameters 1 and $1/\alpha_m$, respectively, and is chosen independently across consumers i and products m. For each consumer i we set $g_i(0) = 1$ deterministically (equivalent to assuming that the default product is not advertised). It is convenient also to define total industry advertising $\alpha_T := \alpha_1 + \alpha_2$ and the share $\hat{\alpha}_m := \alpha_m/\alpha_T$ for each firm m. The constant marginal cost of advertising is $k > 0$ for both firms.

[11] See also Gabszewicz and Thisse [3] and Shaked and Sutton [11].

To complete the specification of our duopoly model, we must fix consumer i's utility threshold $\theta_i(012)$. This we do implicitly by assuming that the shortfall

$$\eta_i(012) := \max f_i[012] - \theta_i(012) \tag{9}$$

between the threshold and the maximum attainable utility at the menu 012 is *constant across consumers*, and we write $\epsilon \geq 0$ for the value of this difference.[12]

We impose a number of constraints on the parameters of our model: Firstly,

$$\Delta c > \underline{v}\Delta q + \epsilon, \tag{10}$$

so that a consumer with the minimum valuation $\underline{v}$ perceives a preference for (the inferior) product 1 over (the superior) product 2 when each is priced at marginal cost. Secondly,

$$\overline{v}\Delta q > \Delta c + \epsilon, \tag{11}$$

so that a consumer with the maximum valuation $\overline{v}$ perceives a preference for 2 over 1 when each is priced at cost. And thirdly,

$$\Delta q \Delta v > \epsilon, \tag{12}$$

an assumption that the consumers' cognitive capabilities are not too weak relative to the utility differences they are called upon to perceive.

4.2 Incentives

Consider first a hypothetical consumer i who perceives no preference between products 1 and 2, but who perceives both to be superior to the default product 0. How likely is he or she to choose firm 1's product? Writing ϕ_m for the p.d.f. and Φ_m for the c.d.f. of the Pareto distribution with scale 1 and shape $1/\alpha_m$, we can compute the answer to be

$$\text{Prob}\left[g_i(1) > g_i(2)\right] = \int_1^\infty \left[1 - \Phi_1(z)\right]\phi_2(z)\mathrm{d}z = \int_1^\infty z^{-1/\alpha_1}\left[\frac{1/\alpha_2}{z^{1+1/\alpha_2}}\right]\mathrm{d}z = \hat{\alpha}_1 \tag{13}$$

(recall that $g_i(1)$ and $g_i(2)$ are independent); and likewise $\text{Prob}\left[g_i(2) > g_i(1)\right] = \hat{\alpha}_2$. It follows that any consumers who are quasi-indifferent between products 1 and 2 choose between them in proportion to the relevant advertising shares.

Who are these quasi-indifferent consumers? Clearly any whose valuations v_i satisfy

$$\Delta p - \epsilon \leq v_i \Delta q \leq \Delta p + \epsilon, \tag{14}$$

of whom there are a mass $2\epsilon/[\Delta q \Delta v]$. Moreover, any consumers with $v_i \Delta q < \Delta p - \epsilon$ will perceive a preference for product 1 over product 2, while any with $v_i \Delta q > \Delta p + \epsilon$ will perceive the opposite preference.

[12] This does *not* involve an assumption that $\max f_i[A] - \theta_i(A)$ is constant with respect to the menu A; indeed, the menu 012 will remain unchanged throughout our analysis.

For simplicity, we shall limit our attention to circumstances in which the market is "covered"; i.e., all consumers purchase from one of the two firms.[13] In this case the above considerations lead to the vector

$$x(p,\alpha) := \begin{bmatrix} x_1(p_1,p_2,\alpha_1,\alpha_2) \\ x_2(p_1,p_2,\alpha_1,\alpha_2) \end{bmatrix} = \frac{1}{\Delta q \Delta v} \begin{bmatrix} 2\epsilon\hat{\alpha}_1 + \Delta p - \epsilon - \underline{v}\Delta q \\ 2\epsilon\hat{\alpha}_2 - \Delta p - \epsilon + \overline{v}\Delta q \end{bmatrix} \tag{15}$$

of demand functions, which we can use to construct the vector

$$\pi(p,\alpha) = \begin{bmatrix} x_1(p,\alpha)[p_1 - c_1] - k\alpha_1 \\ x_2(p,\alpha)[p_2 - c_2] - k\alpha_2 \end{bmatrix} \tag{16}$$

of profit functions for firms 1 and 2.

Observe that the Hessian matrix of firm m's profits with respect to its own choice variables takes the form

$$\begin{aligned} \mathrm{H}_m \circ \pi_m(p,\alpha) &:= \begin{bmatrix} \frac{\partial}{\partial p_m} \\ \frac{\partial}{\partial \alpha_m} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial p_m} & \frac{\partial}{\partial \alpha_m} \end{bmatrix} \pi_m(p,\alpha) \\ &= \frac{1}{\Delta q \Delta v} \begin{bmatrix} -2 & \frac{2\epsilon}{\alpha_T}[1-\hat{\alpha}_m] \\ \frac{2\epsilon}{\alpha_T}[1-\hat{\alpha}_m] & \frac{-4\epsilon}{\alpha_T^2}[1-\hat{\alpha}_m][p_m - c_m] \end{bmatrix}. \end{aligned} \tag{17}$$

Hence, when both $\epsilon > 0$ (the interesting case) and $\langle p,\alpha\rangle > \langle c, 0\rangle$ (the relevant region of the choice variable space), own-price and own-advertising are complements and each yields strictly decreasing returns.

Similarly, the matrix of cross-firm ($n \neq m$) second partials of $\pi_m(p,\alpha)$ takes the form

$$\begin{bmatrix} \frac{\partial}{\partial p_n} \\ \frac{\partial}{\partial \alpha_n} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial p_m} & \frac{\partial}{\partial \alpha_m} \end{bmatrix} \pi_m(p,\alpha) = \frac{1}{\Delta q \Delta v} \begin{bmatrix} 1 & 0 \\ \frac{-2\epsilon\hat{\alpha}_m}{\alpha_T} & \frac{4\epsilon}{\alpha_T^2}[p_m - c_m]\left[\hat{\alpha}_m - \frac{1}{2}\right] \end{bmatrix}. \tag{18}$$

Thus prices are strategic complements and the returns to own-advertising are increasing or decreasing in competitor's-advertising accordingly as a firm's advertising share is greater or less than one half.

4.3 Equilibrium

The system of first order conditions (two per firm) for profit maximization characterizes the equilibrium vectors of prices

$$p^* = \frac{\Delta q \Delta v}{3\Delta q \Delta v - 2\epsilon} \begin{bmatrix} \Delta q[\overline{v} - 2\underline{v}] + 2c_1 + c_2 \\ \Delta q[2\overline{v} - \underline{v}] + c_1 + 2c_2 \end{bmatrix} - \frac{\epsilon}{3\Delta q \Delta v - 2\epsilon} \begin{bmatrix} \Delta q \Delta v + 2c_1 \\ \Delta q \Delta v + 2c_2 \end{bmatrix} \tag{19}$$

and advertising levels

$$\alpha^* = \frac{2\epsilon[\Delta q[\overline{v} - 2\underline{v}] + \Delta c - \epsilon][\Delta q[2\overline{v} - \underline{v}] - \Delta c - \epsilon]}{k\,[3\Delta q \Delta v - 2\epsilon]^3} \begin{bmatrix} \Delta q[\overline{v} - 2\underline{v}] + \Delta c - \epsilon \\ \Delta q[2\overline{v} - \underline{v}] - \Delta c - \epsilon \end{bmatrix}. \tag{20}$$

[13]Wauthy [16] determines when the market will be covered and when each firm will be active in the context of the standard ($\epsilon = 0$) model, and his results are readily generalized to the present setting. He also demonstrates the possibility of "corner solution" equilibria, which we can rule out by imposing the (somewhat stronger than necessary) conditions $2[\underline{v}q_1 - c_1] \geq \Delta q \Delta v + 2\epsilon$ and $\underline{v}q_2 - c_2 \geq \epsilon$.

The second derivative test discriminant for firm m (with $n \neq m$) is

$$|\mathrm{H}_m \circ \pi_m(p^*, \alpha^*)| = \frac{4\epsilon\hat{\alpha}_n^*[2[p_m^* - c_m] - \epsilon\hat{\alpha}_n^*]}{[\alpha_T^*\Delta q \Delta v]^2}, \tag{21}$$

and since it can be shown that

$$\operatorname{sign}\begin{bmatrix} 2[p_1^* - c_1] - \epsilon\hat{\alpha}_2^* \\ 2[p_2^* - c_2] - \epsilon\hat{\alpha}_1^* \end{bmatrix} \geq \operatorname{sign}\begin{bmatrix} \Delta c - \underline{v}\Delta q - \epsilon \\ \overline{v}\Delta q - \Delta c - \epsilon \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{22}$$

we have a local profit maximum for each firm. Note also that

$$p^*|_{\epsilon=0} = \frac{1}{3}\begin{bmatrix} \Delta q[\overline{v} - 2\underline{v}] + 2c_1 + c_2 \\ \Delta q[2\overline{v} - \underline{v}] + c_1 + 2c_2 \end{bmatrix}, \tag{23}$$

the correct equilibrium price vector for the standard case (cf. Tirole [13, p. 296]).

Interestingly, each firm's market share in equilibrium turns out to equal its advertising share; indeed,

$$x(p^*, \alpha^*) = \frac{1}{3\Delta q \Delta v - 2\epsilon}\begin{bmatrix} \Delta q[\overline{v} - 2\underline{v}] + \Delta c - \epsilon \\ \Delta q[2\overline{v} - \underline{v}] - \Delta c - \epsilon \end{bmatrix} = \hat{\alpha}^*. \tag{24}$$

Since the market shares of quasi-indifferent consumers equal the advertising shares by virtue of (13), this means that the equilibrium prices are configured so that the market shares of the remaining (i.e., not quasi-indifferent) consumers take on the same values.

Note also that the advertising cost k is practically irrelevant to the above analysis. The equilibrium prices, advertising shares, and demands for each product are all independent of this parameter, as is the total advertising expenditure

$$k\alpha_T^* = \frac{2\epsilon[\Delta q[\overline{v} - 2\underline{v}] + \Delta c - \epsilon][\Delta q[2\overline{v} - \underline{v}] - \Delta c - \epsilon]}{[3\Delta q \Delta v - 2\epsilon]^2} \tag{25}$$

in equilibrium.

Figure 3 depicts equilibria of the duopoly with $q_1 = 3$, $q_2 = 6$, $c_1 = 1$, $c_2 = 5$, $\underline{v} = 1$, $\overline{v} = 2$, and $k = 77/375$. Panel A shows the (standard) case of $\epsilon = 0$, where the equilibrium prices are $p_1^* = 7/3$ and $p_2^* = 20/3$, neither firm does any advertising, and the resulting demands are

$$x(p^*, \alpha^*) = \frac{1}{\Delta v}\begin{bmatrix} \Delta p^*/\Delta q - \underline{v} \\ \overline{v} - \Delta p^*/\Delta q \end{bmatrix} = \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix}. \tag{26}$$

Panel B illustrates the situation under cognitive constraints. Here $\epsilon = 1/3$ and we have prices $p_1^* = 58/25 < 7/3$ and $p_2^* = 167/25 > 20/3$, advertising levels $\alpha_1^* = 44/125 > 0$ and $\alpha_2^* = 56/125 > 0$, and demands

$$x(p^*, \alpha^*) = \frac{1}{\Delta v}\begin{bmatrix} [\Delta p^* - \epsilon]/\Delta q - \underline{v} \\ \overline{v} - [\Delta p^* + \epsilon]/\Delta q \end{bmatrix} + \frac{2\epsilon}{\Delta q}\begin{bmatrix} \hat{\alpha}_1^*/\Delta v \\ \hat{\alpha}_2^*/\Delta v \end{bmatrix} = \begin{bmatrix} 11/25 \\ 14/25 \end{bmatrix} \begin{matrix} < \\ > \end{matrix} \begin{bmatrix} 4/9 \\ 5/9 \end{bmatrix}. \tag{27}$$

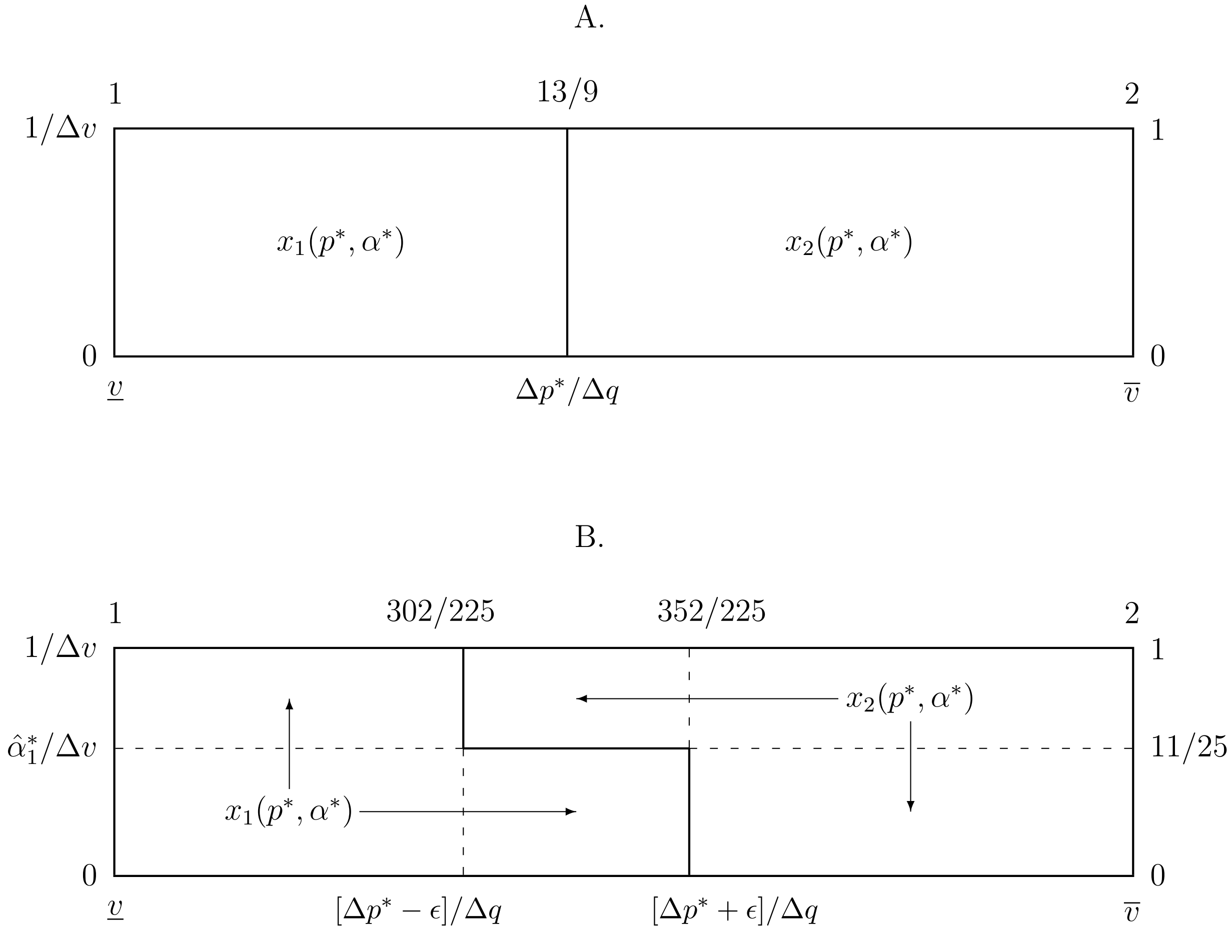

Figure 3: A duopoly with parameter settings $q = \langle 3, 6\rangle$, $c = \langle 1, 5\rangle$, $\underline{v} = 1$, $\overline{v} = 2$, and $k = 77/375$. A. When $\epsilon = 0$ (the standard case), each consumer perceives a preference for one product over the other. The equilibrium prices $p^* = \langle 7/3, 20/3\rangle$ and advertising levels $\alpha^* = \langle 0, 0\rangle$ lead to the demands $x(p^*, \alpha^*) = \langle 4/9, 5/9\rangle$. B. When $\epsilon > 0$, customers with $v_i\Delta q \in [\Delta p - \epsilon, \Delta p + \epsilon]$ are quasi-indifferent between products 1 and 2. For $\epsilon = 1/3$ the equilibrium prices $p^* = \langle 58/25, 167/25\rangle$ and advertising levels $\alpha^* = \langle 44/125, 56/125\rangle$ lead to the demands $x(p^*, \alpha^*) = \langle 11/25, 14/25\rangle$.

A Proofs

Here conditions on the choice function are indicated by their initials; for example, Global Congruence by GC. Moreover, Theorem 1 is referred to as T1, Proposition 1 as P1, etc.

Proposition 1:

A. Given $x, y \in X$, we have $x\mathrm{P}^{\mathrm{g}}y$ only if $C(xy) = x$ and hence $y\overline{\mathrm{P}^{\mathrm{g}}}x$. Therefore P^{g} is asymmetric and so $\mathrm{P}^{\mathrm{g}\,\div}$ is a weak order.

B. Given $x \in A \in \mathcal{D}$, let $x \notin \mathrm{P}^{\mathrm{g}\,\div}\!\uparrow\!(A)$. Then $\exists y \in A$ such that $y\ \mathrm{P}^{\mathrm{g}\,\div}\ x$, and so $y\mathrm{P}^{\mathrm{g}}x$ and $x \notin C(A)$. The result now follows by contraposition.

Lemma 1:

A. Let $C \subseteq \mathrm{R}\!\uparrow$ for some weak order R. The assertion $x\overline{\mathrm{P}^{\mathrm{g}}}y$ means that $\exists A \in \mathcal{D}$ such that $x, y \in A$ and $y \in C(A) \subseteq \mathrm{R}\!\uparrow\!(A)$, and we then have $x\overline{\mathrm{R}}y$. Thus $\overline{\mathrm{P}^{\mathrm{g}}} \subseteq \overline{\mathrm{R}}$, and it follows that $\overline{\mathrm{P}^{\mathrm{g}}}^{*} \subseteq \overline{\mathrm{R}}^{*}$. We have also $\overline{\mathrm{R}}^{*} \subseteq \overline{\mathrm{R}}$ since R is a weak order; hence $\overline{\mathrm{P}^{\mathrm{g}}}^{*} \subseteq \overline{\mathrm{R}}$ and so $\mathrm{R} \subseteq \mathrm{P}^{\mathrm{g}\,\div}$ by contraposition.

B. Let $C = \mathrm{R}\!\uparrow$ for some weak order R. The assertion $x\mathrm{P}^{\mathrm{s}}y$ means that $\exists A \in \mathcal{D}$ such that $x \in C(A) = \mathrm{R}\!\uparrow\!(A) \not\ni y$, and since R is a weak order we have $x\mathrm{R}y$. Thus $\mathrm{P}^{\mathrm{s}} \subseteq \mathrm{R}$.

Lemma 2:

Let GC hold and suppose that $\exists x \in A \in \mathcal{D}$ such that both $x \in \mathrm{P}^{\mathrm{g}\,\div}\!\uparrow\!(A)$ and $x \notin C(A)$. Then $\exists y \in C(A)$, in which case $y\mathrm{P}^{\mathrm{s}}x$ by definition and $y\ \mathrm{P}^{\mathrm{g}\,\div}\ x$ by GC. But this implies that $x \notin \mathrm{P}^{\mathrm{g}\,\div}\!\uparrow\!(A)$, establishing a contradiction.

Theorem 1:

N. The necessity of GC follows from L1.

S. The sufficiency of GC follows from P1 and L2.

Proposition 2:

A. Since $\mathbb{P}^{\mathrm{l}}$ is nested, $\mathbb{P}^{\mathrm{l}\,\div}$ is as well. Given $x, y \in A \in \mathcal{D}$, if $x\mathrm{P}^{\mathrm{l}}_{A}y$ then $C(xy) = x$ and so $y\overline{\mathrm{P}^{\mathrm{l}}_{A}}x$. Hence $\mathbb{P}^{\mathrm{l}}$ is asymmetric, and so $\mathbb{P}^{\mathrm{l}\,\div}$ is a system of weak orders.

B. Given $x \in A \in \mathcal{D}$, let $x \notin \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$. Then $\exists y \in A$ such that $y\ \mathrm{P}^{\mathrm{l}\,\div}_{A}\ x$, and so $y\mathrm{P}^{\mathrm{l}}_{A}x$ and $x \notin C(A)$. The result now follows by contraposition.

Lemma 3:

A. Let $C \subseteq \mathbb{R}\!\uparrow$ for some nested system $\mathbb{R}$ of weak orders. Given $x, y \in B \in \mathcal{D}$, the assertion $x\overline{\mathrm{P}^{\mathrm{l}}_{B}}y$ means that $\exists A \in \mathcal{D}$ such that $x, y \in A \subseteq B$ and $y \in C(A) \subseteq \mathbb{R}\!\uparrow\!(A)$. We then have $x\overline{\mathrm{R}_{A}}y$, and hence $x\overline{\mathrm{R}_{B}}y$ since $\mathbb{R}$ is nested. Thus $\overline{\mathbb{P}^{\mathrm{l}}} \subseteq \overline{\mathbb{R}}$, and it follows that $\overline{\mathbb{P}^{\mathrm{l}}}^{*} \subseteq \overline{\mathbb{R}}^{*}$. We have also $\overline{\mathbb{R}}^{*} \subseteq \overline{\mathbb{R}}$ since $\mathbb{R}$ is a system of weak orders; therefore $\overline{\mathbb{P}^{\mathrm{l}}}^{*} \subseteq \overline{\mathbb{R}}$ and so $\mathbb{R} \subseteq \mathbb{P}^{\mathrm{l}\,\div}$ by contraposition.

B. Let $C = \mathbb{R}\!\uparrow$ for some nested system $\mathbb{R}$ of weak orders. Given $x, y \in A \in \mathcal{D}$, the assertion $x\mathrm{P}^{\mathrm{u}}_A y$ means that $\exists B \in \mathcal{D}$ such that $A \subseteq B$ and $x \in C(B) = \mathbb{R}\!\uparrow\!(B) \not\ni y$. Since $\mathbb{R}$ is a system of weak orders, we have $x\mathrm{R}_B y$ and so $x\mathrm{R}_A y$ since $\mathbb{R}$ is nested. Thus $\mathbb{P}^{\mathrm{u}} \subseteq \mathbb{R}$.

Lemma 4:

Let LC hold and suppose that $\exists x \in A \in \mathcal{D}$ such that both $x \in \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$ and $x \notin C(A)$. Then $\exists y \in C(A)$, in which case $y\mathrm{P}^{\mathrm{u}}_A x$ by definition and $y\;\mathrm{P}^{\mathrm{l}}_A{}^{\div}\; x$ by LC. But this implies that $x \notin \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$, establishing a contradiction.

Theorem 2:

N. The necessity of LC follows from L3.

S. The sufficiency of LC follows from P2 and L4.

Proposition 3:

A. Since $x\overline{\mathrm{P}^{\mathrm{l}}_x}x$ and $x \in C(x)$ imply that $x\overline{\mathrm{S}^{\mathrm{g}}}x$, the relation S^{g} is irreflexive.

B. Given $x \in A \in \mathcal{D}$, let $x \notin \mathrm{S}^{\mathrm{g}\,\div}\!\uparrow \circ\, \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$. If $x \notin \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$, then $x \notin C(A)$ by P2B. If, on the other hand, $x \in \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$, then $\exists y \in \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$ such that $y\;\mathrm{S}^{\mathrm{g}\,\div}\; x$ and hence $y\mathrm{S}^{\mathrm{g}}x$. From $y \in \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$ we have $\neg[x\;\mathrm{P}^{\mathrm{l}}_A{}^{\div}\; y]$, or $x\;\overline{\mathrm{P}^{\mathrm{l}}_A}^{\,*}\; y$. But $y\mathrm{S}^{\mathrm{g}}x$ then implies once again that $x \notin C(A)$. The result now follows by contraposition.

Lemma 5:

A. Let $C \subseteq \mathrm{Q}\!\uparrow \circ \mathbb{R}\!\uparrow$ for some weak order Q and nested system $\mathbb{R}$ of weak orders. The assertion $x\overline{\mathrm{S}^{\mathrm{g}}}y$ means that $\exists A \in \mathcal{D}$ such that $x, y \in A$ and both $y\;\overline{\mathrm{P}^{\mathrm{l}}_A}^{\,*}\; x$ and $y \in C(A) \subseteq \mathrm{Q}\!\uparrow \circ \mathbb{R}\!\uparrow\!(A)$. We then have $\neg[y\;\mathrm{P}^{\mathrm{l}}_A{}^{\div}\; x]$, and hence $y\overline{\mathrm{R}_A}x$ by L3A. Since $y \in \mathbb{R}\!\uparrow\!(A)$ and $\mathbb{R}$ is a system of weak orders, we can conclude that $x \in \mathbb{R}\!\uparrow\!(A)$ and hence $x\overline{\mathrm{Q}}y$. Thus $\overline{\mathrm{S}^{\mathrm{g}}} \subseteq \overline{\mathrm{Q}}$, and it follows that $\overline{\mathrm{S}^{\mathrm{g}}}^{\,*} \subseteq \overline{\mathrm{Q}}^{\,*}$. We have also $\overline{\mathrm{Q}}^{\,*} \subseteq \overline{\mathrm{Q}}$ since Q is a weak order; therefore $\overline{\mathrm{S}^{\mathrm{g}}}^{\,*} \subseteq \overline{\mathrm{Q}}$ and so $\mathrm{Q} \subseteq \mathrm{S}^{\mathrm{g}\,\div}$ by contraposition.

B. Let $C = \mathrm{Q}\!\uparrow \circ \mathbb{R}\!\uparrow$ for some weak order Q and nested system $\mathbb{R}$ of weak orders. The assertion $x\mathrm{S}^{\mathrm{s}}y$ means that $\exists A \in \mathcal{D}$ such that $x, y \in A$ and both $x\;\overline{\mathrm{P}^{\mathrm{l}}_A}^{\,*}\; y$ and $x \in C(A) = \mathrm{Q}\!\uparrow \circ \mathbb{R}\!\uparrow\!(A) \not\ni y$. We then have $\neg[x\;\mathrm{P}^{\mathrm{l}}_A{}^{\div}\; y]$, and hence $x\overline{\mathrm{R}_A}y$ by L3A. Since $x \in \mathbb{R}\!\uparrow\!(A)$ and $\mathbb{R}$ is a system of weak orders, we can conclude that $y \in \mathbb{R}\!\uparrow\!(A)$; and since Q is a weak order it then follows that $x\mathrm{Q}y$. Thus $\mathrm{S}^{\mathrm{s}} \subseteq \mathrm{Q}$.

Lemma 6:

Let EGC hold and suppose that $\exists x \in A \in \mathcal{D}$ such that both $x \in \mathrm{S}^{\mathrm{g}\,\div\triangleright}\!\uparrow \circ\, \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$ and $x \notin C(A)$. Then $\exists y \in C(A)$, and we have $y \in \mathrm{S}^{\mathrm{g}\,\div}\!\uparrow \circ\, \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$ by P3B. It follows that $\neg[y\;\mathrm{S}^{\mathrm{g}\,\div\triangleright}\, x]$, and so (†) either $\neg[y\;\mathrm{S}^{\mathrm{g}\,\div}\, x]$ or $x\;\mathrm{S}^{\mathrm{g}\,\div}\, y$. From $x \in \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$ we have $\neg[y\;\mathrm{P}^{\mathrm{l}}_A{}^{\div}\; x]$, or $y\;\overline{\mathrm{P}^{\mathrm{l}}_A}^{\,*}\; x$. Thus $y\mathrm{S}^{\mathrm{s}}x$ by definition, $y\;\mathrm{S}^{\mathrm{g}\,\div}\, x$ by EGC, and $x\;\mathrm{S}^{\mathrm{g}\,\div}\, y$ by (†). But this implies that $y \notin \mathrm{S}^{\mathrm{g}\,\div}\!\uparrow \circ\, \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow\!(A)$, establishing a contradiction.

Proposition 4:

Let R be irreflexive. Then $\mathrm{R}^{\div\triangleright}$ is a partial order and $\overline{\mathrm{R}}^{*\bowtie}$ is both an equivalence and a congruence with respect to $\mathrm{R}^{\div\triangleright}$.[14] Writing $\mathcal{T} := \{\{y \in X : y\overline{\mathrm{R}}^{*\bowtie}x\} : x \in X\}$ for the partition of X into equivalence classes and $T(x) \in \mathcal{T}$ for the partition element containing a given x, we can define a partial order $\gg$ by $T_1 \gg T_2$ if and only if $\exists$ both $x_1 \in T_1$ and $x_2 \in T_2$ such that $x_1\mathrm{R}^{\div\triangleright}x_2$. By Szpilrajn's [12] Embedding Theorem, $\gg$ can be strengthened to a linear order $\ggg$, and we can proceed to define a weak order Q by $x\mathrm{Q}y$ if and only if $T(x) \ggg T(y)$. We then have $x\mathrm{R}^{\div\triangleright}y$ only if $T(x) \ggg T(y)$ and $x\mathrm{Q}y$, and so $\mathrm{R}^{\div\triangleright} \subseteq \mathrm{Q}$ as desired. Moreover, $x\overline{\mathrm{R}}^{*}y$ only if either $x\overline{\mathrm{R}}^{*\bowtie}y$ or $y\mathrm{R}^{\div\triangleright}x$, in which case either $T(x) = T(y)$ or $T(y) \ggg T(x)$, whence $x\overline{\mathrm{Q}}y$. Thus $\overline{\mathrm{R}}^{*} \subseteq \overline{\mathrm{Q}}$, and so $\mathrm{Q} \subseteq \mathrm{R}^{\div}$ as desired.

Theorem 3:

N. The necessity of EGC follows from L5.

S. Let EGC hold. $\mathbb{P}^{\mathrm{l}\,\div}$ is a nested system of weak orders by P2A. Moreover, S^{g} is irreflexive by P3A, and so by P4 $\exists$ a weak order S^{w} such that $\mathrm{S}^{\mathrm{g}\,\div\triangleright} \subseteq \mathrm{S}^{\mathrm{w}} \subseteq \mathrm{S}^{\mathrm{g}\,\div}$. It follows that $C \subseteq \mathrm{S}^{\mathrm{g}\,\div}\!\uparrow \circ \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow \subseteq \mathrm{S}^{\mathrm{w}}\!\uparrow \circ \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow \subseteq \mathrm{S}^{\mathrm{g}\,\div\triangleright}\!\uparrow \circ \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow \subseteq C$ by P3B and L6, and hence that $C = \mathrm{S}^{\mathrm{w}}\!\uparrow \circ \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow$.

Theorem 4:

N. Suppose that for some $f : X \to \Re$ and $\forall A \in \mathcal{D}$ we have $C(A) = \operatorname{Arg\,max}_{x\in A} f(x)$. Since the weak order represented by f then generates C, GC holds by T1.

S. Let GC hold. By GC and T1, $\exists$ a weak order R that generates C. Since X is finite, R has a representation $f : X \to \Re$. And for each $A \in \mathcal{D}$ we then have $C(A) = \{x \in A : \forall y \in A$ we have $f(x) \geq f(y)\}$ and hence $C(A) = \operatorname{Arg\,max}_{x\in A} f(x)$.

Theorem 5:

N. Suppose that for some $f : X \to \Re$ and $\theta : \mathcal{D} \to \Re$ we have (i)–(iii). Since BO holds under the weaker representation of T6, it holds a fortiori in this case. Define a relation system $\mathbb{R}$ as follows: For each $A \in \mathcal{D}$, let $x\mathrm{R}_A y$ if and only if $\exists B \in \mathcal{D}$ such that $A \subseteq B$ and $f(x) \geq \theta(B) > f(y)$.

- By construction $\mathbb{R}$ is nested, asymmetric, and a lower bound for C.
- Given $x \in A \in \mathcal{D}$, if $x \notin \mathbb{R}\!\uparrow\!(A)$ then $\exists y \in A$ such that $y\mathrm{R}_A x$. It follows that $\exists B \in \mathcal{D}$ such that $A \subseteq B$ and $f(y) \geq \theta(B) > f(x)$, in which case by (ii) we have $\theta(A) \geq \theta(B) > f(x)$ and so $x \notin C(A)$. By contraposition, $C \subseteq \mathbb{R}\!\uparrow$.
- Given $x, y, z \in A \in \mathcal{D}$ for which $x\mathrm{R}_A z$, $\exists B \in \mathcal{D}$ such that $A \subseteq B$ and $f(x) \geq \theta(B) > f(z)$. If $f(y) \geq \theta(B)$ then $y\mathrm{R}_A z$, whereas if $f(y) < \theta(B)$ then $x\mathrm{R}_A y$; so $\mathbb{R}$ is negatively transitive.

Now, since C is generated by a nested system of weak orders, LC holds by T2.

[14] A relation E is an equivalence if it is reflexive ($x = y$ only if $x\mathrm{E}y$), symmetric ($x\mathrm{E}y$ only if $y\mathrm{E}x$), and transitive; and is a congruence with respect to a second relation R if we have $w\mathrm{E}x\mathrm{R}y\mathrm{E}z$ only if $w\mathrm{R}z$.

S. Let LC and BO hold. By LC and T2, $\exists$ a nested system of weak orders $\mathbb{R}$ that generates C. Moreover, P^{b} is a weak order by BO and since X is finite has a representation $f : X \to \Re$. For each $A \in \mathcal{D}$, let $\theta(A) = \min f[C(A)]$.

(i) Let $A \in \mathcal{D}$ be given. It is immediate that $C(A) \subseteq \{x \in A : f(x) \geq \theta(A)\}$, so let $x \in A$ be such that $x \notin C(A) = \mathbb{R}\!\uparrow\!(A)$. Choosing $y \in C(A) = \mathbb{R}\!\uparrow\!(A)$ such that $f(y) = \min f[C(A)]$, we have $y\mathrm{R}_A x$ since $\mathbb{R}$ is a system of weak orders, $y\mathrm{R}_{xy}x$ since $\mathbb{R}$ is nested, $x \notin C(xy)$ and hence $y\mathrm{P}^{\mathrm{b}}x$ since $\mathbb{R}$ generates C, and $f(x) < f(y) = \theta(A)$ since f represents P^{b}. It follows that $\{x \in A : f(x) \geq \theta(A)\} \subseteq C(A)$ by contraposition.

(ii) Let $A, B \in \mathcal{D}$ be such that $A \subseteq B$ and $\max f[A] \geq \theta(B)$. In this case $\exists y \in A$ such that $f(y) \geq \theta(B)$, and so $y \in C(B) = \mathbb{R}\!\uparrow\!(B)$. For each $x \in C(A) = \mathbb{R}\!\uparrow\!(A)$ we have $y\overline{\mathrm{R}_A}x$ and hence $y\overline{\mathrm{R}_B}x$ since $\mathbb{R}$ is nested. Since $\mathbb{R}$ is a system of weak orders, this implies that $x \in \mathbb{R}\!\uparrow\!(B) = C(B)$. But then $C(A) \subseteq C(B)$ and so $\theta(A) = \min f[C(A)] \geq \min f[C(B)] = \theta(B)$.

(iii) Let $x, y \in X$ be such that $f(x) \geq f(y)$. Clearly $f(x) \geq \theta(xy)$. If $f(x) > \theta(xy) = f(y)$ then $x\mathrm{P}^{\mathrm{b}}y$ and $y \notin C(xy)$; in which case $f(y) < \theta(xy)$, a contradiction. Thus $\theta(xy) = f(x)$.

Theorem 6:

N. Suppose that for some $f, g : X \to \Re$ and $\theta : \mathcal{D} \to \Re$ we have (i)–(iii). Let Q be the weak order represented by g, so that $\forall A \in \mathcal{D}$ we have $\mathrm{Q}\!\uparrow\!(A) = \operatorname{Arg\,max}_{x \in A} g(x)$. Define a relation system $\mathbb{R}$ as follows: For each $A \in \mathcal{D}$, let $x\mathrm{R}_A y$ if and only if $\exists B \in \mathcal{D}$ such that $A \subseteq B$ and $f(x) \geq \theta(B) > f(y)$. As shown in the proof of T5, $\mathbb{R}$ is then a nested system of weak orders such that $\forall A \in \mathcal{D}$ we have $\mathbb{R}\!\uparrow\!(A) = \{x \in A : f(x) \geq \theta(A)\}$. Given $A \in \mathcal{D}$, we now have $C(A) = \operatorname{Arg\,max}_{x \in A : f(x) \geq \theta(A)} g(x) = \mathrm{Q}\!\uparrow \circ\, \mathbb{R}\!\uparrow\!(A)$. But then C is generated by a weak order composed with a nested system of weak orders and so EGC holds by T3.

The base relation is inherently asymmetric, so to establish BO we need only show that P^{b} is negatively transitive. Given $x, y, z \in \mathcal{D}$, let $x\overline{\mathrm{P}^{\mathrm{b}}}y\overline{\mathrm{P}^{\mathrm{b}}}z$. Since $y \in C(xy)$ we have both that $\theta(xy) = f(y) \geq f(x)$ and that $f(y) = f(x)$ only if $g(y) \geq g(x)$. Similarly, since $z \in C(yz)$ we have both that $\theta(yz) = f(z) \geq f(y)$ and that $f(z) = f(y)$ only if $g(z) \geq g(y)$. It follows both that $\theta(xz) = f(z) \geq f(x)$ and that $f(z) = f(x)$ only if $g(z) \geq g(x)$. But then $z \in C(xz)$ and so $x\overline{\mathrm{P}^{\mathrm{b}}}z$.

S. Let EGC and BO hold. In proving T3 we have shown that EGC guarantees the existence of a weak order S^{w} such that $C = \mathrm{S}^{\mathrm{w}}\!\uparrow \circ\, \mathbb{P}^{\mathrm{l}\div}\!\uparrow$.

Since $\mathrm{S}^{\mathrm{w}}\!\uparrow$ is generated by a weak order, this function satisfies GC by T1. Similarly, since $\mathbb{P}^{\mathrm{l}\div}\!\uparrow$ is generated by a nested system of weak orders, this function satisfies LC by T2. For any $x, y \in X$ we have $y \notin \mathbb{P}^{\mathrm{l}\div}\!\uparrow\!(xy)$ if and only if $y \notin C(xy)$, and thus the base relations associated with C and $\mathbb{P}^{\mathrm{l}\div}\!\uparrow$ are identical. Since C satisfies BO by assumption, it follows that $\mathbb{P}^{\mathrm{l}\div}\!\uparrow$ satisfies BO as well.

Since $\mathbb{P}^{\mathrm{l}\div}\!\uparrow$ satisfies LC and BO, by T5 it follows that $\exists f : X \to \Re$ and $\theta : \mathcal{D} \to \Re$ such that (ii)–(iii) hold and $\forall A \in \mathcal{D}$ we have $\mathbb{P}^{\mathrm{l}\div}\!\uparrow\!(A) = \{x \in A : f(x) \geq \theta(A)\}$. Similarly, since $\mathrm{S}^{\mathrm{w}}\!\uparrow$ satisfies GC, by T4 we can conclude that $\exists g : X \to \Re$ such that $\forall A \in \mathcal{D}$ we have $\mathrm{S}^{\mathrm{w}}\!\uparrow\!(A) = \operatorname{Arg\,max}_{x \in A} g(x)$.

Given $A \in \mathcal{D}$, we now have $C(A) = \mathrm{S}^{\mathrm{w}}\!\uparrow \circ\, \mathbb{P}^{\mathrm{l}\,\div}\!\uparrow(A) = \mathrm{S}^{\mathrm{w}}\!\uparrow(\{x \in A : f(x) \geq \theta(A)\})$ and so $C(A) = \operatorname{Arg\,max}_{x \in A: f(x) \geq \theta(A)} g(x)$.

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