Balance and the Extended, Generalized Shapley Value

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Abstract

We characterize the Extended, Generalized Shapley Value of McQuillin [6] using an 'efficiency' condition and two 'balance' conditions. One feature of this charateriza-tion is that the conditions only have to apply for a single game.

1 Introduction

The Extended, Generalized Shapley Value described in McQuillin [6] is a Shapley-type solution for games in partition function form, as first described in Thrall and Lucas [12], and with a prior 'coalition structure', as envisaged in Owen [9] and in Hart and Kurz [3]. This note provides a new characterization of the Extended, Generalized Shapley Value, one which we believe to be interesting in at least two respects. First, this characterization imposes requirements only on the value for a single a game: the specific game under consideration. It might therefore be viewed as an alternative, recursive *definition* of the value rather than as an (axiomatic) 'characterization' in the usual sense, except we claim that our 'conditions' look rather like 'axioms'. Second, our 'conditions' are ones of *efficiency* and *balance* (we see this approach as being close to that of Myerson [8]); these are more amenable to interpretation than the Shapley-type axioms used in McQuillin [6].

Our approach can be exemplified as follows. In Figure 1, the left hand table sets up a game in partition function form: in this case, a three-player 'divide-the-dollar-game' with the twist that if no coalition forms then the ‘dollar’ will be given to player a. The right hand table provides two candidate ‘values’. Each value takes a ‘generalized’ form in that it assigns outcomes both to singletons and to prior coalitions embedded in coarser structures, with the intuition being that the value in some way anticipates the outcome of ideally rational bargaining or - perhaps equivalently - of fair allocation whereby the surplus available through further cooperation is shared between cooperative entities that have already formed. The candidate values share two features: first that the outcomes assigned to coalitions across any prior structure are efficient (they sum to the payoff achievable by the grand coalition), and second that the outcomes assigned to coarsely embedded coalitions - where the ‘prior’ structure divides players into only two coalitions - divide the remaining available surplus (zero, in this game) equally. But the values differ in the outcomes they assign to finely embedded singletons. Value 1 confers to a some advantage due to the asymmetry in singleton payoffs in the underlying game, whilst Value 2 (the Extended, Generalized Shapley Value described in McQuillin [6]) does not.

A value can be viewed as ‘unbalanced’ if some player is able to pose objections or threats that cannot be matched by similar threats or objections from other players. Myerson’s well-known, ‘balanced contributions’ characterization of the Shapley value for games in characteristic function form ([8]) equalizes between any two players the effect on one player’s outcome of the other ‘leaving the game’. In a sense, it balances the ‘threats’ of ‘walking away from the negotiating table’. If the original game is in characteristic function form then the new games that arise if original players leave are unambiguous, but our claim here is that if the original game is in partition function form then the new games that arise are unclear. In particular, after more than one player has walked away, we cannot tell whether the players that remain should suppose that the absent players are organized as singletons or are coalesced together. In our example, we cannot tell whether the subgame comprising only player a has a payoff of zero or one. The former supposition leads to the singleton outcomes in Value 1 above, and the latter to the singleton outcomes in Value 2. This problem motivates us to consider a different form of objection or threat, based on the relative effects between any two players of one player amalgamating, for the purposes of remaining negotiations, with some other.

Example Game in Partition Function Form: **Two Example Values:**

Coalition	Partition	Payoff	Value 1	Value 2
$\{a\}$	$\{\{a\},\{b\},\{c\}\}$	1	$\frac{2}{3}$	$\frac{1}{3}$
	$\{\{a\},\{b,c\}\}$	0	0	0
$\{b\}$	$\{\{a\},\{b\},\{c\}\}$	0	$\frac{1}{6}$	$\frac{1}{3}$
	$\{\{a,c\},\{b\}\}$	0	0	0
$\{c\}$	$\{\{a\},\{b\},\{c\}\}$	0	$\frac{1}{6}$	$\frac{1}{3}$
	$\{\{a,b\},\{c\}\}$	0	0	0
$\{a,b\}$	$\{\{a,b\},\{c\}\}$	1	1	1
$\{a,c\}$	$\{\{a,c\},\{b\}\}$	1	1	1
$\{b,c\}$	$\{\{a\},\{b,c\}\}$	1	1	1
$\{a,b,c\}$	$\{\{a,b,c\}\}$	1	1	1

Figure 1: Example Game and Values

Suppose that, at some stage during negotiations over the surplus available in our example game, and with the players currently organized as singletons, Value 1 is posited as a prospective solution to the issue at hand. Player b may then seek to do better, at the expense of a, by pointing to the consequence for a if b amalgamates with c: the *externality* to a in this case is given by $0 - \frac{2}{3} = -\frac{2}{3}$. The only corresponding threat or objection from a, the externality to b if a amalgamates with c ($= -\frac{1}{6}$), is lesser. This form of imbalance arises in any candidate value that shares the two features already described, except Value 2. This note formalizes and generalizes this observation.

2 Preliminaries

Let N denote a finite set of players. We define the set Π to be the set of all partitions of N. We define M, the *set of embedded coalitions*, to be the set $\{(I, \pi) \colon \pi \in \Pi, I \in \pi\}$, and W to be the set of all mappings $w \colon M \to \mathbb{R}$ with $I = \emptyset \to w(I, \pi) = 0$. Any element of W is a transferable utility (TU) *game in partition function form*, on N. We use $(I, \cdot\cdot)$ as an abbreviation for the coarsely embedded coalition $(I, \{I, N \setminus I\}) \in M$ and $\pi \sim IJ$ as an abbreviation for the partition of N that arises when the partition $\pi \in \Pi$ coarsens through the amalgamation of I and J: $\pi \sim IJ \equiv \pi \setminus \{I, J\} \cup \{I \cup J\}$.

Our 'value' approach and notation, $\chi : W \to W$, recollects that of McQuillin [6]; but in this paper we are mainly concerned with the 'value' of some specific game $\hat{w} \in W$. If (I, π) is an embedded coalition, then $\hat{w}(I, \pi)$ and $\chi(\hat{w})(I, \pi)$ are two real numbers of which $\hat{w}(I, \pi)$ is interpreted as the utility *payoff* prescribed to coalition I given partition π in the game $\hat{w}$, and $\chi(\hat{w})(I, \pi)$ is interpreted as the (expected or rightful) utility *outcome* associated with coalition I whenever π is the coalition structure *prior to playing* the game $\hat{w}$. McQuillin [6] describes a particular value, referred to as the *Extended, Generalized Shapley Value*.

Definition 1 (Extended, Generalized Shapley Value) $\forall w \in W$*, the Extended, Generalized Shapley Value of* w*,* $EGSV(w) \in W$*, is defined:* $\forall (I, \pi) \in M$,

$$EGSV(w)(I, \pi) \equiv \sum_{T \subseteq \pi} \frac{(|T| - 1)!(|\pi| - |T|)!}{|\pi|!} \left(w \left(\bigcup_{A \in T} A, \cdot\cdot \right) - w \left(\bigcup_{A \in (T \setminus \{I\})} A, \cdot\cdot \right) \right).$$

3 Theorem

Our theorem characterizes the Extended, Generalized Shapley Value of $\hat{w}$ using three conditions. *Efficiency* is standard: the axiom entails that the total surplus available, which we assume to be the payoff associated with the grand coalition, will be distributed between the coalitions that comprise the prior coalition structure.

Condition 1 (Efficiency) $\forall \pi \in \Pi, \sum_{I \in \pi} \chi(\hat{w})(I, \pi) = \hat{w}(N, \{N\})$.

Balanced Bipartite Dependency applies the familiar idea of 'balanced contributions' restrictively to the circumstance of a prior stucture comprising only two coalitions. In this circumstance, the outcome for one coalition if the other 'leaves the negotiating table' is unambiguous. Our contention is - as explained in our Introduction - that if there are coalitional externalities in $\hat{w}$ and the prior structure comprises more than two coalitions, the nature of mutual dependency is less clear.

Condition 2 (Balanced Bipartite Dependency) $\forall \pi \in \Pi, |\pi| = 2 \rightarrow \forall I, J \in \pi,$ $\chi\left(\hat{w}\right)\left(I, \{I, J\}\right) - \hat{w}(I, \{I, J\}) = \chi\left(\hat{w}\right)\left(J, \{I, J\}\right) - \hat{w}(J, \{I, J\}).$

Given some prior coalition structure comprising three or more coalitions, including coalitions I and J, we say that I's externalities are 'dominated' by J's if for every other prior coalition K the effect on I of J amalgamating with K would be strictly worse than the effect on J of I amalgamating with K. The idea is that in this situation J has a clear upper hand over I. The condition of *Undominated Externalities* represents an absence of this clear advantage.

Condition 3 (Undominated Externalities) $\forall \pi \in \Pi, |\pi| > 2 \rightarrow \forall I, J \in \pi, \exists K \in \pi \setminus \{I, J\}, \chi\left(\hat{w}\right)\left(I, \{I, J\}\right) - \hat{w}(I, \{I, J\}) = \chi\left(\hat{w}\right)\left(J, \{I, J\}\right) - \hat{w}(J, \{I, J\}).$

Theorem 1 $\chi(\hat{w})$ *satisfies Efficiency, Balanced Bipartite Dependency and Undominated Externalities if and only if* $\chi(\hat{w}) = EGSV(\hat{w})$*.*

4 Proof of the Theorem

We start by setting out and proving two lemmas, each of which is also a characterization of the Extended, Generalized Shapley Value of $\hat{w}$. For each lemma we simply substitute an alternative to the Undominated Externalities condition in our theorem.

Condition 4 (Consistency w.r.t. Bilateral Amalgamations)

$$\forall (I, \pi) \in M, \chi\left(\hat{w}\right)\left(I, \pi\right) = \frac{1}{|\pi|\left(|\pi| - 1\right)} \left((|\pi| - 1)\chi\left(\hat{w}\right)\left(I, \pi\right) + \sum_{J, K \in \pi \setminus \{I\}} \chi\left(\hat{w}\right)\left(I, \pi \sim JK\right) \right. \\ \left. + \sum_{J \in \pi \setminus \{I\}} \left(\chi\left(\hat{w}\right)\left(\{I \cup J, \pi \sim IJ\right) - \chi\left(\hat{w}\right)\left(J, \pi\right)\right) \right).$$

Lemma 1 $\chi(\hat{w})$ *satisfies Efficiency, Balanced Bipartite Dependency and Consistency w.r.t. Bilateral Amalgamations if and only if* $\chi(\hat{w}) = EGSV(\hat{w})$*.*

Proof. First note the following proposition (proved, as 'Theorem 3' in McQuillin [6])

Proposition 1 *Suppose* $[\hat{w}_t]_{t=0}^{\infty}$ *is a sequence defined by* $\hat{w}_0 \equiv \hat{w}$ *and* $\forall (I,\pi) \in M$,

$$\hat{w}_t(I,\pi) \equiv \frac{1}{|\pi|\,(|\pi|-1)} \Bigg((|\pi|-1)\hat{w}_{t-1}(I,\pi) + \sum_{J,K\in\pi\setminus\{I\}} \hat{w}_{t-1}\,(I,\pi\sim JK) + \sum_{J\in\pi\setminus\{I\}} \left(\hat{w}_{t-1}(I\cup J,\pi\sim IJ) - \hat{w}_{t-1}(J,\pi)\right) \Bigg).$$

then $\lim_{t\longrightarrow\infty}\ \hat{w}_t$ *is the Extended, Generalized Shapley Value of* $\hat{w}$.

Proof of Lemma 1 follows almost immediately from Proposition 1. (i) (Sufficiency). The Extended, Generalized Shapley Value of $\hat{w}$ satisfies Efficiency and Balanced Bipartite Dependency by construction, and it is immediately obvious from Proposition 1 that it also satisfies Consistency w.r.t. Bilateral Amalgamations.
(ii) (Necessity). We know (see McQuillin [6]) that $EGSV(EGSV(\hat{w})) = EGSV(\hat{w})$. Since $EGSV(\hat{w})$ only takes account of payoffs (in $\hat{w}$) to the grand coalition and to bilaterally embedded coalitions, if $\chi(\hat{w})$ satisfies Efficiency and Balanced Bipartite Dependency then $EGSV(\chi(\hat{w})) = EGSV(\hat{w})$. But we know from Proposition 1 that Consistency w.r.t. Bilateral Amalgamations can only hold when $\chi(\hat{w}) = EGSV(\chi(\hat{w}))$, so if $\chi(\hat{w})$ satisfies all three axioms then $\chi(\hat{w}) = EGSV(\hat{w})$. ■

Condition 5 (Balanced Externalities) $\forall \pi \in \Pi, \forall I,J,K \in \pi$,
$\chi(\hat{w})(I,\pi\sim JK) - \chi(\hat{w})(I,\pi) = \chi(\hat{w})(J,\pi\sim IK) - \chi(\hat{w})(J,\pi)$.

Lemma 2 $\chi(\hat{w})$ *satisfies Efficiency, Balanced Bipartite Dependency and Balanced Externalities if and only if* $\chi(\hat{w}) = EGSV(\hat{w})$.

Proof. (i) (Sufficiency). We show here that $EGSV(\hat{w})$ (which we already know to satisfy Efficiency and Balanced Bipartite Dependency) satisfies Balanced Externalities.

Using Definition 1:

$$\begin{aligned}
\forall\pi &\in \Pi, \forall I, J, K \in \pi, \\
& EGSV(\hat{w})(I, \pi \sim JK) - EGSV(\hat{w})(I, \pi) \\
&= \sum_{\phi \subseteq (\pi \sim JK)} \tfrac{(|\phi|-1)!(|\pi|-|\phi|-1)!}{(|\pi|-1)!} \left(\hat{w}\left(\bigcup_{A \in \phi} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in (\phi \setminus \{I\})} A, \cdot\cdot \right) \right) \\
& - \sum_{\phi \subseteq \pi} \tfrac{(|\phi|-1)!(|\pi|-|\phi|)!}{|\pi|!} \left(\hat{w}\left(\bigcup_{A \in \phi} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in (\phi \setminus \{I\})} A, \cdot\cdot \right) \right) \\
&= \sum_{\phi \subseteq (\pi \setminus \{I,J,K\})} \left(\left(\tfrac{|\phi|!(|\pi|-|\phi|-2)!}{(|\pi|-1)!} - \tfrac{|\phi|!(|\pi|-|\phi|-1)!}{|\pi|!} \right) \left(\hat{w}\left(\bigcup_{A \in \phi \cup \{I\}} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in \phi} A, \cdot\cdot \right) \right) \right. \\
& - \tfrac{(|\phi|+1)!(|\pi|-|\phi|-2)!}{|\pi|!} \left(\hat{w}\left(\bigcup_{A \in \phi \cup \{I,J\}} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in \phi \cup \{J\}} A, \cdot\cdot \right) \right) \\
& - \tfrac{(|\phi|+1)!(|\pi|-|\phi|-2)!}{|\pi|!} \left(\hat{w}\left(\bigcup_{A \in \phi \cup \{I,K\}} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in \phi \cup \{K\}} A, \cdot\cdot \right) \right) \\
& \left. + \left(\tfrac{(|\phi|+1)!(|\pi|-|\phi|-3)!}{(|\pi|-1)!} - \tfrac{(|\phi|+2)!(|\pi|-|\phi|-3)!}{|\pi|!} \right) \left(\hat{w}\left(\bigcup_{A \in \phi \cup \{I,J,K\}} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in \phi \cup \{J,K\}} A, \cdot\cdot \right) \right) \right) \\
&= \sum_{\phi \subseteq (\pi \setminus \{I,J,K\})} \tfrac{(|\phi|+1)!(|\pi|-|\phi|-2)!}{|\pi|!} \left(\hat{w}\left(\bigcup_{A \in \phi \cup \{I,J,K\}} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in \phi \{J,K\}} A, \cdot\cdot \right) \right. \\
& - \hat{w}\left(\bigcup_{A \in \phi \{I,J\}} A, \cdot\cdot \right) + \hat{w}\left(\bigcup_{A \in \phi \{J\}} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in \phi \{I,K\}} A, \cdot\cdot \right) + \hat{w}\left(\bigcup_{A \in \phi \{K\}} A, \cdot\cdot \right) \\
& \left. + \hat{w}\left(\bigcup_{A \in \phi \{I\}} A, \cdot\cdot \right) - \hat{w}\left(\bigcup_{A \in \phi} A, \cdot\cdot \right) \right) \\
&= EGSV(\hat{w})(J, \pi \sim IK) - EGSV(\hat{w})(J, \pi).
\end{aligned}$$

(ii) (Necessity) We show that if $\chi(\hat{w})$ satisfies Efficiency and Balanced Externalities then $\chi(\hat{w})$ satisfies Consistency w.r.t. Bilateral Amalgamations.

If $\chi(\hat{w})$ satisfies Efficiency then

$$\begin{aligned}
\forall\pi &\in \Pi, \forall I, J \in \pi, \chi(\hat{w})(I \cup J, \pi \sim IJ) - \chi(\hat{w})(I, \pi) - \chi(\hat{w})(J, \pi) \\
&= -\sum_{K \in \pi \setminus \{I,J\}} \left(\chi(\hat{w})(K, \pi \sim IJ) - \chi(\hat{w})(K, \pi) \right). \qquad (1)
\end{aligned}$$

If $\chi(\hat{w})$ satisfies Balanced Externalities then

$$\forall K \in \pi \setminus \{I, J\}, \chi(\hat{w})(K, \pi \sim IJ) - \chi(\hat{w})(K, \pi) = \chi(\hat{w})(I, \pi \sim JK) - \chi(\hat{w})(I, \pi). \tag{2}$$

Combining (1) and (2):

$$\forall \pi \in \Pi, \forall I, J \in \pi,$$

$$\chi(\hat{w})(I \cup J, \pi \sim IJ) - \chi(\hat{w})(I, \pi) - \chi(\hat{w})(J, \pi) = -\sum_{K \in \pi \setminus \{I,J\}} (\chi(\hat{w})(I, \pi \sim JK) - \chi(\hat{w})(I, \pi))$$

$$\therefore \chi(\hat{w})(I \cup J, \pi \sim IJ) - \chi(\hat{w})(I, \pi) - \chi(\hat{w})(J, \pi) + \sum_{K \in \pi \setminus \{I,J\}} (\chi(\hat{w})(I, \pi \sim JK) - \chi(\hat{w})(I, \pi)) = 0. \tag{3}$$

Using (3):

$$\forall (I, \pi) \in M,$$

$$\sum_{J \in \pi \setminus \{I\}} (\chi(\hat{w})(I \cup J, \pi \sim IJ) - \chi(\hat{w})(I, \pi) - \chi(\hat{w})(J, \pi)) + \sum_{J,K \in \pi \setminus \{I\}} (\chi(\hat{w})(I, \pi \sim JK) - \chi(\hat{w})(I, \pi)) = 0$$

$$\begin{aligned}
\therefore \chi(\hat{w})(I, \pi) =& \chi(\hat{w})(I, \pi) \\
&+ \frac{1}{|\pi|(|\pi| - 1)} \left(\sum_{J \in \pi \setminus \{I\}} (\chi(\hat{w})(I \cup J, \pi \sim IJ) - \chi(\hat{w})(I, \pi) - \chi(\hat{w})(J, \pi)) \right. \\
&\left. + \sum_{J,K \in \pi \setminus \{I\}} (\chi(\hat{w})(I, \pi \sim JK) - \chi(\hat{w})(I, \pi)) \right) \\
&= \frac{1}{|\pi|(|\pi| - 1)} \left((|\pi| - 1)\chi(\hat{w})(I, \pi) + \sum_{J,K \in \pi \setminus \{I\}} \chi(\hat{w})(I, \pi \sim JK) \right. \\
&\left. + \sum_{J \in \pi \setminus \{I\}} (\chi(\hat{w})(I \cup J, \pi \tilde{} IJ) - \chi(\hat{w})(J, \pi)) \right).
\end{aligned} \tag{4}$$

Equation (4) is Consistency w.r.t. Bilateral Amalgamations. Lemma 1 now completes the necessity aspect of the proof. ■

Balanced Externalities implies Undominated Externalities, so by Lemma 2 if $\chi(\hat{w}) = EGSV(\hat{w})$ then $\chi(\hat{w})$ satisfies Efficiency, Balanced Bipartite Dependency and Undominated Externalities. To prove our Theorem, it remains to show the converse, which we do by showing that if $\chi(\hat{w})$ satisfies Efficiency, Balanced Bipartite Dependency and Undominated Externalities then $\chi(\hat{w})$ satisfies Balanced Externalities. Note first that Undominated Externalities immediately implies Balanced Externalities in a partition of fewer than four coalitions, so if $\chi(\hat{w})$ satisfies Efficiency, Balanced Bipartite Dependency and Undominated Externalities then, by Lemma 2, $\forall(I,\pi) \in M, |\pi| \leqslant 3 \rightarrow \chi(\hat{w})(I,\pi) = EGSV(\hat{w})(I,\pi))$. If $\chi(\hat{w})$ satisfies Efficiency, Balanced Bipartite Dependency and Undominated Externalities and if $(\forall(I,\pi) \in M, |\pi| \leqslant n \rightarrow \chi(\hat{w})(I,\pi) = EGSV(\hat{w})(I,\pi))$ then $\forall(I,\pi) \in M, \forall J,K,L \in \pi \setminus \{I\}, |\pi| = n+1, \chi(\hat{w})(I,\pi \sim JK) - \chi(\hat{w})(I,\pi) < \chi(\hat{w})(J,\pi \sim IK) - \chi(\hat{w})(J,\pi) \rightarrow EGSV(\hat{w})(I,\pi) - \chi(\hat{w})(I,\pi) < EGSV(\hat{w})(J,\pi) - \chi(\hat{w})(J,\pi)$ and $\forall(I,\pi) \in M, \forall J,K,L \in \pi \setminus \{I\}, |\pi| = n+1, EGSV(\hat{w})(I,\pi) - \chi(\hat{w})(I,\pi) < EGSV(\hat{w})(J,\pi) - \chi(\hat{w})(J,\pi) \rightarrow \chi(\hat{w})(I,\pi \sim JL) - \chi(\hat{w})(I,\pi) < \chi(\hat{w})(J,\pi \sim IL) - \chi(\hat{w})(J,\pi)$, so if $\chi(\hat{w})$ satisfies Efficiency, Balanced Bipartite Dependency and Undominated Externalities and if $(\forall(I,\pi) \in M, |\pi| \leqslant n \rightarrow \chi(\hat{w})(I,\pi) = EGSV(\hat{w})(I,\pi))$ then $\forall(I,\pi) \in M, \forall J,K,L \in \pi \setminus \{I\}, |\pi| = n+1, \chi(\hat{w})(I,\pi \sim JK) - \chi(\hat{w})(I,\pi) < \chi(\hat{w})(J,\pi \sim IK) - \chi(\hat{w})(J,\pi) \rightarrow \chi(\hat{w})(I,\pi \sim JL) - \chi(\hat{w})(I,\pi) < \chi(\hat{w})(J,\pi \sim IL) - \chi(\hat{w})(J,\pi)$. By this induction, if $\chi(\hat{w})$ satisfies Efficiency, Balanced Bipartite Dependency and Undominated Externalities then $\chi(\hat{w})$ satisfies Balanced Externalities. This completes the proof of our Theorem.

5 Discussion

Lemmas 1 and 2 in themsleves offer mathematically interesting characterizations of $EGSV(\hat{w})$. Lemma 1 closely recollects, though with a somewhat different formal framework, a characterization of the Shapley value due to Haviv [4]. In effect, Lemma 1 generalizes Haviv's [4] characterization to games in partition function form. Lemma 2 demonstrates a remarkable symmetry property of the Extended, Generalized Shapley Value. In any prior coalition structure that includes coalitions I, J and K, the externality to I if J amalgamates with K is the same as the externality to J if I amalgamates with K. However, though this is a strong form of 'balance', we could not view an absence of this property as an 'imbalance'. An inequality between these two externalities could, for example, be offset by an opposite inequality between the externalities generated by I or J amalgamating with some other coalition L. Only if I's externalities are *dominated* by J's, might we say that a value is unbalanced.

Efficiency and *balance* are more amenable to interpretation, in our view, than axioms such as *linearity* (which derive more directly Shapley's original axioms, [11]). *Efficiency* is a property of *ideal rationality*. *Balance* is a property of bargaining equilibrium So a value that is characterized by *efficiency* and *balance* can be viewed as the outcome that would be achieved through bargaining between the players of the game, if these players were ideally rational.

A particular feature of the characterizations that appear both in our Lemmas and in our Theorem is that the 'conditions' are only applied to the value of a single game, $\hat{w}$. There have by now been many Shapley-type values proposed for games in partition form (for example: Albizuri et al [1]; Clippel and Serrano [2]; Macho-Stadler et al [5]; Myerson [7]; Pham Do and Norde [10]), and it seems likely that no one of these is 'correct' in every sense. One must therefore decide whether the axioms that respective authors have imposed seem appropriate to the particular situation at hand. However, this is usually made difficult by the fact that characterizations are generally dependent - through axioms such as additivity or linearity, or between-game 'symmetry' axioms that are sometimes invoked - on the axioms being appropriate to the present game and also to many others. In effect, therefore, the 'particular situation at hand' is usually some broad class of games. An advantage of the characterizations in this note, is that the 'particular situation at hand' is a single game.

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