THE COST OF SEGREGATION IN SOCIAL NETWORKS

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Abstract. This paper investigates the private provision of public goods in segregated societies. While most research agrees that segregation undermines public goods provision the findings are mixed for private provision: social interactions, being strong within groups and limited across groups, may either increase or impede voluntary contributions. Surprisingly, very little light is shed in the literature on the impact of government intervention on private provision in networks. This paper, first, develops an index, called the Bonacich transfer index, for societies with general network structures of social interactions, which, roughly speaking, measures the impact of redistributive policies. Then, the paper shows that the Bonacich transfer index vanishes in large segregated societies, which suggests an "asymptotic neutrality" of government intervention.

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1. INTRODUCTION

Diversity is becoming a pervasive feature of most societies. Yet, in spite of the numerous gains from cultural differences within society, diversity often breeds segregation, which is detrimental to public goods provision. Segregation may occur along one or a few lines such as ethnicity, religion, language, and income, and its main aspect of limited social interactions across different groups is perceived to undermine the quality of public amenities and hamper public projects. There is robust empirical evidence in the literature; in fact, amongst others, to quote Banerjee, Iyer, and Rohini (2005) *"One of the most powerful hypotheses in political economy…"*.

The literature is furnished with a variety of mechanisms to explore the channels through which segregation operates on public goods provision. Alesina, Baqir, and Easterly (1999) argue that the divergence in preferences across groups for public goods -languages of instruction at school or the location of the highway- sharply dilutes the support for their provision. Fernández and Levy (2008) show that divergence in preferences may affect the choice of optimal funding policies for public goods. Ethnic fragmentation also results in less spending on education in Poterba (1997) and Goldin and Katz (1999) and reduces growth in Easterly and Levine (1999) and Alesina and La Ferrara (2005). Besley, Pande, Rahman, and Rao (2004) show that leaders provide public goods essentially to their ethnic groups, largely excluding others and Vigdor (2004) observes a low demand for public goods due to minimal altruistic preferences.

This paper seeks to understand how segregation impacts the private provision of public goods. In general, public goods are provided by both government and individuals. Private contributions account for the provision of many important public goods ranging from charitable education and health care to essential infrastructure. The access to private contributions, however, may be often constrained by geographical location or social interactions, benefiting neighbors and acquaintances, while effectively excluding others. A recent literature, by Bramoullé and Kranton (2007), Bramoullé, Kranton, and D'Amours (2011), and Allouch (2012), has investigated public goods games, where consumers may benefit only from neighbors provision, generalizing the standard model of private provision of pure public goods. In addition, Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010) incorporate private information, Galeotti and Goyal (2010) investigate issues of network formation, Bloch and Zenginobuz (2007) examine a standard local public good model with spillovers between jurisdictions, Rébillé and Richefort (2012) provide a welfare analysis, while Elliott and Golub (2012) explore decentralized mechanisms for efficient provision. However, unlike the case of public provision, the implication of segregation on private provision is unclear. On one hand, segregation may raise provision due to the strong feeling of solidarity within groups, but on the other hand it may decrease provision due to the weak social attachments across groups.

This paper focuses on the effects of government intervention on private provision in segregated societies. Social interactions in segregated societies, are represented by particular network structure, whereby the density of inward social ties for each group is greater than the density of outward social ties. Segregation can emerge from very different social processes and network formation dynamics. Schelling (1969) provides a simple, yet powerful, model showing that very mild individual preferences for having neighbors of the same type may lead to full segregation, even though no individual prefers the final outcome. The case of strong individual preferences is referred to as the homophily principle in sociology: the tendency of individuals to disproportionally form social ties with others similar to themselves. Homophily is a well-documented pattern of social networks and often called upon to understand various social interactions such as friendship, marriage, job market outcomes, and even social mobility. There is an emerging literature in the economics of social networks, by Currarini, Jackson, and Pin (2009), Bramoullé, Currarini, Jackson, Pin, and Rodgers (2012), and Golub and Jackson (2012) that models a random process of network formation strongly influenced by homophily. Our approach, although it is quite different, takes advantage of the insights of the above mentioned literatures, since we investigate fixed network structures of social interactions that already display segregation and not the matching processes nor the network formation dynamics leading to them.

We begin our analysis by exploring the effects of government intervention on private provision in societies with general network structures of social interactions. In our model, the channel of government intervention is lump-sum income redistribution, which plays a central role in economics for achieving various distributional objectives, and is often employed as a benchmark for other channels of intervention. The scale of income redistribution is crucial to our analysis since, similar to the standard private provision literature, we focus on budget-balanced transfers of relatively small magnitude so that the set of contributors remains unchanged. It is well known from Warr (1983) and Bergstrom, Blume, and Varian (1986) that the private provision model of pure public goods is subject to a strong neutrality result, whereby income redistribution has no effect on either the aggregate provision of public goods or on consumption of private goods. The neutrality result, further analyzed in Bernheim (1986) and Andreoni (1989), is equivalent to a complete crowding-out, "*dollar-for-dollar*" for tax-financed government provision, which has traditionally been the focus of much attention. Crowding-out effects can be a serious problem for public goods that rely mostly on private provision, and may limit the overall effectiveness of income redistribution.

In the case of local public goods, where not all consumers are necessarily linked to each other, it is unclear how much of the income redistribution affects consumers' welfare, or equivalently, how much of government intervention is negated by consumers' actions. We show that, under a standard utilitarian approach, the welfare effect of income redistribution is determined by the Bonacich centrality. Bonacich centrality, due to Bonacich (1987), is a vector that measures power and prestige in social networks and was shown to be related to the Nash equilibrium outcomes of a game by the key contribution of Ballester, Calvó-Armengol, and Zenou (2006). Quite different from the Nash-Bonacich linkage, Allouch (2012) shows that the effect of income redistribution on the aggregate provision is also determined by Bonacich centrality. Hence, the Bonacich centrality vector is key to the analysis of the impact of income redistribution on both aggregate welfare and aggregate provision.

In order to compare the impact of income redistribution across societies of different sizes and different network structures of social interactions, we develop a new network measure, called the Bonacich transfer index. The Bonacich transfer index measures the potential per-capita welfare gains after income redistribution. Geometrically, the Bonacich transfer index is simply the norm of the projection of the Bonacich centrality vector on the hyperplane of budget-balanced transfers normalized by the size of the network. Intuitively, the higher is the Bonacich transfer index, the more welfare gains may be achieved from income redistribution. For instance, when income redistribution is neutral the Bonacich transfer index is zero. Then, we apply the Bonacich transfer index to segregated societies that display particular network structures of social interactions. The computation shows that the Bonacich transfer index vanishes in large segregated societies, which implies an "asymptotic neutrality" of income redistribution. This result mirrors the widely known neutrality result in the case of pure public goods mentioned above. Hence, our approach shows that the network structure of social interactions in segregated societies may limit the effects of income redistribution policies and by the same token a wide range of other closely related policies.

The paper is organized as follows: Section 2 introduces the private provision of public goods on networks. Section 3 shows that various effects of income redistribution are related to the Bonacich centrality vector. Section 4 develops the Bonacich transfer index for societies with general network structure of social interactions. Section 5 computes the index for segregated societies. Section 6 provides an example meant to give an intuitive feel for our results and Section 7 concludes the paper.

2. The model

We consider a society comprising n consumers embedded on a fixed network $\mathbf{g}$ of social interactions. The society is divided into $T \geq 2$ groups of consumers of similar attributes which may involve, amongst other things, ethnicity, religion, language, and income. Let $\Pi = \{C_1, C_2, \ldots, C_T\}$ denote the partition of the set of consumers into T non-empty and pairwise disjoint groups. Le C_{t_i} denote consumer i's own group and $\mathcal{N}_i$ denote consumer i's neighbors. We do not assume that necessarily $\mathcal{N}_i \subset C_{t_i}$ and therefore consumers may have social ties with other consumers from any social group, however, for ease of exposition, we assume that the network $\mathbf{g}$ is connected.

The preferences of each consumer $i = 1, \ldots, n$, are represented by a twice continuously differentiable, strictly increasing, and strictly quasi-concave utility function $u_i(x_i, q_i + Q_{-i})$, where x_i is consumer i's private good consumption, q_i is consumer i's public good provision, and $Q_{-i} = \sum_{j \in \mathcal{N}_i} q_j$ is the sum of public good provisions of consumer i's neighbors in the society. Furthermore, the public good can be produced from the private good via a unit-linear production technology. Therefore, the price of private and public goods can be normalized to $p = (p_x, p_Q) = (1, 1)$. Notice that each consumer may benefit from the public good provision of all his neighbors regardless of their group identities. For each consumer i the utility maximization problem can be written

$$\max_{x_i, q_i} u_i(x_i, q_i + Q_{-i})$$
$$\text{s.t.} \quad x_i + q_i = w_i \text{ and } q_i \geq 0,$$

where w_i is his income (exogenously fixed). The utility maximization problem can be represented equivalently as

$$\max_{x_i, Q_i} u_i(x_i, Q_i) \tag{1}$$
$$\text{s.t.} \quad x_i + Q_i = w_i + Q_{-i} \text{ and } Q_i \geq Q_{-i},$$

where consumer i chooses his (local) public good consumption, $Q_i = q_i + Q_{-i}$. Let γ_i be the Engel curve of consumer i. Then consumer i's local public good demand is

$$Q_i = \max\{\gamma_i(w_i + Q_{-i}), Q_{-i}\},$$

or, equivalently

$$q_i = Q_i - Q_{-i} = \max\{\gamma_i(w_i + Q_{-i}) - Q_{-i}, 0\}. \tag{2}$$

We focus our analysis on a particular preferences.

Gorman polar form preferences The utility function u_i satisfies $\gamma_i'(\cdot) = 1 - a$, for $a \in [0, 1]$.

Although the assumption of Gorman polar form preferences is quite restrictive, it includes some interesting and important classes of preferences, for instance, both Cobb-Douglas preferences and quasi-linear preferences with respect to a common numeraire satisfy this assumption.

Let $G = [g_{ij}]$ denote the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that consumer $i \neq j$ are neighbors and $g_{ij} = 0$ otherwise. The adjacency matrix of the network, G, is symmetric with nonnegative entries and therefore has a complete set of real eigenvalues (not necessarily distinct), denoted by $\lambda_{\max}(G) = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = \lambda_{\min}(G)$, where $\lambda_{\max}(G)$ is the largest eigenvalue and $\lambda_{\min}(G)$ is the lowest eigenvalue of G. By the Perron–Frobenius Theorem, it holds that $\lambda_{\max}(G) \geq -\lambda_{\min}(G) > 0$.

Network normality $a \in]0, -\frac{1}{\lambda_{\min}(G)}[$.

The network normality assumption is equivalent to $\gamma_i'(\cdot) = 1 - a \in]1 - \frac{1}{\lambda_{\min}(G)}, 1[$, which amounts to both the normality of the private good and a strong normality of the public good.

Proposition 2.1. *Assume network normality. Then there exists a unique Nash equilibrium for the private provision.*

Proof. See Bramoullé, Kranton, and D'Amours (2011) and Allouch (2012).□

Bramoullé, Kranton, and D'Amours (2011) investigate the existence and uniqueness of Nash equilibrium in games with strategic substitutes with linear best reply functions on networks. More generally, their contribution shows that the lowest eigenvalues, $\lambda_{\min}(G)$, is key to equilibrium outcomes. Building on the contributions of Bramoullé, Kranton, and D'Amours (2011) and Bergstrom, Blume, and Varian (1986), Allouch (2012) introduces the assumption of network normality, which may also accommodate nonlinear best-reply functions, and establishes the existence and uniqueness of Nash equilibrium in the private provision of public goods on networks.

3. Government intervention and private provision

This section explores the effects of government intervention on private provision. The government aims to achieve socially optimal outcomes by drawing on income redistribution as a policy instrument. Income redistribution takes the form of lump-sum transfers, which are traditionally viewed as as a reference point for other policy instruments. Government intervention in private provision is very much in the spirit of the second welfare theorem although, unlike competitive equilibrium, the Nash equilibrium outcomes will typically be inefficient . We denote by a budget-balanced transfer, a $t = (t_1, t_2, \ldots, t_n)^T \in \mathbb{R}^n$ such that $\sum_{i=1}^n t_i = 0$. Let $(q_1^*, \ldots, q_n^*)$ denote the Nash equilibrium corresponding to the income distribution $w = (w_1, \ldots, w_n)$ and $(q_1^t, \ldots, q_n^t)$ denote the Nash equilibrium corresponding to the income distribution $w + t = (w_1 + t_1, \ldots, w_n + t_n)$. Similar to Warr (1983) and Bergstrom, Blume, and Varian (1986), we will focus our analysis on income redistributions that leaves the set of contributors unchanged, and we will refer to them as "relatively small". In general, there are compelling reasons for presuming that not all consumers will be contributing to the public good. For simplicity, passing to subnetworks if necessary, we assume that all consumers are contributors. Notice that an income distribution almost proportional to the eigenvector centrality, the unique unit eigenvector associated with $\lambda_{\max}$, will always lead to an interior Nash equilibrium.

3.1. **Bonacich centrality.** The key contribution of Ballester, Calvó-Armengol, and Zenou (2006) was first to relate the Nash equilibrium outcomes of a game to the Bonacich centrality vector, due to Bonacich (1987), defined by

$$\mathbf{b}(G, a) = (I - aG)^{-1}\mathbf{1},$$

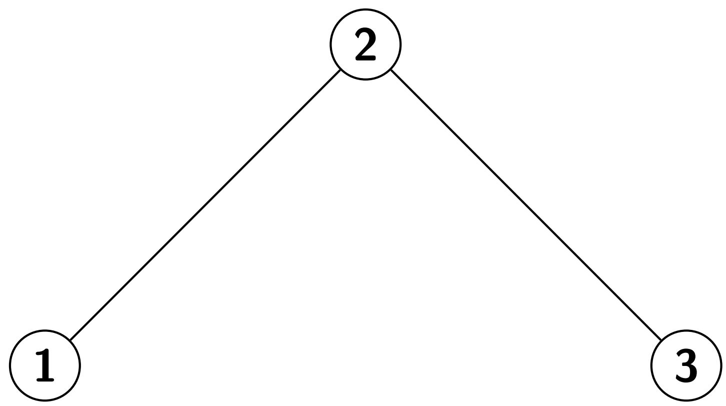
where $\mathbf{1}$ is the n-dimensional vector with all components equal one and a is the attenuation parameter. The Bonacich centrality describes the potential importance, influence, prominence of a consumer in the network. Since for $a < \frac{1}{\lambda_{\max}(G)}$ it holds that

$$\mathbf{b}(G, a) = (I - aG)^{-1}\mathbf{1} = \sum_{k=0}^{+\infty} a^k \mathbf{1}^T G^k,$$

and $(G^k)_{ij}$ counts the total number of walks of length k from i to j. The Bonacich centrality of a consumer i can be interpreted as follows:

$$\mathbf{b}_i(G,a) = \sum_{k=0}^{+\infty} a^k \sum_{j=1}^{n} (G^k)_{ij}.$$

Intuitively, the Bonacich centrality of a consumer i counts the number of walks emanating from i discounted by a to the power of their length. Hence, the attenuation parameter a captures the decay of influence of distant consumers on a particular consumer Bonacich centrality.



$$\mathbf{b}(G,-a) = \begin{pmatrix} \mathbf{b}_1(G,-a) \\ \mathbf{b}_2(G,-a) \\ \mathbf{b}_3(G,-a) \end{pmatrix} = \frac{1}{1-2a^2} \begin{pmatrix} 1-a \\ 1-2a \\ 1-a \end{pmatrix}.$$

Figure 1. Bonacich centrality vector

Quite different from the Nash-Bonacich linkage, Allouch (2012) shows that the effect of income redistribution on the aggregate provision of public goods, $Q = \sum_{i=1}^{n} q_i$ may be related to a generalization of Bonacich centrality. The following proposition is specific to the case of preferences of the Gorman polar form.

Proposition 3.1. *Assume network normality. Then for any relatively small transfer t it holds that*

$$q^t - q^* = (1-a)(I + aG)^{-1}t. \tag{3}$$

Hence, it follows that

$$Q^t - Q^* = (1-a)\ \boldsymbol{b}(G,-a)\cdot t.$$

Proof. First, it follows from (2) that for each consumer $i = 1, 2, \ldots, n$,

$$\begin{aligned} q_i^t - q_i^* &= ((1-a)(w_i + t_i + Q_{-i}^t) - Q_{-i}^t) - ((1-a)(w_i + Q_{-i}^*) - Q_{-i}^*). \\ &= (1-a)t_i - a(Q_{-i}^t - Q_{-i}^*). \end{aligned}$$

Rearranging terms, it follows that $(I + aG)(q^t - q^*) = (1-a)t$, and therefore $q^t - q^* = (1-a)(I+aG)^{-1}t$. Hence, it holds that

$$Q^t - Q^* = (1-a)\ \mathbf{b}(G,-a)\cdot t.\square$$

The assumption of network normality, needed for the uniqueness of a Nash equilibrium, may be relaxed and Proposition 3.1 still holds partially. Indeed, for almost any $a \in [0,1]$ the matrix $I + aG$ is invertible. Therefore, similar to Theorem 1 in Bergstrom, Blume, and Varian (1986), for almost any Nash equilibrium q^* corresponding to the income distribution $w = (w_1, \ldots, w_n)$ and any transfer of relatively small magnitude t, it may be easily checked that $q^* + (1-a)(I+aG)^{-1}t$ is a Nash equilibrium corresponding to the income distribution $w + t = (w_1 + t_1, \ldots, w_n + t_n)$.

3.2. **Welfare effect of income redistribution.** We take a standard utilitarian approach in order to understand the welfare effect of income redistribution. Specifically, we consider the (indirect) social welfare function

$$\mathcal{SW}(w) \stackrel{\text{def}}{=} \sum_{i=1}^{n} u_i(x_i^*, Q_i^*),$$

which is the sum of utilities achieved by consumers at the unique Nash equilibrium with income distribution $w = (w_1, \ldots, w_n)$.

Proposition 3.2. *There exists a positive real number κ such that for any relatively small transfer t it holds that*

$$\mathcal{SW}(w+t) - \mathcal{SW}(w) = -\kappa\ \boldsymbol{b}(G,-a)\cdot t.$$

Proof. When preferences of consumers are of the Gorman polar form, the indirect utility function can be written as

$$v_i(p, w_i) = \alpha(p)w_i + \beta_i(p), \quad \text{for each } i.$$

where $\alpha(p) > 0$ is common to all consumers. From the utility maximization in (1) it follows that at the unique Nash equilibrium each consumer maximizes his utility with respect to the price $p = (p_x, p_Q) = (1, 1)$ and social wealth $w_i + Q^*_{-i}$. Therefore it holds that

$$u_i(x^*_i, q^*_i + Q^*_{-i}) = v_i(p, w_i + Q^*_{-i}) = \alpha(p)(w_i + Q^*_{-i}) + \beta_i(p). \tag{4}$$

Hence, it follows from (3) and (4) that

$$\begin{aligned}
\mathcal{SW}(w+t) - \mathcal{SW}(w) &= \sum_{i=1}^{n}[u_i(x^t_i, q^t_i + Q^t_{-i}) - u_i(x^*_i, q^*_i + Q^*_{-i})] \\
&= \sum_{i=1}^{n}[\alpha(p)(w + t_i + Q^t_{-i}) - \alpha(p)(w + Q^*_{-i})] \\
&= \alpha(p)\sum_{i=1}^{n}[Q^t_{-i} - Q^*_{-i}] = \alpha(p)\sum_{i=1}^{n} G(q^t - q^*) \\
&= \alpha(p)\mathbf{1}^T G(q^t - q^*) = \alpha(p)(1-a)\mathbf{1}^T G(1 + aG)^{-1}t \\
&= \alpha(p)(1-a)\mathbf{1}^T(-\frac{1}{a}I + \frac{1}{a}(I + aG))(1 + aG)^{-1}t \\
&= -\frac{\alpha(p)(1-a)}{a}\mathbf{1}^T((1 + aG)^{-1} + I)t \\
&= -\frac{\alpha(p)(1-a)}{a}\mathbf{1}^T(1 + aG)^{-1}t.
\end{aligned}$$

Therefore, if one sets $\kappa = \frac{\alpha(p)(1-a)}{a} > 0$ the desired result follows, that is,

$$\mathcal{SW}(w+t) - \mathcal{SW}(w) = -\kappa\ \mathbf{b}(G, -a) \cdot t.\square$$

Proposition 3.2 stipulates that an income redistribution carried out from high Bonacich centrality to low Bonacich centrality consumers leads to an aggregate welfare increase.

Remark 1. One of the most deeply ingrained ideas when thinking about public goods is that they are always underprovided by a system of private provision. Surprisingly, Proposition 3.1 shows that the effect of income redistribution on the aggregate provision of public goods is also determined by the Bonacich centrality vector; however, by pulling the income redistribution in the opposite direction of the ones raising aggregate welfare, that is, an income redistribution carried out from low Bonacich centrality to high Bonacich centrality consumers increases the aggregate provision.[1] Therefore, one may conclude that when public goods are provided solely by voluntary contribution, raising aggregate welfare and increasing aggregate provision are sharply conflicting policy objectives.

4. Bonacich transfer index

Our investigation shows that Bonacich centrality vector is key to understand various effects of income redistribution. In the following, we would like to compare the effects of income redistribution across societies of different sizes and different network structures of social interactions. To be able to do so, we develop a new network measure, called the Bonacich transfer index, defined by

$$\mathbf{b}^{\mathcal{TI}}(G,-a) \overset{\text{def}}{=} \frac{1}{n}\|\text{proj}_{\mathbf{1}^\perp}\mathbf{b}(G,-a)\|.$$

The Bonacich transfer index is the norm of the projection of the Bonacich centrality vector on the hyperplane of budget-balanced transfers $\mathbf{1}^\perp$, normalized by the size of the society n.

Proposition 4.1. *Let $\mathcal{B}_{\mathcal{T}} = \{t \in \mathbf{1}^\perp \mid \|t\| \leq 1\}$ denote the unit ball in the hyperplane of budget-balanced transfers $\mathbf{1}^\perp$. Then itt holds that*

$$\max_{t\in\mathcal{B}_{\mathcal{T}}} \frac{\mathcal{SW}(w+t)-\mathcal{SW}(w)}{n} = \frac{\kappa}{n}\,\mathbf{b}^{\mathcal{TI}}(G,-a)$$

[1]For instance, in Figure 1 again, starting from a private provision equilibrium where consumers 1, 2, and 3 are contributors, an income redistribution $t = (t_1, t_2, t_3)$ with $t_2 < 0$ will increase aggregate provision of public goods, while the opposite income redistribution $-t$ will raise aggregate welfare

Proof. From Proposition 3.2 it holds that $\mathcal{SW}(w+t) - \mathcal{SW}(w) = -\kappa\ \mathbf{b}(G,-a)\cdot t$. Therefore, the maximum of $\mathcal{SW}(w+t) - \mathcal{SW}(w)$ for $t \in \mathcal{B}_{\mathcal{T}}$ occurs at

$$t^{\mathcal{SW}} = -\frac{\text{proj}_{\mathbf{1}^{\perp}}\mathbf{b}(G,-a)}{\|\text{proj}_{\mathbf{1}^{\perp}}\mathbf{b}(G,-a)\|}.$$

Hence

$$\max_{t\in\mathcal{B}_{\mathcal{T}}} \frac{\mathcal{SW}(w+t) - \mathcal{SW}(w)}{n} = -\frac{\kappa}{n}\ \mathbf{b}(G,-a)\cdot\left(-\frac{\text{proj}_{\mathbf{1}^{\perp}}\mathbf{b}(G,-a)}{\|\text{proj}_{\mathbf{1}^{\perp}}\mathbf{b}(G,-a)\|}\right) = \frac{\kappa}{n}\ \mathbf{b}^{\mathcal{TI}}(G,-a).\square$$

Proposition 4.1 relates the the Bonacich transfer index to the potential per-capita welfare gain after income redistribution. Hence, the Bonacich transfer index corresponds to an average utilitarian approach to welfare, which in turn is adequate to deal with population change. It is worth noticing that when the the size of the society is fixed, the average utilitarian approach is identical in its policy recommendations to the standard utilitarian approach.

Remark 2. The optimal income redistributions for raising per-capita welfare and raising per-capita public good provision, for $t \in \mathcal{B}_{\mathcal{T}}$, are diametrically opposite to each other. Nonetheless, their impacts may be simultaneously related to the Bonacich transfer index since it may be also shown that

$$\max_{t\in\mathcal{B}_{\mathcal{T}}} \frac{Q^t - Q^*}{n} = (1-a)\ \mathbf{b}^{\mathcal{TI}}(G,-a).$$

This suggests that the Bonacich transfer index is a useful benchmark for various effects of income redistribution.

Now, we establish that the Bonacich transfer index may be expressed from a selection of the spectrum of the network. A similar result was established by Allouch (2012) for the Bonacich centrality vector. The intuition is as follows: Bonacich centrality vector is closely related to the number of walks in the network, which in turn may be computed directly from a selection of the spectrum of the network. An eigenvalue μ of G, which has an associated eigenvector not orthogonal to the vector $\mathbf{1}$, is said to be a main eigenvalue (Cvetković (1970)). By the Perron-Frobenius theorem, the maximum eigenvalue of G has an associated eigenvector with all its entries positive and, therefore, is a main eigenvalue. The distinct main eigenvalues $\mu_1, \mu_2, \ldots, \mu_s$ ($\mu_1 > \mu_2 > \ldots > \mu_s$), of G form the main part of the spectrum, denoted by $\mathcal{M}$ (Harary and Schwenk (1979)). The cosine of the angle between the eigenspace of μ_i, $\mathcal{E}_G(\mu_i)$, and the vector $\mathbf{1}$, denoted by β_i, is called a main angle of G. Obviously, μ_i is a main eigenvalue if and only if $\beta_i \neq 0$. Moreover, it holds that $\sum_{i=1}^{s} {\beta_i}^2 = 1$.

The following proposition shows that the Bonacich transfer index may be expressed from the main part of the spectrum $\mathcal{M}$.

Proposition 4.2.

$$\mathbf{b}^{\mathcal{TI}}(G,-a) = \sqrt{\frac{1}{n}(\sum_{i=1}^{s} \frac{{\beta_i}^2}{(1+a\mu_i)^2} - (\sum_{i=1}^{s} \frac{{\beta_i}^2}{1+a\mu_i})^2)}$$

Proof. Let u_i be the unit eigenvector of the main eigenvalue μ_i orthogonal to $\mathcal{E}_G(\mu_i) \cap \mathbf{1}^{\perp}$. The eigenvector u_i is determined uniquely since we choose $\beta_i = u_i \cdot \frac{\mathbf{1}}{\sqrt{n}}$ to be the cosine of the acute angle between $\mathcal{E}_G(\mu_i)$ and $\mathbf{1}$. Let V be a matrix whose columns, $v_1, v_2, \ldots, v_n$, are eigenvectors of G chosen to extend the eigenvectors $\{u_1, u_2, \ldots, u_s\}$ of G to an orthonormal basis of $\mathbb{R}^n$. Therefore, $G = VDV^T$, where $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Therefore, it holds that

$$\begin{aligned}\mathbf{b}(G,-a) &= (I+aG)^{-1}\mathbf{1} = V(I+aD)^{-1}V^T\mathbf{1} \\ &= \sum_{i=1}^{n} \frac{\mathbf{1}\cdot v_i}{1+a\lambda_i} v_i = \sum_{i=1}^{s} \frac{\mathbf{1}\cdot u_i}{1+a\mu_i} u_i \\ &= \sqrt{n} \sum_{i=1}^{s} \frac{\beta_i}{1+a\mu_i} u_i.\end{aligned}$$

From the Pythagorean theorem, it holds that

$$\|\mathbf{b}(G,-a)\|^2 = \|\mathrm{proj}_{\mathbf{1}^{\perp}}\mathbf{b}(G,-a)\|^2 + \|\mathrm{proj}_{\mathbf{1}}\mathbf{b}(G,-a)\|^2.$$

Hence, it follows that

$$\begin{aligned}
\mathbf{b}^{\mathcal{TI}}(G,-a)^2 &= \frac{1}{n^2}\|\mathrm{proj}_{\mathbf{1}^\perp}\mathbf{b}(G,-a)\|^2 = \frac{1}{n^2}(\|\mathbf{b}(G,-a)\|^2 - \|\mathrm{proj}_{\mathbf{1}}\mathbf{b}(G,-a))\|^2) \\
&= \frac{1}{n^2}(n\|\sum_{i=1}^{s}\frac{\beta_i}{1+a\mu_i}u_i\|^2 - n\|((\sum_{i=1}^{s}\frac{\beta_i}{1+a\mu_i}u_i)\cdot\frac{\mathbf{1}}{\sqrt{n}})\frac{\mathbf{1}}{\sqrt{n}}\|^2) \\
&= \frac{1}{n}((\sum_{i=1}^{s}\frac{\beta_i}{1+a\mu_i}u_i)\cdot(\sum_{i=1}^{s}\frac{\beta_i}{1+a\mu_i}u_i) - (\sum_{i=1}^{s}\frac{{\beta_i}^2}{1+a\mu_i})^2\|\frac{\mathbf{1}}{\sqrt{n}}\|^2) \\
&= \frac{1}{n}(\sum_{i=1}^{s}\frac{{\beta_i}^2}{(1+a\mu_i)^2} - (\sum_{i=1}^{s}\frac{{\beta_i}^2}{1+a\mu_i})^2).
\end{aligned}$$

Therefore, it holds that

$$\mathbf{b}^{\mathcal{TI}}(G,-a) = \sqrt{\frac{1}{n}(\sum_{i=1}^{s}\frac{{\beta_i}^2}{(1+a\mu_i)^2} - (\sum_{i=1}^{s}\frac{{\beta_i}^2}{1+a\mu_i})^2)}.\square$$

The Bonacich transfer index has a natural geometric interpretation, since it is related to the gap in Jensen's inequality[2] for the convex function $\phi(x) = x^2$, applied to the convex combination of $\frac{1}{1+a\mu_1}, \frac{1}{1+a\mu_2}, \ldots, \frac{1}{1+a\mu_s}$ with weights ${\beta_1}^2, {\beta_2}^2, \ldots, {\beta_s}^2$.

[2]Jensen's inequality is a central inequality in the study of convex functions.

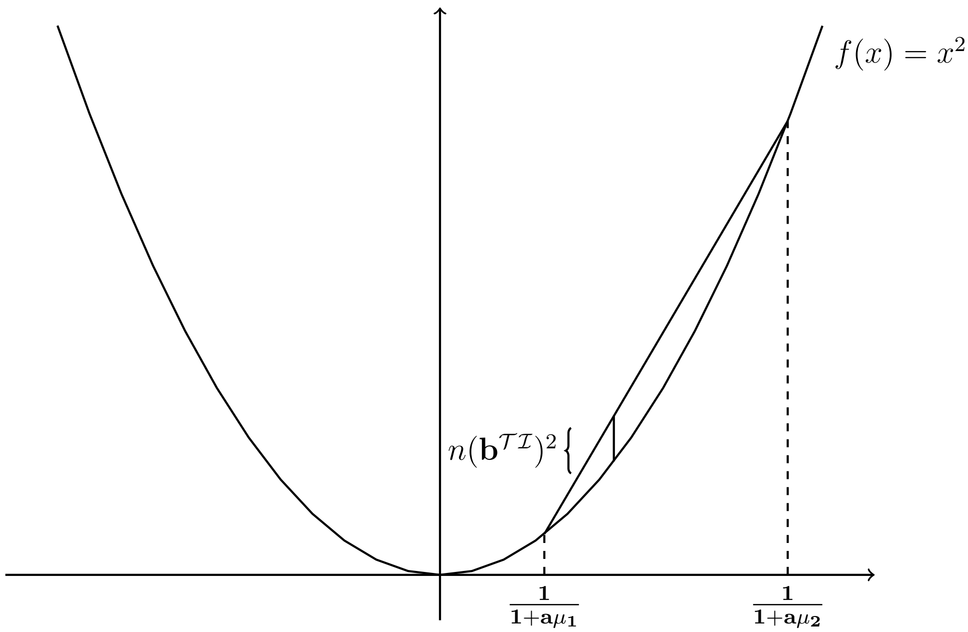


Figure 2. Bonacich transfer index with two main eigenvalues.

Corollary 4.3. $\mathbf{b}^{\mathcal{TI}}(G, -a) = 0$ *if and only if the network is regular.*

Proof. From the above Jensen's gap interpretation of the Bonacich transfer index it follows that $\mathbf{b}^{\mathcal{TI}}(G, a) = 0$ if and only if $s = 1.$, which is equivalent to ${\beta_1}^2 = 1$. By definition of main angles ${\beta_1}^2 = 1$ is equivalent to $\frac{\mathbf{1}}{\sqrt{n}}$ is an eigenvalue of G, which holds if and only if $\mathbf{g}$ is a regular network.□

Corollary 4.3 shows that a zero Bonacich transfer index characterizes regular networks. Hence, in regular networks a relatively small budget-balanced transfer will cause no change to the per-capita welfare.[3]

Corollary 4.4. *If* $\lambda_{\min}(G) = \mu_s$, *then*

$$\lim_{a \uparrow \frac{-1}{\lambda_{\min}(G)}} \mathbf{b}^{\mathcal{TI}}(G, -a) = +\infty.$$

[3]Actually, this is the only instance where the two policy objectives of raising aggregate provision and raising welfare coincide since they are both redundant (see also Allouch (2012)).

Proof. First, observe that if $\lambda_{\min}(G) = \mu_s$ then it follows that $\mu_s < 0$, which implies $s \geq 2$. Moreover,

$$\begin{aligned}\mathbf{b}^{\mathcal{T}\mathcal{I}}(G,-a) &= \sqrt{\frac{1}{n}(\sum_{i=1}^{s}\frac{{\beta_i}^2}{(1+a\mu_i)^2} - (\sum_{i=1}^{s}\frac{{\beta_i}^2}{1+a\mu_i})^2)} \\ &= \frac{1}{1+a\mu_s}\sqrt{\frac{1}{n}(\sum_{i=1}^{s-1}{\beta_i}^2(\frac{1+a\mu_s}{1+a\mu_i})^2 + {\beta_s}^2 - (\sum_{i=1}^{s-1}{\beta_i}^2\frac{1+a\mu_s}{1+a\mu_i} + {\beta_s}^2)^2)}.\end{aligned}$$

Obviously, since $s \geq 2$ implies that $\beta_s < 1$ it holds that

$$\lim_{a\uparrow -\frac{1}{\mu_s}}\sqrt{\frac{1}{n}(\sum_{i=1}^{s-1}{\beta_i}^2(\frac{1+a\mu_s}{1+a\mu_i})^2 + {\beta_s}^2 - (\sum_{i=1}^{s-1}{\beta_i}^2\frac{1+a\mu_s}{1+a\mu_i} + {\beta_s}^2)^2)} = \sqrt{\frac{{\beta_s}^2 - {\beta_s}^4}{n}} > 0.$$

Hence it follows that

$$\lim_{a\uparrow\frac{-1}{\lambda_{\min}(G)}}\mathbf{b}^{\mathcal{T}\mathcal{I}}(G,-a) = \lim_{a\uparrow -\frac{1}{\mu_s}}\frac{1}{1+a\mu_s}\sqrt{\frac{{\beta_s}^2 - {\beta_s}^4}{n}} = +\infty.\square$$

Corollary 4.4 shows that the Bonacich transfer index may be also unbounded, which together with Corollary 4.3 implies that the Bonacich transfer index may take a wide range of values.

5. Segregated society

This section computes the Bonacich transfer index for segregated societies with particular structure of social interactions. We introduce the following assumption about the network of social interactions of the society:

Segregated society:

(i) For each consumer i,

$$|\mathcal{N}_i \cap C_{t_i}| > \sum_{t\neq t_i}|\mathcal{N}_i \cap C_t|,$$

(ii) If consumers i, j belong to the same group, that is $C_{t_i} = C_{t_j}$, then

$$|\mathcal{N}_i \cap C_t| = |\mathcal{N}_j \cap C_t|, \text{ for each } t = 1, 2, \dots, T.$$

The segregated society assumption is about the density of links between the different groups of the society. Condition (i) stipulates that the number of links each consumer has to consumers from his own group exceeds the number of links he has to consumers from different groups. Condition (ii) is merely a network regularity requirement. It stipulates that the number of links a consumer in group C_r has to consumers in group C_l is independent of the choice of the consumer in C_r.

Theorem 5.1. *If the society is segregated then*

$$\mathbf{b}^{\mathcal{TI}}(G, -a) \leq \frac{1}{2\sqrt{n}}.$$

Proof. Condition (ii) of segregated society implies that $\pi = \{C_1, C_2, \dots, C_T\}$ defines an equitable partition of the set of consumers (see Powers and Sulaiman (1982)) An equitable partition gives rise to a quotient graph g/π characterized by the adjacency matrix $G/\pi = [d_{hl}]_{1 \leq h,l \leq T}$, where d_{hl} denotes the number of links from a consumer in group C_h to consumers in group C_l.[4] Notice that the adjacency matrix, G/π, is not necessarily symmetric, since in general $d_{hl} \neq d_{lh}$. The quotient graph plays an important role in the study of the main part of spectrum $\mathcal{M}$ since it holds that

$$\mathcal{M} \subset \operatorname{spec}(G/\pi) \subset \operatorname{spec}(G). \tag{5}$$

Observe that all the eigenvalues of G are real and so that the eigenvalues of G/π are real. Moreover, Condition (i) of segregated society implies that G/π is diagonally dominant. From Gerosgorin Circle Theorem it follows that all eigenvalues of G/π are positive (see Varga (2004), Theorem 1.1). From (5) one obtains $\mu_i \geq 0$, for each $\mu_i \in \mathcal{M}$. This implies that $0 \leq \frac{1}{1+a\mu_i} \leq 1$, for each $\mu_i \in \mathcal{M}$. Since the Jensen's gap

[4]Equitable partitions are referred to as colorations while the quotient graph g/π is also known as divisor.

is less than $\frac{1}{4}(= \max_{x\in[0,1]}\{x - x^2\})$ it holds that

$$\mathbf{b}^{\mathcal{TI}}(G, -a) = \sqrt{\frac{1}{n}(\sum_{i=1}^{s} \frac{{\beta_i}^2}{(1 + a\mu_i)^2} - (\sum_{i=1}^{s} \frac{\beta_i^2}{1 + a\mu_i})^2)} \leq \frac{1}{2\sqrt{n}}.\square$$

Theorem 5.1 shows that the Bonacich transfer index vanishes in large segregated society, which suggests an asymptotic neutrality of income redistribution.

6. Examples

Consider a society comprising two groups of consumers C_1 and C_2 of sizes, respectively, n_1 and n_2 such that $n_1 = 4n_2$. The society has a particular network structure of social interactions G defined as follows: for each consumer in C_1 the number of links to consumers from C_1 is d and the number of links to consumers from C_2 is r. For each consumer in C_2 the number of links to consumers from C_2 is d and the number of links to consumers from C_1 is (obviously) $4r$. We assume that G is connected, that is, $r > 0$. Let us consider the adjacency matrix of the quotient graph g/π corresponding to the partition $\pi = \{C_1, C_2\}$:

$$G/\pi = \begin{pmatrix} d & r \\ 4r & d \end{pmatrix}.$$

Then, G has exactly two main eigenvalues and they are

$$\mu_1 = d + 2r \quad \text{and} \quad \mu_2 = d - 2r$$

Integrated society: $d < r$. For each consumer the number of links to consumers from his own group is smaller than the number of links to consumers from the other group. Then, it holds that

$$\mu_1 = d + 2r \geq 0 \quad \text{and} \quad \mu_2 = d - 2r \leq 0.$$

As a consequence, for $a \geq 0$ it follows that

$$\frac{1}{1+a\mu_1} \leq 1 \text{ and } \frac{1}{1+a\mu_1} \geq 1. \qquad \text{(Figure 3)}$$

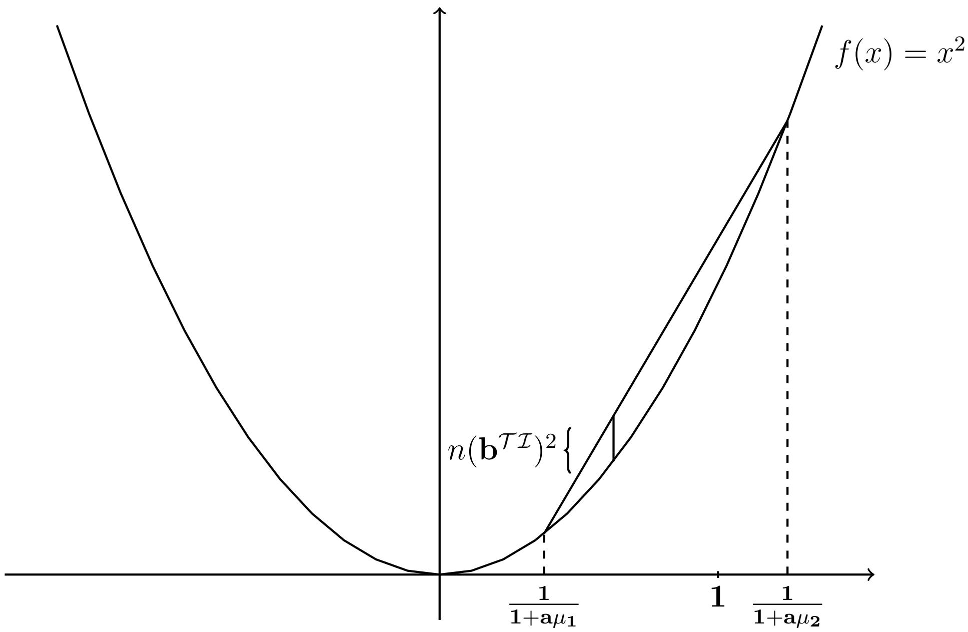


Figure 3. Bonacich transfer index: *(integrated society)*

Observe that if $d = 0$ then we have an extremely integrated society it holds that

$$\mu_2 = -\mu_1 = -2r = \lambda_{\min}.$$

From Corollary 4.4 it follows that

$$\lim_{a \uparrow -\frac{1}{\mu_2}} \mathbf{b}^{\mathcal{TI}}(G, -a) = +\infty.$$

Segregated society: $d > 4r$.

For each consumer the number of links to consumers from his own group is smaller than the number of links to consumers from the other group. Then, it holds that

$$\mu_1 = d + 2r \geq 0 \text{ and } \mu_2 = d - 2r \geq 0.$$

Hence, for $a \geq 0$ it holds that

$$\frac{1}{1+a\mu_1} \leq 1 \quad \text{and} \quad \frac{1}{1+a\mu_1} \leq 1. \qquad \text{(Figure 4)}$$

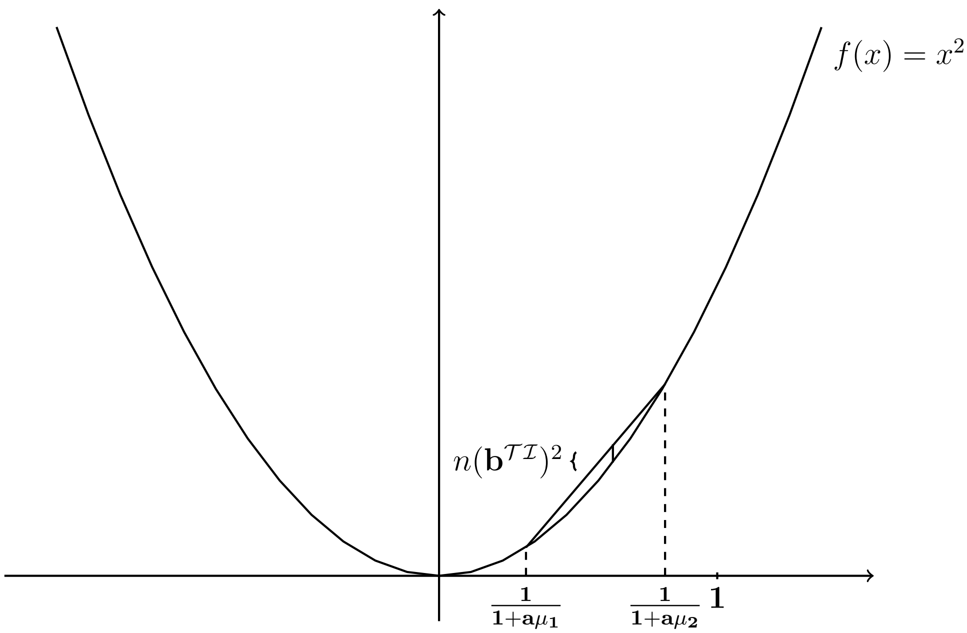


Figure 4. Bonacich transfer index: *(segregated society)*

Observe that if $r = 0$ then we have the limit of an extremely segregated society. Obviously, the network of social interactions is no longer connected and has now two unconnected groups. From Corollary 4.3 one may deduce that the Bonacich transfer index of each component is zero.

7. CONCLUSION

Enhancing private provision of public goods has long been an important policy objective and our paper shows that understanding social networks is a key way to achieve this. To this effect, the Bonacich centrality index, developed in this paper, may be thought of as an instrument to capture the income redistribution multiplier effect within each society. The computation of the Bonacich centrality index shows that redistributive policies may have a normative significance in integrated societies and are ineffective in segregated societies. Surprisingly, this result is obtained only from the underlying network structure of social interactions since preferences of consumers care about their neighbors only insofar as they affect public good provision and not their group identities. Hence, one straightforward implication of our result suggests that the optimal policy to increase the income redistribution multiplier effect within a society by reshuffling the network structure of social interactions stipulates sponsoring bridges between groups while reducing links density within groups.

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