Individual Preference Restrictions in Hedonic Coalition Formation Games*

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January 22, 2013

Abstract

In this paper we study restrictions imposed on individual preferences, as opposed to preference profiles, and we introduce individual preference restrictions that characterize the existence and uniqueness of stable hedonic coalition structures on rich preference domains. We prove that if such a preference domain satisfies an individual preference restriction then it guarantees the existence of a stable coalition structure if and only if it satisfies Intersection Restriction. Intersection Restriction requires that the intersection of any two strongly individually rational coalitions is ranked weakly above at least one of them. We also present a similar characterization result for the uniqueness of a stable coalition structure by the individual preference restrictions of Superset Restriction and Strong Intersection Restriction. Superset Restriction is satisfied if a strongly individually rational coalition that is ranked above another one is always a superset of the latter, while Strong Intersection Restriction requires that the intersection of any two strongly individually rational coalitions is ranked weakly above both coalitions.

Keywords: coalition formation, matching, stability, core

*This research was funded, in part, by the Canada Research Chairs program.

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1 Introduction

We examine the hedonic coalition formation problem, which got its name from Drèze and Greenberg (1980), who referred to the players' subjective ranking of coalitions, based on the identity of the coalition's members, as hedonic preferences. Despite the origin of the name of the hedonic coalition formation model, which is based on direct preferences over coalitions (rather than on coalitional values that are divided up among the members), hedonic coalition formation may be interpreted as one based on objective preferences over coalitions. Given that players only care about the coalition they join, and that each player is exactly in one coalition (which may consist of one player only), it is a simple model of coalition formation that generalizes the well-known two-sided and one-sided matching models, whose ample literature is surveyed by Roth and Sotomayor (1990).

Given strict preferences, if the coalition structure is stable (in the core) then there is no coalition of players that would strictly prefer to form this coalition to staying in their respective coalitions in the coalition structure. It is well known that the core of a hedonic coalition formation problem may be empty, or in other words, there may not exist a stable hedonic coalition structure for a given coalition formation problem. This is already clear from the roommate problem which is not always stable, and is further demonstrated by Banerjee, Konishi, and Sönmez (2001), and Bogomolnaia and Jackson (2002), which offer intuitive sufficient restrictions on preference profiles for the existence of stable coalition structures. Iehlé (2007) provides a characterization of preference profiles at which a stable hedonic coalition exists, using a balancedness condition. Moreover, Pápai (2004) gives a simple characterization of the existence of a unique stable coalition structure in terms of restrictions on coalitions.

In this paper we focus on restrictions on individual preferences, following Alcalde and Romero-Medina (2006), rather than on restrictions on the preference profile (Bogomolnaia and Jackson (2001), Banerjee et. al (2002)) or on feasible coalitions (Pápai (2004)). What makes a preference restriction an individual preference restriction? If each player's preferences satisfy the individual preference restriction, then the resulting preference profile also satisfies the individual preference restriction. Therefore, given the set of preferences that satisfy an individual preference restriction (which is the same for each player[1]), each preference profile in the preference domain given by the Cartesian product of the players' preferences satisfies this individual preference restriction. It is clear that the *top coalition* property of Banerjee et. al (2002) and the *ordinal balancedness* condition of Bogomolnaia and Jackson (2001), for example, are not individual preference restrictions, as the respective preference domains satisfying these axioms are not Cartesian products of sets of individual preferences. We will refer to restrictions of this type as profile restrictions.

Both profile restrictions and individual preference restrictions are important to investigate, as they complement each other. Studying profile restrictions can clarify whether there is a stable coalition structure in a particular situation, given the players' preferences collectively, while studying individual preference restrictions may shed light on what each players' preferences should look like, independently of the others' preferences, in order to guarantee the existence of a stable coalition structure. In addition, one can also see the distinction between these two approaches as descriptive versus normative, since only individual players may be asked to report preferences that satisfy certain pre-specified restrictions, in contrast to preference profiles which cannot be normatively restricted in this way. The normative approach, however, entails some incentive issues that we do not study in this paper. Another difference between the two types of restrictions, as Alcalde and Romero-Medina (2006) has already pointed out, is that individual preference restrictions that guarantee the existence of a stable coalition structure, unlike profile restrictions, are immune to population changes: if some players leave or new players arrive, it is still assured that a stable coalition structure exists, provided that the new players' preferences also satisfy the individual preference

[1] In this context *same* means that for all pairs of players i and j, if we exchange i's and j's positions in the preferences of player i then we get the corresponding preference ordering for player j.

restriction.

We introduce an individual preference restriction called Intersection Restriction, and prove that under appropriate assumptions on what constitutes a rich preference domain that satisfies an individual preference restriction, this property is the only individual preference restriction that guarantees the existence of a stable coalition structure for each preference profile in this domain. Intersection Restriction is satisfied if whenever there are two strongly individually rational coalitions (where strong individual rationality means that the coalition is ranked above remaining single), their intersection is weakly ranked above at least one of them. We also propose two individual preference restrictions, Superset Restriction and Strong Intersection Restriction, which together characterize the existence of a unique stable hedonic coalition structure. Superset Restriction is satisfied if a strongly individually rational coalition that is ranked above another one is always a superset of the latter, and Strong Intersection Restriction requires that for any two strongly individually rational coalitions their intersection is ranked weakly above both of them. It can be seen that these two properties cannot be combined into a weaker property, as the two requirements are essentially the opposite of each other, and we prove that their disjunction gives a characterization for the uniqueness of a stable hedonic coalition structure under our assumption of a rich preference domain.

Cechlárová and Romero-Medina (2001), Alcalde and Revilla (2004), Alcalde and Romero-Medina (2006), and Dimitrov et al. (2006) propose sufficient conditions for the existence of a stable hedonic coalition structure using individual preference restrictions. We compare our proposed conditions for the existence and uniqueness of stable coalition structures, to the conditions proposed in the above mentioned papers, primarily by Alcalde and Romero-Medina (2006). When comparable, the sufficient conditions in the above papers are stronger than our conditions. Moreover, in contrast to these previous papers, which give sufficient conditions only, we provide characterizations. That is, Intersection Restriction is not only a sufficient condition, but it is also a necessary condition for existence when the preference domain is rich and, similarly, Superset Restriction or Strong Intersection Restriction are necessary for the uniqueness of stable hedonic coalition structures. Therefore, our results shed light on how demanding the restriction on individual preferences need to be in order to ensure the existence and uniqueness of a stable coalition structure when restricting individual preferences.

2 Hedonic Coalition Formation

There is a finite **set of players** $N = \{1, \ldots, n\}$, where $n \geq 4$, and a **set of preferences** $\mathcal{R}_i$ for each player $i \in N$. We will refer to each nonempty subset of N as a **coalition**, and let $\mathcal{N} = \{S \subset N \,:\, S \neq \emptyset\}$ denote the set of coalitions.[2] Furthermore, let $\mathcal{N}_i = \{S \in \mathcal{N} \,:\, i \in S\}$ denote the set of coalitions that player i is a member of. Player i's preferences $R_i \in \mathcal{R}_i$ strictly order the elements of $\mathcal{N}_i$. Preferences are complete, transitive, and antisymmetric. We will write SP_iS' to indicate strict preferences, and SR_iS' to indicate that either SP_iS' or $S = S'$. We assume that players only care about the coalition they join. A **(hedonic) coalition formation problem** is defined by a pair (N, R), where $R = (R_1, \ldots, R_n)$ is a **preference profile** in the preference domain $\mathcal{R}$, and $\mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_n$ is the Cartesian product of $\mathcal{R}_i$'s. Throughout this paper N is fixed, and thus a coalition formation problem is simply defined by a preference profile $R \in \mathcal{R}$. Given $R \in \mathcal{R}$ and $S \in \mathcal{N}$, we denote $(R_i)_{i \in S}$ by R_S. We also write R_{-i} to denote $R_{N \setminus \{i\}}$, and R_{-S} to denote $R_{N \setminus S}$. We also use the following notational shorthand: $R_i = (S_1, S_2, \ldots)$ if R_i ranks S_1, first, S_2 second, etc.

A **coalition structure** is a partition $\{S_1, \ldots, S_k\}$ of N, with $n \geq k \geq 1$, that is, $\bigcup_{t=1}^{k} S_t = N$, where all S_t are pairwise disjoint. Let Σ denote the set of all coalition structures, and let $\sigma \in \Sigma$ denote a coalition structure. Moreover, let σ_i be the unique coalition in σ that contains i.

A coalition structure σ is **stable at *R*** if there is no coalition $S \in \mathcal{N}$ such that for all

[2] The binary relations $\subset$ and $\supset$ include the possibility that the sets in question are equal.

$i \in S$, $SP_i\sigma_i$. If there is such a coalition S then we say that $\boldsymbol{S}$ **blocks** $\boldsymbol{\sigma}$. Note that stable coalition structures form the core of a coalition formation problem. A preference domain $\bar{\mathcal{R}}$ **guarantees the existence of a stable coalition structure** if there exists a stable coalition structure for all preference profiles $R \in \bar{\mathcal{R}}$. A preference domain $\bar{\mathcal{R}}$ **guarantees the existence of a unique stable coalition structure** if there exists a unique stable coalition structure for all preference profiles $R \in \bar{\mathcal{R}}$.

A coalition $S \in \mathcal{N}_i$ is **individually rational (for** $\boldsymbol{i \in S}$**) at** $\boldsymbol{R_i}$ if $SR_i\{i\}$. A coalition $S \in \mathcal{N}_i$ is **strongly individually rational (for** $\boldsymbol{i \in S}$**) at** $\boldsymbol{R_i}$ if $SP_i\{i\}$. A coalition S is **individually rational at** $\boldsymbol{R}$ if for all $i \in S$, S is individually rational at R_i. A coalition S is **strongly individually rational at** $\boldsymbol{R}$ if for all $i \in S$, S is strongly individually rational at R_i. Observe that for all $i \in N$, $\{i\}$ is individually rational at R, but not strongly individually rational. Moreover, if $S \in \mathcal{N}$ is not a singleton, then if S is individually rational at R then it is also strongly individually rational at R. Let $\text{IR}(R_i)$ denote the individually rational part of R_i, that is, the ordering of coalitions starting from the top-ranked coalition and ending with $\{i\}$.

3 Rich Preference Domains

We will say that a preference profile R satisfies an individual preference restriction if for all $i \in N$, player i's preferences R_i satisfy the individual preference restriction. Note that if for all $i \in N$, for all $R_i \in \bar{\mathcal{R}}_i$, R_i satisfies an individual preference restriction then for all $R \in \bar{\mathcal{R}}$, R satisfies the individual preference restriction. In this sense, we may refer to $\bar{\mathcal{R}} \subset \mathcal{R}$ as a preference domain which satisfies this particular preference restriction.

For all $i, j \in N$, $R_i \in \mathcal{R}_i$, and $R_j \in \mathcal{R}_j$, we will write that $R_i \sim R_j$ if R_i and R_j are identical after reversing the positions of i and j in the two preference orderings. Note that if R_i satisfies an individual preference restriction, then so does R_j. In the following we will assume that a preference domain $\bar{\mathcal{R}} \in \mathcal{R}$ satisfying an individual preference restriction has the **symmetry** property that if $R_i \in \bar{\mathcal{R}}_i$ and $R_i \sim R_j$ then $R_j \in \bar{\mathcal{R}}_j$.

For the characterizations, which show that the proposed individual preference restrictions are not only sufficient but also necessary conditions for the existence, and for the uniqueness, respectively, of a stable coalition structure, we have to make sure that the preference domain is rich enough, in order to rule out that the existence (or uniqueness, respectively) of a stable coalition structure is satisfied "by accident" on a small domain. For example, if each player only has preference orderings in which N is ranked first while satisfying an individual preference restriction which does not imply that N has to be ranked first, then the resulting preference domain guarantees the existence of a (unique) stable coalition structure by virtue of being small, rendering the individual preference restriction unnecessary. Therefore, if the domain is too small, there is no non-trivial individual preference restriction that is necessary for stability. In order to make necessity meaningful, and thus our characterized property non-trivial, we need to make sure that the preference domain is rich enough. This, however, is not as straightforward as it seems, since we have to ensure that the richness conditions do not contradict the individual preference restriction imposed on the preferences. Therefore, we introduce transformations of preferences which satisfy the same individual preference restriction as the original preferences, and require that a rich domain contain all preferences which satisfy the same individual preference restrictions described by these transformations.

Preferences $\tilde{R}_i \in \mathcal{R}_i$ **satisfy the same individual preference restriction** as $R_i \in \mathcal{R}_i$ if $\tilde{R}_i$ can be obtained from R_i using one of the four transformations TR1-TR4 defined below.

Note that since all coalitions in a stable coalition structure are individually rational, and hence we only focus on the individually rational part of preferences, it is enough to define the transformed preferences $\tilde{R}_i$ by specifying $\mathrm{IR}(\tilde{R}_i)$, rather than all of the orderings of coalitions in $\tilde{R}_i$.

TR1 (indispensable players)

For all $SP_i\{i\}$ such that $j \notin S$, let $\{i\}\tilde{P}_iS$, and let $\text{IR}(R_i)$ and $\text{IR}(\tilde{R}_i)$ be the same otherwise.

This transformation allows players to have preferences that reflect that only coalitions containing all indispensable players are acceptable.

TR2 (intolerable players)

For all $SP_i\{i\}$ such that $j \in S$, let $\{i\}\tilde{P}_iS$, and let $\text{IR}(R_i)$ and $\text{IR}(\tilde{R}_i)$ be the same otherwise.

This transformation allows players to have preferences that reflect that coalitions containing any intolerable members are not acceptable.

TR3 (irrelevant players)

Let $\text{IR}(\tilde{R}_i)$ and $\text{IR}(R_i)$ be the same, except that each S with $j \in S$, for some $j \in N \setminus \{i\}$, that satisfies both $SP_i\{i\}$ and $SP_i\{i,j\}$ is replaced by $S \setminus \{j\}$.

This transformation allows players to have preferences that reflect that some players are considered irrelevant. In the definition player j is irrelevant for i, so j is dropped from each individually rational coalition that contains j. Note that if $\{i,j\}$ is individually rational at R_i then only the (possibly transformed) coalitions that were ranked above $\{i,j\}$ at R_i remain strongly individually rational at $\tilde{R}_i$. Therefore, this transformation may change $\{i\}$'s position in the preference ordering such that fewer coalitions are individually rational than before.

TR4 (substitute players)

If there exist R_i, R_i' such that only the positions of j and l are exchanged in the coalitions ranked in $\text{IR}(R_i)$ versus $\text{IR}(R_i')$, where $l \notin S$ for all $SP_i\{i\}$ and $j \notin S$ for all $SP_i'\{i\}$, then R_i and $\tilde{R}_i$ only differ from each other as follows: for all S with $SP_i\{i\}$ such that $j \in S$, S is replaced by $\tilde{S} = S \bigcup \{l\}$.

This transformation allows players to have preferences that reflect that some members may be substitutes in a coalition, and thus the concurrent presence of such substitute players leaves the preference ordering unchanged. Observe that using the symmetry assumption on $\bar{\mathcal{R}}$, we can set up R_j from R_i and then R_l from R_j, and then $R'_i \in \bar{\mathcal{R}}_i$ follows from R_l by symmetry. Thus, starting from any $R_i \in \mathcal{R}_i$, we can define TR4 alternatively in the following manner: if there exists $j \in N \setminus \{i\}$ such that for all S with $SP_i\{i\}$, $j \notin S$ then let $\text{IR}(R_i)$ and $\text{IR}(\tilde{R}_i)$ be the same, except, for all S with $SP_i\{i\}$ such that $l \in S$ for some $l \in N$, replace S by $S' = S \bigcup \{j\}$.

Example 1 [3] *Illustrations of TR1-TR4*

Let $R_1 = (1235, 12, 12345, 124, 1)$. If we apply TR1 to 4 then we get $\tilde{R}_1 = (12345, 124, 1)$, and if we apply TR2 to 3 then we have $\tilde{R}_1 = (12, 124, 1)$. TR3 applied to 2 yields $\tilde{R}_1 = (135, 1)$. Finally, given $R'_1 = (1236, 12, 12346, 124, 1)$, we can apply TR4 to 5 and 6 in R_1 and R'_1 to get $\tilde{R}_1 = (12356, 12, 123456, 124, 1)$. ■

A domain $\bar{\mathcal{R}}$ is a **rich preference domain** if for all $R_i \in \bar{\mathcal{R}}_i$, for all $\tilde{R}_i \in \mathcal{R}_i$ that satisfies the same individual preference restriction as R_i, $\tilde{R}_i \in \bar{\mathcal{R}}_i$. We will say that a particular individual preference restriction is a **rich preference restriction** if there exists a rich preference domain which satisfies this individual preference restriction.

4 The Existence Characterization

We are now ready to introduce one of the main properties of individual preferences, *Intersection Restriction*, which is used in the characterization of the existence of stable coalition structures.

Intersection Restriction: For all coalitions $S, S' \in \mathcal{N}_i$ such that $SP_iS'P_i\{i\}$, $(S \bigcap S')R_iS'$.

Intersection Restriction requires that the intersection of any two strongly individually

[3] For simplicity we list players consecutively by their numbers and without parenthesis, instead of the conventional set notation.

rational coalitions be ranked weakly above at least one of the two coalitions. Note that if $S \subset S'$ or $S' \subset S$ in the above definition then the property is satisfied.

In our first theorem we characterize the existence of a stable coalition structure. Before stating the theorem, we need to verify that Intersection Restriction is a rich preference restriction, that is, we need to verify that if R_i satisfies Intersection Restriction and the transformed preference profile $\tilde{R}_i$ satisfies the same individual preference restriction, then $\tilde{R}_i$ also satisfies the Intersection Restriction, for each of the four transformations TR1 to TR4. Although a bit tedious, these are straightforward verifications and are therefore left to the reader.

Theorem 1 *A rich individual preference restriction guarantees the existence of a stable coalition structure if and only if it is Intersection Restriction.*

We first introduce some definitions that are used in the proof of the Theorem.

A coalition S is **self-blocked at $\boldsymbol{R}$ (by $\boldsymbol{S'}$)** if there exists $S' \subset S$ such that for all $i \in S'$, $S'P_iS$. Two coalitions S and S' are **independent** if $S \not\subset S'$ and $S' \not\subset S$.

Proof of Theorem 1

Part 1: *Let $\bar{\mathcal{R}} \subset \mathcal{R}$ satisfy Intersection Restriction. Then $\bar{\mathcal{R}}$ guarantees the existence of a stable coalition structure.*

Fix $R \in \bar{\mathcal{R}}$, where $\bar{\mathcal{R}}$ is a preference domain satisfying Intersection Restriction. Let $\hat{\mathcal{N}} \subset \mathcal{N}$ be the set of all strongly individually rational coalitions at R that are not self-blocked, and let $\bar{\mathcal{N}} \subset \hat{\mathcal{N}}$ be the set of coalitions such that for all $S \in \hat{\mathcal{N}}$ if there exists $S' \in \hat{\mathcal{N}}$ such that $S \subset S'$, and for all $i \in S$, $S'P_iS$, then $S \notin \bar{\mathcal{N}}$, and otherwise $S \in \bar{\mathcal{N}}$. Next, let $\check{\mathcal{N}} \subset \bar{\mathcal{N}}$ be a maximal feasible subset of $\bar{\mathcal{N}}$; that is, for all $S, S' \in \bar{\mathcal{N}}$, $S \bigcap S' = \emptyset$, and there does not exist $T \in \bar{\mathcal{N}} \setminus \check{\mathcal{N}}$ such that for all $S \in \check{\mathcal{N}}$, $T \bigcap S = \emptyset$. Finally, add all missing singletons in order to get a partition of N: for all $i \in N$ such that i is not a member of any coalition $S \in \check{\mathcal{N}}$, add $\{i\}$ to $\check{\mathcal{N}}$. Denote the resulting coalition structure by $\sigma \in \Sigma$.

We will show that σ is a stable coalition structure at R. Suppose that σ is not a stable coalition structure at R. Then there exists a coalition that blocks σ. Moreover, there exists a coalition T that blocks σ and is not self-blocked at R. For suppose that T blocks σ and is self-blocked by $T' \subset T$ at R. Then T' also blocks σ at R, and since we have a finite number of players, this implies that we can choose T that blocks σ to be not self-blocked at R. Note that T is not a singleton since all non-singleton coalitions in σ are strongly individually rational at R by the construction of σ. Thus, since T blocks σ and is not a singleton, it follows that T is strongly individually rational at R. In sum, $T \in \hat{\mathcal{N}}$.

Let $S \in \sigma$ such that $T \bigcap S \neq \emptyset$. Observe that $S \in \hat{\mathcal{N}}$, and thus S is both strongly individually rational and not self-blocked at R when S is not a singleton. Thus, S is both individually rational and not self-blocked at R in general.

Case 1 $T \subset S$

Since T blocks σ, for all $i \in T$, TP_iS. This means that S is self-blocked by T, a contradiction.

Case 2 $S \subset T$

Observe that we have $T \in \hat{\mathcal{N}}$ such that $S \subset T$ and for all $i \in S$, TP_iS, given that T blocks σ. This contradicts the fact that $S \in \bar{\mathcal{N}}$.

Case 3 T and S are independent

In this case S is not a singleton and thus S is strongly individually rational at R. Since T blocks σ, for all $i \in (T \bigcap S)$, TP_iS. Then, given that for all $i \in N$, R_i satisfies Intersection Restriction, for all $i \in T \bigcap S$, $(T \bigcap S)P_iS$. This means that S is self-blocked at R by $T \bigcap S$, which is a contradiction.

Part 2: *Let $\bar{\mathcal{R}} \subset \mathcal{R}$ be a rich domain that guarantees the existence of a stable coalition structure. Then $\bar{\mathcal{R}}$ satisfies Intersection Restriction.*

Let $\bar{\mathcal{R}} \subset \mathcal{R}$ be a rich domain that guarantees the existence of a stable coalition structure.

Suppose, by contradiction, that $\bar{\mathcal{R}}$ does not satisfy Intersection Restriction. Then there exist $i \in N$ and $\bar{R}_i \in \bar{\mathcal{R}}_i$ such that $\bar{R}_i$ does not satisfy Intersection Restriction. Thus, there exist $S, S' \in \mathcal{N}_i$ such that $S\bar{P}_i S' \bar{P}_i \{i\}$, and $S' \bar{P}_i (S \bigcap S')$.

Suppose $S \subset S'$. Then $S \bigcap S' = S$ and $S' \bar{P}_i S$, a contradiction. Suppose $S' \subset S$. Then $S \bigcap S' = S'$ and $S' \bar{P}_i S'$, a contradiction. Therefore, S and S' are independent.

We will transform $\bar{R}_i$ in the following six steps. We denote by $\tilde{R}_i^t$ the resulting preferences, and by $\tilde{S}_t$ and $\tilde{S}_t'$ the transformed coalitions S and S', respectively, in step t, $t = 1, \ldots, 6$.

Step 1 Apply TR1 to all $j \in S \bigcap S'$.

Step 2 Apply TR2 to all $j \in N \setminus (S \bigcup S')$.

Observe that as a result of Steps 1 and 2, for all $T \subset N$ such that T is individually rational at $\tilde{R}_i^2$, $S \bigcap S' \subset T \subset S \bigcup S'$.

Step 3 Apply TR3 to all $j \in S \bigcap S'$ such that $j \neq i$.

Choose T, T' that are independent and are strongly individually rational at $\tilde{R}_i^3$ and $T \bigcup T'$ is not a strict superset of $\tilde{T} \bigcup \tilde{T}'$, where $\tilde{T}, \tilde{T}'$ are independent and $\tilde{T}, \tilde{T}'$ are both strongly individually rational at $\tilde{R}_i^3$. Note that T, T' exist since $S' \bar{P}_i (S \bigcap S')$ implies that $\tilde{S}_3 \tilde{P}_i^3 \{i\}$ and $\tilde{S}_3' \tilde{P}_i^3 \{i\}$ and $\tilde{S}_3, \tilde{S}_3'$ are independent.

Step 4 Apply TR1 to all $j \in T \bigcap T'$

Step 5 Apply TR2 to all $j \in N \setminus (T \bigcup T')$.

The only individually rational coalitions at $\tilde{R}_i^5$ are $T, T', T \bigcup T'$, and $T \bigcap T'$. Suppose there exists $\hat{S} \subset N$ such that $\hat{S} \neq T, T', T \bigcap T'$ and $\hat{S} \tilde{P}_i^5 \{i\}$. Then Step 4 implies that $\hat{S} \subset T \bigcup T'$ and Step 5 implies that $\hat{S} \supset T \bigcap T'$. If $\hat{S} \subset T$ or $\hat{S} \supset T'$ then $\hat{S}$ and T are independent and $\hat{S} \bigcup T' \subset T \bigcup T'$, where $\hat{S} \bigcup T' \neq T \bigcup T'$. This contradicts the way we chose T, T' in Step 4. A similar argument shows that $\hat{S} \not\subset T'$ and $\hat{S} \not\supset T'$. Thus, $\hat{S}$ is independent of both T and T' and $\hat{S} \bigcup T \subset T \bigcup T'$, where $\hat{S} \bigcup T \neq T \bigcup T'$, a contradiction.

Step 6 Apply TR3 to all $j \in (T \bigcap T')$ such that $j \neq i$.

At $\tilde{R}_i^6$ we have the following two cases:

Case 1: $\tilde{T}, \tilde{T}'$ are the only individually rational coalitions at $\tilde{R}_i^6$.

Case 2: $\tilde{T}, \tilde{T}'$ and $\tilde{T} \bigcup \tilde{T}'$ are the only individually rational coalitions at $\tilde{R}_i^6$.

Let $\tilde{T}, \tilde{T}'$ denote the transformed coalitions T, T', respectively. Observe that $|\tilde{T}| = |\tilde{T}'| = 2$.[4] Let $R_1 = \tilde{R}_i^6$. Without loss of generality, we can describe R_1 as follows. In Case 1, $R_1 = (12, 13, 1)$ and in Case 2, R_1 satisfies $12 P_i 13 P_i 1$ and $123 P_i 1$. In Case 1, we can specify preferences in $\bar{\mathcal{R}}_2$ and $\bar{\mathcal{R}}_3$, for players 2 and 3, respectively, in order to define a roommate cycle. In particular, let $\hat{R}_2 = (12, 23, 2)$, $R_2 = (23, 12, 2)$, $\hat{R}_3 = (23, 13, 3)$, and $R_3 = (13, 23, 3)$. Note that $R_1 \sim \hat{R}_2$, $R_1 \sim \hat{R}_3, \hat{R}_2 \sim R_3$, and $\hat{R}_3 \sim R_2$, and thus $R_2 \in \bar{\mathcal{R}}_2$ and $R_3 \in \bar{\mathcal{R}}_3$. By TR2, we can let all other agents j, that is, $j \neq 1, 2, 3$, rank $\{j\}$ first in preferences R_j. Since for all $j \in N$, $R_j \in \bar{\mathcal{R}}_j$, we have $R \in \bar{\mathcal{R}}$. However, this preference profile does not have a stable coalition structure, and thus $\bar{\mathcal{R}}$ does not guarantee the existence of a stable coalition structure, which is a contradiction. This completes the proof of Theorem 1 for Case 1. For Case 2, we need a further argument in order to finish the proof.

Note that if $13 P_1 123$ then we can use a similar argument to that of Case 1. In the following we will show the argument for the worst case, where $R_1 = (123, 12, 13, 1)$ but this argument also works for the other case, $(12, 123, 13, 1)$. Let $R_1' = (134, 14, 13, 1)$. Note that $R_1' \in \bar{\mathcal{R}}_i$, by symmetry. Thus, we can apply TR4 to R_1 and R_1' to get $\hat{R}_1 = (1234, 124, 13, 1)$. Let $R_1'' = (124, 12, 14, 1)$. Note that $R_1'' \in \bar{\mathcal{R}}_i$, by symmetry. Therefore, we can apply TR4 to R_1 and R_1'' to get $\check{R}_1 = (1234, 12, 134, 1)$. Now we have $\hat{R}_3 = (1234, 234, 13, 3)$ from $\hat{R}_1$ and $\bar{R}_2 = (1234, 12, 234, 2)$ from $\check{R}_1$. We also have $R_2 = (1234, 234, 12, 2)$ from $\hat{R}_3$ and $R_3 = (1234, 13, 234, 3)$ from $\check{R}_2$. Also, we get $R_4 = (1234, 234, 14, 4)$ from $\hat{R}_3$. Note that

[4] $|X|$ denotes the cardinality of set X.

$\hat{R}_1 \sim \hat{R}_3, \check{R}_1 \sim \check{R}_2, \check{R}_3 \sim \check{R}_2$, $\hat{R}_2 \sim R_3, \hat{R}_3 \sim R_4$, and thus $R_2 \in \bar{\mathcal{R}}_2$, $R_3 \in \bar{\mathcal{R}}_3$, and $R_4 \in \bar{\mathcal{R}}_4$. Now consider the profile $R \in \bar{\mathcal{R}}$, where, given TR2, we let R_j rank j first for $j = 5, \ldots, n$. Note that coalitions $\{1,2,3\}$ and $\{1,2,3,4\}$ are not individually rational at this profile since $\{2\}P_2\{1,2,3\}$ and $\{1\}P_1\{1,2,3,4,\}$, and $\{1,2\}$, $\{2,3,4\}$, and $\{1,3\}$ form a cycle. Therefore, this preference profile does not have a stable coalition structure, and $\bar{\mathcal{R}}$ does not guarantee the existence of a stable coalition structure, which is a contradiction. This completes Case 2 and concludes the proof of Theorem 1. ■

The following example is designed to illustrate Part 1 of the proof of the theorem.

Example 2 *Constructing a stable coalition structure when preferences satisfy Intersection Restriction.*

Let $n = 6$ and consider the following preferences.

$R_1 = (134, 123456, 13, 136, 1)$

$R_2 = (25, 125, 256, 123456, 2)$

$R_3 = (123456, 34, 134, 3)$

$R_4 = (2345, 34, 123456, 134, 4)$

$R_5 = (256, 56, 123456, 2456, 5)$

$R_6 = (256, 123456, 56, 6)$

Note that R satisfies Intersection Restriction. Now we specify the sets that are used in the construction of a stable coalition structure. $\hat{\mathcal{N}} = \{123456, 34, 256, 56\}$. Note that 134 is self-blocked by 34, and hence 134 is not in $\hat{\mathcal{N}}$. $\bar{\mathcal{N}} = \{123456, 34, 256\}$. 56 is eliminated because 56 is a subset of 256, where 256 is in $\hat{\mathcal{N}}$, and both players 5 and 6 prefer 256 to 56. Then $\check{\mathcal{N}} = \{123456\}$ or $\check{\mathcal{N}} = \{34, 256\}$. Finally, we have $\sigma = \{123456\}$ or $\sigma = \{1, 34, 256\}$, where the missing player 1 is added as a singleton in the latter case. Both of these coalition structures are stable at the specified preference profile R. ■

5 The Uniqueness Characterization

The following two individual preference restrictions, Superset Restriction and Strong Intersection Restriction, will be used in the characterization of the uniqueness of stable coalition structures.

Superset Restriction: For all coalitions $S, S' \in \mathcal{N}_i$ such that $SP_iS'P_i\{i\}$, $S' \subset S$.

Superset Restriction requires that for any two strongly individually rational coalitions S and S' such that S is preferred to S', S is a superset of S'. Note that Superset Restriction implies Intersection Restriction, since Intersection Restriction only has implications for independent pairs of strongly individually rational coalitions.

Strong Intersection Restriction: For all coalitions S, $S' \in \mathcal{N}_i$ such that $SP_iS'P_i\{i\}$, $(S \bigcap S')R_iS$.

Strong Intersection Restriction requires that the intersection of any two individually rational coalitions be ranked weakly above both of these coalitions. Clearly, Strong Intersection Restriction is a stronger property than Intersection Restriction. Observe also that Intersection Restriction is thus implied by both Strong Intersection Restriction and Superset Restriction.

Once more, we need to show that Superset Restriction and Strong Intersection Restriction are indeed rich preference restrictions, and these straightforward proofs are left to the reader.

Our second theorem characterizes the existence of a unique stable coalition structure.

Theorem 2 *A rich individual preference restriction guarantees the existence of a unique stable coalition structure if and only if it is either Superset Restriction or Strong Intersection Restriction.*

Why can't we combine the two properties of Superset Restriction and Strong Intersection Restriction? Let's consider an expansion of these two individual preference restrictions to a set of restrictions that in some sense combine the two and includes both as opposite extreme members. This set of individual preference restrictions is given by the following definition. There exists a partition of N into $\{N_U, N_L\}$ such that for all $S \in \mathcal{N}_i$ if $SP_i\{i\}$ then either $S \subset N_U$ or $S \subset N_L$. Moreover, for all $S, S' \in \mathcal{N}_i$ such that $SP_iS'P_i\{i\}$ if $S, S' \subset N_U$ then $S' \subset S$, and if $S, S' \subset N_L$ then $(S \bigcap S')R_iS$. Note that preferences satisfy Superset Restriction if $N_U = N$, and Strong Intersection Restriction is satisfied if $N_L = N$ in this definition. However, the intermediate properties in this class do not satisfy our symmetry assumption on the preference domain. This construction already suggests that the two individual preference restrictions cannot easily be combined if we insist on symmetry, and without symmetry it leads to preference restrictions that are less than compelling.

One argument to convince ourselves that these two conditions don't "mix" goes as follows. Superset Restriction means that whenever $SP_iS'P_i\{i\}$, $S \supset S'$. The latter is equivalent to $S \bigcap S' = S'$, and thus $SP_i(S \bigcap S')$, which is exactly the opposite of what Strong Intersection Restriction requires. Thus, and perhaps not too surprisingly, the two individual preference restrictions exclude each other.

The following equivalent statement of Theorem 2 demonstrates that indeed the disjunction of the properties of Superset Restriction and Strong Intersection Restriction characterizes the uniqueness of a stable coalition structure. If $\bar{\mathcal{R}}$ is rich, for all $R \in \bar{\mathcal{R}}$ there exists a unique stable coalition structure if and only if $\bar{\mathcal{R}}$ satisfies either Superset Restriction or Strong Intersection Restriction. This is a characterization, which shows that there does not exist a unique individual preference restriction in our characterization because stability is a property of preference profiles, and not that of individual preferences.

Proof of Theorem 2:

Part 1: *Let $\bar{\mathcal{R}} \subset \mathcal{R}$ satisfy Superset Restriction or Strong Intersection Restriction. Then*

$\bar{\mathcal{R}}$ *guarantees the existence of a unique stable coalition structure.*

We note that if $\bar{\mathcal{R}} \subset \mathcal{R}$ satisfies either Superset Restriction or Strong Intersection Restriction then it also satisfies Intersection Restriction, and thus Part 1 of Theorem 1 implies that $\bar{\mathcal{R}}$ guarantees the existence of a stable coalition structure, so it would suffice to prove that the existing stable coalition structure for each preference profile is unique. Instead, we provide the existence proof as well for both Superset Restriction and Strong Intersection Restriction, which allows us to base the uniqueness arguments on the much simpler constructions of a stable coalition structure provided here. These constructive proofs are also of interest in their own right, as they give a more intuitive and easier way to identify the unique stable coalition structure, in each of the two cases, than the more general construction used in the proof of Theorem 1.

Superset Restriction

Fix $R \in \bar{\mathcal{R}}$, where $\bar{\mathcal{R}}$ is a preference domain satisfying Superset Restriction. Find all strongly individually rational coalitions $S \in \mathcal{N}$ at R such that for all $S' \in \mathcal{N}$ with $S' \supset S$ and $S' \neq S$, S' is not strongly individually rational at R. Let σ consist of all of these coalitions and the remaining singletons. We will show that σ is a partition of N, and that it is the only stable coalition structure at R.

(a) *σ is a partition of N*

Since the union of all the coalitions in σ is N, we only have to show that for all distinct $S, S' \in \sigma$, $S \bigcap S' = \emptyset$. Suppose that there exist distinct $S, S' \in \sigma$ such that $S \bigcap S' \neq \emptyset$. Note that S and S' are both individually rational at R. If S is a singleton then $S \subset S'$ and $S \in \sigma$ implies that S' is not strongly individually rational at R, a contradiction. Thus, S is not a singleton. We can show similarly that S' is not a singleton. Therefore, we can assume without loss of generality that $S P_i S' P_i \{i\}$ for some $i \in S \bigcap S'$. Then, since R_i satisfies Superset Restriction, $S' \subset S$. But then $S' \in \sigma$ implies that S is not a strongly individually rational at R, which is a contradiction.

(b) *σ is a stable coalition structure at R*

Suppose that σ is not a stable coalition structure at R. Then there exists $T \subset N$ such that T blocks σ. Let $S \in \sigma$ such that $T \bigcap S \neq \emptyset$. Note that $T \neq S$. Since T blocks σ, and for all $\tilde{S} \in \sigma$, $\tilde{S}$ is individually rational at R, T is also individually rational at R. Suppose T is a singleton. Let $T = \{i\}$. Then, since T blocks σ, $\{i\}P_iS$ and S is not individually rational at R, a contradiction. Therefore, T is strongly individually rational at R.

If $S \subset T$ then, given that T is strongly individually rational at R, $S \notin \sigma$, a contradiction. This also implies that S is not a singleton and S is therefore strongly individually rational at R. If $T \subset S$ then, given that both T and S are strongly individually rational at R, Superset Restriction implies that, for all $i \in T$, SP_iT which contradicts the fact that T blocks σ. Therefore, S and T are independent, and since both of them are strongly individually rational at R, Superset Restriction is violated.

(c) *σ is the unique stable coalition structure at R*

Suppose that there exists $\sigma' \in \Sigma$ such that $\sigma' \neq \sigma$ and σ' is stable at R. Let $S \in \sigma$ and $S' \in \sigma'$ such that $S \bigcap S' \neq \emptyset$ and $S \neq S'$. Note that both S and S' are individually rational at R. If S is a singleton then $S \subset S'$ and S' is not a singleton , so S' is strongly individually rational at R. But then $S \in \sigma$ implies that S' is not strongly individually rational at R, a contradiction. If S' is a singleton then let $S' = \{i\}$ and observe that SP_iS', otherwise S' would block σ at R. Suppose that for all $\tilde{S} \in \sigma'$ such that $\tilde{S} \bigcap S \neq \emptyset$, $\tilde{S}$ is a singleton. Then, given the previous argument, S would block σ'. Thus, we can choose S' to be a non-singleton, and both S and S' are strongly individually rational at R. Then, by Superset Restriction, we have either $S \subset S'$ or $S' \subset S$. If $S \subset S'$ then $S \in \sigma$ implies that S' is not strongly individually rational at R, a contradiction. Thus, $S' \subset S$ and for all $i \in S'$, SP_iS', by Superset Restriction. However, this is true for all $S' \in \sigma'$ such that $S \bigcap S' \neq \emptyset$, whether S' is a singleton or not. Therefore, S blocks σ' at R, which is a contradiction.

Strong Intersection Restriction

Fix $R \in \bar{\mathcal{R}}$, where $\bar{\mathcal{R}}$ is a preference domain satisfying Strong Intersection Restriction. Find all strongly individually rational coalitions $S \in \mathcal{N}$ at R such that for all $S' \in \mathcal{N}$ with $S' \subset S$ and $S' \neq S$, S' is not strongly individually rational at R. Let σ consist of all of these coalitions and the remaining singletons. We will show that σ is a partition of N, and that it is the only stable coalition structure at R.

(a) *σ is a partition of N*

Since the union of all the coalitions in σ is N, we only have to show that for all distinct $S, S' \in \sigma$, $S \bigcap S' = \emptyset$. Suppose that there exist distinct $S, S' \in \sigma$ such that $S \bigcap S' \neq \emptyset$. Note that S and S' are both individually rational at R. If S is a singleton then $S \subset S'$, and $S' \in \sigma$ implies that $S \notin \sigma$, by the construction of σ, a contradiction. Thus, S is not a singleton. Similarly, S' is not a singleton, and thus both S and S' are strongly individually rational at R. Therefore, either $SP_iS'P_i\{i\}$ or $S'P_iSP_i\{i\}$ for all $i \in S \bigcap S'$. Then, since R_i satisfies Strong Intersection Restriction, $(S \bigcap S')R_iS$, for all $i \in S \bigcap S'$, which means that $S \bigcap S'$ is strongly individually rational at R. But since $S \bigcap S' \subset S$, $S \in \sigma$ implies that $S \bigcap S'$ is not strongly individually rational at R, unless $S \bigcap S' = S$. Finally, if $S \bigcap S' = S$ then $S \subset S'$ and $S' \in \sigma$ implies that S is not strongly individually rational at R, which is a contradiction.

(b) *σ is a stable coalition structure at R*

Suppose that σ is not a stable coalition structure at R. Then there exists $T \subset N$ such that T blocks σ. Let $S \in \sigma$ such that $T \bigcap S \neq \emptyset$. Note that $T \neq S$ and T is individually rational at R since T blocks σ, and for all $\tilde{S} \in \sigma$, $\tilde{S}$ is individually rational at R. Therefore, T is not a singleton since S is individually rational at R. Thus, T is strongly individually rational at R, and $S \in \sigma$ implies that $T \not\subset S$.

If T and S are independent then S is not a singleton and therefore S is strongly individually rational at R. Then, since both S and T are strongly individually rational at

R, Strong Intersection Restriction implies that for all $i \in T \bigcap S$, $(T \bigcap S) P_i S$ so $T \bigcap S$ is individually rational at R. Given that $T \bigcap S \subset S$ and $T \bigcap S \neq S$, this contradicts the fact that $S \in \sigma$.

Since $T \not\subset S$ and T and S are not independent, $S \subset T$. Then, for all $i \in S$, TP_iS, since T blocks σ. Suppose S is not a singleton. Then S is strongly individually rational at R, and Strong Intersection Restriction implies that for all $i \in S$, $(T \bigcap S) P_i T$, that is, SP_iT, a contradiction. Thus, S is a singleton. This argument holds for all $S \subset N$ such that $S \in \sigma$ and $T \bigcap S \neq \emptyset$ and thus for all $i \in T$, $\{i\} \in \sigma$. Since T blocks σ, for all $i \in T$, $TP_i\{i\}$, and thus it follows from the construction of σ that for some $T' \subset T$, $T' \in \sigma$, a contradiction.

(c) *σ is the unique stable coalition structure at R*

Suppose that there exists $\sigma' \in \Sigma$ such that $\sigma' \neq \sigma$ and σ' is stable at R. Let $S \in \sigma$ and $S' \in \sigma'$ such that $S \bigcap S' \neq \emptyset$ and $S \neq S'$.

Suppose that S is not a singleton. Observe that if all $\tilde{S} \in \sigma'$ such that $S \bigcap \tilde{S} \neq \emptyset$ is a singleton then, given that $S \in \sigma$ and thus S is strongly individually rational at R, S blocks σ. This is a contradiction. Therefore, we can let S' be a non-singleton and S' is strongly individually rational at R. Since S is also strongly individually rational at R, Strong Intersection Restriction implies that for all $i \in S \bigcap S'$, $(S \bigcap S') R_i S$. This means that $S \bigcap S'$ is individually rational at R, and if $S \bigcap S' \neq S$ then $S \notin \sigma$, a contradiction. Thus, $S \bigcap S' = S$ and we have $S \subset S'$. Then, by Strong Intersection Restriction, for all $i \in S$, SP_iS'. Since this argument applies to all $\tilde{S} \in \sigma'$ such that $S \bigcap \tilde{S} \neq \emptyset$, it follows that S blocks σ' at R, a contradiction. Therefore, S is a singleton. This argument holds for all $\bar{S} \in \sigma$ such that $\bar{S} \bigcap S' \neq \emptyset$ and thus, for all $i \in S'$, $\{i\} \in \sigma$. This implies that S' is not a singleton and thus S' is strongly individually rational at R. Then S' blocks σ, a contradiction.

Part 2: *Let $\bar{\mathcal{R}} \subset \mathcal{R}$ be a rich domain that guarantees the existence of a unique stable coalition structure. Then $\bar{\mathcal{R}}$ satisfies Superset Restriction or Strong Intersection Restriction.*

Let $\bar{\mathcal{R}} \subset \mathcal{R}$ be a rich domain that guarantees the existence of a unique stable coalition structure.

Step A *Let* $i \in N$, $R_i \in \bar{\mathcal{R}}_i$, let $S, S' \in \mathcal{N}_i$ *such that* $SP_iS'P_i\{i\}$, *and let* $T, T' \in \mathcal{N}_i$ *such that* $TP_iT'P_i\{i\}$. *Then* $S \subset S'$ *implies that* $T \not\supset T'$.

Let $i \in N$ and $R_i \in \bar{\mathcal{R}}_i$, let $S, S' \in \mathcal{N}_i$ such that $SP_iS'P_i\{i\}$, and let $T, T' \in \mathcal{N}_i$ such that $TP_iT'P_i\{i\}$. Suppose that $S \subset S'$ and $T \supset T'$. We will transform R_i in the following 5 steps. We denote by $\tilde{R}_i^t$ the resulting preferences in step At, $t = 1, \ldots, 1$.

Step A1 Apply TR1 to all $j \in S$.

Step A2 Apply TR2 to all $j \in N \setminus S'$.

Let $\bar{S} \in \mathcal{N}_i$ such that $\bar{S}\tilde{P}_i^2S'$ and for all $\hat{S}$ such that $\hat{S}\tilde{P}_i^2S'$, $\bar{S} \not\subset \hat{S}$, where $\hat{S} \neq \bar{S}$. Note that $\bar{S}$ exists since $S\tilde{P}_i^2S'$.

Step A3 Apply TR1 to all $j \in \bar{S}$.

Since for all $\hat{S} \in \mathcal{N}_i$, $\hat{S} \neq \bar{S}$, such that $\hat{S}\tilde{P}_i^2\bar{S}$, and $\bar{S} \not\subset \hat{S}$, we have $\hat{S} \setminus \bar{S} \neq \emptyset$, and thus $\tilde{R}_i^3 = (\bar{S}, S')$. Note that $S \subset \bar{S} \subset S'$, given Step A1 and Step A2.

Step A4 Fix $j \in S' \setminus \bar{S}$ and apply TR3 to all $h \in S' \setminus \bar{S}$ such that $h \neq j$.

Let S'^4 and $\bar{S}^4$ denote the coalition resulting from S' and $\bar{S}$, respectively, after Step A4. Then $S'^4 = \bar{S}^4 \bigcup \{j\}$.

Step A5 Fix $j \in \bar{S} \setminus \{i\}$ and apply TR3 to all $h \in \bar{S} \setminus \{i\}$ such that $h \neq j$.

Let $i = 1$ and $\tilde{R}_1 = \tilde{R}_1^5$. Then, without loss of generality, $\tilde{R}_1 = (12, 123)$.

We can use similar steps to the above 5 steps, applying them to T, T' instead of S, S', in order to obtain, without loss of generality, $\hat{R}_1 = (145, 14)$, where 4 and 5 may not be distinct from 2 and 3. Note that $\tilde{R}_1, \hat{R}_1 \in \bar{\mathcal{R}}_1$. Observe also that symmetry implies that $\bar{R}_1 = (123, 12)$ is such that $\bar{R}_1 \in \bar{\mathcal{R}}_1$, given $\hat{R}_1$. Let $\tilde{R}_2 = (123, 12)$ from $\bar{R}_1$. Then $\tilde{R}_2 \in \bar{\mathcal{R}}_2$, by symmetry. Let $\tilde{R}_3 = (123, 23)$ from $\bar{R}_1$. Then $\tilde{R}_3 \in \bar{R}_3$, by symmetry. Finally, given

TR2, we can let $\tilde{R}_j$ rank $\{j\}$ first for all $j = 4, \ldots, n$, where $\tilde{R}_j \in \bar{\mathcal{R}}_j$. Now observe that $\tilde{R} \in \bar{\mathcal{R}}$, so $\tilde{R}$ has a unique stable coalition structure. However, both $\{\{1, 2, \}, \{3\}, \ldots, n\}$ and $\{\{1, 2, 3\}, \{4\}, \ldots, n\}$ are stable coalition structures at $\tilde{R}$, a contradiction.

Step B *Let $i \in N$ and $R_i \in \bar{\mathcal{R}}_i$. Let $S, S' \in \mathcal{N}_i$ such that $SP_iS'P_i\{i\}$ and let S* and *S' be independent. Then $(S \bigcap S')P_iS'$.*

Let $S, S' \in \mathcal{N}_i$ such that $SP_iS'P_i\{i\}$ and let S and S' be independent. Since for all $R \in \bar{\mathcal{R}}$ there exists a stable coalition structure, Part 2 of Theorem 1 applies here, and hence R satisfies Intersection Restriction. This implies that $(S \bigcap S')R_iS'$. Since S and S' are independent, this means that $(S \bigcap S')P_iS'$.

Step C *For all $i \in N$, for all $R_i \in \bar{\mathcal{R}}_i$, R_i satisfies either Superset Restriction or Strong Intersection Restriction.*

Let $i \in N$ and $R_i \in \bar{R}_i$. If at most one coalition is strongly individually rational at R_i, then both Superset Restriction and Strong Intersection Restriction are satisfied vacuously. Thus, assume that there exist $S, S' \in \mathcal{N}_i$ such that $SP_iS'P_i\{i\}$.

Case 1 $(S \bigcap S')P_iS'$

Let $T, T' \in \mathcal{N}_i$ such that $TP_iT'P_i\{i\}$. Since $S \bigcap S' \subset S'$ and $(S \bigcap S')P_iS$, Step A implies that $T' \not\subset T$. If $T \subset T'$ then $T \bigcap T' = T$ and thus $(T \bigcap T')R_iT$ holds. If T and T' are independent then suppose $TP_i(T \bigcap T')$. This violates Step B. Therefore, $(T \bigcap T')R_iT$. This proves that R_i satisfies Strong Intersection Restriction.

Case 2 $S'R_i(S \bigcap S')$

Since $S'R_i(S \bigcap S')$, Step B implies that S and S' are not independent. If $S \subset S'$ then $S \bigcap S' = S$ and $S'R_iS$, a contradiction. Therefore, $S' \subset S$. Let $T, T' \in \mathcal{N}_i$ such that $TP_iT'P_i\{i\}$. Then, given that $S' \subset S$, Step A implies that $T \not\subset T'$. If T and T' are independent then $(T \bigcap T')P_iT'$, by Step B. Thus, we can apply the argument in Case 1 to

T, T', and conclude that R_i satisfies Strong Intersection Restriction. But this contradicts the fact that $S' \subset S$. Therefore, T and T' are not independent and $T' \subset T$. This proves that R_i satisfies Superset Restriction.

Step D $\bar{\mathcal{R}}_i$ *satisfies either Superset Restriction or Strong Intersection Restriction.*

Given Step C, this statement follows from symmetry of $\bar{\mathcal{R}}$. More specifically, fix $i, j \in N$ and $R_i, R_j \in \bar{\mathcal{R}}_j$. By Step C, R_i satisfies either Superset Restriction or Strong Intersection Restriction. Similarly, R_j satisfies either Superset Restriction or Strong Intersection Restriction. Suppose R_i satisfies Superset Restriction and R_j satisfies Strong Intersection Restriction. Then an argument similar to that of Step A establishes that we have a contradiction and thus either both R_i and R_j satisfy Superset Restriction, or both satisfy Strong Intersection Restriction. This argument applies to all players and preferences, and thus we can conclude that $\bar{\mathcal{R}}$ satisfies either Superset Restriction or Intersection Restriction. ■

6 Discussion

Consider the following property.

Union Restriction: For all coalitions S, $S' \in \mathcal{N}_i$ such that $SP_iS'R_i\{i\}$, for all $\bar{S} \supset S \bigcup S'$, we have $\bar{S}R_iS'$.

Union Restriction requires that the union, and each superset of the union, of any two strongly individually rational coalitions is ranked weakly above at least one of the two coalitions, and that all supersets of any strongly individually rational coalition is also strongly individually rational.

One might expect that Union Restriction or a similar individual preference restriction would be included as an alternative to Intersection Restriction in Theorem 1, to serve as a counterpart to Superset Restriction in Theorem 2. Indeed, Union Restriction is sufficient for the existence of a stable coalition structure at each preference profile. To see this, fix a preference profile R. Observe that if a strongly individually rational coalition S is self-blocked, there is a unique coalition T such that S is self-blocked by T. Thus, if there is at least one strongly individually rational coalition at R, we can let $\sigma = \{N\}$ if N is not self-blocked. If N is self-blocked then we can let $\sigma = \{S_1, N \setminus S_1\}$ if N is self-blocked by S_1 and $N \setminus S_1$ is not self-blocked. If $N \setminus S_1$ is also self-blocked then let $\sigma = \{S_1, S_2, N \setminus (S_1 \bigcup S_2)\}$ if N is self-blocked by S_1, $N \setminus S_1$ is self-blocked by S_2, and $N \setminus (S_1 \bigcup S_2)$ is not self-blocked, and so on. It is easy to verify that the coalition partition σ defined this way is a stable coalition structure at R, when there is at least one strongly individually rational coalition, and otherwise the coalition structure consisting of all singletons is stable at R.

Union Restriction, however, is not a rich individual preference restriction. In particular, it does not satisfy TR2 and TR3. These violations are fundamental since Union Restriction requires, among other things, that *all* supersets of strongly individually rational coalitions are strongly individually rational, which renders the domain quite small. Specifically, observe that Union Restriction implies that the grand coalition is always strongly individually rational, and in fact it is always ranked first or second, as long as there is at least one strongly individually rational coalition. Indeed, since in Theorem 1 our characterization only uses Intersection Restriction, there is no individual preference restriction (a property like Union Restriction) that satisfies TR2 and TR3 while guaranteeing the existence of a stable coalition structure. It is also worthwhile to remark that Union Restriction and Superset Restriction turn out to be logically independent properties. At the same time, Intersection Restriction is implied by Superset Restriction (as well as strong Intersection Restriction), which is a logical implication of Theorems 1 and 2.

Next, we would like to discuss the individual preference restrictions introduced by the paper closest to ours, Alcalde and Romero-Medina (2006). Other conditions on individual preferences that guarantee stability have been proposed by Cechlárová and Romero-Medina (2001), and Dimitrov et al. (2006). The first one assumes that preferences are based on the most preferred and least preferred member of a coalition, while the latter bases the evaluation of coalitions on the existence of a set of "friends" and "enemies," and introduces two individual preference restrictions in which preferences are entirely determined by these sets. All of these properties require indifferences in preferences, and thus are not directly comparable to our properties since we consider strict preferences only.

Alcalde and Romero-Medina (2006) proposes four sufficient properties of individual preferences. In order to state them, we will need the following definition. For all $i \in N$, for all $S \in \mathcal{N}_i$, let $Ch_i(S)$ denote player i's first ranked choice among coalitions only containing members of S.

Union Responsiveness Condition: R_i satisfies the Union Responsiveness Condition if for all coalitions $S, S' \in \mathcal{N}_i$ such that $S' \subset S$ and $S' \neq Ch_i(S)$, SP_iS'.

The Union Responsiveness Condition implies that the first or second ranked coalition is the grand coalition. This is more demanding than Union Restriction, because it has requirements for all coalitions, not just for individually rational ones.

Intersection Responsiveness Condition: For all coalitions $S, S' \in \mathcal{N}_i$, SP_iS' implies that $(S \bigcap S')R_iS'$.

The Intersection Responsiveness Condition is very closely related to Intersection Restriction, but it is stronger, since it has requirements for coalitions ranked below $\{i\}$, unlike Intersection Restriction, which only imposes requirements on individually rational coalitions.

Singularity: For each coalition $S \in \mathcal{N}_i$, $SP_i\{i\}$ implies that $S = Ch_i(N)$.

This property is restrictive, as it implies that the first or second ranked coalition has to be $\{i\}$, and it is immediately clear that Singularity guarantees that there is a unique stable coalition structure. Singularity is a stronger property than both Superset Restriction and Strong Intersection Restriction.

Essentiality: There exists an essential coalition $\bar{S}$ for i such that $\bar{S}$ is the first-ranked coalition, and a coalition is strongly individually rational if and only if it is a superset of $\bar{S}$. Moreover, for all coalitions S, S' that are strongly individually rational at R_i, $S \subset S'$ implies that SP_iS'.

Essentiality, which guarantees the existence of a unique stable coalition structure, implies Strong Intersection Restriction. Thus, it is also a stronger property than Intersection Restriction. A weaker version of Essentiality, *Top Responsiveness,* is used in Alcalde and Revilla (2004), which implies Intersection Restriction and allows for multiple stable coalition structures.

In sum, all of our corresponding individual preference restrictions are weaker than the ones proposed by Alcalde and Romero-Medina (2006), and therefore our sufficiency results (Part 1 of both theorems) strengthen and unify their results.

Finally let us note that three of the above individual preference restrictions, Union Responsiveness, Intersection Responsiveness, and Essentiality, are not rich individual preference restrictions while, somewhat surprisingly, Singularity is rich. The first three properties all violate TR1, TR2, and TR3, but all of the properties satisfy TR4. We would like to remark, however, that Intersection Responsiveness is easy to modify in order to turn it into a rich individual preference restriction, since for this property all violations of richness come from preference orderings below remaining single, that is, from the part of the preference orderings that are not individually rational. Indeed, if we modify Intersection Responsiveness appropriately, then we get Intersection Restriction, which is a rich individual preference restriction. On the other hand, the failure of being a rich individual preference restriction is more intrinsic to Union Responsiveness and Essentiality (together with Union Restriction). The problem stems mainly from the fact that these individual preference restrictions require that all supersets of some coalitions are individually rational.

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