Network Games with Perfect Complements

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Abstract

The model presented here examines the case of games played on networks where an agent's decision to undertake some costly investment is perfectly complementary with the decisions of his neighbors. Equilibrium actions of individual agents are found to depend on a measure of their network centrality known as their *coreness*. High levels of investment depend on the existence of densely connected subgroups within the network known in the sociology literature as *k-cores*. Peripheral nodes who inhabit sparsely connected areas of the network will be unable to support high levels of investment, even if they themselves have low costs and high degrees. Supplementary results which describe the structure of optimal networks and the relationship between coreness and Bonacich centrality are also presented.

Potential applications of this framework include models of human capital investment within firms, costly information transmission, user engagement in social networking products, and bandwidth allocation decisions in large autonomous networks.

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1 Introduction

A feature which characterizes many of the day-to-day decisions which are made by individuals is the dependence on coordination of our action with the actions of others with whom we frequently meet. These so-called 'strategic complementarities' in action can often be imperfect, such as in the case of investing effort in a team project, since extra effort by one person is a imperfect substitute for another person's lack of effort. In many situations however, strict coordination of decisions is absolutely necessary. For example, the benefit gained from socializing with co-workers when starting a new job or moving to a foreign country is strictly limited by the socializing efforts of other individuals with whom you interact. Such phenomena are generally classified as *peer effects* in the literature and so this paper adds to the growing body of work on peer effects in networks.

Traditional models have often relied on Rousseau's description of the stag hunt as a prototypical example of such a coordination problem. More recent models which examine the relationship between the interaction structure of a society and such phenomena have however often framed the decision as a binary one e.g. adopt new technology/don't adopt, revolt/stay at home, withdraw savings/don't withdraw etc. The current paper differs from this tradition by examining how the structure of social interaction may influence decisions in the case where one-off investments are made in a continuous 'capital' variable by individuals at private cost, and benefits are realized through pairwise interaction with other individuals' capital. In that respect, this paper is most closely aligned with the model of Ballester et al. (2006), but considers neighboring levels of investment to be perfectly complementary with one's own level (and hence there no substitutability in effort).[1] In terms of their best response functions, individuals decide to increase or decrease investment depending on the number of neighbors who have put in weakly higher levels of investment, since benefits will be determined by the least investment made by either party in a pairwise relationship.

One potential application of this framework is to model the emergence of meaningful communication within large groups when there is some privately borne cost.

[1] In most interpretations of the stag hunt scenario described by Rousseau we have that the payoff to hunting hare is independent of the action of the other player, so this setup is aligned with the spirit of the stag hunt game (see Crawford (1991), Skyrms (2003))

For example, the learning of new technical terminology in environments of high complexity (e.g. computer programming, options trading, scientific inquiry) requires some upfront investment cost and increases in attractiveness as others also invest. However, any two individuals can only communicate at the competence level of the least able, implying that the minimum investment level determines the transmission rate of information. Despite the existence of no communication (or 'babbling') equilibria in many such games, the experimental literature has often noted that meaningful communication tends to emerge, even in situations where no prior common language exists to facilitate coordination (e.g. Blume et al. (1998), Selten and Warglien (2007)).

Another example of an application of this framework could be to model users of a social networking site who have to decide how much time to spend using the product. In this case, the benefit from time spent is perfectly complementary with other friends who also use it, as the pairwise surplus gained from interaction is constrained by the individual contributing the minimal amount of content on the site. Yet another possible application is to bandwidth allocation problems in autonomous wireless or peer-to-peer networks where nodes can divert costly resources to maximize the throughput of information.

The model itself bears an initial similarity to a 'threshold game', but subtle difference in the payoff structure give rise to qualitatively different predictions. The main difference between the setup of this model and that of Young (1998) for example, is that costs are only incurred once at the node level rather than at the edge level, and so thresholds for action depend on the absolute number of neighbors playing a given strategy, rather than a proportion. There will be a marked difference in equilibrium patterns of play as agents now wish to coordinate, not with a small cabal of insiders, but with larger 'core' groups of the network.

The paper now proceeds with a review of the relevant literature on network games before moving on to a description of the model in a simple case of pure complementarity. The following sections then generalize the setting and examine the construction and stability of efficient networks.

2 Related Literature

The growing literature looking at games played on networks has been surveyed by Jackson (2008) and more recently by Jackson and Zenou (2012), so I present a brief summary of papers directly relevant to the present model.

The bulk of the literature examines the relationship between specific structural characteristics of the network and Nash equilibria of complete information games.[2] Several key contributions have identified a link between the network position of agents and their equilibrium actions, with a particular focus on different measures of network centrality. A fundamental contribution to this literature is Ballester et al. (2006), which was the first model to clearly identify the link between the equilibria of games played on networks and some variant of the eigenvector centrality of agents. In their model the utility from exerting effort is a multiplicative function of their neighbors' efforts and costs are quadratic and privately borne. This gives rise to best response functions which are linear in the actions of other agents and can be solved uniquely by agents selecting their Bonacich centralities as efforts. Calvó-Armengol et al. (2011) consider a different setting in which agents may invest in active and passive communication and have quadratic loss functions which are minimized when actions are matched with their local state and the actions of others. Equilibrium actions and communication efforts in their model are found to depend on a measure of centrality named the Invariant Method index, which bears strong similarities to the Bonacich centrality. These eigenvector centrality results do not hold in the setting investigated by Bramoullé and Kranton (2007) and Bramoullé et al. (2012) where actions are considered as strategic substitutes and have a low decay rate along paths in the network. In the case where network effects are particularly pronounced, the link between Bonacich centrality and effort disappears and instead we have that equilibria are characterized by a partition of agents into active and inactive sets, such that the links from active agents supply all inactive agents with the public good.

A second strand of the literature on network games of strategic complementarity which is related to the results presented in this paper is that associated with so-called

[2] A notable exception is Galeotti et al. (2009) who develop a tractable model of equilibrium play when the network itself is unknown.

'threshold games'. A typical setting is one similar to that described by Granovetter (1978) where a group of individuals face a collective action problem in the form of a binary choice, e.g. either to strike or not to strike, but prefer only to take the action if at least some threshold percentage of the group do the same. Differences in thresholds can lead to cascading behaviors, where one individual switching strategy forces others to switch, leading to yet more switching until a new equilibrium is reached. Similar problems were also formally analyzed in an earlier paper by Schelling (1973), who also discusses some effects of the spatial 'configuration' on cascades and equilibrium outcomes. Since the communication structure of a society might enable individuals with heterogeneous thresholds to more easily coordinate on their preferred equilibrium, these issues are addressed in Chwe (1999) and Chwe (2000). Chwe finds that optimal networks can be formed from a series of interlocking cliques of agents, which allow all agents to take the risky action by making all locally important thresholds common knowledge, since they are linked to all of their neighbours' neighbors.

The model presented in the current paper is perhaps more closely related to that of Morris (2000), who investigates the role of network structure in a binary decision threshold game when the modeler is concerned about the robustness of decision making with respect to contagion. Morris shows that only sufficiently inward looking groups of nodes can be resilient to an invading (perhaps detrimental) social norm which has occurred in other areas of the network. In a similar respect to the results of Chwe (2000), segregated networks prevent best response dynamics from triggering a cascade of revisions arising from some small local perturbation. The models of Young (1998) and Young (2001) are also concerned with the structural conditions which enable different regimes of play in different areas of the network and arrive at a similar conclusion to Morris (2000).

Young (1998) also highlights a negative result with regard to the ability of any network structure to prevent a risk-dominant equilibrium from prevailing as the unique stochastically stable state of play. This in turn builds on an earlier paper by Ellison (1993), which also examines the robustness of equilibria in a stag hunt game to trembles in decision making for some simple (exogenously given) structures. The Ellison (1993) model has also been extended to the setting where the network is endogenous by Jackson and Watts (2002), who find that stochastically stable equilibria may arise which are neither risk-dominant nor Pareto-dominant when agents can select who they are linked to.

3 Model

The model considers the actions of a set of individuals $N = \{1, 2, \dots, n\}$ where $n \geq 3$. These individuals are located on an undirected and possibly weighted network which can be represented by the triple $G = \langle V, E, w \rangle$ where $V = N$ is the set of vertices, E is the set of edges and w is a weighting function from E to the real numbers. Since the network is undirected, edges are unordered pairs (i, j) where $\{i, j\} \in E$ (equivalently $ij \in E$) implies not only that node i is linked to j but also that j is linked to i. Additionally, g_{ij} will be used to denote the weight associated with the edge $w(ij)$ where applicable. Let $\mathcal{N}_i = \{j \neq i \mid g_{ij} > 0\}$ denote the neighborhood of agent i, which is the subset of N with whom an agent i interacts, and let $d_i(G) := |\mathcal{N}_i|$ be the degree of agent i in network G.

Each individual selects a level of investment $x_i \in [0, L]$ which will be perfectly complementary with the investment levels of neighboring individuals.[3] We can suppose that this setup represents an information transmission game where players as employees of a firm who must choose capacities, as in Sobel (2012), which represent their competence at communicating highly complex messages. The investment is now a form of human capital, which is perfectly complementary with other employees of the firm. Each individual may also have an idiosyncratic benefit from the investment $\alpha_i x_i$, but shall incur a private cost which is quadratic in x_i, as in Ballester et al. (2006). Unlike Ballester et al. (2006), who focus on linear best response functions where actions are local complements but global substitutes, this paper considers the case where there is no substitutability in action across individuals. Utility functions therefore take the form

$$u_i = \alpha_i x_i + \sum_{j \in \mathcal{N}_i} g_{ij} \min(x_i, x_j) - \frac{1}{2} x_i^2$$

[3]The assumption that the strategy space is compact plays no role in formal results and is a simplification which permits the application of tools from the literature of supermodular games as in Belhaj et al. (2012).

This function is strictly concave and continuous in x_i, although not continuously differentiable. Consider for now the special case where $\alpha_i = 0$ for all i and $g_{ij} \in \{0, 1\}$. Define the set of neighbors of i who are choosing weakly higher levels of investment as $\mathcal{M}_i(x_i) := \{j \in \mathcal{N}_i \mid x_i \leq x_j\}$. The subnetwork of individuals with $x_j \geq x_i$ will be denoted by $H(x_i)$, and in addition, let $d_i(H(x_i)) := |\mathcal{M}_i(x_i)|$. For a given action profile $\mathbf{x}_{-i}$ let $x_i^*(\mathbf{x}_{-i})$ denote agent i's best response. This best response $x_i^*(\mathbf{x}_{-i})$ must firstly satisfy the condition

$$x_i^* \leq d_i(H(x_i^*)) \tag{1}$$

To see why, notice that to sustain an investment level x_i^* as a best response to some $\mathbf{x}_{-i}$ we need that at least x_i^* neighbors of i are playing a weakly higher action[4]. Since each agent is constrained in what the can obtain by the investment levels of others, increasing x_i brings benefits only along those links for which $x_i < x_j$. Whilst the marginal cost of effort is x_i, a reduction in x_i by i is pivotal only along links to neighbors in $\mathcal{M}_i(x_i)$, implying that x_i cannot exceed $d_i(H(x_i^*))$ as lowering investment increases utility.

When $u_i(x_i^*)$ is differentiable at x_i^* this condition (1) met with equality. When $u_i(x_i^*)$ lies in kinked sections of its range, the largest x_i satisfying this condition is the unique best response since $d_i(H(x_i))$ is weakly decreasing in x_i. The best response x_i^* must therefore also satisfy

$$\tilde{x}_i > d_i(H(\tilde{x}_i)) \text{ for all } \tilde{x}_i > x_i^* \tag{2}$$

This second condition ensures that marginally raising x_i cannot be beneficial as we are now increasing utility along a lower number of links. These best response conditions bear some similarity to those seen in 'threshold games' (e.g. Young (1998), Morris (2000) and Chwe (2000)) as they depend directly on the number of neighbors playing a given strategy. This model differs from the above mentioned as optimal actions depend on the absolute number of neighbors playing a (weakly) higher action, rather than a proportion of a neighborhood, since investment cost is split across all neighbors.

[4] When x_i^* is not an integer this implies that $x_i^* < d_i(H(x_i^*))$, although $\lceil x_i^* \rceil \leq d_i(H(x_i^*))$ will hold in any equilibrium.

Moreover, the equilibrium level of investment for i will depend not only on $d_i(H(x_i^*))$ but, by association, also on $d_j(H(x_i^*))$ for $j \in \mathcal{M}_i(x_i^*)$. Since these $j \in \mathcal{M}_i(x_i^*)$ should also be best responding this implies that $x_j^* \leq d_j\left(H\left(x_j^*\right)\right)$ and, since $x_i^* \leq x_j^*$ for all $j \in \mathcal{M}_i(x_i^*)$ by definition, that $x_i^* \leq d_j(H(x_i^*))$. We can extend this logic to $j' \in \mathcal{M}_j(x_i^*)$, and to their neighbors, and so on.

To summarize, in order to sustain an investment level of x_i^* in equilibrium, agent i must have at least x_i^* neighbors playing $x_j^* \geq x_i^*$, who in turn must have at least x_i^* neighbors playing $x_{j'}^* \geq x_i^*$, etc. Clearly this condition implies that for a given agent to sustain high levels of investment in equilibrium we require not only that they be highly connected, but that their neighbors and neighbours' neighbors be highly connected. The finite nature of the network will however ensure that this subset of highly connected agents cannot grow arbitrarily large as we iterate this condition along paths from i. As later results will demonstrate, for groups of agents to sustain some positive investment level in equilibrium they must be sufficiently connected but also sufficiently inward looking. This discussion necessitates a more precise definition of cohesive subgroups in the context of the model.

3.1 Cohesive Subgroups

Notions of group cohesiveness in networks have long been studied in the sociology literature and there are many seemingly natural definitions, such as cliques, clans and clubs which are given in standard texts such as Wasserman and Faust (1994). A variant of these which has been used in the economics literature is the notion of a p-cohesive subset, which is defined in Morris (2000). Formally, a subset of nodes is said to be p-cohesive if every node within that subset has (at least) a proportion p of their neighbors within that subset. A related idea is also found in Young (1998) and Young (2001) where a subset of nodes S is called r-close-knit if for every $S' \subseteq S$ the proportion of links originating in S' and ending in S is at least r. Intuitively, p-cohesiveness is a condition on the degrees of nodes, whereas r-close-knittedness is a condition on links within a subgroup and therefore a p-cohesive subgroup is $p/2$-close-knit.

This model will utilize a particularly useful concept originally defined by Seidman (1983) known as the *k-core*. Seidman (1983) considers subgraphs of G which can

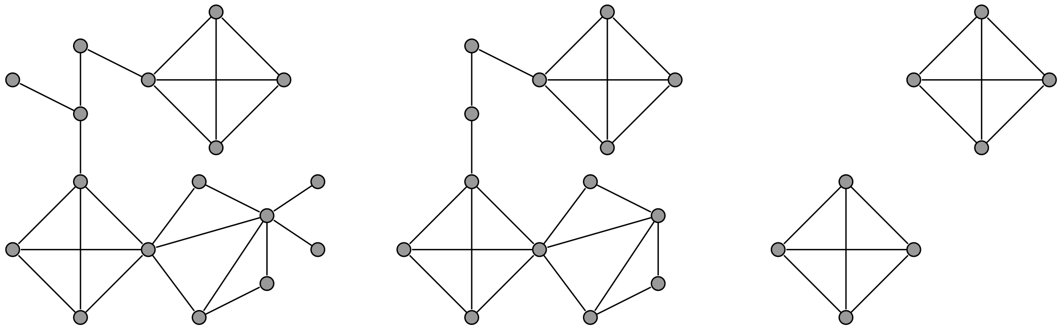


Figure 1: k-cores of a network

be induced by repeatedly pruning nodes of low degrees from the network in order uncover groups of densely connected individuals. The graph which is obtained by iteratively removing all nodes of degree less than k is known as a *core* of order k, or a k-core.[5] A precise definition of a k-core of a graph G follows:

Definition 1. A *k-core* of a graph G is a subgraph $H \subseteq G$ such that $d_i(H) \geq k$ for each $i \in H$

The notation $d_i(H)$ in the above definition denotes the degree of agent i within the subgraph H. When I state that a group of nodes 'form' a k-core, this means that the subgraph consisting of these nodes and links between them is itself a k-core.

Every connected graph trivially supports a 1-core, whilst the 2-core which can be formed using the least possible number of edges is the ring network. Also note that the definition implies that nodes belonging to k-cores of high orders are also members of lower orders, permitting a nested$k$-core decomposition of any given network. Figure 1 shows such a k-core decomposition of a graph for $1 \leq k \leq 3$.

If an agent i is contained within a k-core then this would imply that they have at least k neighbors of degree k or greater. There is an intuitive similarity between the notion of a k-core and the h-index, a citation metric used to assess the impact of published academic authors.[6] There have been a number of applications of the

[5]This is a slight abuse of terminology as Seidman (1983) refers to the k-core as maximal subgraph which can be obtained by iteratively removing nodes of lower degree. I follow Wasserman and Faust (1994) and the more recent literature by referring to any core of order k as a k-core.

[6]The h-index of an author is the maximal integer h such that h of their papers have at least h citations.

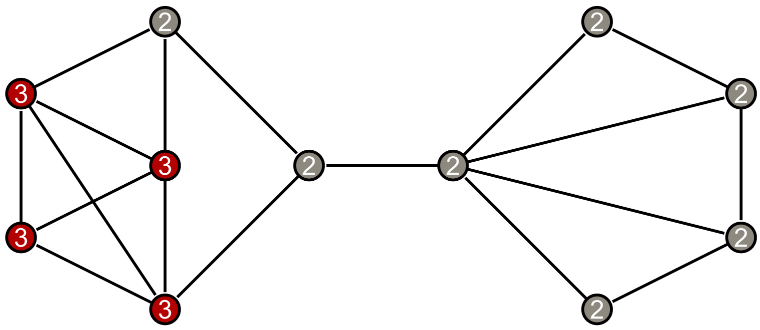


Figure 2: Coreness profile of a bridge network

concept of the k-core outside of economics, for example, in the analysis of protein networks in bioinformatics (e.g. Bader and Hogue (2003) and Wuchty and Almaas (2005)) and in the visualization of large networks in computer science (e.g. Baur et al. (2004)).

Since cores of successive orders are nested within the previous core we can define a *coreness* value for each $i \in N$:

Definition 2. An individual $i \in N$ has *coreness* $c_i = k$ if it is contained in a core of order k but not in a core of order k' for $k' > k$.

The coreness of a node can be interpreted as a measure of its centrality, as we can view it as a condition on its degree centrality and the degree centrality of other nodes reachable along paths of different lengths starting from i. Nodes with high coreness are likely to have important roles in the network since they have neighbors with large degrees (who in turn have neighbors with large degrees, etc). High coreness can also signal that a given node is a member of a dense and cohesive subset of the network, since cliques of size n immediately form an $(n-1)$-core. Although the coreness profile of a network, denoted by $\mathbf{c}(G)$, can give an indication of dense subsets of G, more information will be needed in general to establish cohesiveness of the network as a whole (e.g. the links between different cores).[7]

As can be seen in the example in Figure 2, the coreness of individual nodes can depend on structural characteristics of the network which are relatively 'far away'. In this example, the addition of a single link between the remaining pair of nodes

[7] I also write $c_i(H)$ when it is convenient to distinguish between the coreness of a node under different subgraphs or supergraphs of G.

with degree 2 would raise the coreness of all nodes to 3. Although adding a link can increase the corenesses of distant nodes, the following lemma shows that increases in the coreness of any given node can be at best be directly proportional to increases in their degree.

Lemma 1. *Adding an edge between i and j can increase c_i and c_j by at most* 1. *Moreover, $c_{j'}$ can increase by at most* 1 *for any other $j' \neq i, j$.*

Proof. First note that adding a edge cannot lower the coreness of any node in the network. Now suppose that edge ij increased c_i from k to $k+m$ for some $m \geq 2$, this implies that i now has at least $k+m$ neighbors with coreness $k+m$ in the network $G+ij$. However if we remove this newly added edge this would leave i with $k+m-1$ neighbors with at least coreness $k+m-1$, contradicting our assumption that c_i was initially k in G.

To prove for $j' \neq i, j$, suppose again that edge ij raised $c_{j'}$ from k to $k+m$ for some $m \geq 2$. Focusing on the subset $\mathcal{K} \subseteq N$ who form the k-core in G, we notice that if either i or j were members, their degrees have only increased by 1 in $G+ij$ and, as established, their coreness increases by at most 1. Since their new coreness is at best $k+1$ they cannot form part of the new $(k+m)$-core needed to support $c_{j'}(G+ij) = k+m$. If i and j were not members of $\mathcal{K} \subseteq N$ then $d_\kappa(G) = d_\kappa(G+ij)$ for all $\kappa \in \mathcal{K}$ and hence $c_{j'}(G) = c_{j'}(G+ij)$ due to the fact that the k-cores are nested. □

Although the removal of one link can have a cascading effect which influences all nodes (e.g the transition from ring to line network), we can also interpret Lemma 1 as saying that the removal of an edge ij cannot lower the coreness of any agent in the network by more than 1. With these definitions in hand, I now discuss the properties of equilibrium action profiles in a special case of the model.

3.2 Equilibrium

Now that the coreness of nodes has been defined, the model can be solved for action profiles $\mathbf{x} \in X = [0, L]^n$ which constitute a Nash equilibrium of the game $\Gamma = \langle N, X, u \rangle$. I will focus for now on the Pareto superior Nash equilibrium of this game, before considering the set of all equilibria. An observation which can be made immediately is that Γ is a supermodular game.[8] With this we can establish two facts using Lemma 2 below:

Lemma 2. *The game* Γ *is supermodular and hence:*

1. *A greatest and least equilibrium must exist*
2. *If* $\mathbf{x}$ *and* $\mathbf{x}'$ *are both equilibrium profiles of* Γ *such that* $\mathbf{x} \geqslant \mathbf{x}'$ *then* $\mathbf{x}$ *Pareto dominates* $\mathbf{x}'$

The proof of this statement follows directly from Theorem 5 and 7 (respectively) of Milgrom and Roberts (1990).[9] The Lemma immediately implies that the greatest equilibrium is also the Pareto dominant one, so with the existence of this equilibrium established, we can now characterize it in terms of the coreness of agents.

Proposition 1. *The Pareto efficient Nash equilibrium in the case where* $\alpha_i = 0$ *and* $g_{ij} \in \{0, 1\}$ *is* $\mathbf{x}^* = \mathbf{c}(G)$

Proof. I first show that $\mathbf{x}^* = \mathbf{c}(G)$ is an equilibrium. Partition the agents into subsets $\{S_1, S_2, \ldots, S_K\}$ based on their coreness such that $c_i = k$ for all $i \in S_k$. Take the set of agents with the largest coreness S_K, there must exist a connected graph of at least $K+1$ such agents. Since $d_i(H(x_i^*)) \geq K = x_i^*$ for all $i \in S_K$ we just need to show that $d_i(H(\tilde{x}_i)) < \tilde{x}_i$ for any $\tilde{x}_i > K$, but since the subgraph composed of agents playing a strictly higher action is empty this condition is satisfied. Now take S_{K-1} and note again that $d_{i'}(H(x_{i'}^*)) \geq K - 1 = x_{i'}^*$ for all $i' \in S_{K-1}$ since all agents have coreness $K-1$. To show that $d_{i'}(H(\tilde{x}_{i'})) < \tilde{x}_{i'}$ for any $\tilde{x}_{i'} > K-1$ note that the subgraph $H(\tilde{x}_{i'})$ is formed from all the individuals in S_K, who we assumed had strictly higher corenesses. Therefore, by the definition of coreness, $x_{i'}^* = c_{i'}$ is the maximal $x_{i'}$ satisfying $d_{i'}(H(x_{i'})) \geq x_{i'}$, otherwise $i' \in S_K$. This reasoning holds for all subsets of lower coreness and so the investment profile $\mathbf{x}^* = \mathbf{c}$ is a Nash equilibrium.

[8] Firstly, the strategy set $X = [0, L]^n$ is a complete lattice using the usual partial order $\mathbf{x} \geqslant \mathbf{x}'$ if $x_k \geq x_k'$ for all $k = 1, \ldots, n$. By the definition of Milgrom and Roberts (1990), the game is supermodular since u_i has increasing differences in (x_i, x_{-i}), u_i is supermodular in x_i for fixed x_{-i}, and u_i is upper semi-continuous in X_i and order continuous in X_{-i} with a finite upper bound.

[9] The existence of equilibrium can be trivially established since $x_i = 0$ for all i is always an equilibrium.

Now to show that it is equilibrium is maximal, assume that there exists another equilibrium vector of actions $\hat{\mathbf{x}}$ such that $\hat{\mathbf{x}} \geqslant \mathbf{x}^*$. Take any $i \in N$ playing $\hat{x}_i > x_i^* = c_i$ in equilibrium $\hat{\mathbf{x}}$, the best response condition (1) implies that i has at least $\lceil \hat{x}_i \rceil$ neighbors playing $\hat{x}_j \geq \hat{x}_i$, in other words $d_i (H(\hat{x}_i)) \geq \hat{x}_i$. For investments $\hat{x}_j$ to be equilibrium best responses this further implies that all $j \in \mathcal{M}_i(\hat{x}_i)$ have at least $\lceil \hat{x}_j \rceil$ neighbors playing $\hat{x}_{j'} \geq \hat{x}_j$, and so on for $j' \in \mathcal{M}_j(\hat{x}_j)$ etc. However, continuing with this reasoning contradicts the assumption that the coreness of node i was $c_i = x_i^* < \hat{x}_i$ since we can now construct a subgraph containing i (using these nodes and links only) where each node has at least degree $\lceil \hat{x}_i \rceil$ within that subgraph. □

Intuitively, this result tells us that in order for agents to play the Pareto efficient levels of investment in equilibrium, the network must be sufficiently cohesive in each agent's 'local' neighborhood. The proof of this proposition does not directly use the fact that this is a supermodular game, but supermodularity allows us to infer that the equilibrium where $\mathbf{x}^* = \mathbf{c}$ is Pareto dominant since it is maximal in terms of investment. The reader can use the supermodularity to verify that beginning with a action profile $\mathbf{x} = (L, \ldots, L)$ and iterating along the best responses of agents we can arrive at $x_i = c_i$ for all i, implying that $\mathbf{x}^* = \mathbf{c}(G)$ is the maximal equilibrium (see Jackson (2008)).

Despite the fact that the Pareto optimal equilibrium has this simple characterization in terms of degree coreness, the complementarity of action gives rise to other equilibria which are Pareto dominated by $\mathbf{x}^* = \mathbf{c}(G)$. Unlike the model of Ballester et al. (2006), the assumption that $\alpha_i > 0$ is not enough to guarantee uniqueness of equilibrium in this model as there will in general be a multiplicity of interior equilibria. Before characterizing the set of equilibria in the pure complementarity case some additional definitions are needed.

In order to describe all equilibria of the game in terms of coreness we need to introduce definitions which allow us to induce subgraphs on the network as functions of the action profile $\mathbf{x}$. Recall that $H(x)$ denotes the subgraph of G such that $x_i \geq x$ for all $i \in H(x)$, and now let $V_H(x) \subseteq N$ denote the set $\{i \in N \mid x_i \geq x\}$, (or the vertices) which comprise the subgraph $H(x)$. In addition, let $H'(x)$ denote the subgraph constructed using the verticies $V_{H'}(x) = V_H(x)$ and edges $E_{H'}(x) =$

$\{ij \in H(x) \mid x_i > x \text{ or } x_j > x\}$, this can be thought of as the graph $H(x)$ less the edges $E(x) = \{ij \in H(x) \mid x_i = x_j\}$. We can also view the remaining edges in $E_{H'}(x)$ as the those along which only one member of $V_H(x)$ is pivotal in increasing surplus. Any edge in $H(x)$ which requires a joint deviation to increase surplus can be considered as 'inactive' in $H'(x)$.

Proposition 2. *An action profile* $\mathbf{x}^*$ *is a Nash equilibrium in the case where* $\alpha_i = 0$ *and* $g_{ij} \in \{0, 1\}$ *if and only if, for any* $x_i \in \mathbf{x}^*$*,* $H(x_i)$ *is* x_i*-core and* $d_i(H'(x_i)) \leq x_i$ *for each* $i \in N$ *.*

Proof. First note that $\mathbf{x}^*$ being an equilibrium implies the inequality in (1) must hold for each i and so $d_i(H(x_i)) \geq x_i$ for each $i \in S(x)$, implying that $H(x_i)$ is an x_i-core. For this condition to become sufficient we need that no agent can join a higher core by playing a higher action. Since the subnetwork $H'(x)$ gives higher cores of active links which can be reached if investment is increased marginally from x_i to $\tilde{x}_i$, the condition that $d_i(H'(x_i))$ is no larger than x_i ensures that such an increase is never beneficial given the action profile of others $\mathbf{x}_{-i}$ □

To explain the reasoning intuitively, the condition that $H(x_i)$ is an x_i-core means that no agent will want to lower their investment, since it is supported by higher actions from at least x_i other agents. However, it must also be the case that no agent can unilaterally deviate to join a higher core in the network of 'active' links $H'(x_i)$ given the actions $\mathbf{x}_{-i}$ of other agents. To check that a given profile is an equilibrium one must first verify, for all investment levels x_i^*, that (1) the set of agents playing $x_i^* \geq x$ form an x-core, and (2) their degrees in $H'(x_i)$ does not exceed x_i^*.

The proposition also implies that playing $x_i^* = 0$ in equilibrium can only occur if every member of the the component which contains i also plays $x_j = 0$. The least equilibrium is therefore susceptible to trembles in play, as an exogenous shock to preferences which induces one node to increase their action to 1 will cascade through the whole component via the best response dynamics. The same is not necessarily true for the Pareto efficient equilibrium, the stability of which depends upon the presence of supplementary links which do not directly contribute to coreness but can act as a buffer against such shocks. Although a fully fledged analysis of the stochastic stability would prove useful, these considerations are left for future work.

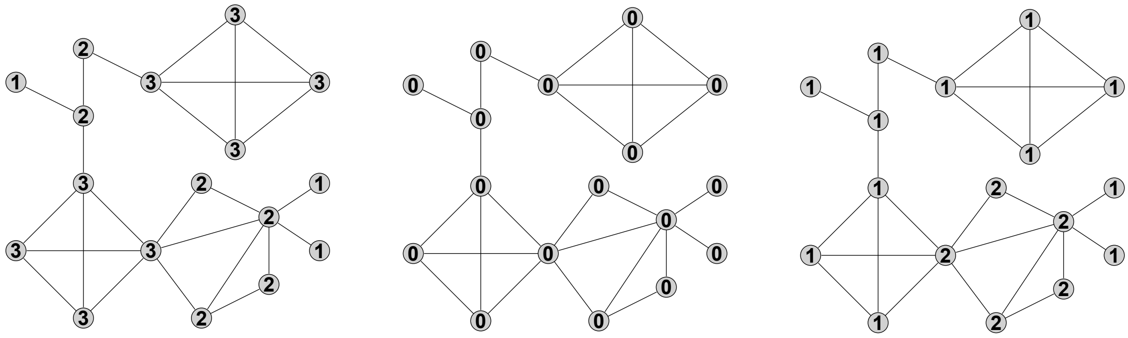


Figure 3: Some Equilibria of the Game for the Network in Figure 1

3.3 Coreness and Bonacich Centrality

Before considering the more general case of this model, I first make a comparison between coreness and the centrality measure proposed by Bonacich (1987). The Bonacich centrality of a node measures the number of paths of all lengths which originate at node i, weighted by a decay factor δ which increases with the length l of the path. Formally, let $\mathbf{G}$ be the adjacency matrix of G and define the Bonacich centrality of a node i as $b_i = \sum_{l=0}^{\infty} \delta^l \sum_{j \in N} g_{ij}^l$ where $\sum_{j \in N} g_{ij}^l$ is the sum of all paths of length l from i, (i.e. the sum across the i^{th} row of $\mathbf{G}^l$). The Bonacich centrality is well defined whenever $\delta < \frac{1}{|\lambda_{max}(G)|}$ where $|\lambda_{max}(G)|$ is the absolute value of the largest eigenvalue of $\mathbf{G}$. This condition ensures that the sum does not grow too quickly as we iterate on powers of $\mathbf{G}$.

If we take coreness as a measure of centrality, a question which naturally arises is whether there is some relationship between coreness and the Bonacich centrality measures which play a key role in the models of Ballester et al. (2006), Ballester and Calvó-Armengol (2010) and Bramoullé et al. (2012). In particular, is it the case that if a node i has a higher coreness than a node j , that i also has a higher Bonacich centrality? The answer turns out to be 'no', but since the coreness of a node places limitations on paths leading from it to others in the network, we will be able to put a lower bound on the Bonacich centralities of nodes.

Proposition 3. *If node i has coreness $c_i(G) = k$ then*

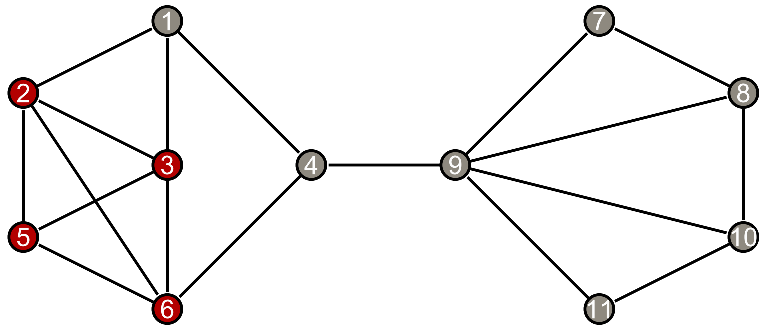
$$b_i(G, \delta) \geq \frac{1}{1 - \delta k} \tag{3}$$

Proof. Since the Bonacich centrality of a node is increasing in the number of paths emanating from it, I focus on the case where all nodes reached on paths from i have exactly degree k. When this is the case I minimize the number of possible paths from i under the constraint that $c_i = k$. If $c_i = k$ then there are at least k^l paths of length l from i to other nodes in the network. Summing paths of all lengths we get $b_i(G, \delta) = \sum_{l=0}^{\infty} \delta^l \sum_{j \in N} g_{ij}^l \geq \sum_{l=0}^{\infty} (\delta k)^l$ and provided $|\delta k| < 1$ this implies that $\sum_{l=0}^{\infty} (\delta k)^l = \frac{1}{1-\delta k}$. To show that this limit is well defined (i.e. $\delta < 1/k$) we rely on an elementary result from spectral graph theory to arrive at an expression for the largest eigenvalue of $\mathbf{G}$.

I assume without loss of generality that the graph is connected since both the coreness and the Bonacich centrality of a node can only depend on structural properties within the same component. Since we assumed that $d_j = k$ for all nodes reachable from i, these nodes form a k-regular graph . Due to this fact, we conclude that the largest eigenvalue of the adjacency matrix $\mathbf{G}_k$ is k (see Brualdi, 2011) . The condition that $\delta < \frac{1}{|\lambda_{max}(G)|}$ now means that $\frac{1}{1-\delta k}$ is well defined as $\frac{1}{1-\delta k} > 1$ for any δ where b_i is itself well defined. □

This bound is tighter when δ is low and the proof of Proposition 3 demonstrates that when G itself is a regular graph then (3) is met with equality. Although nodes with higher coreness will usually have higher Bonacich centralities this is not always the case, as demonstrated in Figure 4. Since the degree of node 9 in Figure 3 is greater than any other $j \in N$ we find that $b_9(G, \delta) > b_j(G, \delta)$ for any other $j \in N$ when the decay factor $\delta = 0.1$, since the paths of length 2 are weighted less by a factor of 10 when compared with paths of length 1.

Finding an upper bound for $b_i(G, \delta)$ as a function $c_i(G)$ remains an open problem, however, it seems possible that one may not exist, as we can increase the Bonacich centrality of a node i arbitrarily whilst holding their coreness fixed by simply appending cliques of order $m > c_i$ to neighbors of i. The remaining issue is to ascertain whether the relative Bonacich centralities of other nodes in the network grow at a fast enough rates to counterbalance the increase in $b_i(G, \delta)$, which in turn depends on the structural properties of the network.



Node	Coreness	Bonacich Centrality ($\delta = 0.1$)	Bonacich Centrality ($\delta = 0.25$)
3	3	1.620	10.298
5	3	1.486	8.696
6	3	1.621	10.186
9	2	1.700	7.615

Figure 4: Coreness and Bonacich Centralities of Nodes in a Bridge Network

4 Agent Heterogeneity and Weighted Links

Although the assumption of identical costs helps to clarify the role of degree coreness in equilibrium action, it also imposes an undue degree of symmetry on the problem.I now consider the general case, where for each $i \in N$ we assume that $\alpha_i \in \mathbb{R}_+$ and $g_{ij} \in [0, 1]$, and then solve for Nash equilibrium in this new setting. In order to characterize equilibria in this case the coreness of a node must be generalized to include properties other than the degree coreness in an associated subgraph of $H \subseteq G$. I borrow terminology from Batagelj and Zaveršnik (2002) to define the 'generalized coreness' of a node via the use of a node property function. A node property function is a function $p(i, H)$ such that for each $i \in N$ and some $H \subseteq G$ the function $p(i, H)$ assigns a value $p_i \in \mathbb{R}$ to node i.

Definition 3. A *generalized k-core* of a graph G is a subgraph H such that each $i \in H$ has $p(i, H) \geq k$

For the concept of degree coreness used in the previous section it is possible to define the property function as $p(i, G) = d_i(G)$.[10] I define the relevant node

[10]Similarly, the characterization of all equilibria in Proposition 2 used the same function defined

property function for Proposition 4 to be $p(i, H(x_i)) = \alpha_i + \sum_{j \in H(x_i)} g_{ij}$. The function $p(i, H(x_i))$ gives, for any $i \in N$ and subgraph $H(x_i)$, the weighted degree of i in that subgraph $H(x_i)$ plus that individual's cost heterogeneity parameter α_i. The property function $p(i, H(x_i))$ captures what is essentially the marginal utility that an agent i gets from decreasing x_i having fixed $\mathbf{x}_{-i}$. On the other hand, $p(i, H'(x_i))$ captures the marginal gains which are realized by i from raising their investment level. To simplify notation, I follow Batagelj and Zaveršnik (2002) in referring to generalized cores of order k which formed using the vertex property function $p(i, H)$ as k-p-cores.

The investments made in all equilibria of the game are now given in the following proposition:

Proposition 4. *Action profile $\mathbf{x}^*$ is an equilibrium if and only if, for any $x_i \in \mathbf{x}^*$, $H(x_i)$ is x_i-p-core and $p(i, H'(x_i)) \leq x_i$ for each $i \in N$.*

Proof. The proof is a fairly straightforward extension of the proof of Proposition 2. If $\mathbf{x}^*$ is an equilibrium then $p(i, H(x_i)) \leq x_i$ holds for each i, otherwise nodes prefer to reduce x_i, implying that $H(x_i)$ is x_i-p-core. As before, sufficiency is provided by the condition $p(i, H'(x_i)) \leq x_i$ for each $i \in N$ □

Agents which have higher α_i parameters need lower weighted degrees to sustain a given level of effort since these parameters appear as substitutes in the vertex property function. Applying Proposition 4 we can see that the Pareto dominant equilibrium has all agents playing their p-coreness, that is, the maximal k such that i is in a generalized core of order$k$ but not one of order $\tilde{k}$ for $\tilde{k} > k$, as we increase the order by pruning nodes from $H \subseteq G$ which have $\alpha_i + \sum_{j \in H} g_{ij} < k$.

A brief comment relating these results to those obtained by Morris (2000) is now necessary. In the case where $\alpha_i = \alpha$ but G is weighted we can see that it is not the degree but rather the weighted degree of i in $H_i(x_i)$ which will determine differences in equilibrium investment across nodes. If we further assume that the matrix $\mathbf{G}$ of weights is row stochastized so that weights are now $g'_{ij} = \frac{g_{ij}}{\sum_j g_{ij}}$ we can observe that high levels of equilibrium effort occur when agents are clustered and cohesive. The maximal K-cores can indeed look very different if we consider the case of row

on $H(x_i)$ and $H'(x_i)$

stochastic weights, since dyads with large α_i could now have the highest generalized coreness. By row stochasizing we are now effectively dividing a node's benefits by their degree, thus putting this model more in line with the original literature on threshold games.

5 Efficient Networks

To consider the properties of optimal networks I focus again on the Pareto dominant equilibrium $\mathbf{x}^* = \mathbf{c}(G)$ and add a link cost γ which is linear in the degree of each agent. Utility functions now take the form

$$u_i = \alpha_i x_i + \sum_{j \in \mathcal{N}_i} g_{ij} \min(x_i, x_j) - \frac{1}{2} x_i^2 - \gamma d_i$$

Substituting in equilibrium investments and again assuming that $\alpha_i = 0$ for all i and $g_{ij} \in \{0, 1\}$, the problem for the network designer is

$$\max_{G \in \mathcal{G}} \sum_{i \in N} \sum_{j \in \mathcal{N}_i} \min(c_i, c_j) - \frac{1}{2} c_i^2 - \gamma d_i$$

Whilst networks in which agents have high corenesses will in general produce larger amounts of surplus, higher cores will require a larger number of links to construct. The question which immediately arises is how to allocate to some $S \subseteq N$ a coreness of k using the minimal number of links. Moreover, is the marginal cost of adding extra individuals to that k-core increasing or decreasing in $n_k = |S|$?

The answer to the first question is simple, we must construct a k-regular graph using $\frac{n_k \cdot k}{2}$ links where possible. In this case all individuals have exactly degree k within group and no agent $i \in S$ is linked to someone of a lower coreness (this link is wasted since it does not alter the coreness of i). If both k and n_k are odd then we cannot construct a k-regular graph using a non-integer number of links and so we instead take the ceiling of $\frac{n_k \cdot k}{2}$ denoted by $\lceil \frac{n_k \cdot k}{2} \rceil$. Whilst the marginal cost of adding an extra individual to this regular graph is clearly increasing in k, the cost of adding extra individuals to a given core is essentially linear in n_k (ignoring the integer problems) and is given by equation (4) below[11]:

$$\lceil\frac{(n_k+1)\,k}{2}\rceil - \lceil\frac{n_k\cdot k}{2}\rceil = \begin{cases} \frac{k}{2} & \text{if } k \text{ even} \\ \lceil\frac{k}{2}\rceil & \text{if } k \text{ odd and } n_k\text{even} \\ \lfloor\frac{k}{2}\rfloor & \text{if } k \text{ odd and } n_k\text{odd} \end{cases} \tag{4}$$

Since the utility functions u_i exhibit increasing differences in x_i and x_{-i} this suggests that if adding one individual to a core of order k increases the utility surplus then it should be beneficial to have all n agents in this core. As Proposition 5 shows, complete networks will be optimal provided the linking cost is low enough:

Proposition 5. *If $a_i = 0$ for all i with linear link costs the efficient network is the complete network if $\gamma \leq \frac{(n-1)}{2}$ or the empty network if $\gamma \geq \frac{(n-1)}{2}$*

Proof. To find the optimal network G in the feasible set of graphs $\mathcal{G}$ we are given the problem

$$\max_{G\in\mathcal{G}} \sum_{i\in N}\sum_{j\in\mathcal{N}_i} \min\left(c_i, c_j\right) - \frac{1}{2}c_i^2 - \gamma d_i$$

The utility surplus generated by all nodes in the complete network is $n\left((n-1)\left(\frac{1}{2}\left(n-1\right)-\gamma\right)\right)$ since $d_i = c_i = n-1$. This surplus is non-negative when $\frac{1}{2}\left(n-1\right) \geq \gamma$. Now consider some other network G' and an agent i in that network. Agent i has coreness c_i and therefore has at least c_i neighbors with coreness c_i. The maximum possible surplus generated at a given node in any other network is $d_ic_i - \frac{1}{2}c_i^2 - \gamma d_i = c_i\left(d_i - \frac{1}{2}c_i\right) - d_i\gamma$. Since $c_i \leq d_i$ in any network this surplus cannot exceed $\frac{1}{2}d_i^2 - d_i\gamma$, which ensures that each node has non-negative surplus only if $\frac{1}{2}d_i \geq \gamma$. If $\gamma > \frac{1}{2}\left(n-1\right)$ this upper bound tells us that no node can have non-negative surplus in any non-empty network. If $\gamma \leq \frac{1}{2}\left(n-1\right)$ we have that $\frac{1}{2}d_i^2 - d_i\gamma < \frac{1}{2}\left(n-1\right)^2 - \left(n-1\right)\gamma$ for any other G' and so the complete network is optimal. □

Corollary 1. *If $\alpha_i = \alpha$ for all i then the efficient network is the complete network if $\gamma \leq \alpha + \frac{(n-1)}{2}$ or the empty network if $\gamma \geq \alpha + \frac{(n-1)}{2}$*

[11]If k is even $\lceil\frac{(n_k+1)k}{2}\rceil - \lceil\frac{n_kk}{2}\rceil = \frac{k}{2}$ for any $n_k \in \mathbb{N}$, while if k is odd then either $\lceil\frac{n_kk}{2}\rceil = \frac{n_kk}{2}$ or $\lceil\frac{(n_k+1)k}{2}\rceil = \frac{(n_k+1)k}{2}$. In this first case $\frac{n_kk}{2}$ is even and so $\lceil\frac{(n_k+1)k}{2}\rceil - \lceil\frac{n_kk}{2}\rceil = \frac{n_kk}{2} + \lceil\frac{k}{2}\rceil - \frac{n_kk}{2} = \lceil\frac{k}{2}\rceil$, while in the second case $\lceil\frac{n_kk}{2}\rceil = \frac{n_kk}{2} + \frac{1}{2}$ and so $\lceil\frac{(n_k+1)k}{2}\rceil - \lceil\frac{n_kk}{2}\rceil = \frac{n_kk}{2} + \frac{k}{2} - \frac{n_kk}{2} - \frac{1}{2} = \lfloor\frac{k}{2}\rfloor$

The above corollary follows from the proof of Proposition 5 as the cost γ must satisfy the condition $n\left(\frac{1}{2}\left(\alpha+(n-1)\right)^2\right) \geq n\left(n-1\right)\gamma + n\frac{1}{2}\alpha^2$. Efficient networks will either be complete, if they can be constructed with positive surplus, or else empty. Intuitively, this comes from the fact that increases in coreness deliver a weakly positive externality to all others in the neighborhood of the increased node. This externality increases with the coreness k (as nodes must be linked to more individuals for higher k) and hence with linear costs we are driven to a boundary solution.

Since the complete network maximizes total surplus and utility functions exhibit increasing differences it will also be pairwise stable in the case where $\alpha_i = \alpha$ for some $\alpha \geq 0$. An issue which may be of interest for future study would be to consider a network formation game when agents have different α_i and select weighted degrees, as in the setting of Section 4.[12]

6 Conclusion

This paper has provided an analysis of the equilibrium properties of games played on networks where agents' actions are perfect complements. Although the setting is related to that of a 'threshold game', the results presented here are qualitatively different from those of Morris (2000) and Young (1998), and give new insights into the structural factors which may influence equilibrium decisions. The main contribution is a characterization of all equilibria in terms of the *coreness* of agents in subgraphs of G. In particular, the action played by each agent i in the Pareto optimal equilibrium is equal to their degree coreness in G for the pure complementarity case.

In an exogenously given network, agents who are located in dense but large subgroups will select high levels of investment. Peripheral nodes who inhabit sparsely connected areas of the network will be unable to support high levels of investment even if they themselves have low costs and high degrees. From a designers perspective, the optimal networks in this framework are always either empty or complete

[12] The interplay between the substitutability of α_i and g_{ij}, and the positive assortativity which will be present should lead to structures where nodes with high types refuse to link to those with low types, leaving low types with little opportunity to form a sizable core.

due to the strong positive externalities present when increasing the coreness of nodes.

Although this model is the first to highlight the direct link between the equilibrium behavior of agents in games played on networks and their coreness, there has been recent work in computer science which has focused on the concept of coreness in relation to network games. In particular, Bhawalkar et al. (2012) and Manshadi and Johari (2009) provide recent examples from outside the economics literature of models which highlight the link between the k-core of a network and maximal levels of equilibrium action in network games. The paper of Bhawalkar et al. (2012) finds that the sensitivity of equilibrium actions to exogenous changes in cost are dependent on the the maximal k-core of the network and present an algorithm to select nodes whose choices should be "anchored" to prevent unraveling of equilibrium in sparsely connected graphs. Manshadi and Johari (2009) on the other hand find that the coreness of individuals puts a lower bound on the maximal equilibrium actions in a supermodular game where agents benefit from the aggregate actions of their neighbors.

Future work is required to examine (1) the stability of equilibria in this framework, and (2) the properties of optimal networks in other settings such as heterogeneous types and convex link costs.

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