Loss calibrated rationing methods

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Abstract

The *standard* problem of rationing a single overdemanded commodity ([11], [2], [16]) has a natural *bipartite* extension with multiple types of a one-dimensional commodity (e.g., jobs with different skill requirements), and each agent can only consume some types of the resource (e.g., workers have different skills).

We define the new standard *loss calibrated* rationing methods, that equalize across agents the ratio of shares to (calibrated) losses (demand minus share). We extend them to bipartite methods that 1) are not affected by the elimination of an edge and the corresponding flow (Consistency), and 2) treat resource types with identical connectivity as a single type. They are essentially the only standard methods with a bipartite extension meeting the two properties above. Most of the parametric methods discussed in the literature ([16], [17]) do not admit such extension.

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1 Introduction

In our earlier paper [10] we consider the problem of selecting a fair max-flow in a rich class of flow problems on a bipartite graph. There are finitely many sources and sinks, each with a finite capacity to send or receive some homogenous commodity (the *resource*), and the edges can carry arbitrarily large flows. To each source is attached an agent i, and we interpret the capacity of this source as the amount of resource i can consume, his *demand*; each sink represents a different *type of resource*, and the capacity of sink a is the available amount of type a resource; the exogenous bipartite graph G specifies which agent can consume which type of resource (the connectivity constraints).

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Assuming that the capacity of the sinks is overdemanded[1], a max-flow fills the sinks to capacity, but it must ration the sources (ship less than their capacity). We look for a fair rationing method.

The special case with a single type of resource and a set of agents N (a single sink and multiple sources) corresponds to the much studied problem of rationing fairly an amount t of some commodity according to individual demands or liabilities $x_i, i \in N$, when total demands exceed the resources, $\sum_N x_i > t$. This model is also known as the bankruptcy or estate division problem ([11], [2], [16]). The literature develops a rich axiomatic discussion of a variety of rationing methods, from the simple and familiar Proportional method to exotic ones like the Talmudic method: see the surveys [9], [14].

Here we speak of the model with a single resource-type as the **standard** rationing model, to distinguish it from the more general **bipartite** model with a finite set Q of resource-types, and an arbitrary bipartite graph G of connectivity constraints. We ask to what extent the theory of standard rationing methods generalizes to the bipartite context. We provide a fairly complete answer for the family of (standard) *parametric* methods, that plays a central role in the normative discussion of rationing methods. First, with the exception of the three benchmark methods Proportional, Uniform Gains, and Uniform Losses, we find that most parametric methods discussed in the literature cannot be consistently extended to the bipartite context (once "consistently" is given a precise meaning). On the other hand we discover a new family of reasonable parametric methods, called *loss calibrated*, that can.

We recall first some basic facts about standard parametric methods.

2 Standard parametric methods

We use the notation $x_S = \sum_{i \in S} x_i$. For any profile of demands $x = (x_i)_{i \in N}$ and available resource t such that $0 \leq t \leq x_N$, we speak of the (standard) rationing problem (N, x, t). A standard rationing method h selects a profile of shares $y = h(N, x, t) = (y_i)_{i \in N}$ such that $0 \leq y \leq x$ and $y_N = t$.

The *Consistency* property is the defining property of standard parametric methods. Fix a method h, a problem (N, x, t), and $y = h(N, x, t)$. Then Consistency requires that for any subset S of N, we have $(y_i)_{i \in S} = h(S, (x_i)_{i \in S}, y_S)$. Although Consistency is not a test of fairness[2], it is a powerful rationality statement about the way a rationing method deals with related problems. In the words of Balinski and Young ([3]): "every part of a fair division must be fair".

[1] This means that the total capacity of any subset of sinks is not larger than that of all the sources connected to these sinks. In general the graph can be decomposed in two disjoint subgraphs, one where sinks are overdemanded as above, the other where sources are overdemanded: see [10].

[2] For instance the following *priority method* is consistent. Fix a priority ordering of N, and to divide t units among a subset S of agents, try first to meet the demand of the highest priority agent i in S; if any resource is left (if $t > x_i$), use those toward the demand of the next highest priority agent, and so on.

Combined with the compelling requirements of Symmetry (symmetric treatment of agents) and Continuity (of the mapping $(x,t) \rightarrow y$), Consistency characterizes the family of *parametric methods* ([16]). Every such method h^θ is defined by a continuous real valued function $\theta(v,\lambda)$ on $\mathbb{R}^2_+$, weakly increasing in λ, and such that $\theta(v,0)=0, \theta(v,\infty)=v$. Given a problem (N,x,t), the allocation $y=h^\theta(N,x,t)$ is the unique solution of the following system in $(y,\lambda)\in\mathbb{R}^N_+\times\mathbb{R}_+$:

$$\sum_{i\in N}\theta(x_i,\lambda)=t \text{ ; } y_i=\theta(x_i,\lambda) \text{ for all } i \tag{1}$$

(note that the design constraint $0\leq y\leq x$ is satisfied)

Although rarely discussed in the literature, the alternative representation of parametric methods as the solution of a minimization program (introduced in [16]) is key to the generalization we are looking for.

Fix a parametric method h^θ as above, and an arbitrary continuous, strictly increasing real valued function f on $\mathbb{R}_+$. The following property defines[3] a function $u(v,w)$ on the cone $C=\{(v,w)\in\mathbb{R}^2_+|0\leq w\leq v\}$, continuous in both variables, and convex in w. For all $(v,w)\in C$ and all $\lambda\geq 0$:

$$w=\theta(v,\lambda)\overset{def}{\Leftrightarrow}\frac{\partial u}{\partial_- w}(v,w)\leq f(\lambda)\leq\frac{\partial u}{\partial_+ w}(v,w) \tag{2}$$

where we use the convention $\frac{\partial u}{\partial_- w}(v,0)=-\infty$ when the left derivative is not defined, and similarly $\frac{\partial u}{\partial_+ w}(v,v)=+\infty$, when the right one is not defined.

Note that if θ is strictly increasing in λ, this simplifies to $w=\theta(v,\lambda)\Leftrightarrow\frac{\partial u}{\partial w}(v,w)=f(\lambda)$.

By construction $\frac{\partial u}{\partial w}(v,w)$ is strictly increasing in w:

$$w<w'\Rightarrow\lambda<\lambda'\Rightarrow f(\lambda)<f(\lambda')\Rightarrow\frac{\partial u}{\partial_- w}(v,w)<\frac{\partial u}{\partial_+ w}(v,w')$$

therefore for any problem (N,x,t) the following minimization program has a unique solution

$$y=\arg\min_{0\leq y\leq x, y_N=t}\sum_{i\in N}u(x_i,y_i) \tag{3}$$

This solution is precisely $y=h^\theta(N,x,t)$. For completenes we prove in the Appendix the equivalence of (1) and (3). Because f is arbitrary, we have many choices of u for each parametric method.[4]

We illustrate this representation for the three benchmark methods standing out in the microeconomic axiomatic literature, as well as the social psychology literature ([12], [7], [6]): Proportional method h^{pro}, Uniform Gains (aka Equal

[3] Note that $u(z,w)$ is given up to an arbitrary function of z, but such a function plays no role in the minimization program (3).

[4] Of course we also have many choices for θ, because for any increasing bijection of $\mathbb{R}_+$ into itself, $\theta(z,\lambda)$ and $\theta(z,g(\lambda))$ represent the same method.

Awards) h^{ug}, and Uniform Losses (aka Equal Losses) h^{ul}. We choose $f(\lambda) = \ln(\lambda)$, of which the primitive is (up to a constant) the entropy function $En(v) = v\ln(v)$.

The (simplest) parametrization of h^{pro} is $\theta(v,\lambda) = \frac{\lambda}{\lambda+1}v$. It increases strictly in λ, and $w = \theta(v,\lambda) \Leftrightarrow \lambda = \frac{w}{v-w}$. Therefore (2) is $\frac{\partial u}{\partial w}(v,w) = \ln(w) - \ln(v-w)$, and we choose $u(v,w) = En(w) + En(v-w)$:

$$h^{pro}(x,t) = \arg \min_{0 \le y \le x, y_N = t} \sum_{i \in N} En(y_i) + En(x_i - y_i)$$

The (simplest) parametrization of h^{ug} is $\theta(v,\lambda) = \min\{\lambda, v\}$, strictly increasing when $w < v$, hence (2) gives $\frac{\partial u}{\partial w}(v,w) = \ln(w)$, and $\frac{\partial u}{\partial_- w}(v,v) = \ln(v)$. We choose $u(v,w) = En(w)$:

$$h^{ug}(x,t) = \arg \min_{0 \le y \le x, y_N = t} \sum_{i \in N} En(y_i)$$

The (simplest) parametrization of h^{ul} is $\theta(v,\lambda) = \max\{v - \frac{1}{\lambda}, 0\}$, so that $w = \theta(v,\lambda) \Leftrightarrow \lambda = \frac{1}{v-w}$ as long as $w > 0$. Therefore $\frac{\partial u}{\partial w}(v,w) = -\ln(v-w)$ for $w > 0$, while $\frac{\partial u}{\partial_+ w}(v,0) = -\ln(v)$:

$$h^{ul}(x,t) = \arg \min_{0 \le y \le x, y_N = t} \sum_{i \in N} En(x_i - y_i)$$

3 Overview of the results

The generalization of the Consistency property (thereafter CSY) to the bipartite model is straightforward. For a given initial problem, say a certain method selects a flow φ_{ia} between agent i and the resource of type a (φ_{ia} can only be positive if ia is an edge of the connectivity graph G). We can reduce the inital problem by removing the edge ia from the graph, while substracting φ_{ia} from both the demand of agent i and the available type a resource. Then Consistency requires that the solution to the reduced problem be unchanged on all other edges of G.

As for any application of the Consistency axiom (of which [15] gives a survey), with a consistent bipartite rationing method the "global" argument that the method selects a fair allocation for the entire graph, is confirmed by many (in the order of $2^{|G|}$) similar arguments on smaller, "local" problems. This is especially valuable when G is large and individual agents are only aware of its local structure. We also refer the reader to section 1.1 of [10], explaining why CSY is a compelling requirement when agents are held responsible for their own connectivity constraints.

Another critical requirement of our model is that two resource-types with exactly the same connectivity can be treated as a single resource. Two types can only be distinguished if some agents do so themselves. We call this axiom *Merging Identically Connected Resource-types* (MIR). For instance if all agents are connected to all resource-types, we simply add the available resources and apply the standard method we wish to generalize.

We give a fairly complete description of which standard parametric methods are extendable to a bipartite method meeting CSY and MIR. The fact that our three benchmark methods h^{pro}, h^{ug}, h^{ul}, are indeed extendable is the main finding of [10]. Their extensions, written H^{α} for $\alpha = pro, ug, ul$, solve three minization programs entirely similar to those discussed in the previous section.

Write $y_i = \sum_{a:ia\in G} \varphi_{ia}$ for agent i's total share of resources in the feasible max-flow φ, and $\mathcal{F}$ for the set of such flows. Then

$$H^{pro} \text{ selects } \arg\min_{\varphi\in\mathcal{F}} \sum_{ia\in G} En(\varphi_{ia}) + \sum_{i\in N} En(x_i - y_i)$$

$$H^{ug} \text{ selects } \arg\min_{\varphi\in\mathcal{F}} \sum_{ia\in G} En(\varphi_{ia})$$

$$H^{ul} \text{ selects in } \arg\min_{\varphi\in\mathcal{F}} \sum_{i\in N} En(x_i - y_i)$$

After defining bipartite rationing methods and our key axioms in Section 4, we show in section 5 that a standard method extendable to the bipartite context must satisfy a property that we dub Convexity*: if y is selected for problem (x,t), then for all $\delta, 0 \le \delta \le 1$, the solution for problem $(x-(1-\delta)y, \delta t)$ is δy. In words the final allocation is unchanged if we proceed in two steps: first we distribute a fraction of the final allocation, then we divide the rest of the resources according to the reduced demands. This property is not normatively compelling, however it reveals that many familiar parametric methods are not extendable to the bipartite context; these include the Talmudic method ([2]), Young's equal sacrifice methods ([17]) and most of their dual methods: Lemmas 1 and 2.

We introduce in section 6 the new family of *loss calibrated* standard methods , all of which are parametric and have a bipartite extension meeting CSY and MIR. Such a method h^{β} is defined by a continuous and weakly increasing function β from $\mathbb{R}_+$ into itself, such that $\beta(z) > 0$ for $z > 0$. Write $B(z) = \int_1^z \ln(\beta(t))dt$ then define h^{β} by means of the following program

$$h^{\beta}(x,t) = \arg\min_{0\le y\le x, y_N = t} \sum_{i\in N} En(y_i) + B(x_i - y_i)$$

This is program (3) for the function $u(v,w) = En(w) + B(v-w)$, therefore by choosing $f(\lambda) = \ln(\lambda)$ and applying (2), a parametric representation θ^{β} of h^{β} is as follows:

$$w = \theta^{\beta}(v,\lambda) \Leftrightarrow \ln(\lambda) = \frac{\partial u}{\partial w}(v,w) \Leftrightarrow w = \lambda\beta(v-w)$$

A key result (in section 8) is that a bipartite extension H^{β} of h^{β} again selects the max-flow

$$\arg\min_{\varphi\in\mathcal{F}} \sum_{ia\in G} En(\varphi_{ia}) + \sum_{i\in N} B(x_i - y_i)$$

(this is strictly true only if $B(0)$ is finite; if $B(0)$ is infinite, describing a bipartite extension is a bit more complicated). Furthermore the bipartite extension is unique if $\beta(0) = 0$.

Theorem 1 in Section 7 has two statements. When we impose a strict version of Resource Monotonicity (every individual allocation increases strictly in the total resource to be divided), the set of extendable standard methods is the subset of loss calibrated methods for which $\beta(0) = 0$. When we impose Scale Invariance (multiplying all demands and all resources by a common factor does the same thing to the selected allocation), we find the one-dimensional family of loss calibrated methods where β is a power function, $\beta(z) = z^p$ for some $p > 0$.

Theorem 2, still in Section 7 captures by weaker axiomatic requirements a richer set of "hybrid" loss calibrated methods, combining uniform losses h^{ul} when the resources are close enough to total demand, and a general loss calibrated for smaller levels of resources.

Theorems 3, 4 in Section 8 explain the bipartite extension(s) of the standard methods uncovered in Section 7.

The Appendix contains the long proofs of Theorems 1,2, and a little more. We actually describe the full family of standard methods extendable to the bipartite context under CSY and MIR. It is more complicated than even the hybrid family in Theorem 2, but not more interesting.

4 Bipartite rationing problems and methods

We write $\mathcal{N}$ for the set of potential agents and $\mathcal{Q}$ for that of potential resource-types. A rationing problem specifies a set N of n agents, a set Q of q types, and a bipartite graph $G \subseteq N \times Q$: $(i, a) \in G$ means that agent i can consume the type a. Note that G may not be connected. For $i \in N$ and $a \in Q$, let $f(i) = \{a \in Q | (i, a) \in G\}$ and $g(a) = \{i \in N | (i, a) \in G\}$, both assumed to be non empty.

Agent i demands the total amount $x_i, x_i \geq 0$, of resources, and type a can supply r_a units, $r_a \geq 0$ (its capacity). The profiles of demands and of capacities are x and r, respectively.

A *bipartite allocation problem* is $P = (N, Q, G, x, r)$ or simply $P = (G, x, r)$ if the sets N and Q are unambiguous. A *flow* in problem P is $\varphi \in \mathbb{R}_+^G$ such that

$$\varphi_{g(a)a} \leq r_a \text{ for all } a \in Q; \text{ and } \varphi_{if(i)} \leq x_i \text{ for all } i \in N,$$

where $\varphi_{g(a)a} \overset{def}{=} \sum_{i \in g(a)} \varphi_{ia}$, and $\varphi_{if(i)} \overset{def}{=} \sum_{a \in f(i)} \varphi_{ia}$. The flow φ is a *max-flow* if it maximizes $\sum_i \varphi_{if(i)}$ (equivalently $\sum_a \varphi_{g(a)a}$). Define $\mathcal{F}(P)$, also written $\mathcal{F}(G, x, r)$, as the set of max-flows for problem $P = (G, x, r)$; any $\varphi \in \mathcal{F}(P)$ is a *solution* to the problem P. Agent i's allocation, or share, is $y_i = \varphi_{if(i)}$.

We say that P is a *rationing problem* if in every $\varphi \in \mathcal{F}(P)$ all resources are fully allocated: $\varphi_{g(a)a} = r_a$ for all $a \in Q$. Then we call agent i *rationed* if $\varphi_{if(i)} < x_i$. A well known consequence of the max-flow min-cut theorem ([1]) is that any problem P can be decomposed in at most two disjoint subproblems, one in which resources are overdemanded (a rationing problem), the other in which resources are underdemanded; that is, in every max-flow, all agents receive exactly their demand, while resource-types may have unused capacity.[5] Therefore we restrict attention throughout to overdemanded problems P. This restriction is captured by the following system of inequalities:

$$\textit{for all } B \subseteq Q\text{: } r_B \leq x_{g(B)}. \tag{4}$$

We write $\mathcal{P}$ for the set of bipartite rationing problems.

A rationing problem P^0 is *standard* if it involves a single resource type to which all agents are connected: $P^0 = (N, x, t)$, where $x \in \mathbb{R}_+^N$ and $0 \leq t \leq x_N$. We let $\mathcal{P}^0$ be the set of standard problems.

Definition 1 *A bipartite rationing method H associates to each problem $P \in \mathcal{P}$ a max-flow $\varphi = H(P) \in \mathcal{F}(P)$.*

A standard rationing method h is a method applying only to standard problems. Thus $h(P^0) = h(N, x, t)$ is a division of t among the agents in N such that $0 \leq h_i(N, x, t) \leq x_i$ for all $i \in N$.

The following four additional properties of fairness and regularity for bipartite rationing methods are required in most of the literature on standard methods (see [9], [14]).

Symmetry (SYM). A method H is *symmetric* if the labels of the agents and types do not matter.[6]

Continuity (CONT). A method H is *continuous* if the mapping $(x, r) \to H(G, x, r)$ is continuous in the subset of $\mathbb{R}_+^N \times \mathbb{R}_+^Q$ defined by (4).

Ranking (RKG). A method satisfies Ranking if for any $P = (G, x, r)$ and any $i, j \in N$: $\{f(i) = f(j) \text{ and } x_i \geq x_j\} \Rightarrow y_i \geq y_j$.

Ranking*(RKG*). A method satisfies Ranking* if for any $P = (G, x, r)$ and any $i, j \in N$: $\{f(i) = f(j) \text{ and } x_i \geq x_j\} \Rightarrow x_i - y_i \geq x_j - y_j$

Definition 2 *We write $\mathcal{H}$ (resp. $\mathcal{H}^0$) for the set of symmetric, continuous bipartite (resp. standard) rationing methods meeting Ranking and Ranking*. We use the notation $\mathcal{H}(A, B, \cdots), \mathcal{H}^0(A, B, \cdots)$ for the subset of methods in $\mathcal{H}$ or $\mathcal{H}^0$ satisfying properties $A, B, \cdots$.*

A bipartite method $H \in \mathcal{H}$ defines a standard rationing method $h \in \mathcal{H}^0$ by the way it deals with an arbitrary single resource-type a and the complete graph $G = N \times \{a\}$:

$$h(N, x, t) = H(N \times \{a\}, x, r_a = t)$$

(this definition is independent of a by Symmetry)

The first of our two key axioms for bipartite methods is the natural generalization of Consistency for standard methods.

[5] This is a rationing problem when we exchange the roles of agents and resources.

[6] For any permutation π of N and σ of Q, define $G^{\pi,\sigma}$ by $(\pi(i), \sigma(a)) \in G^{\pi,\sigma} \Leftrightarrow (i, a) \in G$. SYM requires: $\{H(G, x, r) = \varphi; H(G^{\pi,\sigma}, x^\pi, r^\sigma) = \varphi'\} \Rightarrow \{\varphi_{ia} = \varphi'_{\pi(i)\sigma(a)}$ for all $(i, a) \in G\}$.

Notation: for a given graph $G \subseteq N \times Q$, and subsets $N' \subseteq N$, $Q' \subseteq Q$, the *restricted graph* of G is $G(N', Q') \overset{def}{=} G \cap \{N' \times Q'\}$, and the *restricted problem* obtains by also restricting x to N' and r to Q'.

Consistency (CSY). Fix a problem $P = (N, Q, G, x, r) \in \mathcal{P}$, an agent $i \in N$ and an edge $ia \in G$. Given a method $H \in \mathcal{H}$, the *reduced* problem after removing ia is $P' = (N^*, Q^*, G \diagdown \{ia\}, x^H(-ia), r^H(-ia))$ where

$$N^* = N \text{ if } f(i) \neq \{a\} \text{ , } N^* = N \diagdown \{i\} \text{ if } f(i) \neq \{a\}$$
$$Q^* = Q \text{ if } g(a) \neq \{i\} \text{ , } Q^* = Q \diagdown \{a\} \text{ if } g(a) = \{i\}$$
$$x_i^H(-ia) = x_i - \varphi_{ia}; \; x_j^H(-ia) = x_j \text{ for } j \neq i$$
$$r_a^H(-ia) = r_a - \varphi_{ia}; \; r_b^H(-ia) = r_b \text{ for } b \neq a$$

Then the two flows $H(P) = \varphi$ and $H(P') = \varphi'$ must coincide on $G \diagdown \{ia\}$.

For a standard method h, CSY takes the following form. For all $P^0 = (N, x, t)$ and all $i \in N$:

$$y = h(N, x, t) \Rightarrow y_j = h_j(N \diagdown \{i\}, x_{-i}, t - y_i) \text{ for all } j \neq i$$

(repeated applications of which give the formulation at the beginning of Section 2).

Our second key axiom has no counterpart in the standard problem.

Merging Identically Connected Resource-types (MIR). If in problem $P \in \mathcal{P}$ two types a_1, a_2 are such that $g(a_1) = g(a_2)$, then we can merge those two types in a single type with capacity $r_{a_1} + r_{a_2}$, without affecting the flow selected by the method H.[7]

MIR says that an artificial split of a resource-type into subtypes, that does not affect the connectivity pattern, should have no impact either on the optimal flow. In particular if $G = N \times Q$ is the complete graph, the allocation $y_i = \varphi_{if(i)}$ is simply $h_i(x, r_Q)$, where h is the standard method associated to H.

5 Convexity* and an impossibility result

Juarez ([8]) introduces the following property for a standard method h:

Convexity (CVX). For all $(N, x, t) \in \mathcal{P}^0$ and all $\delta \in [0, 1]$

$$y = h(x, t) \Rightarrow h(\delta x + (1 - \delta) y, t) = y.$$

Interpretation: if we lower claims by a fraction of the losses, the allocation does not change. Convexity is a natural axiom that plays no role in our results, but it is a natural way to introduce its dual axiom[8], that, on the contrary, plays a central role here.

[7] That is, for all i, the flow φ_{ia} is unchanged for all $a \neq a_1, a_2$, and the flow to the merged node is $\varphi_{ia_1} + \varphi_{ia_2}$.

[8] The dual h^* of the standard method h is defined by $h^*(x, t) = x - h(x, x_N - t)$. A rationing method h satisfies the dual A^* of an axiom A, iff the dual method h^* satisfies A. We omit the straightforward proof that CVX and CVX* are dual axioms.

Convexity* (CVX*). For all $(N, x, t) \in \mathcal{P}^0$ and all $\delta \in [0, 1]$

$$y = h(x, t) \Rightarrow h(x - (1 - \delta)y, \delta t) = \delta y. \tag{5}$$

Interpretation: if we distribute a fraction of the gains, and lower claims accordingly, the allocation does not change.

We do not view CVX* (or CVX) as normatively compelling, in particular because many familiar methods in $\mathcal{H}^0(CSY)$ fail CVX* (and CVX). Lemma 1 below explains these incompatibilities, then Lemma 2 shows the relevance of CVX* to the methods in $\mathcal{H}(CSY, MIR)$.

One of the earliest standard methods in the literature, the *Talmudic* method h^{tal} ([2]) is a self-dual compromise between *uniform gains* h^{ug} and *uniform losses* h^{ul}. With the notation $a \wedge b = \min\{a, b\}$ and $b_+ = \max\{b, 0\}$, define

$$h^{tal}(x, t) = h^{ug}(\tfrac{x}{2}, t \wedge \tfrac{x_N}{2}) + h^{ul}(\tfrac{x}{2}, (t - \tfrac{x_N}{2})_+)$$

The Equal Sacrifice and Dual Equal Sacrifice methods are introduced in [17]. Besides their empirical relevance to tax schedules, they connect elegantly the three benchmark methods. Pick a concave[9], strictly increasing function $s : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{-\infty\}$ (so $s(z) = -\infty$ and $s'(z) = +\infty$ can only happen at $z = 0$). The *equal s-sacrifice* method selects $y = h(x, t)$ by budget balance ($y_N = t$) and the system

$$\text{for all } i\text{: } y_i > 0 \Rightarrow s(x_i) - s(y_i) = \max_N \{s(x_j) - s(y_j)\}$$

For the same function s, the *dual equal s-sacrifice* method selects $y = h(x, t)$ by budget balance and the system

$$\text{for all } i\text{: } y_i < x_i \Rightarrow s(x_i) - s(x_i - y_i) = \max_N \{s(x_j) - s(x_j - y_j)\}$$

We let the reader check that these methods are in $\mathcal{H}^0$; in particular concavity of s ensures RKG* for an equal sacrifice method, and RKG for its dual. Note also that h^{pro} and h^{ul} are equal sacrifice methods, for $s(z) = \ln(z)$ and $s(z) = z$ respectively, while h^{pro} and h^{ug} are dual equal sacrifice methods for the same two functions s.

Lemma 1

i) The Talmudic method fails CVX*;

ii) The only equal sacrifice methods meeting CVX* are h^{pro} and h^{ul};

iii) The dual $s-$equal sacrifice method meets CVX* if and only if it is h^{pro}, or h^{ug}, or (up to normalization) $s(z) = \ln(1 + Cz)$ for some positive number C.

Proof

Statements *i*) and *ii*) are Corollary 1 to Lemma 2 in [10]. We only prove statement *iii*).

[9] Concavity of s is needed to ensure RKG* for the equal s-sacrifice method, and RKG for the dual equal s-sacrifice one.

Fix s and assume the dual $s-$method meets CVX*. Fix a, b, a', b', all positive and such that $a > b, a' > b'$, and

$$s(a) - s(b) = s(a') - s(b') \tag{6}$$

In the problem $N = \{1, 2\}$, $x = (a, a')$, $t = a + a' - b - b'$ the dual $s-$method picks $y = (a - b, a' - b')$. By CVX* for any $\delta \in [0, 1]$, the method chooses δy in the problem with demands $x - (1 - \delta)y = (\delta a + (1 - \delta)b, \delta a' + (1 - \delta)b')$; the profile of losses $(x - (1 - \delta)y) - \delta y = (b, b')$ is unchanged; therefore

$$s(\delta a + (1 - \delta)b) - s(b) = s(\delta a' + (1 - \delta)b') - s(b') \tag{7}$$

In the rest of the proof we assume that s is differentiable, omitting for brevity the details of the argument when the concave function s only has a left and a right derivative at some kink points. Note that $s'(z)$ is positive for all positive z (and possibly infinite at zero).

We just showed that equality (6) implies (7) for any $\delta \in [0, 1]$. We fix now $a, b, 0 < b < a$, and $\delta \in [0, 1]$. For $\varepsilon > 0$ small enough there exists $\varepsilon' > 0$ such that $s(a) - s(b) = s(a + \varepsilon) - s(b + \varepsilon')$, therefore

$$s(\delta a + (1 - \delta)b) - s(b) = s(\delta(a + \varepsilon) + (1 - \delta)(b + \varepsilon')) - s(b + \varepsilon')$$

When ε goes to zero, the ratio $\frac{\varepsilon'}{\varepsilon}$ converges to $\frac{s'(a)}{s'(b)}$, and the equality above converges to

$$s'(\delta a + (1 - \delta)b) \cdot (\delta\varepsilon + (1 - \delta)\varepsilon') = s'(b)\varepsilon'$$
$$\Rightarrow s'(\delta a + (1 - \delta)b) \cdot (\delta s'(b) + (1 - \delta)s'(a)) = s'(a)s'(b) \tag{8}$$

Thus on the interval $[b, a]$, the positive function $z \to s'(z)$ takes the form $s'(z) = \frac{\alpha}{\beta + \gamma z}$ with

$$\alpha = (a - b)s'(a)s'(b), \beta = as'(a) - bs'(b), \gamma = s'(b) - s'(a)$$

Note that $\alpha > 0$, $\gamma \geq 0$, and $\beta + \gamma b > 0$. Upon normalization of s we can choose $\alpha = 1$. On two strictly overlapping intervals $[b, a]$ and $[c, d]$ the two expressions $\frac{1}{\beta + \gamma z}$ must coincide, therefore

$$s'(z) = \frac{1}{\beta + \gamma z} \text{ for all } z > 0 \text{ , and } \beta, \gamma \geq 0 \text{ , } \beta + \gamma > 0$$

We conclude that our dual $s-$method is h^{ug} if $\gamma = 0$; if $\gamma > 0$, after one more normalization, it takes the form $s'(z) = \frac{1}{1 + Cz}$ for some positive C, as desired.■

The rationing methods uncovered by statement *iii*) are new. See more comments after Proposition 1 in the next section.

Together with Lemma 1, our next result shows that many of the most debated consistent standard methods , cannot be consistently extended to the bipartite context.

Remark 1 The familiar axioms *Lower Composition (LC)* and *Upper Composition (UC)*, (aka *Composition Up*, and *Composition Down*; see [9], [14]) are critical to the characterization of equal sacrifice methods and their dual ([17], [9]). For all x, t, t', such that $t < t' \leq x_N$, LC requires $h(x, t') = h(x, t) + h(x - h(x, t), t' - t)$, and UC requires $h(x, t) = h(h(x, t'), t)$.

We conjecture that the following slightly stronger version of statement *ii*) holds true: *the only standard methods meeting CVX^* and Lower Composition (LC) are h^{pro} and h^{ul}.* Similarly, statement *iii*) would become: *the only standard methods meeting CVX^* and Upper Composition (UC) are h^{pro}, h^{ug},* and *the dual $s-$equal sacrifice methods where (up to normalization) $s(z) = \ln(1+Cz)$ for some positive number C.*

Lemma 2: *Assume the set $\mathcal{Q}$ of potential resource-types is infinite. If $H \in \mathcal{H}(CSY, MIR)$, the associated standard method h satisfies Consistency and Convexity*.*

Proof There is nothing to prove for CSY. For CVX* we reproduce for completeness the argument in Lemma 2 of [10].

Fix $H \in \mathcal{H}(CSY, MIR)$ and a standard problem $(N, x, t) \in \mathcal{P}^0$. Fix any two integers $p, q, 1 \leq p < q$, and a set Q of types with cardinality q. Consider the problem $P = (N \times Q, x, r)$ with complete graph, where $r_a = \frac{t}{q}$ for all $a \in Q$, and let y be the associated profile of shares at $\varphi = H(P)$. By MIR $y = h(x, t)$ and by symmetry $\varphi_{ia} = \frac{y_i}{q}$ for all $i \in N$. Drop now p of the nodes and let Q' be the remaining set of types. Applying CSY successively to all edges connecting these p nodes gives $H(N \times Q', x', r') = \varphi'$, where $x' = x - \frac{p}{q}y$, $r'_a = \frac{t}{q}$ for all $a \in Q'$, and φ' is the restriction of φ to $N \times Q'$. Therefore $y' = \frac{q-p}{q}y$. MIR in the reduced problem gives $y' = h(x', \frac{q-p}{q}t)$. This shows $\frac{q-p}{q}y = h(x - \frac{p}{q}y, \frac{q-p}{q}t)$, precisely (5) for $\delta = \frac{p}{q}$. Continuity implies (5) for other real values of δ.■

Remark 2 We introduce in [10] the property of *Node-Consistency*, weaker than CSY. It only considers reduced games in which either an agent i or a resource-type is deleted (all edges in $f(i)$, or all in $g(a)$). Similarly *Reduction of Complete Graph* (RCG) weakens MIR in that it only allows to merge *all* resource-types when *all* types are identically connected, i.e., the graph G is complete. It turns out that Lemma 2 is preserved under these two weaker assumptions: if $H \in \mathcal{H}$ meets CSY and RCG, the standard method h still meets CSY and CVX*.

In the next section we focus on the set $\mathcal{H}^0(CSY, CVX^*)$ of standard methods, which we now know contains all the methods that can be extended to $\mathcal{H}(CSY, MIR)$. By Young's theorem ([16]), any such method h is parametric: $h = h^\theta$. Without loss of generality, we consider only those parametrizations θ that distinguish all parameter values in the following sense: for all $\lambda < \lambda'$ there is some v such that $\theta(v, \lambda) < \theta(v, \lambda')$. We say that such a parametrization is *clean*.

The definition of $\mathcal{H}^0$ includes the properties Ranking and Ranking*. It is easy to check that h^θ meets RKG iff $\theta(v, \lambda)$ is weakly increasing in v as well, and RKG* iff $\theta(v,\lambda)$ is 1-Lipschitz in v: $\theta(v',\lambda)-\theta(v,\lambda)\leq v'-v$ for all $v<v'$. Our next result describes the impact on θ of Convexity*.

Lemma 3: *A standard method $h\in\mathcal{H}^0$ is in $\mathcal{H}^0(CSY,CVX^*)$ if and only if it any one of its clean parametrization θ satisfies the following. For all $\lambda\geq 0$ and all $\delta\in[0,1]$, there exists $\lambda'\geq 0$ such that*

$$\theta(v-\delta\theta(v,\lambda),\lambda')=(1-\delta)\theta(v,\lambda)\text{ for all }v\geq 0 \tag{9}$$

Proof
Only if. We fix $\lambda\geq 0,\delta\in[0,1]$, and define $J(v)=\{\lambda'\geq 0|\theta(v-\delta\theta(v,\lambda),\lambda')=(1-\delta)\theta(v,\lambda)\}$ for all $v\geq 0$. If $v>\theta(v,\lambda)$, $J(v)$ is a non empty compact interval, because $\theta(v-\delta\theta(v,\lambda),0)=0$ and $\theta(v-\delta\theta(v,\lambda),\infty)>(1-\delta)\theta(v,\lambda)$. If $v=\theta(v,\lambda)$ the definition of λ' becomes $\theta((1-\delta)v,\lambda')=(1-\delta)v$, therefore $J(v)$ is a possibly empty closed half-line $[a,\infty[$ (recall $\theta(v,\cdot)$ increases weakly and $\theta(v,\cdot)\leq v$). Note that $v=\theta(v,\lambda)$ cannot hold for all $v\geq 0$, for this would imply $v=\theta(v,\lambda")$ for all $\lambda"\geq\lambda$ and all v, contradicting the cleanliness of θ. Thus at least one $J(v)$ is non empty and compact.

Next we claim that $J(x_1)\cap J(x_2)$ is non empty for any two x_1,x_2. Define $t=\theta(x_1,\lambda)+\theta(x_2,\lambda)$ and apply CVX* to the problem $(x=(x_1,x_2),t)$:

$$h^\theta(x-\delta\cdot(\theta(x_1,\lambda),\theta(x_2,\lambda)),(1-\delta)t)=(1-\delta)\cdot(\theta(x_1,\lambda),\theta(x_2,\lambda))$$

Hence for some parameter λ' we have $\theta(x_i-\delta\theta(x_i,\lambda),\lambda')=(1-\delta)\theta(x_i,\lambda)$ for $i=1,2$, as claimed.

We conclude that $\cap_{v\geq 0}J(v)$ is non empty, and any λ' in this set satisfies (9).
If. Fix $(N,x,t)\in\mathcal{P}^0$ and $\delta\in[0,1]$. Let λ be a solution of system (1), and λ' be such that (9) holds true. Applying (9) to $v=x_i$ for each $i\in N$ shows that $(1-\delta)\cdot h^\theta(x,t)$ is precisely $h^\theta(x-\delta\cdot h^\theta(x,t),(1-\delta)t)$.■

6 Loss calibrated standard methods

In this Section we define the large family of loss calibrated standard methods, all in $\mathcal{H}^0(CSY,CVX^*)$. We use these methods in the next Section to characterize several subsets of $\mathcal{H}^0(CSY,CVX^*)$ under mild additional properties. A full description of $\mathcal{H}^0(CSY,CVX^*)$ obtains as a by-product of the proofs in the Appendix.

Definition *We call the function β, from $\mathbb{R}_+$ into itself, a* ***calibration*** *function if it is continuous, weakly increasing, $\beta(0)\in\{0,1\}$, and $\beta(z)>0$ for all $z>0$. If $\beta(0)=0$, β is a* ***strict*** *calibration; if $\beta(0)=1$, it is a* ***non-strict*** *calibration. The sets of strict and non strict calibrations are denoted $\mathcal{B}^s$ and $\mathcal{B}^{ns}$ respectively, and $\mathcal{B}=\mathcal{B}^s\cup\mathcal{B}^{ns}$ is the set of all calibration functions.*

Proposition 1
i) Fix $\beta\in\mathcal{B}^s$. For all $(N,x,t)\in\mathcal{P}^0$ such that $t<x_N$, the system in $y\in\mathbb{R}_+^N$

$$0\leq y\leq x\ ,\ y_N=t$$

$$\frac{y_i}{\beta(x_i - y_i)} = \frac{y_j}{\beta(x_j - y_j)} \text{ for all } i, j \text{ such that } x_i, x_j > 0 \tag{10}$$

has a unique solution $y = h^\beta(x,t)$ *defining a rationing method in* $\mathcal{H}^0(CSY, CVX^*)$, *and parametrized by* θ^β *as follows. For all* $(v,\lambda) \in \mathbb{R}^2_+$

$$\theta^\beta(v,\lambda) \text{ is the unique solution } y \in [0,v] \text{ of } y = \lambda\beta(v-y) \tag{11}$$

ii) Fix $\beta \in \mathcal{B}^{ns}$. *For all* $(N,x,t) \in \mathcal{P}^0$ *the system in* $y \in \mathbb{R}^N_+$

$$0 \le y \le x \ , \ y_N = t$$

$$\{y_i < x_i \Rightarrow \frac{y_i}{\beta(x_i - y_i)} = \max_j \frac{y_j}{\beta(x_j - y_j)}\} \text{ for all } i \tag{12}$$

has a unique solution $y = h^\beta(x,t)$ *defining a rationing method in* $\mathcal{H}^0(CSY, CVX^*)$, *and parametrized by the following* θ^β. *For all* $(v,\lambda) \in]0,\infty[\times\mathbb{R}_+$

$$\theta^\beta(v,\lambda) \text{ is the unique solution } y \in [0,v] \text{ of } \frac{y}{\beta(v-y)} = \lambda \wedge v \tag{13}$$

We write $\mathcal{LC}^s$ (resp. $\mathcal{LC}^{ns}$) for the set of strict (resp. non-strict) loss-calibrated methods corresponding to the two types of calibration functions, and $\mathcal{LC} = \mathcal{LC}^s \cup \mathcal{LC}^{ns}$ is the set of all loss-calibrated methods.

Proof

If $t = x_N$ or $x_i = 0$ the corresponding shares are clear by definition of rationing methods, hence statement $i)$ is a complete definition of a rationing method.

For $v = 0$, both (11) and (13) give $\theta^\beta(0,\lambda) = 0$. For $v > 0$, the function $y \to \frac{y}{\beta(v-y)}$ is continuous and strictly increasing for $y \in [0, v[$; in the strict case this ratio goes to ∞ as y approaches v; in the non strict case it reaches v. Hence both systems (11) and (13) have a unique solution $\theta^\beta(v,\lambda)$ in the corresponding intervals. The function θ^β is clearly continuous on $\mathbb{R}^2_+$ and weakly increasing in both variables. The corner conditions $\theta^\beta(v,0) = 0$, and $\theta^\beta(v,\infty) = v$ are equally clear.

Next we fix $(N,x,t) \in \mathcal{P}^0$ with $t < x_N$, and let $y_i = \theta^\beta(x_i,\lambda)$ be the allocation selected by the parametric method θ^β, i.e., λ solves $\sum_{i\in N} \theta^\beta(x_i,\lambda) = t$. In the strict case it is clear from (11) that $\frac{y_i}{\beta(x_i-y_i)}$ is independent of i whenever $x_i > 0$. In the non-strict case (13) implies $\frac{y_i}{\beta(x_i-y_i)} = \lambda \wedge x_i$ for all i. If $y_i < x_i$ we have $\frac{y_i}{\beta(x_i-y_i)} = \lambda$ (recall $\beta(x_i - y_i) \ge 1$), and if $y_i = x_i$ we have $\frac{y_i}{\beta(x_i-y_i)} \le \lambda$, completing the proof that y is a solution of (12).

Conversely in the strict case a solution y of (10) clearly takes the form $y_i = \theta^\beta(x_i,\lambda)$ for some λ. In the non-strict case, if y is a solution of (12), we set $\lambda = \max_j \frac{y_j}{\beta(x_j-y_j)}$. For i such that $y_i = x_i$, inequality $\frac{y_i}{\beta(x_i-y_i)} \le \lambda$ implies $\frac{y_i}{\beta(x_i-y_i)} = \lambda \wedge x_i$; for i such that $y_i < x_i$ we have $\frac{y_i}{\beta(x_i-y_i)} = \lambda$ and $\frac{y_i}{\beta(x_i-y_i)} \le y_i \le x_i$, so $\frac{y_i}{\beta(x_i-y_i)} = \lambda \wedge x_i$ as well.

We have shown that (10) defines a *strict* method h^β parametrized by (11), and (12) defines a *non-strict* one h^β parametrized by (13).

The properties CONT, SYM, and CSY are clear. Next we check RKG, i.e., $\theta^\beta(v,\lambda)$ is weakly increasing in v. In the non-strict case, fix λ and assume to the contrary $v < v'$ and $y' < y$, where y, v, λ, as well as y', v', λ, meet (13). Then

$$\lambda \wedge v = \frac{y}{\beta(v-y)} \geq \frac{y}{\beta(v'-y')} > \frac{y'}{\beta(v'-y')} = \lambda \wedge v'$$

contradiction. The proof in the strict case is similar.

To prove RKG*, i.e., $\theta^\beta(v,\lambda)$ is 1-Lipschitz in v, start with the strict case. Fix λ and assume to the contrary $v < v'$ and $v - y > v' - y'$: then $y' > y$, so $\frac{y}{\beta(v-y)} = \frac{y'}{\beta(v'-y')}$ yields a contradiction. In the non-strict case, pick λ, $v < v'$, and assume $\frac{y}{\beta(v-y)} = \lambda \wedge v$, $\frac{y'}{\beta(v'-y')} = \lambda \wedge v'$. If $\lambda \leq v$ then we are back to the previous case; if $v < \lambda$ then $y = v$ (by (13)) and $y' \leq v'$ so we are done.

We check finally CVX*. Fix $\beta \in \mathcal{B}^s \cup \mathcal{B}^{ns}$, a problem $(N,x,t) \in \mathcal{P}^0$ and $\delta \in [0,1]$. Direct inspection shows that if the triple (x,t,y) meets system (10) in the strict case(or (12) in the non-strict case), then so does the triple $(x - (1-\delta)y, \delta t, \delta y)$.■

We give some examples of loss calibrated methods.

The identity function $\beta(z) = z$ is a strict calibration ($\beta \in \mathcal{B}^s$), corresponding to the proportional method $h^{pro} \in \mathcal{LC}^s$; equation (11) gives the parametrization $\theta^{pro}(v,\lambda) = \frac{\lambda}{1+\lambda}v$ mentioned in the introduction

The constant function $\beta(z) = 1$ is non-strict ($\beta \in \mathcal{B}^{ns}$), it corresponds in (12) to the uniform gain method $h^{ug} \in \mathcal{LC}^{ns}$, parametrized by $\theta^{ug}(v,\lambda) = \lambda \wedge v$.

It is easy to see that h^{ul} is not loss calibrated ($h^{ul} \notin \mathcal{LC}$), except in a limit sense explained below.

A familiar and powerful requirement for standard methods is that they should ignore the scale of the problem. This is compelling when the context does not include any benchmark level of demands or resources.

Scale Invariance (SI). For all $(N,x,t) \in \mathcal{P}^0$ and all $\lambda > 0$

$$h(\lambda x, \lambda t) = \lambda h(x,t)$$

This axiom cuts a simple one-dimensional family in $\mathcal{LC}$, where the calibration function is a power function $\beta(z) = z^p$ for some $p \geq 0$. For $p = 0$ this is simply uniform gains h^{ug}. For $p > 0$ the corresponding method $y = h^p(x,t)$ obtains by solving the system

$$0 \leq y \leq x \ , \ \frac{y_i}{(x_i - y_i)^p} = \frac{y_j}{(x_j - y_j)^p} \text{ for all } i,j, \text{ and } y_N = t \tag{14}$$

For $p = 1$, this is simply h^{pro}. For $p = 2$ and $p = \frac{1}{2}$, the parametrization is in closed form:

$$\theta^2(v,\lambda) = v\left(1 - \frac{2}{1 + \sqrt{\lambda v + 1}}\right)$$

$$\theta^{\frac{1}{2}}(v,\lambda) = v\frac{2}{1 + \sqrt{\frac{v}{\lambda} + 1}}$$

The family (14) $\{h^p, p \geq 0\}$ is new to the literature on standard methods. It connects the three benchmark methods as follows: $h^0 = h^{ug}$, $h^1 = h^{pro}$, and $h^{ul} = \lim_{p\to\infty} h^p$, where the limit is in the pointwise sense (for any problem (x,t) $\lim_{p\to\infty} h^p(x,t) = h^{ul}(x,t)$).

Theorem 1 in the next Section implies that the power functions are the *only* scale invariant methods in $\mathcal{LC}$. In fact it says much more: the only scale invariant methods in $\mathcal{H}^0(CSY, CVX^*)$ are h^{ul} and h^p for some $p \geq 0$.

In statement *iii*) of Lemma 1 we discovered another new one-dimensional, family of methods connecting this time h^{ug} and h^{pro}. For any $C > 0$, the dual equal s-sacrifice method h^C with $s(z) = \ln(1 + Cz)$ is (non-strictly) loss calibrated by $\beta^C(z) = 1 + Cz$. Indeed for all x, x', y, y', such that $y < x, y' < x'$ we have

$$\ln(1 + Cx) - \ln(1 + C(x - y)) = \ln(1 + Cx') - \ln(1 + C(x' - y'))$$

$$\Longleftrightarrow \frac{y}{1 + Cx} = \frac{y'}{1 + Cx'} \Leftrightarrow \frac{y}{1 + C(x - y)} = \frac{y'}{1 + C(x' - y')}$$

The method h^C is parametrized by as follows:

$$\theta^C(v, \lambda) = \frac{\lambda \wedge v}{1 + C(\lambda \wedge v)}(1 + Cv)$$

Moreover $h^0 = h^{ug}$ and $\lim_{C\to\infty} h^C = h^{pro}$. We omit the easy proof of these two claims.

Finally we construct yet another new one-dimensional family connecting h^{ug} and h^{pro} in $\mathcal{LC}^s$ by choosing the strict calibration $\beta^D(z) = z \wedge D$, where D is an arbitrary positive constant. The corresponding parametrization is

$$\theta^D(v, \lambda) = \frac{\lambda}{1 + \lambda} v \wedge \lambda D$$

and one checks that $\lim_{D\to 0} h^D = h^{ug}$, while $\lim_{D\to\infty} h^D = h^{pro}$.

7 Characterization of (hybrid) loss calibrated rationing

The long proof of Theorems 1,2 below starts by the analysis of the set $\mathcal{H}^0(CSY, CVX^*)$ of consistent standard methods satisfying CVX*: subsection 10.1. This set contains uninterestingly complicated methods combining countable families of loss calibrated submethods: see subsection 10.1.?. In Theorems 1,2 we add to CSY and CVX^* one of three mild and natural axiomatic requirements, and obtain relatively simple subfamilies of methods, either directly loss calibrated (in $\mathcal{LC}$), or close variants of loss calibrated methods, that we dub *hybrid* methods. Moreover the bipartite extension of the latter methods is easy to describe, which is not the case for more general methods in $\mathcal{H}^0(CSY, CVX^*)$.

The first of the three axioms just mentioned is Scale Invariance, defined at the end of the previous section. The second one is well known in the literature.

Strict Resource Monotonicity (SRM): for all $(N,x,t),(N,x,t') \in \mathcal{P}^0$ such that $y = h(x,t)$, $y' = h(x,t')$

$$t < t' \Rightarrow y_i < y'_i \text{ for all } i \in N$$

The proportional method h^{pro} is strictly resource monotonic, but neither h^{ug} nor h^{ul} is. The equal $s-$sacrifice method (resp. its dual) meets SRM if and only if $s(0) = -\infty$, for instance $s(z) = -\frac{1}{z^p}$ for $p > 0$.

Theorem 1

i) The rationing method h is in $\mathcal{H}^0(CSY, CVX^*, SI)$ if and only if it is h^{ug}, h^{ul}, or it is strictly calibrated by a power function $\beta(z) = z^p$ for some $p > 0$.

ii) The rationing method h is in $\mathcal{H}^0(CSY, CVX^*, SRM)$ if and only if it is a strictly loss calibrated: $\mathcal{LC}^s = \mathcal{H}^0(CSY, CVX^*, SRM)$.

The long proofs of Theorems 1 and 2 are in the Appendix.

Our second main result captures a subset of $\mathcal{H}^0(CSY, CVX^*)$ that is larger than $\mathcal{LC}$, in particular it contains h^{ul}. The additional requirement we impose is a mild strengthening of RKG*, which holds true for the three benchmark methods, all equal sacrifice methods and their dual, and the Talmudic method.

∂Ranking* (∂RKG*): for all $(N,x,t),(N,x,t') \in \mathcal{P}^0$ such that $t \leq t'$, $y = h(x,t)$, and $y' = h(x,t')$

$$\{x_i - y_i = x_j - y_j \text{ for all } i,j\} \Rightarrow \{x_i - y'_i = x_j - y'_j \text{ for all } i,j\}$$

This says that if at some level of the resources t all agents experience the same loss, then for higher levels of resources the losses remain identical across agents. It is strictly weaker than the strict version of RKG*, $x_i < x_j \Rightarrow x_i - y_i < x_j - y_j$.

The set $\mathcal{H}^0(CSY, CVX^*, \partial RKG^*)$ contains $\mathcal{LC}$, h^{ul}, as well as new hybrid methods combining h^{ul} and a loss calibrated method.

Proposition 2 *Fix a calibration function $\beta \in \mathcal{B}$, and a constant A, $0 \leq A \leq \infty$. The following system defines a standard method $h^{\beta,A}$ in $\mathcal{H}^0(CSY, CVX^*)$. For all $(N,x,t) \in \mathcal{P}^0$*

$$h^{\beta,A}(x,t) = h^{\beta}((x-A)_+, t) \text{ for all } t \text{ s.t. } t \leq \sum_N (x_i - A)_+ \tag{15}$$

$$h^{\beta,A}(x,t) = h^{ul}(x,t) \text{ for all } t \text{ s.t. } \sum_N (x_i - A)_+ \leq t \leq x_N \tag{16}$$

This method is parametrized by

$$\theta^{\beta,A}(v,\lambda) = \theta^{\beta}((v-A)_+, \frac{\lambda}{1-A\lambda}) \text{ for all } v \geq 0, \text{ all } \lambda \leq \frac{1}{A} \tag{17}$$

$$\theta^{\beta,A}(v,\lambda) = (v - \frac{1}{\lambda})_+ \text{ for all } v \geq 0, \text{ all } \lambda \geq \frac{1}{A} \tag{18}$$

We write $\widetilde{\mathcal{LC}}$ for the set of hybrid loss calibrated methods thus defined.

Proof

By Proposition 1, equation (17) defines a continuous, weakly increasing (in both variables) function $\theta^{\beta,A}(v,\lambda)$ in the subset $\{0 \leq \lambda < \frac{1}{A}\}$ of $\mathbb{R}^2_+$. Equation (18) defines a function with the same properties in the subset $\{\lambda \geq \frac{1}{A}\}$; continuity on the line $\{\lambda = \frac{1}{A}\}$ follows from $\lim_{\lambda' \to \infty} \theta^{\beta}(v', \lambda') = v'$. The equalities $\theta^{\beta,A}(v,0) = 0$ and $\theta^{\beta,A}(v,\infty) = v$ are equally clear. Thus $\theta^{\beta,A}(v,\lambda)$ is a genuine parametrization, moreover it is weakly increasing and 1-lipschitz in v, so it defines a rationing method in $\mathcal{H}^0(CSY)$. Checking CVX* is easy: for each $\lambda < \frac{1}{A}$ and $\delta \in [0,1]$ we apply Lemma 3 to θ^{β} and the parameter $\frac{\lambda}{1-A\lambda}$ to find $\frac{\lambda'}{1-A\lambda'}$ satisfying (9) for some $\lambda' < \frac{1}{A}$, so that (9) holds for $\theta^{\beta,A}$ when $\lambda < \frac{1}{A}$; the argument for $\lambda \geq \frac{1}{A}$ is similar but for θ^{ul}.

It remains to show that the method parametrized by $\theta^{\beta,A}$ is given by system (15),(16). Fix a problem (N,x,t) and distinguish two cases.

Case 1: $x_i \leq A$ for all i. Then $\theta^{\beta,A}(x_i,\lambda) = 0$ for $\lambda \leq \frac{1}{A}$, so we have $\theta^{\beta,A}(x_i,\lambda) = (x_i - \frac{1}{\lambda})_+ = \theta^{ul}(x_i,\lambda)$ for all $\lambda \geq 0$, hence the parametric method selects with $h^{ul}(x,t)$, just like (16) does.

Case 2: $\sum_N (x_i - A)_+ > 0$. Then $\lambda \to \sum_N \theta^{\beta,A}(x_i,\lambda)$ reaches $\sum_N (x_i - A)_+$ at $\lambda = \frac{1}{A}$ (and possibly before $\frac{1}{A}$), and is strictly larger when $\lambda > \frac{1}{A}$. Therefore the equation $\sum_N \theta^{\beta,A}(x_i,\lambda) = t$ with unknown λ, has its solution before $\frac{1}{A}$ if $t < \sum_N (x_i - A)_+$, and after $\frac{1}{A}$ if $t > \sum_N (x_i - A)_+$. If the former this equation is $\sum_N \theta^{\beta}((x_i - A)_+, \frac{\lambda}{1-A\lambda}) = t$, so it delivers the allocation $h^{\beta}((x-A)_+, t)$. If the latter the equation is $\sum_N (x_i - \frac{1}{\lambda})_+ = t$, and the allocation is $h^{ul}(x,t)$.■

For $A = 0$, our method is simply loss-calibrated: $h^{\beta,0} = h^{\beta} \in \mathcal{LC}$. For $A = \infty$, it is h^{ul}. For $0 < A < \infty$, $h^{\beta,A}$ works as follows: apply to the profile x a uniform reduction by the fixed amount A (possibly canceling some demands); then apply the loss calibrated method to these reduced demands, if they still exceed the available resources; otherwise pick the uniform losses allocation.

Theorem 2 *The rationing method h is in $\mathcal{H}^0(CSY, CVX^*, \partial RKG^*)$ if and only if it is an hybrid loss calibrated method : $\widetilde{\mathcal{LC}} = \mathcal{H}^0(CSY, CVX^*, \partial RKG^*)$.*

Remark 2 The following is axiom strengthens the ∂RKG* property:

∂∂**Ranking (∂∂RKG):** for all i,j,

$$\{x_i < x_j, t < t'\} \Rightarrow h_i(x,t') - h_i(x,t) \leq h_j(x,t') - h_j(x,t)$$

In words, a bigger claim warrants a weakly bigger share of any resource increase. This is a self-dual property, implying at once RKG, RKG*, and ∂RKG*. For the parametric method $\theta(x_i,\lambda)$, ∂∂Ranking amounts to a cross monotonicity property:

$$\{x_i < x'_i, \lambda < \lambda'\} \Rightarrow \theta(x'_i,\lambda) + \theta(x_i,\lambda') \leq \theta(x_i,\lambda) + \theta(x'_i,\lambda')$$

or $\partial_{x\lambda}\theta \geq 0$, for short.

All equal s-sacrifice and dual equal s-sacrifice methods satisfy $\partial\partial$RKG, because the function s is concave. For a loss calibrated method h^β, or the hybrid methos $h^{\beta,A}$, it is easily checked that $\partial\partial$RKG holds if and only if the calibration function β is concave. This in turn gives a variant of Theorem 2, where $\partial\partial$RKG replaces ∂RKG*, and we characterize the hybrid methods with concave calibration.

8 Bipartite (hybrid) loss calibrated methods

We show finally that all standard methods uncovered in Theorems 1 and 2 can be exended to $\mathcal{H}(CSY, MIR)$. For strictly loss calibrated methods (in particular all methods in Theorem 1 except h^{ug} and h^{ul}), this extension is unique: Theorem 3. For h^{ug},h^{ul}, other non strict loss calibrated methods, and all hybrid loss calibrated methods (with $A > 0$ in Theorem 2), it is not unique: Theorem 4.

Given $\beta \in \mathcal{B}$, we define $B(z) = \int_1^z \ln(\beta(t))dt$; this integral is finite for $z > 0$, and possibly infinite ($+\infty$) for $z = 0$ (e.g., $B(0) = +\infty$ if $\beta(z) = e^{-\frac{1}{z}}$). We also write the entropy function $En(z) = z\ln(z) = \int_1^z (\ln(t) + 1)dt$.

Theorem 3 *For any strict calibration $\beta \in \mathcal{B}^s$, the corresponding standard method $h^\beta \in \mathcal{LC}^s$ has a unique extension H^β to $\mathcal{H}(CSY, MIR)$, defined as follows. Given the rationing problem $P = (N, Q, G, x, r) \in \mathcal{P}$, let $N_0 \subseteq N$ be the subset of agents i with $y_i = \varphi_{i,f(i)} = x_i$ for every $\varphi \in F(G, x, r)$. Then,*

$$\varphi^\beta(P) = \arg \min_{\varphi \in \mathcal{F}(G,x,r)} \sum_{ia \in G} En(\varphi_{ia}) + \sum_{i \in N \setminus N_0} B(x_i - y_i). \tag{19}$$

We say that P is *balanced* if $x_N = r_Q$. In this case every max-flow meets individual demands in full, but we still have many choices for the actual flow along the edges. Theorem 3 implies that for a balanced problem, all extended strict calibration methods select the same flow

$$\varphi^\beta(P) = \arg \min_{\varphi \in \mathcal{F}(G,x,r)} \sum_{ia \in G} En(\varphi_{iz}) \tag{20}$$

which coincides in particular with the solution of the bipartite proportional method ([10]).

Note that if $B(0)$ is finite, definition (19) takes the simpler form

$$\varphi^\beta(P) = \arg \min_{\varphi \in \mathcal{F}(G,x,r)} \sum_{ia \in G} En(\varphi_{ia}) + \sum_{i \in N} B(x_i - y_i) \text{ for all } P \in \mathcal{P} \tag{21}$$

Proof *Step 1:* The entropy function is strictly convex and B is convex, therefore the program (19) has a unique solution. Fix $P \in \mathcal{P}$ and N_0 as defined above, and set $N_1 = N \diagdown N_0$. Then Q is uniquely partitioned as $Q_0 \cup Q_1$ such that $g(Q_0) = N_0$, $x_{N_0} = r_{Q_0}$, and $f(N_1) = Q_1$. The restriction of P to (N_0, Q_0) is a balanced problem P_0, while its restriction to (N_1, Q_1) is a *strictly* overdemanded problem P_1, i.e., for all $B \subseteq Q_1$ we have $r_B < x_{g(B) \cap N_1}$. Program

(19) decomposes into two independent programs, namely (20) on P_0 and (21) on P_1.

Check first that for a standard problem (x,t) this defines the loss calibrated method h^β. Indeed $N_0 = \emptyset$ or $N_0 = N$ for such a problem, and the KKT conditions for

$$\arg \min_{y_N = t, y \leq x} \sum_{i \in N} En(y_i) + \sum_{i \in N} B(x_i - y_i)$$

are that $y_i < x_i$ for all i unless the problem is balanced, and $\frac{y_i}{\beta(x_i - y_i)}$ is independent of i.

Next we need to check that H^β meets the four properties in Definition 2, as well as CSY and MIR. SYM is obvious. For RKG and RKG*, note that $f(i) = f(j)$ implies that i and j are both in N_0 or both in N_1; if the former, there is nothing to prove; if the latter assume for instance $x_i \geq x_j$ and $y_i < y_j$, then we improve the objective by averaging the flows to i and j (giving $\frac{1}{2}(\varphi_{ia} + \varphi_{ib})$ to both), because En is strictly convex. Checking MIR is equally easy as two nodes with identical connectivity are also in the same element of the partition $Q_0 \cup Q_1$. Similarly for CSY note that upon dropping an edge, the partitions of N and Q do not change (except if an agent or a type disappears, in which case the corresponding set $N_{0,1}$ and/or $Q_{0,1}$ shrinks), so we can check CSY separately for P_0 and P_1. This is clear because $x_i - y_i$ does not change when we drop an edge ia.

Step 2: The only property requiring a global argument is CONT, because when (x^k, r^k) converges to (x, r), the partitions may shift at the limit problem. In the case where $B(0)$ is finite, the flow $H^\beta(x, r)$ solves the program (21), independent of the partitions of N, Q, and we can apply Berge's maximum theorem under convexity [13]. The correspondence $(x, r) \to \mathcal{F}(G, x, r)$ is convex and compact valued, and both upper and lower hemi-continuous (see, for instance, Bohm [5]), and we minimize a strictly convex function, so the arg min is a continuous function.

A more complex argument is needed to accomodate the case $B(0) = \infty$, where the program (21) is not defined for balnced problems. Fix N, Q, G, and consider a converging sequence in $\mathcal{P}$: $P = \lim_k P^k$. It is clearly enough to show $H(P) = \lim_k H(P^k)$ when the partitions $N^k_{0,1}, Q^k_{0,1}$ are the same for all k. If this is also the partition of P, we can still invoke Berge's Theorem because H solves the strictly convex program (19) over the same correspondence $(x, r) \to \mathcal{F}(G, x, r)$. When the partition changes in the limit, the balanced component must expand. Omitting some straightforward details, it will be enough to prove CONT for a sequence (x^k, r^k) converging to (x, r) such that:

- $P^k = (N, Q, G, x^k, r^k)$ is strictly overdemanded
- $P = (N, Q, G, x, r)$ is balanced and *irreducible* ([10]), i.e., $B \subsetneq Q_1$ we have $r_B < x_{g(B) \cap N_1}$

Assume first that G is the complete graph $N \times Q$. Then the solution $\varphi^k = H^\beta(P^k)$ of (21) is characterized by the KKT conditions $\frac{\varphi^k_{ia}}{\beta(x^k_i - y^k_i)} = \frac{\varphi^k_{ja}}{\beta(x^k_j - y^k_j)}$

for all i, j, a, in particular $y_i^k < x_i^k$ and $\varphi_{ia}^k > 0$ for all i, a, k. This implies $\frac{\varphi_{ia}^k}{\varphi_{ib}^k} = \frac{\varphi_{ja}^k}{\varphi_{jb}^k} = \frac{r_a^k}{r_b^k}$ and in turn $\varphi_{ia}^k = \frac{r_a^k}{r_Q^k} y_i^k$ (even for those types such that $r_a^k = 0$). Now $y_i^k < x_i^k$, $y_N^k = r_Q^k$, the convergence assumptions and the balancedness of P imply $\lim_k y_i^k = x_i$ then $\lim_k \varphi_{ia}^k = \frac{r_a}{r_Q} x_i$, i.e., φ^k converges to the proportional flow in P, precisely the solution of (20).

Next assume G is an arbitrary graph. The solution $\varphi^k = H^\beta(P^k)$ of (21) still has $y_i^k < x_i^k$, $\varphi_{ia}^k > 0$, and is still characterized by the KKT conditions $\frac{\varphi_{ia}^k}{\beta(x_i^k - y_i^k)} = \frac{\varphi_{ja}^k}{\beta(x_j^k - y_j^k)}$ for all i, j, a such that $ia, ja \in G$. Set $\beta_i^k = \beta(x_i^k - y_i^k)$ and define as follows a flow $\overline{\varphi}^k$ in the complete graph $\overline{G} = N \times Q$, augmenting φ^k:

$$\overline{\varphi}_{ia}^k = \varphi_{ia}^k \text{ if } ia \in G;\ \overline{\varphi}_{ia}^k = \frac{\beta_i^k}{\beta_j^k} \varphi_{ja}^k \text{ for any } j \in g(a) \tag{22}$$

Consider the new problem $\overline{P}^k = (\overline{G}, \overline{x}^k, \overline{r}^k)$ where

$$\overline{x}_i^k = x_i^k + \sum_{a \in Q \setminus f(i)} \overline{\varphi}_{ia}^k \ ;\ \overline{r}_a^k = r_a^k + \sum_{i \in N \setminus g(a)} \overline{\varphi}_{ia}^k$$

One checks easily the following facts: $\overline{P}^k$ is strictly overdemanded; $\overline{\varphi}^k$ is a max-flow in $\overline{P}^k$; $\overline{x}_i^k - \overline{y}_i^k = x_i^k - y_i^k$ so that $\overline{\beta}_i^k = \beta(\overline{x}_i^k - \overline{y}_i^k) = \beta_i^k$; and finally $\overline{\varphi}^k$ meets the KKT conditions of (21), namely $\frac{\overline{\varphi}_{ia}^k}{\beta_i^k} = \frac{\overline{\varphi}_{ja}^k}{\beta_j^k}$ for all i, j, a. Therefore $\overline{\varphi}^k = H^\beta(\overline{P}^k)$.

The sequence φ^k is bounded, we check now that $\overline{\varphi}^k$ is bounded as well. The binary relation in N: $i \sim j$ iff $A < \frac{\beta_i^k}{\beta_j^k} < B$ for some A, B, s.t. $A > 0$, is an equivalence relation. If N itself is an equivalence class, each $\overline{\varphi}_{ia}^k$ is bounded as well. If there are two or more equivalence classes, we can take a subsequence of k (denoted k for simplicity) and partition N as $N^+ \cup N^-$ such that $\frac{\beta_i^k}{\beta_j^k} \to 0$ (limit w.r.t. k) for all $i \in N^+$ and $j \in N^-$. Then we set $Q^+ = f(N^+), Q^- = Q \setminus Q^+$, so $g(Q^-) = N^-$. For any $a \in Q^+$, $j \in g(a) \cap N^+$, and $i \in N^-$, the equation $\overline{\varphi}_{ia}^k = \frac{\beta_i^k}{\beta_j^k} \varphi_{ja}^k$ implies $\overline{\varphi}_{ia}^k \to 0$, therefore $\overline{\varphi}_{N^- \times Q^+}^k \to 0$, and in turn $\varphi_{G(N^-, Q^+)}^k \to 0$. Because (x^k, r^k) converges to (x, r) and P is balanced, we have $\varphi_{G(N^-, Q)}^k \to x_{N^-}$. Combining the last two limit statements gives $\varphi_{G(N^-, Q^-)}^k \to x_{N^-}$; on the other hand $G(N^-, Q^-) = G(N, Q^-)$ because $g(Q^-) = N^-$, therefore $\varphi_{G(N^-, Q^-)}^k \to r_{Q^-}$. This contradicts the irreducibility of P, and completes the proof that the sequence $\overline{\varphi}^k$ is bounded.

We check now that any convergent subsequence of φ^k (written φ^k for simplicity) converges to $\varphi = H^\beta(P)$, i.e., is the solution of (20) at P. Take a converging subsequence (written the same) of φ^k such that $\overline{\varphi}^k$ converges as well, and let $\overline{\varphi}$ be its limit. Define the problem $\overline{P} = (\overline{G}, \overline{x}, \overline{r})$ on the complete graph:

$$\overline{x}_i = x_i + \sum_{a \in Q \setminus f(i)} \overline{\varphi}_{ia} \ ;\ \overline{r}_a = r_a + \sum_{i \in N \setminus g(a)} \overline{\varphi}_{ia}$$

so that $\overline{P}$ is balanced. We showed above that $\overline{\varphi}$ is the solution of (20) at $\overline{P}$ (the proportional flow). The solution of (20) for balanced problems meets CSY; by removing successively all edges outside G we get problem P, therefore the restriction φ^k of $\overline{\varphi}^k$ to G is also the solution of (20) for problem P.

Step 3: To show uniqueness, fix an extension H of h^β in $\mathcal{H}(CSY, MIR)$, and a problem $P \in \mathcal{P}$, with corresponding subproblems P_0, P_1 (see Step1) restricted respectively to $N_0 \times Q_0$ and $N_1 \times Q_1$. By repeated applications of CSY, $H(P)$ is the union of the two flows $H(P_0)$ and $H(P_1)$. We show first $H(P_1) = H^\beta(P_1)$.

Set $H(P_1) = \varphi$ and $z_i = x_i - y_i$, so that $\sum_N z_i > 0$. Fix a resource type a, drop all edges in $G(N_1 \times Q_1)$ except those ending at a, and reduce accordingly individual demands as in the definition of CSY. The z_i-s do not change, and the remaining standard problem is $((z_i + \varphi_{ia})_{i \in g(a)}, r_a)$, strictly overdemanded. Invoking CSY and Proposition 1 we have:

$$\varphi_{\cdot a} = h^\beta((z_i + \varphi_{ia})_{i \in g(a)}, r_a) \Leftrightarrow \frac{\varphi_{ia}}{\beta(z_i)} = \frac{\varphi_{ja}}{\beta(z_j)} \text{ for all } i, j \in g(a)$$

As a was arbitrary, these are precisely the KKT conditions for problem (19) restricted to $N_1 \times Q_1$.

Now $H(P_0) = H^\beta(P_0)$ follows by choosing a sequence of strictly overdemanded problems $P^k = (G(N_0 \times Q_0), x^k, r^k)$ converging to P_0: the previous argument gives $H(P^k) = H^\beta(P^k)$ for all k and CONT concludes the proof.■

The proof above shows that for a strictly overdemanded problem $P = (G, x, r)$, the flow $\varphi = H^\beta(P)$ is determined by the profile of losses $z_i = x_i - y_i$ as follows

$$\varphi_{ia} = \frac{\beta(z_i)}{\sum_{g(a)} \beta(z_j)} r_a$$

and z is the unique solution of the system

$$x_i = z_i + \sum_{a \in f(i)} \frac{\beta(z_i)}{\sum_{g(a)} \beta(z_j)} r_a$$

This system appears in [10] for the special case of the proportional method, $\beta(z) = z$.

Theorem 4 *Any standard method $h^{\beta,A} \in \widetilde{\mathcal{LC}}$ can be extended to $\mathcal{H}(CSY, MIR)$.*

Proof sketch

Step 1: Given a non strict loss calibrated method $h^\beta \in \mathcal{LC}^{ns}$ we define one of its extensions H^β to $\mathcal{H}(CSY, MIR)$ as the solution of program (21), well defined because $B(0)$ is finite. Given $P = (G, x, r)$, and for a flow φ the usual notation $y_i = \varphi_{if(i)}$ and $z_i = x_i - y_i$, the KKT conditions for $\varphi = H^\beta(G, x, r)$ are

$$\varphi_{\cdot a} = h^\beta((z_i + \varphi_{ia})_{i \in g(a)}, r_a) \Leftrightarrow \{z_i > 0 \Rightarrow \frac{\varphi_{ia}}{\beta(z_i)} = \max_j \frac{\varphi_{ja}}{\beta(z_j)} \text{ for all } i, j \in g(a)$$

To check that H^β is well defined and in $\mathcal{H}(CSY, MIR)$ proceeds exactly as in steps 1,2 of the previous proof (CONT (step2) is easier because $B(0)$ is finite). We omit the details.

We also need an extension H^{ul} of the uniform losses method h^{ul}, which was introduced in [10]. Given $P = (G, x, r)$ we determine first the profile of shares y as follows

$$y = \arg \min_{\varphi \in \mathcal{F}(G,x,r)} \sum_{i \in N} W(x_i - y_i)$$

where W is an arbitrary strictly convex function ($x - y$ is Lorenz dominant among all profiles of losses achievable in a max-flow). Then the flow φ is

$$\varphi = \arg \min_{\varphi \in \mathcal{F}(G,y,r)} \sum_{ia \in G} En(\varphi_{ia})$$

Note that replacing En by any strictly convex function V would still give an extension of h^{ul}, albeit a different one.

Step 2: We now extend a general hybrid loss calibrated method $h^{\beta,A}$ (Proposition 2). Fix a problem $P = (N, Q, G, x, r) \in \mathcal{P}$, and consider the max-flow problem in the bipartite graph $(N, Q, G, (x - A)_+, r)$ (where $(x - A)_+ = ((x_i - A)_+)_{i \in N}$). It is not necessarily in $\mathcal{P}$. The Gallai Edmunds decomposition of this flow graph is a pair of partitions $N = N_+ \cup N_-$ and $Q = Q_+ \cup Q_-$ such that:

- $G(N_-, Q_-) = \varnothing$
- $(G(N_-, Q_+), x, r)$ is overdemanded: $r_B \leq \sum_{g(B) \cap N_-} (x_i - A)_+$ for all $B \subseteq Q_+$
- $(G(N_+, Q_-), x, r)$ is underdemanded: $\sum_M (x_i - A)_+ \leq r_{f(M) \cap Q_-}$ for all $M \subseteq N_+$

Observe that this last problem is underdemanded when the claims are decreased by A, but remains overdemanded if we use the original claims: this is because the agents in N_- have no link to any resources in Q_-, and the original problem P is overdemanded.

We define the extension $H^{\beta,A}$ of $h^{\beta,A}$ by applying the extension H^β of h^β in step 1 to the (N_-, Q_+)-subproblem, and an extension H^{ul} of h^{ul} to the (N_+, Q_-)-subproblem. We write $\varphi_{[S]}$ for the restriction of φ to $G(S, f(S))$.

$$\varphi^{\beta,A}_{[N_-]} = H^\beta(N_-, Q_+, G(N_-, Q_+), (x - A)_+, r)$$

$$\varphi^{\beta,A}_{[N_+]} = H^{ul}(N_+, Q_-, G(N_+, Q_-), x, r)$$

That this extension satisfies CSY and MIR follows directly. Continuity comes from the continuity properties of these extensions, and from the fact that the edge-flows in any balanced subproblem are identical and do not depend on the extension used.■

References

[1] Ahuja R.K., Magnati T.L., and Orlin J.B., (1993). "Network Flows: Theory, Algorithms and Applications," Prentice Hall.

[2] Aumann, R.J., and Maschler, M., (1985). "Game-theoretic analysis of a bankruptcy problem from the Talmud", *Journal of Economic Theory*, 36, 195-213.

[3] Balinski, M., and H.P. Young, (1982). *Fair Representation:Meeting the Ideal of One Man one Vote, New Haven, Conn.: Yale University press.*

[4] Berge, C., (1963). *Topological Spaces,* London: Oliver and Boyd.

[5] Bohm, V., "On the continuity of the optimal policy set for linear programs", SIAM, J. Appl. Math., 28, 2, 303-306, 1975.

[6] Cook, K., and K. Hegtvedt, (1983). "Distributive justice, equity and equality", *Annual Reviews of Sociology,* 9, 217-241.

[7] Deutsch, M., (1975). "Equity, equality and need: what determines which value will be used as the basis of distributive justice?", *Journal of Social Issues,* 3, 31, 137-149.

[8] Juarez, R., (2005). "Composition, scale invariance and convex rationing solutions", mimeo, Rice University.

[9] Moulin, H., (2002). "Axiomatic Cost and Surplus-Sharing," in *Handbook of Social Choice and Welfare*, K. Arrow, A, Sen and K. Suzumura eds, North-Holland.

[10] Moulin, H., and J. Sethuraman. "The bipartite rationing problem", forthcoming, *Operations Research*.

[11] O'Neill, B., (1982). "A problem of rights arbitration from the Talmud", *Mathematical Social Sciences,* 2, 345-371.

[12] Rescher, N., (1966). *Distributive Justice,* Indianapolis: Bobbs-Merill.

[13] Sundaram, R. K., *A first course in optimization theory*, Cambridge University Press, 1996.

[14] Thomson, W., (2003). "Axiomatic and game-theoretic analysis of bankruptcy and taxation problems", *Mathematical Social Sciences,* 45, 3, 249-297.

[15] Thomson, W., (2005). "Consistent allocation rules", mimeo, University of Rochester.

[16] Young, H. P., (1987). "On dividing an amount according to individual claims or liabilities," *Mathematics of Operations Research*, 12, 397-414.

[17] Young, H.P., (1987). "Distributive justice in taxation", *Journal of Economic Theory*, 48, 321-335.

9 Appendix 1: equivalence of (1) and (3)

Fix a problem (N, x, t) throughout. The solution y of (3) is characterized by the Kuhn-Tucker optimality conditions, i.e., for all i:

$$y_i > 0 \Rightarrow \frac{\partial u}{\partial_- w}(x_i, y_i) \leq \min_{j \in N} \frac{\partial u}{\partial_+ w}(x_j, y_j)$$

$$y_i = 0 \Rightarrow \frac{\partial u}{\partial_+ w}(x_i, y_i) \geq \max_{j \in N} \frac{\partial u}{\partial_- w}(x_j, y_j)$$

Let (y, λ) be the solution of (1): by definition of $\frac{\partial u}{\partial_{+,-} w}$ we have for all i:

$$\frac{\partial u}{\partial_- w}(x_i, y_i) \leq f(\lambda) \leq \frac{\partial u}{\partial_+ w}(x_i, y_i)$$

implying the KT system at once.

Conversely let y be the solution of (3). For any two distinct i, j, we assume $[\frac{\partial u}{\partial_- w}(x_i, y_i), \frac{\partial u}{\partial_+ w}(x_i, y_i)] < [\frac{\partial u}{\partial_- w}(x_j, y_j), \frac{\partial u}{\partial_+ w}(x_j, y_j)]$ and derive a contradiction. The KT conditions imply $y_j = 0$ and $y_i = x_i$. From $y_j = \theta(x_j, 0)$ and (2) we get $\frac{\partial u}{\partial_- w}(x_j, y_j) \leq f(0)$, while $y_i \geq \theta(x_i, 1)$ and (2) give $f(1) \leq \frac{\partial u}{\partial_+ w}(x_i, \theta(x_i, 1)) \leq \frac{\partial u}{\partial_+ w}(x_i, y_i)$, contradiction because f increases strictly.

Thus the intervals $[\frac{\partial u}{\partial_- w}(x_i, y_i), \frac{\partial u}{\partial_+ w}(x_i, y_i)]$ are pairwise overlapping, which means that they contain a common element $f(\lambda)$. Then (y, λ) is a solution of (1), as was to be proved.$\blacksquare$

10 Appendix 2: proof of Theorems 1,2

10.1 Structure of $\mathcal{H}^0(CSY, CVX^*)$

We fix in this subsection a method $\mathcal{H}^0(CSY, CVX^*)$: by Lemmas 2,3, it is a parametric method θ meeting (9); we also assume that θ is clean ($\lambda \neq \lambda' \Rightarrow \theta(\cdot, \lambda) \neq \theta(\cdot, \lambda')$).

10.1.1 the correspondence B

The following correspondence B from $\mathbb{R}_+^2$ into $\mathbb{R}_+$ is the key tool in our proof:

$$B(z, \lambda) = \{y \geq 0 | y = \theta(z + y, \lambda)\} \text{ for all } (z, \lambda) \in \mathbb{R}_+^2 \tag{23}$$

Note first that B is an alternative representation of θ in the following sense: fix any $(v, \lambda) \in \mathbb{R}_+^2$ then for all $y \in \mathbb{R}_+$ we have

$$y = \theta(v, \lambda) \Leftrightarrow y \in B(v - y, \lambda)$$

The fact that the equation $y \in B(v - y, \lambda)$ with unknown y always has a unique solution places many constraints on B, which are the object of this and the next two subsections of this proof.

Note first that $B(z,\lambda)$ can be empty: e.g., for h^{ul} we need to solve $y = (z+y-\frac{1}{\lambda})_+$ which is only possible if $z \leq \frac{1}{\lambda}$. Then we have $B(z,\lambda) = \{0\}$ if $z < \frac{1}{\lambda}$, and $B(\frac{1}{\lambda},\lambda) = \mathbb{R}_+$.

If $B(z,\lambda)$ is non empty, it is closed because θ is continuous. It is also convex: if $y_1, y_2 \in B(z,\lambda)$ and $y_1 < y_2$, we have

$$\theta(z+y_2,\lambda) - \theta(z+y_1,\lambda) = (z+y_2) - (z+y_1)$$

As $y \to \theta(z+y,\lambda)$ is 1-Lipschitz, this function is affine in $[y_1,y_2]$, therefore $[y_1,y_2] \subset B(z,\lambda)$, proving the claim.

Thus $B(z,\lambda)$ is a compact interval $[\underline{b}(z,\lambda), \overline{b}(z,\lambda)] \subset \mathbb{R}_+$ or a half-line $[\underline{b}(z,\lambda), \infty[$, in which case we set $\overline{b}(z,\lambda) = \infty$.

Next we define $M(\lambda) = \{z | B(z,\lambda) \neq \varnothing\}$ for any $\lambda \geq 0$. We claim that $\{B(z,\lambda) \neq \varnothing$ and $z' < z\}$ imply $B(z',\lambda) \neq \varnothing$. Pick y such that $y = \theta(z+y,\lambda)$; then $y' - \theta(z'+y',\lambda)$ is non positive at $y' = 0$ and non negative at $y' = y$, so the continuity of θ gives the claim. As $M(\lambda)$ always contains 0 (because $0 \in B(0,\lambda)$), we see that $M(\lambda)$ is an interval $[0, m(\lambda)]$ or $[0, m(\lambda)[$, where in the second case $m(\lambda) \leq \infty$.

We illustrate these three possible configurations. We already showed that for h^{ul} we have $M(\lambda) = [0, \frac{1}{\lambda}]$. For h^{pro} and h^{ug} we have $M(\lambda) = \mathbb{R}_+$ but very different behavior for B: $B^{pro}(z,\lambda) = \{\lambda z\}$; $B^{ug}(z,\lambda) = \{\lambda\}$ if $z > 0$, $B^{ug}(0,\lambda) = [0,\lambda]$. Finally consider the dual s-equal sacrifice method with $s(z) = -\frac{1}{z}$: we have $\theta(v,\lambda) = \frac{\lambda v^2}{1+\lambda v}$, $B(z,\lambda) = \frac{\lambda z^2}{1-\lambda z}$, and $M(\lambda) = [0, \frac{1}{\lambda}[$.

We conclude this step by some properties of m and M. Clearly $M(0) = \mathbb{R}_+$ so $m(0) = \infty$. Next $m(\lambda) > 0$ for all λ. By contradiction, assume $M(\lambda) = \{0\}$ for some $\lambda > 0$: this implies $0 < \theta(v,\lambda)$ for all v, hence by continuity $y < \theta(z+y,\lambda)$ for all $y, z \geq 0$; fixing y and letting z go to 0 gives $y \leq \theta(y,\lambda)$, which must be an equality by the 1-Lipschitz property. Now $\theta(y,\lambda) = \theta(y,\infty)$ for all y contradicts the cleanliness of θ.

The next fact is that $M(\lambda)$ weakly decreases in λ, and so does $m(\lambda)$. Indeed $\lambda' < \lambda$ and $y = \theta(z+y,\lambda)$ implies that $\theta(z+y',\lambda') - y'$ is non negative at $y' = 0$ and non positive at $y' = y$, so that $B(z,\lambda') \neq \varnothing$.

We note finally that B has full range, in the following sense

$$\textit{for all } z > 0, y \geq 0, \textit{ there is some } \lambda \geq 0 \textit{ such that } y \in B(z,\lambda) \qquad (24)$$

Indeed $\theta(z+y,\lambda) - y$ is continuous in λ, non positive for $\lambda = 0$, and positive for $\lambda \to \infty$.

10.1.2 properties of $\underline{b}(z,\lambda), \overline{b}(z,\lambda)$

We analyze first the limit behavior of $\overline{b}(z,\lambda)$. For all $\lambda, z \in \mathbb{R}_+$ we have

- 1) $\theta(z+y,\lambda) < y$ for all $y > \overline{b}(z,\lambda)$
- 2) $z < m(\lambda) \Rightarrow \overline{b}(z,\lambda) < \infty$

- 3) if $m(\lambda) < \infty$: $\overline{b}(m(\lambda), \lambda) = \infty$ if $m(\lambda) \in M(\lambda)$; $\lim_{z \to m(\lambda)} \overline{b}(z, \lambda) = \infty$ if $m(\lambda) \notin M(\lambda)$

For 1), the 1-Lipschitz property gives for any $\varepsilon > 0$: $\theta(z + \overline{b}(z, \lambda) + \varepsilon, \lambda) \leq \overline{b}(z, \lambda) + \varepsilon$, which must be a strict inequality by definition of $\overline{b}(z, \lambda)$. For 2) assume $z < m(\lambda)$ and $\overline{b}(z, \lambda) = \infty$. Pick $z', z < z' < m(\lambda)$ and some y' such that $\theta(z' + y', \lambda) = y'$. From $\overline{b}(z, \lambda) = \infty$ we have $\theta(z + y, \lambda) = y$ for all y large enough; pick such y above y' and apply the 1-Lipschitz property:

$$\theta(z' + y, \lambda) \leq \theta(z' + y', \lambda) + (y - y') = y$$

On the other hand

$$\theta(z' + y, \lambda) = \theta(z + (y + z' - z), \lambda) = y + z' - z$$

a contradiction. For 3) assume first $m(\lambda) \in M(\lambda)$ and $\overline{b}(m(\lambda), \lambda) < \infty$: then for any $y > \overline{b}(m(\lambda), \lambda)$, 1) gives $\theta(m(\lambda) + y, \lambda) < y$. Setting $\delta = y - \theta(m(\lambda) + y, \lambda)$ we have $\theta((m(\lambda) + \delta) + (y - \delta), \lambda) = y - \delta$, contradicting the definition of $m(\lambda)$. In the case $m(\lambda) \notin M(\lambda)$, consider the identity

$$\theta((y - \theta(y, \lambda)) + \theta(y, \lambda), \lambda) = \theta(y, \lambda) \tag{25}$$

where $z = y - \theta(y, \lambda)$ is non negative (1-Lipschitz property). As $\overline{b}(z, \lambda)$ increases weakly in z (property 6 below), it is enough to derive a contradiction from the assumption $\sup_{z \geq 0} \overline{b}(z, \lambda) = C < \infty$. The latter assumption implies $\theta(y, \lambda) \leq C$ for all y. Taking $y > m(\lambda) + C$ gives $z = y - \theta(y, \lambda) > m(\lambda)$, and identity (25) reads $\theta(y, \lambda) \in B(z, \lambda)$, contradicting the definition of $m(\lambda)$.

We prove next a continuity and two monotonicity properties:

- 4) the graph of B is closed on its domain $\{(z, \lambda) | z \in M(\lambda)\}$, i.e., $B(z, \lambda)$ is upper-semi-continuous;
- 5) λ-monotonicity: $\lambda \to \underline{b}(z, \lambda)$ and $\lambda \to \overline{b}(z, \lambda)$ are weakly increasing;
- 6) z-monotonicity: $z < z' \Rightarrow \overline{b}(z, \lambda) \leq \underline{b}(z', \lambda)$;

Continuity of θ gives statement 4).

For statement 5), fix z and assume $\lambda' < \lambda$. Setting $y = \underline{b}(z, \lambda)$ we have $y = \theta(z + y, \lambda) \geq \theta(z + y, \lambda')$ and $0 \leq \theta(z + 0, \lambda)$; therefore $B(z, \lambda')$ intersects $[0, y]$ (again by continuity of θ), hence $\underline{b}(z, \lambda') \leq \underline{b}(z, \lambda)$. Next we fix z and $\lambda < \lambda'$, set $y = \overline{b}(z, \lambda)$ and we must prove that $B(z, \lambda')$ contains at least one $y' \geq y$. Suppose it doesn't: then it contains some $y' < y$ (recall we assume $B(z, \lambda') \neq \varnothing$), and we apply the 1-Lipschitz property:

$$y' = \theta(z + y', \lambda') \Rightarrow \theta(z + y, \lambda') \leq \theta(z + y', \lambda') + (y - y') = y$$

Because $y \notin B(z, \lambda')$ this implies $\theta(z + y, \lambda') < y = \theta(z + y, \lambda)$, contradicting the monotonicity of θ in λ.

For statement 6) we fix λ and $z < z'$, pick $y \in B(z,\lambda)$, $y' \in B(z',\lambda)$ and show $y \leq y'$. Indeed $y' < y$ would imply $y' = \theta(z'+y',\lambda) < y = \theta(z+y,\lambda)$, hence $z'+y' < z+y$; this contradicts the 1-Lipschitz property because $y - y' > (z+y) - (z'+y')$.

Note that z-monotonicity of $\overline{b}, \underline{b}$, implies that $B(z,\lambda)$ is not a singleton for at most countably many values of z.

10.1.3 brackets and non brackets

In this step we discuss a special shape of the graph of $B(\cdot,\lambda)$.

For all $\lambda \geq 0$

$$d(\lambda) = \sup\{z | \underline{b}(z,\lambda) = 0\} \leq m(\lambda)$$

is well defined because $\underline{b}(0,\lambda) = 0$. Clearly $d(0) = \infty$, moreover d is weakly decreasing in λ by λ-monotonicity(property 6).

If $d(\lambda) = m(\lambda) = z_0$, then $z_0 > 0$ (recall $m(\lambda) > 0$ for all λ), and the graph of $B(\cdot,\lambda)$ is the following z_0-*bracket*

$$B(z,\lambda) = \{0\} \text{ for } z < z_0 \ ; \ B(z_0,\lambda) = \mathbb{R}_+ \ ; \ B(z,\lambda) = \varnothing \text{ for } z > z_0 \tag{26}$$

Indeed we have $\underline{b}(z_0,\lambda) = 0$ by upper semi continuity of $B(\cdot,\lambda)$ (property 4), and for all z,$z < z_0$, $\overline{b}(z,\lambda) \leq \underline{b}(z_0,\lambda) = 0$, hence $B(z,\lambda) = \{0\}$. If $m(\lambda) \notin M(\lambda)$ property 3 implies a contradiction, therefore $m(\lambda) \in M(\lambda)$ and property 3 give $B(z_0,\lambda) = \mathbb{R}_+$ as claimed.

If $d(\lambda) < m(\lambda)$ then we have

$$0 < \underline{b}(z,\lambda) \leq \overline{b}(z,\lambda) < \infty \text{ for all } z, \ d(\lambda) < z < m(\lambda) \tag{27}$$

The right-hand inequality is property 2 above, the left-hand one is by definition of $d(\lambda)$. These inequalities imply the following alternative characterization of brackets. For any $\lambda \geq 0$:

$$B(z,\lambda) = \{0\} \text{ or } \mathbb{R}_+ \text{ for all } z \in M(\lambda)$$

$$\Leftrightarrow \{B(\cdot,\lambda) \text{ is some } z_0\text{-bracket, or } \lambda = 0\} \tag{28}$$

By (27) the right-hand assumption implies $d(\lambda) = m(\lambda)$; if this number is finite we saw above that $B(\cdot,\lambda)$ is the corresponding bracket; if $d(\lambda) = m(\lambda) = \infty$ then $B(z,\lambda) = \{0\}$ for all z, which implies $\lambda = 0$ by cleanliness of the parametrization.

10.1.4 exploiting CVX*

We fix $\widehat{\lambda} > 0$ in this step and the next, and assume that $B(\cdot,\widehat{\lambda})$ is not a bracket. We pick $\delta \in]0,1]$ and set $\delta' = 1 - \delta$. Then we apply CVX* in parametric form (Lemma 3) for $\widehat{\lambda}$ and δ': equation (9) holds for some λ. Apply it to $v = z + y$ where $z \in \widehat{M} = M(\widehat{\lambda})$ and $y \in B(z,\widehat{\lambda})$, this gives

$$\theta(z+y-\delta' y,\lambda)=(1-\delta')y \Leftrightarrow \theta(z+\delta y,\lambda)=\delta y \Leftrightarrow \delta y \in B(z,\lambda)$$

Therefore $\widehat{M} \subseteq M(\lambda)$ and $\delta B(z,\widehat{\lambda}) \subseteq B(z,\lambda)$ for all $z \in \widehat{M}$. We claim that this inclusion is in fact an equality

$$\delta B(z,\widehat{\lambda}) = B(z,\lambda) \text{ for all } z \in \widehat{M} \tag{29}$$

By contradiction, suppose $\delta B(z,\widehat{\lambda}) \subsetneq B(z,\lambda)$. Then $B(z,\lambda)$ is a true interval (not a singleton) and for some $\varepsilon > 0$ at least one of the two following statement is true:

- $\bar{b}(z,\widehat{\lambda}) < \infty$ and $\bar{b}(z,\lambda) \geq \delta\bar{b}(z,\widehat{\lambda}) + \varepsilon$
- $\delta\underline{b}(z,\widehat{\lambda}) \geq \underline{b}(z,\lambda) + \varepsilon$

Assume the first statement. Then $z < \widehat{m} = m(\widehat{\lambda})$, because $\bar{b}(\widehat{m},\widehat{\lambda}) = \infty$ (property 3 above). For any $z', z < z' < \widehat{m}$, we now apply z-monotonicity (property 6):

$$\delta\underline{b}(z',\widehat{\lambda}) \geq \underline{b}(z',\lambda) \geq \bar{b}(z,\lambda) \geq \delta\bar{b}(z,\widehat{\lambda}) + \varepsilon$$

When z' converges to z, this contradicts the fact that the graph of B is closed (property 4). Deriving a contradiction from the second statement is entirely similar. First $\underline{b}(0,\widehat{\lambda}) = 0$, hence $z > 0$. For $z' < z$ we have $\delta\bar{b}(z',\widehat{\lambda}) \leq \bar{b}(z',\lambda) \leq \underline{b}(z,\lambda) \leq \delta\underline{b}(z,\widehat{\lambda}) - \varepsilon$, and a similar contradiction of the upper-semi-continuity of B. Equality (29) is proven.

We check next that the inclusion $\widehat{M} \subseteq M(\lambda)$ (proven at the end of step 10.1.1: $M(\lambda)$ is weakly decreasing) is in fact an equality. There is nothing to prove if $\widehat{M} = \mathbb{R}_+$. If $\widehat{M} = [0,\widehat{m}]$ then property 3 and (29) imply $\bar{b}(\widehat{m},\lambda) = \infty$, hence $\widehat{m} = m(\lambda)$ by z-monotonicity. If $\widehat{M} = [0,\widehat{m}[$ then property 3 again and (29) imply $\lim_{z\to\widehat{m}} \bar{b}(z,\lambda) = \infty$, and the desired conclusion by z-monotonicity again.

Now we check that λ, satisfying (29) and $\widehat{M} = M(\lambda)$, is uniquely determined by $\widehat{\lambda}$ and δ. Otherwise we have λ^1, λ^2 satisfying (29) and such that $M(\lambda^1) = M(\lambda^2)$ and $B(\cdot,\lambda^1) = B(\cdot,\lambda^2)$. This implies $\theta(z+y,\lambda^1) = \theta(z+y,\lambda^2)$ for every $z \in M(\lambda^i)$ and $y \in B(z,\lambda^i)$. If $M(\lambda^i) = \mathbb{R}_+$ the range of such $z+y$ is clearly $\mathbb{R}_+$, by z-monotonicity and upper-semi-continuity of B. This range is still $\mathbb{R}_+$ when $\widehat{m}$ is finite by property 3 above, and upper-semi-continuity. We conclude $\lambda^1 = \lambda^2$ by cleanliness of θ.

The next fact is $\lambda \leq \widehat{\lambda}$, with equality only if $\delta = 1$. Assume $\delta < 1$. Because $B(\cdot,\widehat{\lambda})$ is not a bracket, we can choose z such that $d(\widehat{\lambda}) = \widehat{d} < z < \widehat{m}$ and $0 < \underline{b}(z,\widehat{\lambda}) \leq \bar{b}(z,\widehat{\lambda}) < \infty$ ((27)). Now (29) implies $\bar{b}(z,\lambda) < \bar{b}(z,\widehat{\lambda})$, and we can apply property 1 above:

$$\theta(z+\bar{b}(z,\widehat{\lambda}),\lambda) < \bar{b}(z,\widehat{\lambda}) = \theta(z+\bar{b}(z,\widehat{\lambda}),\widehat{\lambda})$$

and the conclusion $\lambda < \widehat{\lambda}$ follows.

Next we analyze, for a fixed $\widehat{\lambda}$, the mapping g: $]0,1] \ni \delta \to \lambda = g(\delta)$. It is one-to-one, otherwise (29) implies $\delta^1 B(z,\widehat{\lambda}) = \delta^2 B(z,\widehat{\lambda})$ for all $z \in \widehat{M}$, where $\delta^1 \neq \delta^2$: this means that each $B(z,\widehat{\lambda})$ is either $\{0\}$ or $\mathbb{R}_+$, which is excluded because $B(\cdot,\widehat{\lambda})$ is not a bracket (see (28)). Moreover g is continuous: the equality $\lambda = g(\delta)$ is equivalent to the system of equalities (29), and each equality $\delta B(z,\widehat{\lambda}) = B(z,\lambda)$ (for fixed $z \in \widehat{M}$) means

$$\delta y = \theta(z+y,\lambda) \text{ for all } y \text{ such that } y = \theta(z+y,\widehat{\lambda})$$

therefore the graph of g is an intersection of closed sets, hence it is closed. Finally g is increasing, because $\delta^1 < \delta^2$ and $\lambda^i = g(\delta^i), i = 1,2$, implies $B(\cdot,\lambda^1) = \frac{\delta^1}{\delta^2} B(\cdot,\lambda^2)$, and $\lambda^1 < \lambda^2$ follows exactly like in the previous paragraph.

We conclude that g is an increasing homeomorphism from $]0,1]$ into an interval $]\widehat{\ell}_-,\widehat{\lambda}]$ where $\widehat{\ell}_-$ depends on $\widehat{\lambda}$. We write this function $g^{\widehat{\lambda}}$ to remind us that it depends on $\widehat{\lambda}$. For all $\delta \in]0,1]$, $g^{\widehat{\lambda}}(\delta)$ is defined by the system of equalities

$$\delta B(z,\widehat{\lambda}) = B(z, g^{\widehat{\lambda}}(\delta)) \text{ for all } z \in \widehat{M} = M(g^{\widehat{\lambda}}(\delta)) \tag{30}$$

Note that for any $\delta \in]0,1]$, $B(\cdot, g^{\widehat{\lambda}}(\delta))$ is not a bracket and moreover

$$d(g^{\widehat{\lambda}}(\delta)) = \widehat{d} \text{ and } \ell_-(g^{\widehat{\lambda}}(\delta)) = \widehat{\ell}_- \tag{31}$$

The first equality because for any $z \in \widehat{M}$, $\underline{b}(z,\lambda)$ is positive if and only if $\underline{b}(z,\widehat{\lambda})$ is. For the second we set $\lambda = g^{\widehat{\lambda}}(\delta)$ and compute

$$B(\cdot, g^{\lambda}(\delta')) = \delta' B(\cdot,\lambda) = \delta\delta' B(\cdot,\widehat{\lambda}) = B(\cdot, g^{\widehat{\lambda}}(\delta\delta'))$$

This means $g^{\lambda}(\delta') = g^{\widehat{\lambda}}(\delta\delta')$ for all $\delta,\delta' \in]0,1]$, proving the second equality.

10.1.5 extending g^λ to $]0,\infty[$

As in the previous substep, we fix $\widehat{\lambda} > 0$ such that $B(\cdot,\widehat{\lambda})$ is not a bracket, and we use the notation $\widehat{m} = m(\widehat{\lambda})$, etc...

We prove first some simple boundary conditions on $\widehat{\ell}_- = \ell_-(\widehat{\lambda})$:

$$d(\widehat{\ell}_-) \geq \widehat{m} \tag{32}$$

$$\widehat{m} = \infty \Rightarrow \widehat{\ell}_- = 0 \tag{33}$$

$$\widehat{m} < \infty \Rightarrow \widehat{\ell}_- > 0 \text{ and } B(\cdot,\widehat{\ell}_-) \text{ is the } \widehat{m}\text{-bracket} \tag{34}$$

For (32) it is enough to observe that (30) and (27) together imply $\inf_{\widehat{\ell}_- < \lambda'} \underline{b}(z,\lambda') = 0$ for $z \in \widehat{M}$. Next if $\widehat{m} = \infty$ (32) implies $d(\widehat{\ell}_-) = \infty$ therefore $B(z,\widehat{\ell}_-) = \{0\} \Leftrightarrow \theta(z,\widehat{\ell}_-) = 0$, for all z. This proves (33).

Assume next $\widehat{m} < \infty$ and $\widehat{\ell}_- = 0$. Then for any $z > \widehat{m}$ we have $B(z, \lambda) = \varnothing$ for all $\lambda > \widehat{\ell}_- = 0$, and $B(z, 0) = \{0\}$, therefore $1 \in B(z, \lambda)$ cannot hold for any λ, in contradiction of (24). This prove the implication in (34).

We show now that $B(\cdot, \widehat{\ell}_-)$ is the $\widehat{m}$-bracket. We know from (32) $B(z, \widehat{\ell}_-) = \{0\}$ for $z < \widehat{m}$, which implies $\theta(z, \widehat{\ell}_-) = 0$ for the same z (because the identity (25) applied to z and $\widehat{\ell}_-$ gives $\theta(z, \widehat{\ell}_-) \in B(z - \theta(z, \widehat{\ell}_-), \widehat{\ell}_-)$); by continuity $\theta(\widehat{m}, \widehat{\ell}_-) = 0$. We want to show $\theta(z, \widehat{\ell}_-) = z - \widehat{m}$ for all $z \geq \widehat{m}$ (implying $B(\widehat{m}, \widehat{\ell}_-) = \mathbb{R}_+$ as desired). The 1-Lipschitz property gives $\theta(z, \widehat{\ell}_-) \leq z - \widehat{m}$, so we need to derive a contradiction if $\theta(z, \widehat{\ell}_-) < z - \widehat{m}$ for some $z > \widehat{m}$. The identity (25) applied to such z and to $\lambda > \widehat{\ell}_-$ gives $\theta(z, \lambda) \in B(z - \theta(z, \lambda), \lambda)$, implying $z - \theta(z, \lambda) \leq m(\lambda) \leq \widehat{m}$ (recall m is weakly decreasing): thus $\theta(z, \lambda) \geq z - \widehat{m}$ for all $\lambda > \widehat{\ell}_-$, whereas $\theta(z, \widehat{\ell}_-) < z - \widehat{m}$, contradiction. The proof of (34) is complete.

We check now that if for λ^1, λ^2, are such that $]d(\lambda^1), m(\lambda^1)[$ and $]d(\lambda^2), m(\lambda^2)[$ have a non empty intersection, they coincide and $\ell_-(\lambda^2) = \ell_-(\lambda^1)$. Assume $\lambda^1 < \lambda^2$ and pick z in this intersection. Then $\bar{b}(z, \ell_-(\lambda^2)) = 0$ (as $z \leq m(\lambda^2) \leq d(\ell_-(\lambda^2)))$ and $\underline{b}(z, \lambda^1) > 0$ ((27)), implying $\ell_-(\lambda^2) < \lambda^1$, hence the claim by (30) and (31).

We go back to a fixed $\widehat{\lambda} > 0$ such that $B(\cdot, \widehat{\lambda})$ is not a bracket, and we extend the domain of $g^{\widehat{\lambda}}$ to an interval $]\widehat{\ell}_-, \widehat{\ell}^+[$ containing λ, and g^λ to an homeomorphism from $]\widehat{\ell}_-, \widehat{\ell}^+[$ to $]0, \infty[$, defined as in (30) by the equality $\delta B(\cdot, \widehat{\lambda}) = B(\cdot, g^{\widehat{\lambda}}(\delta))$, and such that $m(\lambda), d(\lambda)$, and $\ell_-(\lambda), \ell^+(\lambda)$ are constant for all $\lambda \in]\widehat{\ell}_-, \widehat{\ell}^+[$.

We fix $z \in]\widehat{d}, \widehat{m}[$ and $\delta > 1$. The construction of $g^{\widehat{\lambda}}(\delta)$ takes two steps. We first pick λ such that $\delta \bar{b}(z, \widehat{\lambda}) = \theta(z + \delta \bar{b}(z, \widehat{\lambda}), \lambda)$: this is possible because we have two strict inequalities in opposite directions at $\lambda = 0$ and $\lambda = \infty$ ($\delta \bar{b}(z, \widehat{\lambda})$ and z are both positive). Then we have $m(\lambda) \geq z$ (by definition of m), $\lambda > \widehat{\lambda}$ (as $\bar{b}(z, \lambda) \geq \delta \bar{b}(z, \widehat{\lambda}) > \bar{b}(z, \widehat{\lambda})$), and $d(\lambda) \leq z$ (because $\bar{b}(z, \lambda) > 0$). In fact $d(\lambda) = z$ would imply $\underline{b}(z, \widehat{\lambda}) \leq \underline{b}(z, \lambda) = 0$, contradicting (27). Therefore $d(\lambda) < z$ so $]\widehat{d}, \widehat{m}[$ and $]d(\lambda), m(\lambda)[$ overlap, hence they coincide and so do $\widehat{\ell}_-$ and $\ell_-(\lambda)$.

By (30) we have $B(\cdot, \widehat{\lambda}) = \delta' B(\cdot, \lambda)$ for some $\delta' \in]0, 1]$, and by construction of λ we have $\delta \bar{b}(z, \widehat{\lambda}) \leq \bar{b}(z, \lambda)$; together these two facts imply $\delta\delta' \leq 1$. Now we can pick $\lambda^" < \lambda$ such that $\lambda^" > \widehat{\ell}_-$ and $B(\cdot, \lambda^") = \delta\delta' B(\cdot, \lambda)$: then $B(\cdot, \lambda^") = \delta B(\cdot, \widehat{\lambda})$. This concludes the construction of $g^{\widehat{\lambda}}(\delta)$ for $\delta > 1$.

Checking that $g^{\widehat{\lambda}}$ is an increasing homeomorphism of $]1, \infty[$ into an interval $[\widehat{\lambda}, \widehat{\ell}^+[$ proceeds exactly as in substep 10.1.4. That $m(\lambda), d(\lambda)$, and $\ell_-(\lambda)$, are constant on $]\widehat{\ell}_-, \widehat{\ell}^+[$ is clear from the construction above; the same holds true for $\widehat{\ell}^+$ by the same argument used to prove (31).

The final statement of this subsection is used in the proof of Theorem 2:

$$\widehat{\ell}^+ < \infty \Rightarrow m(\widehat{\ell}^+) = \widehat{d} \text{ and } B(\cdot, \widehat{\ell}^+) \text{ is the } \widehat{d}\text{-bracket} \tag{35}$$

Indeed (30) and (27) together imply $\sup_{\lambda<\widehat{\ell}^+} \underline{b}(z,\lambda) = \infty$ for $z \in]\widehat{d}, \widehat{m}[$; if for some such z the set $B(z,\widehat{\ell}^+)$ is non empty, then $\underline{b}(z,\widehat{\ell}^+)$ is finite, which contradicts the fact that $\underline{b}(z,\lambda)$ is weakly increasing in λ. This proves $m(\widehat{\ell}^+) \leq \widehat{d}$. If $m(\widehat{\ell}^+) < \widehat{d}$ we pick $z \in]m(\widehat{\ell}^+), \widehat{d}[$ so that $B(z,\widehat{\ell}^+) = \varnothing$, and $B(z,\lambda) = \varnothing$ for all $\lambda \geq \widehat{\ell}^+$, whereas $B(z,\lambda) = \{0\}$ for all $\lambda < \widehat{\ell}^+$; this contradicts (24). Thus $m(\widehat{\ell}^+) = \widehat{d}$. By continuity of θ we have $\theta(\widehat{d},\widehat{\ell}^+) = 0$ (because $B(z,\lambda) = \{0\}$ for $z < \widehat{d}$ and λ close to $\widehat{\ell}^+$). It remains to check $\theta(z,\widehat{\ell}^+) = z - \widehat{d}$ for all $z \geq \widehat{d}$. By the 1-Lipschitz property we only need to show that $\theta(z,\widehat{\ell}^+) < z - \widehat{d}$ is impossible: it would imply $z - \theta(z,\widehat{\ell}^+) > \widehat{d}$ and the identity $\theta(z,\widehat{\ell}^+) \in B(z - \theta(z,\widehat{\ell}^+), \widehat{\ell}^+)$ would establish the desired contradiction.

10.1.6 the structure of $\mathcal{H}^0(CSY, CVX^*)$

We gather the results of the two previous steps. Fix an arbitrary method h in $\mathcal{H}^0(CSY, CVX^*)$, with associated correspondence B ((23)). For any $\widehat{\lambda} > 0$, if $B(\cdot,\widehat{\lambda})$ is not a bracket, there exists an interval $]\ell_-, \ell^+[$ containing $\widehat{\lambda}$, an interval $]d,m|$ for the variable z (where $]d,m|$ can be either $]d,m[$ or $]d,m]$), and a correspondence $\widetilde{B} : [0,m| \to \mathbb{R}_+$, such that $B(\cdot,\lambda)$, and as follows in $]\ell_-, \ell^+[$:

$$B(z,\lambda) = f(\lambda)\widetilde{B}(z) \text{ in } [0,m| \times]\ell_-, \ell^+[\text{ ; } B(z,\lambda) = \varnothing \text{ in }]m,\infty[\times]\ell_-, \ell^+[\quad (36)$$

where $f :]\ell_-, \ell^+[\to]0,\infty[$ is an increasing homeomorphism, and the correspondence $\widetilde{B}$ is such that

- $\widetilde{B}(z) = \{0\}$ for all $z < d$
- $\widetilde{B}$ is upper semi continuous and z-monotonic (property 6)
- if $]d,m| =]d,m]$, then $\widetilde{B}(m) = [\underline{b}(m), \infty[$; if m is finite and $]d,m| =]d,m[$, then $\lim_{z \to m} \bar{b}(z) = \infty$

Because a continuous and strictly increasing transformation of λ does not change the method it parametrizes, if ℓ^+ is finite we can always choose the canonical homeomorhism $f(\lambda) = \frac{\lambda - \ell_-}{\ell^+ - \lambda}$. Then the parametrization $\theta(\cdot,\lambda)$ is as follows in $]\ell_-, \ell^+[$:

$$\text{for all } v \text{ and all } \lambda, \ell_- < \lambda < \ell^+ : \theta(v,\lambda) = y \Leftrightarrow y \in \frac{\lambda - \ell_-}{\ell^+ - \lambda}\widetilde{B}(v - y) \quad (37)$$

(existence and uniqueness of y follow from the properties of $\widetilde{B}$).

Any two such *partial parametrizations* $\Delta = (\ell_-, \ell^+, d, m; \widetilde{B})$ and Δ' of h have disjoint intervals $]\ell_-, \ell^+[$ and $]\ell'_-, \ell^{'+}[$ as well as disjoint $]d,m|$ and $]d',m'|$. Thus we have a finite or countable set of partial parametrizations, and for every parameter λ outside the union of the corresponding intervals, $B(\cdot,\lambda)$ is a bracket. For instance (34) says that whenever m is finite, $B(\cdot,\ell_-)$ is the m-bracket, i.e., $\theta(v,\ell_-) = (v-m)_+$ for all v. Similarly if $\ell^+ < \infty$, (35) means $\theta(v,\ell^+) = (v-d)_+$ for all λ.

We illustrate the general construction for the standard methods characterized in section 7.

If $B(\cdot,\lambda)$ is a bracket for all λ, then there is a function $\lambda \to g(\lambda)$ such that $\theta(\cdot,\lambda) = (\cdot - g(\lambda))_+$, and or method is h^{ul}.[10]

If $]\ell_-, \ell^+[=]0,\infty[$, and $\widetilde{B}$ is single valued, we find the loss calibrated method $h^{\widetilde{B}}$.

The hybrid method $h^{\beta,A}$ has $\ell_- = 0$, $\ell^+ = \frac{1}{A}$ (for instance), $d = A$ and $m = \infty$. Finally $\widetilde{B}(z) = \beta((z-A)_+)$. Then $B(\cdot, \frac{1}{A})$ is the A-bracket ((35)) and $B(\cdot,\lambda)$ is a bracket for $\lambda \geq \frac{1}{A}$; we can choose the parametrization $\theta(\cdot,\lambda) = (\cdot - \frac{1}{\lambda})_+$ for such λ. For $\lambda < \frac{1}{A}$ we get from (37):

$$\theta(v,\lambda) = y \Leftrightarrow y = \frac{\lambda}{\frac{1}{A} - \lambda}\beta((v - A - y)_+) = \frac{A\lambda}{1 - A\lambda}\beta((v-A)_+ - y)$$

exactly as in (17), (18), up to the innocuous change of parameter $\frac{A\lambda}{1-A\lambda} = \frac{\lambda'}{1-A\lambda'}$ in the interval $[0, \frac{1}{A}[$.

Whether and how a fully general method in $\mathcal{H}^0(CSY, CVX^*)$ can be extended into $\mathcal{H}(CSY, CVX^*)$ is an interesting open question.

10.2 Proof of Theorem 1

10.2.1 statement *i)*

The "if" statement was proven in section 6, in the discussion following Proposition 1. We prove "only if". Fix a parametric method $\theta \in \mathcal{H}^0(CSY, CVX^*)$ satisfying SI. It is easy to check that SI (for any method in $\mathcal{H}^0(CSY)$) amounts to the following: for all $\lambda, \gamma > 0$ there exists $\lambda' > 0$ such that

$$\theta(\gamma v, \lambda) = \gamma\theta(v, \lambda') \text{ for all } v \geq 0$$

Fixing $\lambda, \gamma > 0$, this implies

$$\text{for all } y, z\text{: } y \in \theta(z + y, \lambda) \Leftrightarrow \gamma y \in \theta(\gamma z + \gamma y, \lambda')$$
$$\Rightarrow \gamma B(z,\lambda) = B(\gamma z, \lambda') \text{ for all } z \geq 0 \quad (38)$$

In particular $m(\lambda') = \gamma m(\lambda)$ and $d(\lambda') = \gamma d(\lambda)$.

If for all $\lambda > 0$, $B(\cdot,\lambda)$ is a bracket, θ is the uniform losses method (subsection 6.1.6).

Assume next that $B(\cdot,\lambda)$ is not a bracket for some $\lambda > 0$. Then for γ close enough to 1, $]d(\lambda), m(\lambda)[$ and $]d(\lambda'), m(\lambda')[$ overlap, so $m(\lambda') = m(\lambda)$ and $d(\lambda') = d(\lambda)$. Thus the only possibility is $d(\lambda) = 0$ and $m(\lambda) = \infty$.

Therefore in the analysis of subsection 6.1.6 and property (36), there is a single interval $]\ell_-, \ell^+[=]0,\infty[$, an increasing homeomorphism f from $\mathbb{R}_+$ into itself, and a correspondence $\widetilde{B}$ from $\mathbb{R}_+$ into itself, such that

$$B(z,\lambda) = f(\lambda)\widetilde{B}(z) \text{ for all } \lambda, z \geq 0$$

[10] It is easy to show that g is a bijection from $\mathbb{R}_+$ into itslef.

and $\widetilde{B}$ has the following properties:

- $\widetilde{B}(0) = [0, \bar{b}(0)]$
- for $z > 0$: $\widetilde{B}(z) \subset [\underline{b}(z), \bar{b}(z)]$ where $0 < \underline{b}(z) \leq \bar{b}(z) < \infty$
- $\widetilde{B}$ is upper semi continuous and z-monotonic (properties 4 and 6)

Applying Scale Invariance ((38)) to $\lambda = 1$ and arbitrary positive γ, gives

$$\widetilde{B}(\gamma z) = \gamma' \widetilde{B}(z) \text{ for all } z \geq 0$$

for some positive $\gamma' = F(\gamma)$ that depends upon γ but not on z. If $\widetilde{B}(z)$ is a true interval at some $z > 0$ ($\underline{b}(z) < \bar{b}(z)$), it is also a true interval for any positive z', which is clearly ruled out by z-monotonicity. Thus $\widetilde{B}$ is a continuous single-valued function everywhere except perhaps at 0, positive on $]0, \infty[$.

The above equation $\widetilde{B}(\gamma z) = F(\gamma)\widetilde{B}(z)$ for all positive γ, z, implies

$$\overline{B}(zz') = \overline{B}(z)\overline{B}(z') \text{ for all } z, z' > 0$$

where $\overline{B}$ is a constant multiple of $\widetilde{B}$, that clearly represents the same rationing method as $\widetilde{B}$. It is well known that a continuous function $\overline{B}$ meeting the above equation is a power function $\overline{B}(z) = z^p$. As $\overline{B}$ is weakly increasing, p is non negative. We conclude that, up to replacing λ by the equivalent parameter $\widetilde{\lambda} = f(\lambda)$, B is the function $B(z, \widetilde{\lambda}) = \widetilde{\lambda} z^p$ for all $z > 0$ and $\widetilde{\lambda} \geq 0$. If $p > 0$, upper semi continuity implies this equality holds for $z = 0$ as well. If $p = 0$, upper semi continuity implies $B(0, \widetilde{\lambda}) = [0, \widetilde{\lambda}]$.

Finally we use (23) to derive the parametrization of our method. For $p = 0$, and any $(v, \widetilde{\lambda}) \in \mathbb{R}_+^2$, the unique solution y of $y \in B(v - y, \widetilde{\lambda})$ is clearly $y = v \wedge \widetilde{\lambda}$, so we have h^{ug}. For $p > 0$ it solves $y = \widetilde{\lambda}(v - y)^p$, precisely like (11) when β is the p-power function.

10.2.2 statement *ii*)

The parametric method θ meets SRM if and only if for all $v > 0$, $\theta(v, \lambda)$ is strictly increasing in λ.

For the "if" statement: the parametric representation θ^β of a method h^β in $\mathcal{LC}^s$ is given by (11) in Proposition 1. It is clearly strictly increasing in λ. On the other hand for h^β in $\mathcal{LC}^{ns}$, θ^β defined by (13) is weakly but not strictly increasing in λ.

We prove now "only if". Pick a parametric method $\theta \in \mathcal{H}^0(CSY, CVX^*)$ satisfying SRM. This implies $d(\lambda) = 0$ for any $\lambda > 0$. Assume $d(\lambda) > 0$: then $B(z, \lambda) = \{0\}$ for $z \in [0, d(\lambda)[$, implying $\theta(v, \lambda) = 0$ for $v \in]0, d(\lambda)[$. Therefore $\theta(v, \lambda') = 0$ as well for $\lambda' < \lambda$, contradicting SRM. In particular $B(\cdot, \lambda)$ cannot be a z_0-bracket.

Next we show $m(\lambda) = \infty$ for any $\lambda > 0$. Suppose not, and recall $m(\lambda) > 0$ (proven at the end of subsection 6.1.1), so we have $0 < m(\lambda) < \infty$ for some positive λ. Then (32) implies $\ell_-(\lambda) > 0$ then $d(\ell_-(\lambda)) \geq m(\lambda) > 0$, contradiction.

Just like in the proof of statement i), there is a single interval $]\ell_-, \ell^+[=]0,\infty[$, an increasing homeomorphism f from $\mathbb{R}_+$ into itself, and a correspondence $\widetilde{B}$ from $\mathbb{R}_+$ into itself, such that

$$B(z,\lambda) = f(\lambda)\widetilde{B}(z) \text{ for all } \lambda, z \geq 0$$

and $\widetilde{B}$ is upper semi continuous and z-monotonic.

We check that $\widetilde{B}$ must be everywhere single-valued. Fix any $z \geq 0$, and suppose $y \in B(z,\lambda) \cap B(z,\lambda')$ for some $\lambda \neq \lambda'$: then $y = \theta(z+y,\lambda) = \theta(z+y,\lambda')$ contradicts SRM because θ increases strictly in λ. So the sets $B(z,\lambda)$ are pairwise disjoint when λ varies in $\mathbb{R}_+$. This is clearly impossible if $\widetilde{B}(z)$ is a true interval.

Thus after changing parameter to $\widetilde{\lambda} = f(\lambda)$, B is a function $B(z,\widetilde{\lambda}) = \widetilde{\lambda}\beta(z)$ where β, from $\mathbb{R}_+$ into itself, is continuous and weakly increasing. The fact that B has full range (property (24)) implies $\beta(z) > 0$ for all $z > 0$. Moreover $\beta(0) = 0$: if $\beta(0) = a > 0$, and $f(\widetilde{\lambda}) = 1$, we have $B(0,\widetilde{\lambda}) = \{a\}$ implying $a = \theta(a,\widetilde{\lambda})$; then $\theta(0,\widetilde{\lambda}) = 0$ and the 1-Lipschitz property of θ implies $[0,a] \subseteq B(0,\widetilde{\lambda})$, contradiction. Thus β is a calibration function in $\mathcal{B}^s$.

As in the previous proof we use now (23) to derive the parametrization of our method: the unique solution y of $y = B(v-y,\widetilde{\lambda}) = \widetilde{\lambda}\beta(v-y)$ is precisely θ^β in (11).

10.3 Proof of Theorem 2

10.3.1 statement "if"

Fix a method $h^{\beta,A}$ as in Proposition 2, and a problem (N,x,t) such that $y = h(x,t)$ satisfies $x_i - y_i = x_j - y_j$ for all i,j. The ∂RKG* property requires that the same equalities jhold whenever the resources increase to t'. This is always true by Symmetry if all coordinates of x are equal, or if $t = x_N$, so we can assume $x_1 \neq x_2$ and $t < x_N$.

Suppose first $t \leq \sum_N (x_i - A)_+$. If $\beta \in \mathcal{B}^s$ the system (10) implies $y_i = y_j$ for all i,j, and a contradiction of $x_1 \neq x_2$. If $\beta \in \mathcal{B}^{ns}$, for the same reason the system (12) implies $y_i = x_i$ for some i, hence $y_j = x_j$ for all j, contradicting $t < x_N$.

Suppose next $\sum_N (x_i - A)_+ \leq t$. Using the parametric representation of h^{ul}, there is some λ such that $x_i - (x_i - \frac{1}{\lambda})_+ = x_j - (x_j - \frac{1}{\lambda})_+$ for all i,j. In view of $x_1 \neq x_2$ this gives $x_i \geq \frac{1}{\lambda}$ for all i, and this inequality is preserved for higher values of λ, corresponding to larger amounts of resources to divide.

10.3.2 statement "only if"

We fix $\theta \in \mathcal{H}^0(CSY, CVX^*)$ satisfying ∂RKG*. We can assume that $B(\cdot,\lambda)$ is not a bracket for at least one $\lambda > 0$, because otherwise $\theta = \theta^{ul}$ (subsection 6.1.6). We fix such $\widehat{\lambda}$ such that $B(\cdot,\widehat{\lambda})$ is not a bracket, and we prove $\widehat{m} = \infty$ by contradiction.

If $\widehat{m} < \infty$ we know (property (34)) that $\widehat{\ell}_- > 0$ and $B(\cdot, \widehat{\ell}_-)$ is the $\widehat{m}$-bracket. For any distinct $x_1, x_2 > \widehat{m}$, this means $\theta(x_i, \widehat{\ell}_-) = x_i - \widehat{m}$ for $i = 1, 2$. Therefore for $t = x_1 + x_2 - 2\widehat{m}$, the solution of problem $(\{1,2\}, x, t)$ is $y_i = x_i - \widehat{m}$, precisely what h^{ul} recommends. Thus for any $z < \widehat{m}$, property ∂RKG* says that our method θ and h^{ul} still coincide for all problems $(\{1,2\}, x, t')$ with $t' = x_1 + x_2 - 2z > t$. Hence for each $z < \widehat{m}$ there is some $\lambda > \widehat{\ell}_-$ such that for $i = 1, 2$:

$$x_i - z = \theta(x_i, \lambda) \Leftrightarrow x_i - z \in B(z, \lambda)$$

For $z \in]\widehat{d}, \widehat{m}[$, the parameter λ must be in $]\widehat{\ell}_-, \widehat{\ell}^+[$ (because $B(z, \lambda) = \varnothing$ for $\lambda \geq \widehat{\ell}^+$), and as the x_i can be arbitrarily large, $B(z, \lambda)$ is some $[\underline{b}(z, \lambda), \infty[$. In view of (36) this means that $\widetilde{B}(z)$ is a true interval for all $z \in]\widehat{d}, \widehat{m}[$, contradicting z-monotonicity ($\widetilde{B}(z)$ can only be a true interval for countably many values of z). This proves $\widehat{m} = \infty$, hence $\widehat{\ell}_- = 0$ ((33)).

Moreover a similar argument shows that $\widetilde{B}(z)$ must be single-valued on $]\widehat{d}, \infty[$. Suppose it is not: $f(\widehat{\lambda})\widetilde{B}(z)$ contains two distinct y_1, y_2; setting $x_i = z + y_i$ we have $x_i - z = \theta(x_i, \lambda)$ for $i = 1, 2$, therefore ∂RKG* implies that our solution coincides with h^{ul} on all problems $(\{1,2\}, x, t')$ with $t' \geq y_1 + y_2$, and this means that for all $z' \in]\widehat{d}, z]$, if we set $y_i' = x_i - z'$ then there is $\lambda \in]0, \widehat{\ell}^+[$ such that $y_i' \in f(\lambda)\widetilde{B}(z')$ for $i = 1, 2$, therefore $\widetilde{B}(z')$ is also a true interval, and this holds true for too many choices of z' as above. We have shown the existence of a function $\widetilde{\beta}$ on $]\widehat{d}, \infty[$, contiuous and weakly increasing, such that $B(z, \lambda) = f(\lambda)\widetilde{\beta}(z)$ for all $\lambda \in]0, \widehat{\ell}^+[$ and $z \in]\widehat{d}, \infty[$. We distinguish two cases.

Case 1: $\widehat{\ell}^+ = \infty$. Then $\widehat{d} = 0$ otherwise we have $B(\cdot, \lambda) = \{0\}$ on $[0, \widehat{d}[$ for all λ, contradicting the full range of B ((24)). Thus $B(z, \lambda) = f(\lambda)\widetilde{\beta}(z)$ for all $\lambda, z > 0$, where $\widetilde{\beta} :]0, \infty[\rightarrow]0, \infty[$, is continuous and weakly increasing. The full range of B ((24)) implies $\widetilde{\beta}(z) > 0$ for all $z > 0$. Setting $\widetilde{\beta}(0) = \lim_{z \to 0} \widetilde{\beta}(z)$ completes the definition of $\widetilde{\beta}$; note that $\widetilde{\beta} \in \mathcal{B}^s$ if $\widetilde{\beta}(0) = 0$, and $\frac{1}{\widetilde{\beta}(0)}\widetilde{\beta} \in \mathcal{B}^{ns}$ if $\widetilde{\beta}(0) > 0$. The upper semi continuity of B implies $B(0, \lambda) = f(\lambda)[0, \widetilde{\beta}(0)]$.

As above we use (23) to show the method θ is in $\mathcal{LC}$. If $\widetilde{\beta}(0) = 0$, the unique solution y of $y = B(v - y, \lambda) = f(\lambda)\widetilde{\beta}(v - y)$ is $\theta^{\widetilde{\beta}}(v, f(\lambda))$, so the change of parameter $\widetilde{\lambda} = f(\lambda)$ gives $\theta \in \mathcal{LC}^s$. If $\widetilde{\beta}(0) > 0$, we change the parameter to $\widetilde{\lambda} = f(\lambda)\widetilde{\beta}(0)$, so the property $y \in B(v - y, \lambda)$ means either $\{y = v \leq \widetilde{\lambda}\}$ or $\{y < v \text{ and } y = \widetilde{\lambda}\beta(v - y)\}$ where $\beta = \frac{1}{\widetilde{\beta}(0)}\widetilde{\beta}$. This is clearly the same as (13) so $\theta = \theta^{\beta}$.

Case 2: $\widehat{\ell}^+ < \infty$. Then we set $\widehat{d} = A$ and apply (35): A is positive (as $A = m(\widehat{\ell}^+)$ and $m(z) > 0$ for all z), and $B(\cdot, \widehat{\ell}^+)$ is the A-bracket. In fact $B(\cdot, \lambda)$ is also a bracket for all $\lambda \geq \widehat{\ell}^+$, otherwise the previous argument shows $m(\lambda) = \infty$, impossible because $]d(\lambda), m(\lambda)[$ and $]\widehat{d}, \widehat{m}[$ are pairwise disjoint (subsection 6.1.5).

Thus for $\lambda < \widehat{\ell}^+$ we have

$$B(z, \lambda) = \{0\} \text{ for } z < A \; ; \; B(A, \lambda) = f(\lambda)[0, \widetilde{\beta}(A)]$$

$$B(z,\lambda) = f(\lambda)\widetilde{\beta}(z) \text{ for } z > A$$

where $\widetilde{\beta}(A) = \lim_{z\to A} \widetilde{\beta}(z)$.

For $\lambda \geq \widehat{\ell}^{+}$, $B(\cdot,\lambda)$ is the $\varphi(\lambda)$-bracket, where $\varphi : [\widehat{\ell}^{+}, \infty[\to [A, 0[$ is a decreasing homeomorphism (it is onto by full range of B, one-to-one by cleanliness, and clearly decreasing).

As usual we solve the description of B gives the solution y of $y \in B(v-y,\lambda)$, which by (23) is precisely $\theta(v,\lambda)$.

For $\lambda < \widehat{\ell}^{+}$

$$\begin{aligned} y &= 0 \text{ if } v < A \\ y &= v - A \text{ if } A \leq v \leq A + f(\lambda)\widetilde{\beta}(A) \\ y &= f(\lambda)\widetilde{\beta}(v-y) \text{ if } v > A + f(\lambda)\widetilde{\beta}(A) \end{aligned}$$

For $\lambda \geq \widehat{\ell}^{+}$

$$\begin{aligned} y &= 0 \text{ if } v < \varphi(\lambda) \\ y &= v - \varphi(\lambda) \text{ if } v \geq \varphi(\lambda) \end{aligned}$$

If $\lambda \geq \widehat{\ell}^{+}$ this is $\theta^{\beta,A}(v,\widetilde{\lambda})$ in (18) upon a change of parameter $\lambda \to \widetilde{\lambda}$ taking $\widehat{\ell}^{+}$ to $\frac{1}{A}$ and $\varphi(\lambda)$ to $\frac{1}{\widetilde{\lambda}}$. If $\lambda < \widehat{\ell}^{+}$ and $\widetilde{\beta}(A) = 0$, this is $\theta^{\beta,A}(v,\widetilde{\lambda})$ in (17) for $\beta(z) = \widetilde{\beta}(z+A)$, and provided $\lambda \to \widetilde{\lambda}$ sends $f(\lambda)$ to $\frac{\widetilde{\lambda}}{1-A\widetilde{\lambda}}$. And for $\widetilde{\beta}(A) > 0$, this is $\theta^{\beta,A}(v,\widetilde{\lambda})$ in (17) for $\beta(z) = \frac{1}{\widetilde{\beta}(A)}\widetilde{\beta}(z+A)$, and provided $\lambda \to \widetilde{\lambda}$ sends $f(\lambda)\widetilde{\beta}(A)$ to $\frac{\widetilde{\lambda}}{1-A\widetilde{\lambda}}$. The proof of Theorem 2 is complete.