Coalitions, tipping points and the speed of evolution☆

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Abstract

This study considers pure coordination games on networks and the waiting time for an adaptive process of strategic change to achieve efficient coordination. Although it is in the interest of every player to coordinate on a single globally efficient norm, coalitional behavior at a local level can greatly slow, as well as hasten convergence to efficiency. For some networks, parameter values exist at which the effect of coalitional behavior changes abruptly from a conservative effect to a reforming effect. These effects are confirmed for a variety of stylized and empirical social networks found in the literature. For coordination games in which the Pareto efficient and risk dominant equilibria differ, polymorphic states can be the only stochastically stable states.

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JEL: C71, C72, C73

1. Introduction

Why do some innovations spread rapidly and others slowly? Why are some innovations never adopted, even though they are inexpensive and methods of equilibrium selection would point to their universal adoption? An answer to these questions should consider waiting times. It takes time for any novelty or innovation to be adopted. If the expected waiting time to adoption of an innovation is long, it may be superceded and rendered redundant before it has become widespread. The question of whether it would have eventually been adopted, had the world remained the same in all other respects following its

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invention, is then moot. Given the importance of waiting times, it is natural to query how they are affected by sensible behavioral assumptions. The behavioral assumption we make in the current paper is the following: from time to time, players with interdependent payoffs come together and adjust their actions to their mutual benefit. That is to say, they form temporary coalitions.

This paper examines the effect of coalitional behavior on expected waiting times for processes to reach long run equilibria. We focus on two action pure coordination games with one efficient action and one inefficient action. The relative efficiency of the efficient action to the inefficient action is given by a parameter α. The set of players with whom any given player interacts is governed by an underlying network. The long run equilibrium (stochastically stable state) is the state in which every player plays the efficient action. The waiting time for the process to reach the long run equilibrium can thus be understood as the delay before a society converges to an efficient social norm. In line with the theoretical predictions of Olson (1971) and much of the subsequent literature on collective action, we are particularly interested in the effect of joint strategic switching by coalitions which are small relative to the total population size.[2]

Two possible effects of coalitional behavior are discovered, a *reforming* effect and a *conservative* effect. For some networks and values of α, we observe a reforming effect: convergence to the long run equilibrium is much faster when coalitional behavior is allowed. Less obviously, for other networks and values of α, there is a conservative effect: convergence to the long run equilibrium is much slower in the presence of coalitional behavior.

The model can be considered a stylized representation of multiple phenomena. These include the dissemination of ideas and socio-cultural memes, the spread of process innovation within and between firms, and the choice of consumer technology (e.g. mobile phone network providers). It can also be considered a model of online platforms used for purposes such as photo sharing (e.g. Flickr, My Opera, Fotki, Fotolog), microblogging (e.g. Twitter, Tumbler, Plurk, Jaiku), or internet telephony (e.g. Skype, Google talk, Oovoo). Essentially, it can be considered a model of any setting in which (i) the main benefit or cost of the choice of action by a player arises through his interaction with others, and (ii) switching costs between actions are relatively low. For example, the direct cost of changing the political position one advocates are small compared to the costs that arise through social interaction as a consequence of making such a change. The same can sometimes be said for the choice of technology to be used for a specific purpose within a firm.[3] In all of these situations, it can be the case

[2]See also Poteete and Ostrom (2004). There also exist important provisos to such predictions (Chamberlin, 1974), particularly in the presence of punishment (Mathew and Boyd, 2011; Hwang, 2009).

[3]To give an example, employees may choose to keep track of appointments via a paper diary

that coalitional behavior can facilitate a move towards more efficient choices by players. As would be expected, coordination in strategic choice implied by the concept of coalition can help move the process towards efficiency for the players involved in such a coalition. However, the reverse can also be true. Imagine that, at some point in time, there is a group of players who play an action different to that played by the majority of the population. Even when it is individually rational for each of these players to remain in the minority, the set of such players acting as a coalition may be able to improve payoffs for every coalition member by jointly switching strategies to the majority action. The coalition has acted conservatively and returned the population to a homogeneous state.[4]

Several network types display tipping point effects. For values of α below some threshold $\underline{\alpha}$, as error probabilities become infinitesimal, convergence to the efficient social norm becomes infinitely slower for the process with coalitional behavior than for the process without coalitional behavior. For values of α above some threshold $\bar{\alpha}$, the opposite happens: the process with coalitional behavior becomes infinitely faster than the process without coalitional behavior. In some instances $\underline{\alpha}$ and $\bar{\alpha}$ take the same value. These speed-of-convergence tipping points are driven by preferences in a similar way to those of Granovetter (1978) and not by informational concerns such as in, for example, Bikhchandani et al. (1992). However, in comparison to Granovetter (1978) or Ellison (1993), additional tipping points are created at the values of α above or below which certain coalitional deviations become optimal. For example, it only makes sense for a group to coordinate a break away from a current norm if the additional payoffs the members of the group generate amongst themselves outweigh their losses from miscoordinating with the rest of the population.

Reforming effects are possible for any network. This is not true for conservative effects: the existence of a conservative effect is dependent on the local structure of the network. For coalitional behavior to make a qualitative difference to outcomes, it has to be the case that some subsets of players should share some neighbors in the network, or share some neighbors of neighbors, and so on. There may exist players who are easily 'isolated' in the sense of being easily separated completely from members of the population playing the initial social norm. The existence of such players facilitates the robust formation of groups of

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or via particular software integrated with a system of electronic mail. Another example is the choice faced by academic researchers of whether to use a TEX editor or WYSIWYG software such as Scientific Workplace (or even Microsoft Word). Such a choice generates significant payoff externalities for coauthors, as the default source code generated by such software often has to be adjusted before it can be compiled in a TEX editor.

[4]Terminological sticklers may point out that, given that the minority action is already played, the switch back may be better described as reactionary than as conservative. However, as the reaction in question is by the deviating players themselves, the authors prefer to see the effect as a dynamic type of conservatism.

players who play an incipient, new social norm.

The theoretical results in the paper are confirmed as robust for non-vanishing parameter values via simulations on both stylized and empirical networks. These include coauthorship networks, workplace networks and social networks, including fragments of friendship networks on FACEBOOK.

The question arises as to what the model of the paper gives when instead of a pure coordination game, the players play a game in which one of the Nash Equilibria is Pareto efficient and the other is risk dominant. In these circumstances, it is no longer the case that the state in which every player chooses the efficient action is always stochastically stable. Neither need the state in which all play the risk dominant action be stochastically stable. In fact, long run equilibria may involve heterogeneous action choices by different parts of a population. Consider a population composed of cliques of players such that players within any given clique are densely connected, but each clique is only loosely connected to other cliques. For some parameter values, the stochastically stable states of such a population involve small cliques coordinating on the efficient equilibrium action and large cliques coordinating on the risk dominant equilibrium action. This gives a justification for the use of small teams in large organizations: efficient behavior may not be long-run stable in large operating units.

2. Relation to existing literatures

2.1. Stochastic stability

Whereas previous concepts of equilibrium stability such as asymptotic stability or evolutionary stable strategies (Smith and Price (1973)) focus on robustness to single errors (mutations) in strategies, Foster and Young (1990), Young (1993) and Kandori et al. (1993) use the methods of Freidlin and Wentzell (1998) to measure the robustness of equilibria of an adaptive strategy revision process to multiple errors in players' choice of strategies. They show that although there may be several stationary states in a dynamic process, some of them may be more robust to such errors than others, and that if the probability of a random error becomes very small, then in the long run some nonempty subset of stationary states which are relatively robust to such errors will be observed almost all of the time. These are the *stochastically stable* states. Bergin and Lipman (1996) prove a kind of folk theorem for stochastic stability, that is they show that any stable state of the unperturbed dynamic process can be selected with appropriately chosen state-dependent mutation rates. Therefore, the structure given to error probabilities is crucial to the predictions of the model.[5] Naidu et al. (2010) analyze a model in which transitions between stochastically stable states are driven

[5] van Damme and Weibull (2002) give conditions on error probabilities under which the results of Young are recovered.

by errors on the part of the players who stand to gain from the move and arrive at different predictions to those of Young (1998) for games of contracting.[6] Logit models, in which more costly errors occur with lower probability, are also common in the literature[7].

A pervasive criticism of stochastic stability as a tool of equilibrium selection has been the large lengths of time it can take for the perturbed process, starting at a non-stochastically stable equilibrium of the unperturbed dynamic, to reach a stochastically stable state (Ellison, 2000). This calls into question the empirical validity of stochastic stability: if it takes a billion periods to reach a stochastically stable state, the concept may not be predictively useful on human timescales. The current paper finds that this problem can be considerably worsened or mitigated by orders of magnitude when coalitional behavior is introduced, and whether a worsening or a mitigation occurs depends on knife edge parameter values.

2.2. Coalitional behavior

There exists a large literature in cooperative game theory on the behavior of coalitions.[8] Concepts include 'strong equilibrium' (Aumann, 1959), coalition proof Nash equilibrium (Bernheim et al., 1987), farsighted coalitional stability (Konishi and Ray, 2003), and coalitional rationalizability (Ambrus, 2009). There is a small literature on coalitional behavior in perturbed evolutionary models. Newton (2012a) introduces a model of *coalitional stochastic stability* in which the 'errors' in the dynamic process are actually small probabilities of payoff improving behavior by coalitions of players, and shows that this can lead to significant differences in equilibrium selection when compared to random error driven selection. Sawa (2012) adapts coalitional stochastic stability for logit-style dynamics. The model of Sawa (2012) also features coalitional behavior as part of the unperturbed dynamic. Serrano and Volij (2008) and Newton (2012b) do similarly, applying stochastic stability to models of coalitional recontracting. Matching models such as those found in Jackson and Watts (2002) and Klaus et al. (2010), in which coalitions are pairs of recontracting agents, also fall into this category.

2.3. Networks and local interaction

Some coalition structures can be considered more reasonable than others. Suggestions have been made in the cooperative game theory literature that subsets of coalitions which are allowed to deviate should also be allowed to deviate, as the possibility of communication between players in the initial coalition could imply the possibility of communication between a proper subset of the players involved.[9] Alternatively, it has been suggested that the union of coalitions which

[6]See Binmore et al. (2003) for a good survey of results in evolutionary bargaining models.

[7]See for example Blume (1993); Alós-Ferrer and Netzer (2010).

[8]For a survey the reader is referred to Peleg and Sudholter (2003).

[9]Algaba et al. (2004).

are allowed to deviate and have a nonempty intersection should also be allowed to deviate, the justification being that players who belong to both potential coalitions could act as intermediaries.[10]

Networks are a natural way to represent payoff effects in games and are also a natural way to delineate feasible coalition structures, for instance by assuming that any coalitional activity is undertaken by connected subgraphs of a graph representing a wider social network.[11] This is the approach taken in the current paper, in which it is assumed that aside from payoff effects, the network ties represent the potential for communication and thus coalition formation between sets of players. The authors believe that coalitional effects are very natural in a local interaction setting such as those analysed in Ellison (2000, 1993); Eshel et al. (1998). In fact, often the motivations for players' being connected to one other in a network representing payoff effects can double as reasons why coalitional behavior between the players is plausible. However, although networks can facilitate the formation of coalitions, the two are distinct concepts. To quote from the International Encyclopedia of Civil Society (Anheier and Toepler, 2009) in the context of transnational organization: "Sometimes these networks generate the shared goals, mutual trust, and understanding needed to form coalitions capable of collaborating.... But networks do not necessarily coordinate their actions, nor do they necessarily come to agreement on specific joint actions (as implied by the concept of coalition)."

The current paper studies interaction on given networks, and not network formation. This is another reason that waiting times for convergence to a stochastically stable state are important. For very long waiting times, it may not be plausible to assume that the underlying network structure remains static long enough for the long run equilibrium to be reached. Waiting times in such models without coalitional behavior have been the subject of recent studies by Montanari and Saberi (2010) and Young (2011). Montanari and Saberi (2010) examine the effect of network structure on the order of magnitude of convergence times as networks increase in size. Young (2011) bounds convergence times for fixed, non-vanishing error probabilities.

2.4. Homophily

Finally, we relate the paper to the literature on 'homophily' - the desire of people to associate with those similar to themselves. This is a well documented phenomenon in the sociology literature. For a survey the reader is referred to McPherson et al. (2001). The economic literature on the topic, presaged by Schelling (1969), has been growing of late. For example, Currarini et al. (2009)

[10] Algaba et al. (2001).

[11] Myerson (1977), Jackson and Wolinsky (1996), Jackson (2005), Kets et al. (2011).

explain data on friendships via direct assumptions about people's preference to associate with those of a similar race. The most relevant paper in the homophily literature is that of Golub and Jackson (2012), which defines homophily as the level of preferential linking to vertices of the same colour in a random network, then relates this level to convergence times for a simple averaging process. They conclude that homophily slows convergence. This result follows because for an averaging process the important factor is the size of the channels through which innovation can spread, rather than the probability with which any given innovation gains a foothold in the population. To illustrate this point, consider a complete network. An averaging process will quickly converge on such a network, whereas for a two action coordination game such as the one in the current paper it will take many errors (and therefore in expectation a very long time) for the process to begin a move from an inefficient equilibrium to an efficient one.

From the perspective of the current paper and its emphasis on joint strategic switching, we note that aside from the network formation and informational (different types of player access different information) interpretations of homophily analysed in Golub and Jackson (2012), there may exist a further effect: players of the same type may find it easier to coordinate their changes in action. This could be due to underlying cultural norms or even the perception of similar 'sunspots'. The location of players of the same type close to one another in a network (i.e. homophily) would then facilitate coalitional behavior. The implications of this for convergence times would be ambiguous and follow the proceeding analysis.

3. Model

Let N be a finite set of players. Players are arranged in a network, which we represent as an graph $\mathbf{g}$, where $g_{ij} = 1$ if there exists a link between players i and j, and $g_{ij} = 0$ otherwise. We assume that the graph is undirected: $g_{ij} = g_{ji}$. Set $g_{ii} = 0$. The network will affect players in two ways:

(i) The network determines the structure of payoffs, in particular which players' actions impose externalities on other players.

(ii) The network mediates joint action by *coalitions* of players.

Let $x_i^t \in \{A, B\}$ denote player i*'s* action at time t. Let $\mathbf{x}_S^t = \prod_{i \in N} x_i^t$ denote the action profile of all players in $S \subseteq N$ at time t. Let N_i denote the set of *neighbors* of player i: $N_i = \{j \in N : g_{ij} = 1\}$. For $S \subseteq N$, let $N_S = \bigcup_{i \in S} N_i \setminus S$. Payoffs of player i in period t are given by:

$$u_i(\mathbf{x}^t) = \chi(x_i^t) \sum_{j \in N_i} \delta(x_i^t, x_j^t)$$

	A	B
A	α, α	$0, 0$
B	$0, 0$	$1, 1$

Figure 1: A two player pure coordination game.

where δ is the Kronecker delta function and for some positive real constant $\alpha > 1$:

$$\chi(x_i^t) = \begin{cases} \alpha, & \text{if } x_i^t = A \\ 1 & \text{if } x_i^t = B \end{cases}.$$

That is, the players play a pure coordination game on the network. If a player chooses action B, his payoff is the number of his neighbors who play action B. If an player chooses action A, his payoff is the number of his neighbors who play action A multiplied by some constant α which is strictly greater than 1. Effectively, the players play their chosen action against each of their neighbors in the game in figure 1. The constant α can be understood to represent some technological superiority of action A over action B, with the magnitude of α representing the magnitude of this superiority.

The underlying dynamic process of this paper is one in which coalitions of players adjust their strategies in a coordinated manner. The sets of players which can do this are determined by the underlying network.

Definition. A coalition of players $S \subseteq N$ is *feasible* in g, denoted $S|_g$ if and only if for all $i, j \in S$ there exists $\{s_m\}_{m=1}^{m=l}$ such that $s_1 = i$, $s_l = j$, $\prod_{m=1}^{l-1} g_{s_m s_{m+1}} \mathbb{1}_S(s_m) = 1$.

That is, S is a feasible coalition if and only if there is a path between any two players in S that only uses edges between players in S. That is, players in a feasible coalition either directly interact with one another, or have interactions mediated by other players in the coalition. Another way of stating this is that the network restricted to players in S forms a connected subgraph. It is assumed that N is feasible: $\mathbf{g}$ is a connected network. This is without loss of generality: if the network comprised more than one component, analysis of each component would proceed independently of the other components.

When a coalition chooses its actions, we mandate that it chooses a *better response*. That is, players in the coalition adjust their actions in a coordinated manner such that no member of the coalition loses from the adjustment. Note that the addition of a Pareto condition to define a form of coalitional best response would complicate definitions without changing the results of the paper. Define the set of better responses for a set of players S:

$$A_S(\mathbf{x}^t) := \Big\{\mathbf{x}_S : u_i(\mathbf{x}_S, \mathbf{x}^t_{N\setminus S}) \geq u_i(\mathbf{x}^t)\ \forall\, i \in S\Big\}.$$

Let $G_{A_S(\mathbf{x}^t)}(.)$ be a probability distribution over $A_S(\mathbf{x}^t)$. $G_{A_S(\mathbf{x}^t)}(.)$ will determine the actions chosen by a coalition S when it is called upon to better respond. We assume full support on the set of better responses.

Assumption 1. *Each $G_{A_S(\mathbf{x}^t)}(.)$ has full support on $A_S(\mathbf{x}^t)$.*

We are particularly interested in the effect of coalitional behavior by coalitions which are small relative to the total size of the population. It is natural to assume that there are limits to how large a coalition can be. Such a limit could be a consequence of higher costs of communication for larger coalitions. One approach would be to bound the maximum path length between coalition members on the network. Another approach, the one taken here, is to limit the maximum number of players in a coalition. Let $\mathcal{N}(k)$ be the set of feasible coalitions of size k or smaller:

$$\mathcal{N}(k) = \{S \subseteq N : (S|_g \text{ and } |S| \leq k)\}.$$

For given k, let $F_k(.)$ be a distribution on $\mathcal{N}(k)$. $F_k(.)$ will determine which coalition gets the opportunity to update its actions in any given period. The process is one of asynchronous updating: only one coalition at a time updates its actions.

Assumption 2. *$F_k(.)$ has full support on $\mathcal{N}(k)$*

The process of strategy updating $\Phi_{k,\alpha,\varepsilon}$ is constructed as follows. Each period, a coalition S is chosen according to $F_k(.)$. The coalition decides on an intended new action profile for its members. Denote this intended action profile by y^{t+1}_S. This profile is chosen from the set of better responses $A_S(\mathbf{x}^t)$:

$$y^{t+1}_S \sim G_{A_S(\mathbf{x}^t)}(.).$$

Following the decisions on which actions to take, each player will play his choice. This is the unperturbed dynamic. A perturbed dynamic is generated by considering the possibility that a player makes a mistake when attempting to play the action that he has agreed to play. Each player in the coalition, independently of the other players, with a small probability ε makes an *error* and chooses an action at random. That is, independently for each $i \in S$:

$$\text{With probability } 1-\varepsilon : \quad x^{t+1}_i = y^{t+1}_i$$

With probability ε : $\quad x_i^{t+1} \sim U[\{A, B\}]$.

Finally, all players who are not part of the chosen coalition for period t do not update their actions. For all $i \in N \setminus S$:

$$x_i^{t+1} = x_i^t$$

So the change in the action profile is determined by a Markov process, $\Phi_{k,\alpha,\varepsilon}$, on state space $X := \{A, B\}^{|N|}$, with transition probabilities $P_{k,\alpha,\varepsilon}(.,.)$ derived from the above description of the process. Each period, this process involves feasible coalitions of players changing their strategies in a payoff improving manner. When choosing his strategy, each player in the coalition independently makes an error with probability ε and chooses an action at random. The process with $\varepsilon = 0$ corresponds to an unperturbed dynamic in which players do not make errors.

Note that for $\varepsilon > 0$, $\Phi_{k,\alpha,\varepsilon}$ is irreducible and aperiodic and therefore has a unique invariant distribution $\pi_{k,\alpha,\varepsilon}$ which is ergodic. Denote the expected time for the process $\Phi_{k,\alpha,\varepsilon}$ to reach state y starting from x by $W_{k,\alpha,\varepsilon}(x, y)$.

Definition.

$$\tau_y = \min\{t \geq 0 : \Phi_{k,\alpha,\varepsilon}^t = y\}; \qquad W_{k,\alpha,\varepsilon}(x, y) = \mathbb{E}[\tau_y | \Phi_{k,\alpha,\varepsilon}^0 = x]$$

We shall occasionally be interested in the set of absorbing states under the process with $\varepsilon = 0$. Denote this set $\Lambda_{k,\alpha}$.

$$\Lambda_{k,\alpha} := \{x \in X : \ P_{k,\alpha,0}(x, x) = 1\}$$

4. Leading example: the square lattice

Consider the square lattice with von Neumann neighborhood, pictured in figure 2, embedded on a torus as in figure 3. We examine $W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|})$, the expected time for the process to move from an inefficient social norm in which every player plays B to an efficient social norm in which every player plays A.

First, consider the case $\alpha < 3/2$; the relative benefits of the better technology are not so great. If $k = 1$ then $W_{1,\alpha,\varepsilon}(B^{|N|}, A^{|N|}) \in \Theta(\varepsilon^{-2})$. To see this, consider that two errors (figure 2.ii) are necessary to move to a state C_2 (figure 2.iii) in which a block of four players play A and every other player plays B. From C_2, it takes a single error to move to a state such as C_3 (figure 2.iv) in which a larger block of players plays A. However, at least one error is required to move backwards from C_2 to $B^{|N|}$. That is, the probability of moving to a state in which a larger block of players plays A *conditional on* leaving C_2 is of order 1. This means that the waiting time until $A^{|N|}$ is reached involves the wait for the initial two errors of order ε^{-2}, followed by subsequent waits of order ε^{-1}. These

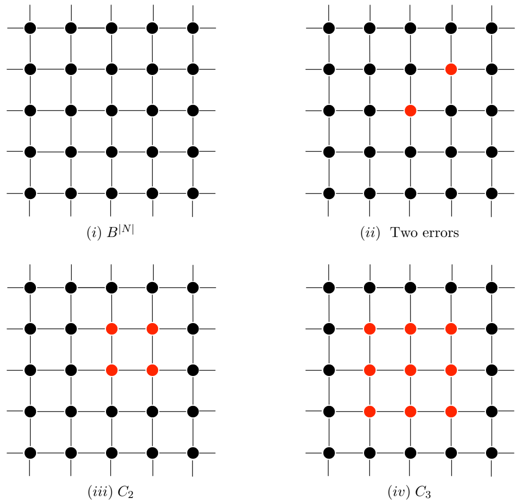


Figure 2: Square lattice in various states. Black vertices play B, red vertices play A.

terms combine additively and so the wait for the initial two errors dominates as ε becomes small.[12]

Now consider $k = 4$, coalitions including up to four players can form. Errors are no longer required to leave C_2. If two adjacent A players form a coalition, they can gain by switching to B and achieving a payoff of 3, which is higher than the payoff of 2α they attain in state C_2. In the absence of random errors, C_2 collapses and the process returns to $B^{|N|}$. More than two errors are required to exit the basin of attraction of $B^{|N|}$. Following three errors on a diagonal, the process can attain the state C_3 which has a 3 by 3 block of players playing A. This block of players playing A can expand with the aid of a single error. It is not possible to leave C_3 without the help of errors. To see this, first consider the player in the centre of the square. He attains his maximum possible payoff of 4α, so he will not intentionally change his action as part of a coalition or otherwise. Secondly, consider the neighbors of the central player. Their payoffs at C_3 are 3α, so they will never intentionally change their action unless the central player also changes his, which he will not. The players at the corners of the block of A players cannot earn more than their C_3 payoff of 2α unless some non-corner player in the square changes his action, which will not occur. Similar arguments to those in the case $k = 1$ lead us to conclude that $W_{4,\alpha,\varepsilon}(B^{|N|}, A^{|N|}) \in \Theta(\varepsilon^{-3})$. Convergence to the efficient social norm is an order of magnitude slower in the presence of coalitional behavior: small outbreaks of innovation are snuffed out as the players involved in the outbreak collaborate to recoordinate with the population as a whole. The

[12] This argument is due to Ellison (1993).

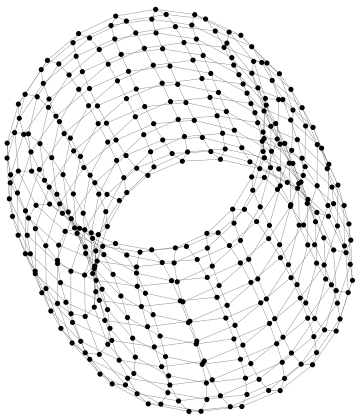


Figure 3: Square lattice on a torus.

possibility of coalitional behavior has led to a *conservative* effect.

Now, consider the case $3/2 \leq \alpha < 2$, $k = 4$. From state C_2, pairs of players who play strategy B in C_2, and who each have a neighbor playing A, can switch together to playing A and attaining a payoff of 2α which is greater than their payoff of 3 in C_2. In this way, the set of players playing A can expand without errors. This speeds up the process of moving to the efficient norm. However, to reach C_2 the initial two errors are still necessary so the order of magnitude of the wait is the same as that for the case without coalitional behavior, $W_{4,\alpha,\varepsilon}(B^{|N|}, A^{|N|}) \in \Theta(\varepsilon^{-2})$.

Finally, consider the case $2 \leq \alpha$, $k = 4$. From state $B^{|N|}$, a square of 4 players can form a coalition. By switching to playing A, thus reaching state C_2, they can attain a payoff of 2α which is greater than their payoff at $B^{|N|}$ of 4. This can happen for any such set of players on the grid. Therefore, the process can move to $A^{|N|}$ without the aid of any errors. $W_{4,\alpha,\varepsilon}(B^{|N|}, A^{|N|}) \in \Theta(1)$. Convergence to the efficient social norm is orders of magnitude faster in the presence of coalitional behavior: coalitions coordinate innovation in the population. The possibility of coalitional behavior has led to a *reforming* effect.

The reasoning of the preceding three paragraphs leads to the following proposition.[13]

Proposition 1. *Let* $\boldsymbol{g}$ *be the square lattice on a torus with von Neumann neigh-*

[13]These results giving existence of reforming and conservative effects dependent on the value of α extend readily to hyper-cubic lattices.

borhoods, $4 \leq k \ll N$*. Then, as* $\varepsilon \to 0$*:*

$$\alpha < \frac{3}{2} \implies \frac{W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|})}{W_{1,\alpha,\varepsilon}(B^{|N|}, A^{|N|})} \to \infty$$

$$\frac{3}{2} \leq \alpha < 2 \implies \frac{W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|})}{W_{1,\alpha,\varepsilon}(B^{|N|}, A^{|N|})} \in \Theta(1)$$

$$2 \leq \alpha \implies \frac{W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|})}{W_{1,\alpha,\varepsilon}(B^{|N|}, A^{|N|})} \to 0$$

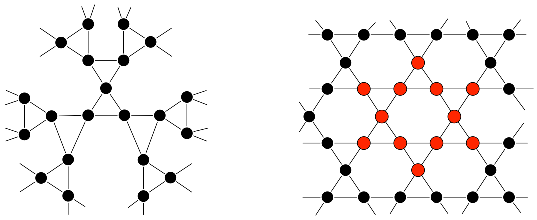
Proof. See appendix for proof of this and all propositions the proofs of which are not included in the text. □

Proposition 1 tells us that coalitional behavior can have either a *conservative* or a *reforming* effect. For low values of α, it has a conservative effect: when small groups of players start to play A and form a configuration which is stable under an individual best response dynamic, coalitions of players tear apart the cluster of deviant behavior, taking the process back to the state in which B is played by all. For large values of α, coalitional behavior has a reforming effect: groups of players can coordinate their choice to play A, increasing their payoffs from the change by ensuring that it occurs simultaneously to that of their neighbors. This speeds up the process of convergence to the efficient social norm.[14]

Note that $W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|})$ is not necessarily monotonic in k. If $k = |N|$, there is the possibility of the coalition $S = N$ being chosen to respond, and the players in N choosing to play $A^{|N|}$ in a single step without the aid of any errors. Convergence is fast and the formation of the grand coalition has a reforming effect.

Other networks display tipping point effects even starker than those of the square lattice. Consider the regular graph of overlapping triangles with local structure shown in figure 4(i). Action profiles in which a large majority of players play B, but there exist triangles of A players, are stable under the unperturbed dynamic for $k = 1$ when $\alpha < 3$. However, when $k = 2$, $\alpha < 3/2$, pairs of A players at the edge of such a set of triangles gain from agreeing to switch simultaneously to B to earn payoffs of 3 which are higher than their existing payoffs of 2α. Such a configuration thus unravels and the process returns to $B^{|N|}$. When $\alpha > 3/2$, pairs of B players who have a common neighbor playing A gain from switching

[14] Montanari and Saberi (2010) would consider all of these parameter specifications to give fast convergence as the order of magnitude of the waiting time does not increase in population size. Given that period length is undefined, for fixed small ε this could encompass massive differences in actual waiting times. The focus of the current paper is not whether convergence is 'fast' or 'slow' as such (although simulation results can be read this way), but on the effects of coalitional behavior relative to the baseline process without coalitional behavior.



(*i*) Overlapping triangles (*ii*) Kagome lattice at state Kag_3

Figure 4: Kagome lattice and graph of overlapping trianges, with minimum group of A players robust to coalitions highlighted for the Kagome lattice.

to A, so the set of players who play A can expand rapidly following just a single error.

The Kagome lattice is slightly different. Limiting convergence times in this case depend on the number of errors required to attain a configuration such as that in figure 4(ii), which can be considered analagous to the 3 by 3 square of players in the square lattice. Players who play A in this configuration will not switch to B without the aid of random errors. Similar arguments to those for the square lattice show that for both the regular graph of overlapping triangles and the Kagome lattice, coalitional behavior slows convergence to $A^{|N|}$ by an order of magnitude when $\alpha < \frac{3}{2}$ and speeds it by orders of magnitude when $\alpha \geq \frac{3}{2}$

The question arises as to how far the results of the preceding paragraphs can be extended to general network architectures. The answer is sometimes unambiguously positive and sometimes not.

5. General results

Here we give some general observations for any network. We start by noting that if α is large enough then there exists a reforming effect: in the limit, the process with coalitional behavior has an infinitely faster transition to $A^{|N|}$. The reasoning behind this result is simple. In state $B^{|N|}$, the players who obtain the highest payoffs are those with the largest number of neighbors. Let one of these players be player i. Let $\alpha > |N_i|$. Then any pair of neighbors in the network can switch to action A and obtain payoffs of α, which is higher than their payoffs in state $B^{|N|}$. So, without any errors occuring, the process can transit to $A^{|N|}$.

Proposition 2. *For any $k \geq 2$, there exists $\bar{\alpha}$ such that, as $\varepsilon \to 0$:*

$$\forall\, \alpha \geq \bar{\alpha}: \quad \frac{W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|})}{W_{1,\alpha,\varepsilon}(B^{|N|}, A^{|N|})} \to 0$$

Proof. Choose $\bar{\alpha} = \max_{i \in N} |N_i|$. Then for $\alpha > \bar{\alpha}$, $\Lambda_{k,\alpha} = \{A^{|N|}\}$. □

Take any set of players S which is small enough to engage in coalitional behavior, and is isolated in the sense that every member of S has 'enough' of his neighbors within S. All players in such a set will play A in any absorbing state of the unperturbed process.

Proposition 3. *If $S \subseteq N$, $|S| \leq k$, is such that:*

$$\forall\, i \in S: \quad |N_i \cap S| > |N_i|\alpha^{-1}$$

then for all $x \in \Lambda_{k,\alpha}$, $i \in S$, we have that $x_i = A$.

Proof. Assume S is feasible. If not, then analyse each component of S in isolation. If S is chosen to better respond, for any $x_N^t \in X$, we have $A^{|S|} \in A_S(x_N^t)$, as either a player is already playing A and cannot be harmed by others switching to A, or is playing B and earning a payoff lower than $|N_i|$ which is lower than the payoff obtained from $A^{|S|}$ which is at least $\alpha|N_i \cap S|$. So a state in which all players in S play A is reached without errors with some probability. The same inequality implies that any player in S would strictly lose were he to switch to B in a later period. □

This result implies that a reforming effect is obtained if k is large enough. If $k = N$, the process can move to $A^{|N|}$ in a single step without errors, as every player agrees together to play A. For $k = 1$, at the very least a single error is required to move from $B^{|N|}$ to $A^{|N|}$.

Corollary 1. *For any $\alpha > 1$, there exists $\bar{k}$ such that, as $\varepsilon \to 0$:*

$$\forall\, k \geq \bar{k}: \quad \frac{W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|})}{W_{1,\alpha,\varepsilon}(B^{|N|}, A^{|N|})} \to 0$$

Proof. Choose $\bar{k} = |N|$. Choose $S = N$. This gives $|N_i \cap S| = |N_i \cap N| = |N_i| > |N_i|\alpha^{-1}$, so proposition 3 and its proof apply. □

The *basin of attraction* of an absorbing state of the unperturbed dynamic $x \in \Lambda_{k,\alpha}$ is the set of states from which, in the absence of errors, convergence to x is guaranteed. The *basin of possible attraction* is the set of states from which, in the absence of errors, convergence to x is possible. Allowing coalitions of a greater size reduces the set of absorbing states. This is equivalent to the fact that for any game, the set of $(k+1)$-strong Nash equilibria is a subset of the set of k-strong Nash equilibria. Furthermore, an expansion of the set of allowable coalitions also (weakly) expands the basins of possible attraction of absorbing states, as any path which was possible with a lower k, is also possible with a higher k. No such monotonicity exists for basins of attraction.

Definition. The basin of attraction of $x \in \Lambda_{k,\alpha}$ is defined as:

$$D_{k,\alpha}(x) = \left\{ y \in X : \ P^t_{k,\alpha,0}(y,x) \to 1 \ \text{ as } t \to \infty \right\}$$

Definition. The basin of possible attraction of $x \in \Lambda_{k,\alpha}$ is defined as:

$$\bar{D}_{k,\alpha}(x) = \left\{ y \in X : \ \sum_{t=1}^{\infty} P^t_{k,\alpha,0}(y,x) > 0 \right\}$$

Lemma 1. *Assume $k_1 \leq k_2$. Then $\Lambda_{k_1,\alpha} \supseteq \Lambda_{k_2,\alpha}$. Moreover, $x \in \Lambda_{k_2,\alpha}$ implies that $\bar{D}_{k_1,\alpha}(x) \subseteq \bar{D}_{k_2,\alpha}(x)$.*

Furthermore, we can directly characterize the basin of possible attraction of $A^{|N|}$. This is a result along the lines of Morris (2000) in that it characterizes the states from which 'contagion', the spread of A across the entire population under the unperturbed ($\varepsilon = 0$) dynamic, can occur.[15]

Lemma 2. *For any $x \in X$, $x \in \bar{D}_{k,\alpha}(A^{|N|})$ if and only if there does not exist $S \subseteq \{i : x_i = B\}$ such that:*

$$\forall\, T \subseteq S,\ |T| \leq k : \ \exists\, i \in T : \quad \frac{|N_i \setminus S| + |N_i \cap T|}{|N_i| + |N_i \cap T|} < \frac{1}{1+\alpha}$$

That is, contagion cannot occur if there exists a set of players which (i) is insular enough to protect it from being 'infected' by A by the rest of the population but (ii) does not contain subsets of players which are themselves insular enough to coordinate their switch to A. When $k = 1$, T must be a singleton, so $N_i \cap T$ is empty and we have, in essence, the result of Morris (2000). The proof, however, is more similar to that of Easley and Kleinberg (2010). Note that lemma 1 implies that the larger is k, the larger is the basin of possible attraction of $A^{|N|}$, from which contagion can occur. However, the size of the basin of possible attraction is only part of the story, and we have already seen in section 4 that although larger k increases the size of $\bar{D}_{k,\alpha}(A^{|N|})$, it can also increase the waiting time until it is reached.

6. Conservative effects

It was seen in section 5 that a reforming effect of coalitional behavior is always possible for large enough values of α. Is a conservative effect similarly always possible? The answer is no, as can be seen if we consider ring networks. In such

[15] Morris (2000) pays specific attention to countably infinite populations for which the basin of possible attraction contains states in which a finite number of players play A.

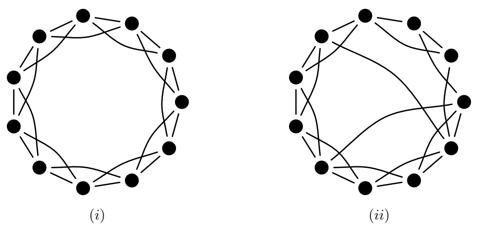


Figure 5: (i) Ring, connected to 2 neighbors on either side. (ii) Ring with rewiring.

a network each vertex is connected to m neighbors on either side. For $k = 1$, the only absorbing states of the unperturbed dynamic are $A^{|N|}$ and $B^{|N|}$. No intermediate equilibria exist for any m, α. The only possible effect of coalitional behavior is then to speed the transition. This is a general result in the absence of intermediate equilibria. From any state, without random errors, the process will converge to either $A^{|N|}$ or $B^{|N|}$. Any number of errors that is enough to move the process into the basin of possible attraction of $A^{|N|}$ when $k = 1$ is also enough to move the process into the basin of possible attraction of $A^{|N|}$ when $k > 1$.

Proposition 4. *If for* $k = 1$, $\Lambda_{k,\alpha} = \{A^{|N|}, B^{|N|}\}$, *then for any* $k \geq 1$*:*

$$\frac{W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|})}{W_{1,\alpha,\varepsilon}(B^{|N|}, A^{|N|})} \in O(1)$$

So, rings and square lattices give very different results when it comes to predicting the impact of coalitional behavior on adaptive dynamics. This is important as each is a commonly used model of local interaction. Moreover, they are common starting points for the construction of small world networks. The question of whether such small worlds retain the properties derived for square lattices and ring networks is addressed in the next section. Another important network that satisfies the conditions of proposition 4 is the complete network, in which every player neighbors every other player. The complete network and the ring are very different networks: in the class of connected networks with $|N|$ vertices, the complete network has the greatest number of links ($\frac{|N|(|N|-1)}{2}$); the ring with $m = 1$ has one more link than the lowest possible number of links ($|N| + 1$).

Now we turn our attention to a method for showing the existence of a conservative effect for small enough α. This requires us to define the notion of a *parochial* set of players.

Definition. For $S \subseteq N$, define:

$$
\begin{aligned}
I_0(S) &= \{i \in S : \quad N_i \subseteq S\}, \\
I_m(S) &= \{i \in S : \quad |N_i \setminus S| \leq |N_i \cap I_{m-1}(S)|\}, \qquad m \geq 1.
\end{aligned}
$$

Note that $I_{m-1}(S) \subseteq I_m(S)$. We say that S is *parochial* if there exists $m \geq 0$ such that $S = I_m(S)$.

Note that unlike the concept of isolation used in the statement of proposition 3, the definition of a parochial set does not depend on the value of α. Every member of a parochial set will have at least half of his neighbors within the set, but this fact alone does not suffice to make a set parochial. In fact, any set of players in which every player has at least one neighbor outside of the set cannot be parochial. Define $\mathcal{P}_A$ as the set of states such that the set of players playing A contains a parochial subset.

Definition.
$$\mathcal{P}_A = \{x \in X : \quad \exists\, S \subseteq \{i \in N : \ x_i = A\} \ \text{ such that } S \text{ is parochial.}\}$$

If S is the set of players playing A and S is not parochial then there exist, for some k, α, nonempty sets of A players who are not in $I_m(S)$ for any m, and who can gain by coordinating a switch back to B. Iterating, the process can return to $B^{|N|}$. Conversely, if some parochial set of players is playing A, for any values of k and α, no member of the set will never switch to B unless at least one member of the set makes an error.

Lemma 3. *$x \in \mathcal{P}_A$ if and only if $\nexists\, k, \alpha$ such that $x \in \bar{D}_{k,\alpha}(B^{|N|})$.*

As the process is time homogeneous and has a finite state space, this implies that under parameters satisfying lemma 3, the process $P_{k,\alpha,0}$ will either enter $\mathcal{P}_A$ or reach state $B^{|N|}$. Leaving the basin of attraction of $B^{|N|}$ implies entering the basin of possible attraction of $\mathcal{P}_A$. This bounds the waiting time to reach $A^{|N|}$ from below by the waiting time to reach $\mathcal{P}_A$. That is, the waiting time to reach $A^{|N|}$ is at least of the order of ε to the power of the number of errors required to reach a state in which some parochial subset of players play A.

The use of this result can be seen in that for the square and Kagome lattices, states C_3 and Kag_3 respectively are in the set $\mathcal{P}_A$. For α close to 1, three errors are required to reach these states from $B^{|N|}$. As the waiting times to reach $A^{|N|}$ for these lattices without coalitional behavior is $O(\varepsilon^{-2})$, this immediately implies the existence of a conservative effect. The same cannot be said for the regular network of connected cliques (figure 6). Assume α is close to 1. Although it is true that $Cli_2 \in \Lambda_{1,\alpha}$ and $Cli_2 \notin \Lambda_{k\geq 4,\alpha}$, there is no conservative effect. The reason for this is that when $k = 1$, two errors are required to reach Cli_2. When

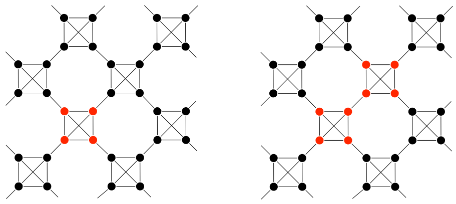


Figure 6: Interconnected cliques.

$k \geq 3$, only a single error in which a player plays A is required, following which the three other members of the clique can better respond by switching to A. It then requires only one more error to move the process to $Cli_3 \in \mathcal{P}_A$. From Cli_3 the process can expand via single error driven steps between states in $\Lambda_{k \geq 4, \alpha}$ until $A^{|N|}$ is reached. Therefore the waiting time with and without coalitional behavior is $\Theta(\varepsilon^{-2})$.

Moreover, the introduction of coalitional behavior can completely alter the paths by which A spreads in the population. Take a square lattice on a torus and a clique of ten players and construct a connected network $\mathbf{g}$ via the following step: select a player i on the torus, remove his exiting edges to the right and below, and replace them with edges to player j and l in the clique. The network is shown in figure 7. When $k = 1$, the fastest paths from $B^{|N|}$ to $A^{|N|}$ involve action A invading the torus (including player i), before errors by four players other than j in the clique make it possible for j to better respond by playing A (for this we require that he has five neighbors playing A). After j switches to A, all of the other players in the clique have 5 neighbors playing A and can individually better respond by playing A. The adoption of A on the clique is the hardest step, requiring 4 errors, and the waiting time until $A^{|N|}$ is reached is $\Theta(\varepsilon^{-4})$. When $k = 9$, the fastest way for A to spread is when it begins in the clique. A single player in the clique who errs and plays A can result in the other players in the clique forming a coalition and better responding by also switching to A. This in turn leads to player i on the torus switching to A. This state, Hyb_1, is in $\mathcal{P}_A$ and it requires at least two errors to leave its basin of attraction. Two such errors can prompt a move to Hyb_3 which is in $\mathcal{P}_A$. From here the process can move between states in $\mathcal{P}_A$ via single errors, culminating at $A^{|N|}$. The waiting time is $\Theta(\varepsilon^{-2})$. So the presence of a single clique on the edge of the network has changed what, in its absence, would have been a conservative effect of coalitional behavior into a reforming effect. It does this by making possible joint deviations by a highly interconnected group of players, amongst whom, in the absence of coalitional behavior, the efficient action A struggles to gain a foothold.

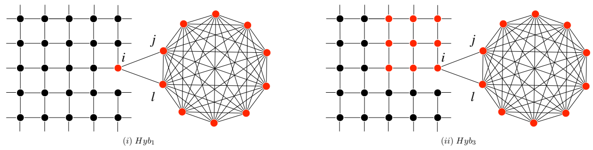


Figure 7: Square lattice on a torus connected to a clique.

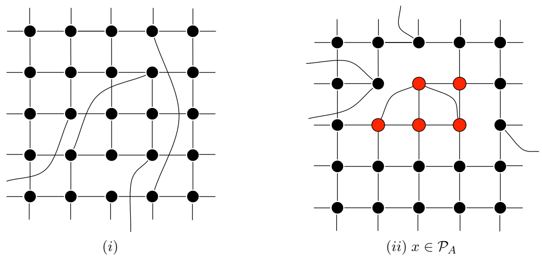


Figure 8: Square lattices with randomly rewired edges.

7. Small worlds

An important class of networks are small worlds: networks with small average shortest path lengths between players. Social networks often exhibit small world properties, such as the six degrees of separation conjectured in Friyges Karinthy's short story 'Chain Links' and famously examined in Milgram's small world experiments. Other small worlds include neural networks and electric power grids. Two common ways to construct small worlds involve small amounts of rewiring of square lattices or ring networks (Szabo and Fath, 2007).

7.1. Small world, square lattice

Consider small world networks formed by rewiring edges of a square lattice. Specifically, consider square lattices with a small number of edges rewired so that every player still has four neighbors. Such a network will not necessarily exhbit a conservative effect for small enough α. For some rewirings, a state in $\mathcal{P}_A$ can be reached via only two errors. Such a rewiring is exhibited in figure 8(ii). If, from such a state, there exists an onward path to $A^{|N|}$ between states in $\mathcal{P}_A$ with each step caused by a single error, then a conservative effect will not exist for any α. Networks like this one, in which several rewired edges affect a small neighborhood, will clearly be a small proportion of the class of regular rewired square lattices when the number of rewired edges is small enough relative to the network size. Moreover, if a proportion p of all the edges in the network are rewired, as the population becomes large, the proportion of networks for which a state in $\mathcal{P}_A$ can be reached by two errors or fewer goes to zero. For a fixed neighborhood size, the

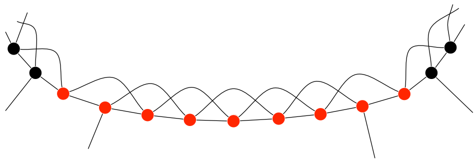


Figure 9: Rewired ring in state Rin_2.

proportion of networks which include a neighborhood with two or more rewirings that remain locally linked within said neighborhood decreases with $|N|^2$, whereas the number of such neighborhoods in the graph increases with $|N|$.

7.2. Small world, rings

Ring networks do not display limiting conservative effects. Small amounts of relinking will not usually alter this property. Consider the graph of figure 9. A small amount of relinking compared to the network size means that new links are likely to join nodes separated by a significant distance on the original ring. The state Rin_2 illustrated in figure 9 can be reached from $B^{|N|}$ after two errors. Note that although Rin_2 is an intermediate state which did not exist before relinking, $Rin_2 \in \mathcal{P}_A$ so coalitional behavior does not return the process to $B^{|N|}$. At least one error is required to leave Rin_2, but only a single error suffices to continue the spread of A around the ring.

So, the existence or otherwise of a conservative effect of coalitional behavior on square lattices and rings is extended to small worlds constructed from said networks. Simulation results demonstrate that behavior for even large (10 percent) amounts of link rewiring is similar to that obtained for the original networks. How the small world is constructed has important ramifications for predictions regarding the effect of coalitional behavior on convergence times. It is interesting to note that in ring derived graphs, there is a large amount of clustering: the neighbor of a neighbor is likely to be a neighbor. This differs from a square lattice derived graph, for which there is almost no clustering. Yet, in the sense of there being a conservative effect, the square lattice derived graph is more sensitive to coalitional behavior.

8. Random regular graph

As $p \to 1$ the graph in section 7.1 approximates a random regular graph. Note that for all connected regular graphs of degree m, when $\alpha \geq m-1$, a single error will be enough to lead to convergence to $A^{|N|}$. High values of α always lead to fast convergence. The interesting case is when $\alpha < m-1$. Let $Q_{m,|N|}$

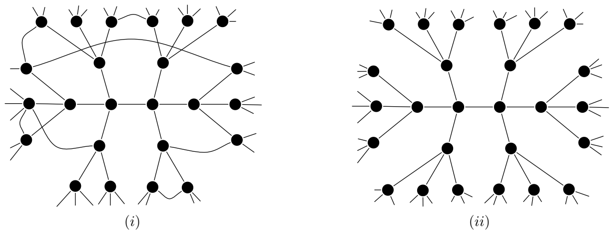


Figure 10: (i) Random regular graph. (ii) Bethe lattice.

be the set of connected m-regular graphs of size $|N|$. Let $\mathcal{Q}_{m,|N|}$ be the uniform distribution over $Q_{m,|N|}$. For any $a \in \mathbb{N}_+$, $k \geq 1$, define:

$$\mathcal{R}^{a,k}_{m,|N|} = \left\{ q \in Q_{m,|N|} : W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|}) \in O(\varepsilon^{-a}). \right\}.$$

$\mathcal{R}^{a,k}_{m,|N|}$ is the set of connected m-regular networks which give convergence times which are shorter or of the same order as ε^{-a}. For large $|N|$, random regular graphs are locally like the Bethe lattice. Any small set of players playing action A will then have players on its edge who have only one neighbor playing A. A player in this situation will benefit from switching to B and earning a payoff of 3 rather than a payoff of α. Thus, the set is rolled back from its edges. The order of magnitude of convergence times grows with population size whether or not coalitional behavior is allowed.[16]

Proposition 5. *For all a, k:*

$$\alpha < m - 1 \quad \implies \quad \lim_{|N| \to \infty} \mathcal{Q}_{m,|N|}(\mathcal{R}_{m,|N|}) \to 0$$

This implies that for large random graphs, convergence will be very slow, with or without coalitional behavior. This does not preclude coalitional behavior from slowing or speeding convergence, and we refer you to the simulation results of the following section for random graphs of size $|N| = 256$. What it does indicate is that for any $k \ll |N|$, convergence on random graphs will become qualitatively slow as the population size becomes large, regardless of the presence or otherwise of coalitional behavior.

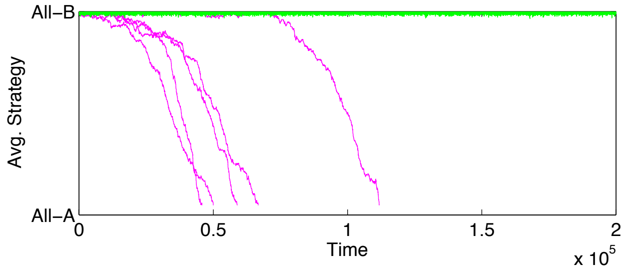


Figure 11: Example runs on square lattice, $\alpha = 1.1$, for $k = 1$ (pink) and $k = 8$ (green).

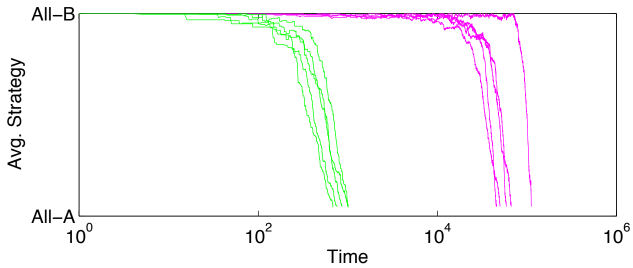


Figure 12: Example runs on square lattice, $\alpha = 2.0$, for $k = 1$ (pink) and $k = 8$ (green).

9. Simulations

Simulations were carried out for a variety of networks to analyze the effect of coalitional behavior on waiting times under small, positive values of ε. The networks fall into two categories. The first category is *generated* networks: squares, rings, associated small worlds and random graphs. The second category, *empirical* networks, includes several examples of real life networks from academic literature.

We apply the model to five empirical networks, visualisations of which are given in Fig. 13.[17] Zachary's karate club describes social relationships between 34 club members of a university karate club, observed over the period 1970-1972. In the network, an edge indicates that two members were observed to maintain significant interactions outside of the formal classes and meetings of the club. Zachary interpreted these interactions as indicating friendship outside of club activities. The network is of interest since within the study period, a fission occured in the club's social structure due to a disagreement over tuition fees. As

[16]The condition $\alpha < m - 1$ imposes the condition that any player with only a single neighbor playing A would like to play B. This plays a role in the proof by ensuring that any absorbing state other than $B^{|}N|$ features cycles in the graph on which every player plays A.

[17]Graph manipulation, measures, visualisation, and sub-graph formation performed with Gephi: `http://gephi.org/`, an open source graph visualisation and analysis platform.

such, the network has been of interest to developers of community identification algorithms for some time (Bagrow and Bollt, 2005; Girvan and Newman, 2002; Newman, 2006b). Alternatively, Newman (2006b,a) compiled a very large scientific collaboration network for the network theory scientific community. Here, we only consider the largest connected component, comprising 379 vertices (authors). Edges indicate that two authors have co-authored a paper. Schwimmer's taro exchange network represents gift-giving, via taro-exchange, amongst 22 households in a Papuan village (Schwimmer, 1973). Kapferer's tailor shop network (Kapferer, 1972) represents observed 'sociational' interactions amongst workers in a Zambian tailor factory during a 10 month period. Kapferer's network is of interest since during the period of observation, the workers collectively negotiated for higher wages. The network we use in this study is the first network (of two) which Kapferer described, after which an unsuccessful strike occured.

Finally, we consider a sub-graph of the FACEBOOK social-network. To obtain a manageable size sub-graph for analysis we consider first the principle, connected, component of the graph '0' from Stanford's SNAP database[18](McAuley and Leskovec, 2012) (size 324 nodes) before taking three connected communities from this graph, comprising 95 nodes in total as identified by the modularity algorithm of Blondel et al. (2008). The FACEBOOK network was collected by the 'Social circles' FACEBOOK APP (for details, see reference). Taken together, we have a diverse set of real networks amongst human actors for either social, exchange, or collaborative purposes.

The simulations were conducted for $\varepsilon = 0.01$ unless convergence results for comparison could not be readily obtained using the computational resources at hand. In the latter case, a *fast-convergence* setting with $\varepsilon = 0.1$ was used, and is indicated for a row of results in the tables by a superscript '$*$' in the α column. In Tables 1 and 2 we describe the parameters used under each setting. γ is the proportion of B players in the population below which the process is considered to have converged.

Table 1: The *default* simulation parameter settings used in the study.

Parameter	**Symbol**	**Value**
Tremble probability	ε	0.01
Convergence limit	γ	5%
Replicates	R	20

In the simulation results tables in the paper, waiting times are given as averages of all replicates for a given experimental condition. Where a result is given

[18]See: `http://snap.stanford.edu/data/egonets-Facebook.html` .

Table 2: The *fast-convergence* parameter settings used in the study.

Parameter	**Symbol**	**Value**
Tremble probability	ε	0.10
Convergence limit	γ	20%
Replicates	R	5

as $> x$ this indicates that some fraction of the replicates converged within the maximum waiting time set for the experiment. Those replicates which did not converge were assigned the maximal waiting time prior to averaging across all replicates, and the average, in this case, is treated as a lower bound.

As expected, reforming effects of coalitional behavior were found for all networks for large enough α.[19] Conservative effects for small α were also observed for several networks. Consider the small world networks of table 9. A large conservative effect is seen for rewired square lattices, whereas no effect is observed for rewired rings. Random graphs, including scale-free networks, also exhibit strong conservative effects for $\varepsilon = 0.1$ treatments.[20] Ratios of convergence times for different α vary markedly in the presence and absence of coalitional behavior. For a square lattice with von-Neumann neighborhood, the ratio of convergence times for $\alpha = 1.1$ and $\alpha = 2.0$ for $k = 1$ is 1, whereas for $k = 8$ the ratio is over 200.

Results for the empirical networks indicate some conservative effect for all networks other than the 'Network theory coauthorship' graph and 'Schwimmer taro exchange'.[21] Kapferer's tailor shop network and the Facebook sub-graph have much higher average degree than the other networks and exhibited much longer convergence times, hence most reported results for these networks are for $\varepsilon = 0.1$.

[19]Simulations, by their nature, do not deal with limiting results. This introduces a measure of imprecision into the reading of the results. Aside from the effects of small ε considered in the theory of the paper, there can also be other effects of coalitional behavior. Higher k could, for example, lead to faster convergence due to multiple switches per period. Conversely, if a single player needs to move to continue on a path to convergence, higher k can reduce the probability of that player being selected in any given period. How any normalization of results for such effects should be carried out is unclear, therefore raw results are presented. Only results displaying order of magnitude differences can be understood to be due to different error requirements on paths to convergence.

[20]For $\varepsilon = 0.01$, convergence in these networks was too slow to make meaningful comparisons.

[21]The latter is a small network and we have checked directly that the theory of the paper predicts reforming effects of coalitional behavior for all values of α.

Table 3: Average wait-times (steps) for convergence to the efficient strategy profile for generated networks over selected values of α, with ($k > 1$) and without ($k = 1$) coalitional updating. See Appendix for details of networks and simulations.

α	**Square lattice**			**Ring**		
	$k = 1$	$k = 4$	$k = 8$	$k = 1$	$k = 4$	$k = 8$
Regular						
1.1	71,964	>200,000	>200,000	46,239	72,002	>102,484
1.6	71,964	23,542	>63,603	46,239	6,612	7,485
2.0	71,964	1,272	848	46,239	850	392
3.0	3613	977	652	6,973	729	350
Small-world						
1.1*	8,106	54,422	>200,000	8,508	6,695	9,715
1.6	96,505	25,307	45,204	62,353	7,723	6,480
2.0	34,617	1,803	1,382	24,026	1,013	536
3.0	3,942	929	543	5,474	708	369
Random						
1.1*	14,438	>200,000	>200,000	12,214	>200,000	>200,000
1.6*	5,139	3,584	>79,268	5,388	3,467	18,322
2.0	18,587	4,430	6,081	19,495	4,646	9,790
3.0	4,478	764	500	4,659	734	491

Table 4: Average wait-times (steps) for convergence to the efficient strategy profile for a scale-free network over selected values of α, with ($k > 1$) and without ($k = 1$) coalitional updating. See Appendix for details of networks and simulations.

α	**Scale-Free**		
	$k = 1$	$k = 4$	$k = 8$
1.1*	11,850	>200,000	>200,000
1.6	47,097	>200,000	>200,000
2.0	15,138	2,209	2,117
3.0	6,318	747	485

Table 5: Average wait-times (steps) for convergence to the efficient strategy profile for real networks over elected values of α, with ($k = 4$) and without ($k = 1$) coalitional updating. See Appendix for details of networks and simulations. Size of each real network given in parentheses. The column 'Conv.' gives the convergence rate where not 100%.

α	$k = 1$	Conv.	$k = 4$	Conv.
Zachary's Karate Club ($n = 34$)				
1.1	48,051		>192,324	10%
1.6	13,673		15,796	
2.0	5,465		476	
3.0	1,503		175	
Network Theory Co-authorship ($n = 379$)				
1.1*	42,106		49,130	
1.6	>200,000	0%	80,224	
2.0	131,104		4,599	
3.0	22,419		1,293	
Schwimmer Taro Exchange ($n = 22$)				
1.1	>172,908	25%	15,681	
1.6	41,968		367	
2.0	1,058		134	
3.0	656		34	
Kapferer's Tailor Shop ($n = 39$)				
1.1*	85,528		>200,000	0%
1.6*	1,405		28,930	
2.0*	1,006		1,810	
3.0	17,694		2,186	
Facebook sub-graph ($n = 95$)				
1.1*	>711,855	80%	>1,842,243	20%
1.6*	8,279		10,765	
2.0*	2,545		1,256	
3.0*	1,328		255	

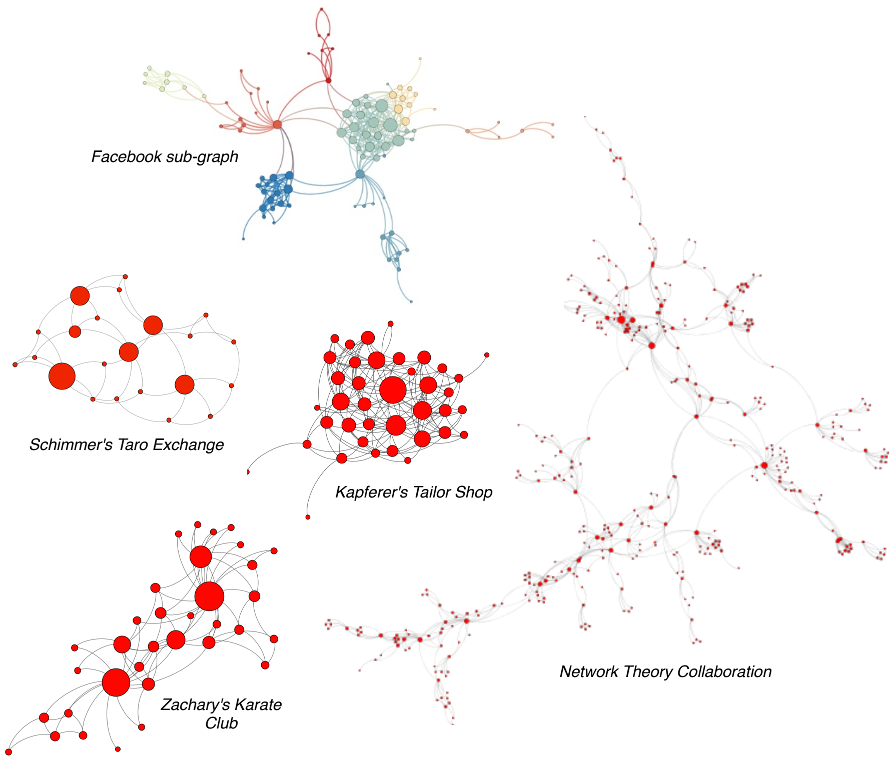


Figure 13: Visualisations of the five real networks considered in this study: the FACEBOOK sub-graph McAuley and Leskovec (2012); Schwimmer's taro exchange network Schwimmer (1973); Kapferer's tailor shop Kapferer (1972); the co-author collaboration network in network theory science Newman (2006a); and Zachary's Karate Club network Zachary (1977).

	A	B
A	α, α	$0, 1$
B	$1, 0$	$1, 1$

Figure 14: A two player coordination game, $1 < \alpha < 2$.

10. Pareto efficiency versus risk dominance

Consider the model adapted so that rather than play a pure coordination game, the players instead play the 'stag hunt' game in figure 14 with each of their neighbors. Note that for this game, there is never a conservative effect of coalitional behavior, as in every situation in which a player would wish to return to playing B as part of a coalition, he would also wish to return to playing B when acting as an individual. As long as $\alpha < 2$, (B, B) is the risk dominant equilibrium of this game, whereas (A, A) is the Pareto efficient equilibrium. Individual adaptive dynamics select the equilibrium in which every player plays the risk dominant action B as a stochastically stable state. This is not the case for the coalitional dynamics of this paper. Instead, it can be the case that all stochastically stable states are polymorphic states in which different players choose different actions depending on their location in the network.

We consider a class of networks Γ_{Cli} composed of interconnected cliques such that every member of a clique is connected to every other member of the clique, and one player outside of the clique. An example in which every clique comprises four players is given in figure 6. Here we allow cliques to be of any size.

Proposition 6. *Consider a network $\boldsymbol{g} \in \Gamma_{Cli}$. For a given clique S, $|S| > k$, in any stochastically stable state x:*

$$|S| \geq \tfrac{(k+1)\alpha}{2-\alpha} \quad \Longrightarrow \quad x_S = B^{|S|},$$

$$|S| \leq \tfrac{(k-3)\alpha}{2-\alpha} \quad \Longrightarrow \quad x_S = A^{|S|}.$$

So in networks composed of lightly interconnected cliques, stochastically stable states involve medium-size cliques playing A and large cliques playing B. In long run equilibria, efficiency is attained within medium-size cliques, but not within larger ones.[22] This means that on the margin, an additional player being

[22]Further information about the neighbors of the clique allows these bounds to be tightened. If there is no B player in N_S then the $(k+1)$,$(k-3)$ factors can be replaced by $(k+1)$,$(k-1)$. If the number of A players in N_S is less than k then the relevant factors are $(k-1)$,$(k-3)$. If neither of these hold then the factors are (k),$(k-2)$. Furthermore, if we replace 'in any stochastically stable state' with 'in some stochastically stable state' then the upper and lower bounds on $|S|$ can be tightened by 1.

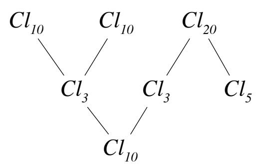


Figure 15: Fragment of network of interconnected cliques. Cl_m indicates a clique of size m.

added to a clique may decrease total payoffs. Despite the positive payoff externalities enjoyed by clique members on the addition of an extra player, the long run stable outcome may switch to one in which the clique in question coordinates on the inefficient action. Now consider small cliques such that the whole clique can form a coalition. For a stochastically stable state x, if no player in the clique has a neighbor outside the clique who plays B, then members of the clique all play A in x. If at least one member of the clique has a neighbor who plays B, then all members of the clique will play A if every member does better from this than he would do if he played B. Otherwise, all members of the clique play B.

Proposition 7. *Consider a network $\boldsymbol{g} \in \Gamma_{Cli}$. For a given clique S, $|S| \leq k$, in any stochastically stable state x:*

$$x_{N_S} = A^{|N_S|} \quad \Longrightarrow \quad x_S = A^{|S|},$$

$$x_{N_S} \neq A^{|N_S|},\ |S| > \tfrac{\alpha}{\alpha-1} \quad \Longrightarrow \quad x_S = A^{|S|},$$

$$x_{N_S} \neq A^{|N_S|},\ |S| < \tfrac{\alpha}{\alpha-1} \quad \Longrightarrow \quad x_S = B^{|S|}.$$

To see how this works in an example, consider the partial network illustrated in figure 15. Cl_m indicates a clique of size m, and the lines between cliques indicate a link between one of the players in each of the cliques. $k = 8$ and $\alpha = \frac{4}{3}$. Then proposition 6 implies that members of a clique of size $m \geq 18$ play B in any stochastically stable state. This applies to the members of the clique of size twenty in figure 15. Similarly, members of a clique of size $8 < m \leq 10$ will play A in any stochastically stable state. This applies to the three Cl_{10} in figure 15. The clique Cl_3 of size three to the left of the figure will then have no neighbors who play B, and thus by proposition 7 will play A in any stochastically stable state. The cliques Cl_3 and Cl_5 of size three and five to the right of the figure have at least one neighbor who plays B, so their behavior depends on whether their size is greater or lower than the value $\alpha/(\alpha-1) = 4$. So by proposition 7, members of Cl_3 on the right hand side play B and members of Cl_5 play A in any stochastically stable state.

So coalitional behavior can lead to heterogeneous choices by cliques within a population depending on their size. This is not necessarily monotonic, with large cliques playing the risk dominant action, medium-size cliques the Pareto efficient action. In the absence of neighbors, small cliques would also play the Pareto efficient action, but the presence of neighbors playing the risk dominant action pushes them to do likewise. An interpretation of this from the perspective of organizational design could be that teams should not be so large that the internal pressure against risky yet efficient behavior dominates, but neither should they be so small that external pressure has the same effect.

11. Conclusion

In settings in which every member of a population has a common interest in coordinating on a given efficient action, it might be expected that overt cooperation by coalitions of players in their choice of action would facilitate the spread of that action in the population. This paper has shown that this is not always the case. Aside from the existence of such reforming effects of coalitional behavior, there can also exist conservative effects by which coalitions slow the spread of efficient behavior in a population. As the relative efficiency of the efficient action increases, the effect of coalitional behavior on a dynamic can switch abruptly from a conservative effect to a reforming effect. Not all networks exhibit conservative effects and their existence or otherwise depends on the structure of the underlying network.

Simulation results appear consistent with the theory of the paper and indicate that we should expect conservative and reforming effects to manifest themselves in dynamics of social and technological change on a variety of network types, including several empirical social networks analyzed in the literature. The introduction of coalitional behavior is seen to greatly increase the sensitivity of convergence speeds to the relative efficiency of competing technologies. Finally, for games in which the Pareto efficient and risk dominant equilibria differ, we see that play of both actions can coexist in the population in the long run and that the network design determines which actions survive in which parts of the network.

This paper shows the effects of coalitional behavior on networked coordination games to be important and non-obvious. The assumptions of the paper are not strong, the departure from existing literature being that small groups of connected players coordinate their action choice some of the time. Implications for the study of social dynamics and network design clearly exist and merit subsequent study.

Appendix A. Proofs

We use the concepts of and similar notation to Ellison (2000). For $x, y \in X$, define:

$$r(x,y) = \min \left\{ r \in \mathbb{R}_+ : \exists\, t \in \mathbb{N}_+ : \lim_{\varepsilon \to 0} \frac{P^t_{k,\alpha,\varepsilon}(x,y)}{\varepsilon^r} > 0 \right\}$$

and for $x_1, \ldots, x_T \in X$, sets $W, Y \subseteq X$:

$$r(x^1, \ldots, x^T) = \sum_{t=1}^{t=T-1} r(x^t, x^{t+1}); \qquad r(W,Y) = \min_{\substack{(x^1,\ldots,x^T)\\ x^1 \in W \\ x^T \in Y}} r(x^1, \ldots x^T).$$

Extend the notion of basin of attraction to sets:

$$D_{k,\alpha}(W) = \left\{ y \in X : \ P^t_{k,\alpha,0}(y, W) \to 1 \ \text{ as } t \to \infty \right\}$$

then the *radius* of a set W is:

$$R(W) = r(W, X \setminus D_{k,\alpha}(W)).$$

Define modified resistance:

$$r^*(x^1, \ldots, x^T) = r(x^1, \ldots, x^T) - \sum_{t=2}^{t=T-1} R(x^t)$$

and

$$r^*(x, W) = \min_{\substack{(x^1,\ldots,x^T)\\ x = x^1 \\ x^T \in W}} r^*(x^1, \ldots x^T).$$

Finally, define the *modified coradius*:

$$CR^*(W) = \max_{x \notin W} r^*(x, W)$$

For purposes of comparison, these quantities will sometimes be given subscripts k, α.

Proof of proposition 1. For $k = 1$, $\alpha \geq 3$, a single error is enough to move the process to $A^{|N|}$. For $\alpha < 3$, two errors move the process to C_2. A single error then suffices to move the process to C_3, and so on. Noting that from any state not equal to $B^{|N|}$, a single error is enough to give an expanding set of squares, and that $R_{1,\alpha}(C_i) = 1$, we have:

$$\alpha < 3: \quad CR^*_{1,\alpha}(A^{|N|}) = 2$$
$$\alpha \geq 3: \quad CR^*_{1,\alpha}(A^{|N|}) = 2$$

For $4 \leq k \ll |N|$, for $\alpha < 3/2$, two errors do not suffice to leave $D_{k,\alpha}(B^{|N|})$. Three errors, however, can take the process to C_3, from where a single error can take the process to C_4 and so on. Note that $R_{k,\alpha}(C_i) = 1$ for $i \geq 3$. For $3/2 \leq \alpha < 2$, two errors are required to move outside $D_{k,\alpha}(B^{|N|})$ to C_2. $C_2 \in \bar{D}_{k,\alpha}(A^{|N|})$. For $\alpha \geq 2$, $B^{|N|} \in D_{k,\alpha}(A^{|N|})$. We have:

$$\alpha < 3/2: \quad CR^*_{k,\alpha}(A^{|N|}) = 3$$
$$3/2 \leq \alpha < 2: \quad CR^*_{k,\alpha}(A^{|N|}) = 2$$
$$\alpha \geq 2: \quad CR^*_{k,\alpha}(A^{|N|}) = 0$$

Results follow from $W_{k,\alpha,\varepsilon}(B^{|N|}, A^{|N|}) \geq W_{k,\alpha,\varepsilon}(x, A^{|N|})$ for all $x \in X$; and Theorem 2 of Ellison (2000). □

Proof of lemma 1. The set of feasible coalitions is independent of k, so $k_1 \leq k_2$ implies:

$$\mathcal{N}(k_1) \subseteq \mathcal{N}(k_2) \quad \Longrightarrow \quad \mathrm{supp}(F_{k_1}) \subseteq \mathrm{supp}(F_{k_2})$$

so for all $x, y \in X$:

$$P_{k_1,\alpha,0}(x, y) > 0 \quad \Longrightarrow \quad P_{k_2,\alpha,0}(x, y) > 0$$

and

$$P_{k_1,\alpha,0}(x, X \setminus \{x\}) > 0 \quad \Longrightarrow \quad P_{k_2,\alpha,0}(x, X \setminus \{x\}) > 0$$

which implies

$$x \notin \Lambda_{k_1,\alpha} \quad \Longrightarrow \quad x \notin \Lambda_{k_2,\alpha}$$

which implies $\Lambda_{k_1,\alpha} \supseteq \Lambda_{k_2,\alpha}$. Furthermore, for $x \in \Lambda_{k_2,\alpha}$:

$$y \in \bar{D}_{k_1,\alpha}(x) \quad \Longrightarrow \quad P^t_{k_1,\alpha,0}(y, x) > 0 \text{ for some } t \in \mathbb{N}_+$$
$$\Longrightarrow \quad P^t_{k_2,\alpha,0}(y, x) > 0 \quad \Longrightarrow \quad y \in \bar{D}_{k_2,\alpha}(x).$$

□

Proof of lemma 2. For given $x \in X$, assume there exists $S \subseteq \{i : x_i = B\}$ such that:

$$\forall T \subseteq S,\ |T| \leq k : \ \exists i \in T : \quad \frac{|N_i \setminus S| + |N_i \cap T|}{|N_i| + |N_i \cap T|} < \frac{1}{1+\alpha}.$$

Take such an S. Then for all $T \subseteq S$, there exists $i \in T$ such that:

$$\alpha(|N_i \setminus S| + |N_i \cap T|) < |N_i \cap S| \leq u_i(x).$$

The left hand side is the maximum payoff attainable by player i from playing A if all players in $S \setminus T$ play B. So no subset of players in S will switch to A unless some subset of players in S have already switched to A. Therefore $x \notin \bar{D}_{k,\alpha}(A^{|N|})$. This proves the 'only if' part of the lemma.

To prove the 'if' part of the lemma assume that $x \notin \bar{D}_{k,\alpha}(A^{|N|})$. Starting from state $x = x^1$, if there is any subset of players $U \subseteq N$ such that for all $i \in U$, $u_i(x_U = A^{|U|}, x^t_{-U}) \geq u_i(x^t)$, then with some probability U better responds and the state moves to $x^{t+1}_S = A^{|U|}$, $x^{t+1}_{-U} = x^t_{-U}$. Iterate until there is no such subset of players, say at time τ. Let $S = \{i : x^\tau_i = B\}$. This set must be nonempty or else $x^\tau = A^{|N|}$, which would contradict $x \notin \bar{D}_{k,\alpha}(A^{|N|})$. Now, for all $T \subseteq S$, as at least one player, say player i, in T would be strictly worse off if T switched to action A:

$$\alpha(|N_i \setminus S| + |N_i \cap T|) < u_i(x^\tau) = |N_i \cap S| = |N_i| - |N_i \setminus S|.$$

Rearranging gives:

$$\frac{|N_i \setminus S| + |N_i \cap T|}{|N_i| + |N_i \cap T|} < \frac{1}{1+\alpha}.$$

and we have our result. □

Proof of proposition 4. $\Lambda = \{B^{|N|}, A^{|N|}\}$. Note that $\bar{D}_{1,\alpha}(A^{|N|}) \subseteq \bar{D}_{k,\alpha}(A^{|N|})$. So:

$$CR^*_{1,\alpha}(A^{|N|}) = R_{1,\alpha}(B^{|N|}) = r_{1,\alpha}(B^{|N|}, \bar{D}_{1,\alpha}(A^{|N|}))$$

$$\geq r_{1,\alpha}(B^{|N|}, \bar{D}_{k>1,\alpha}(A^{|N|})) \geq r_{k>1,\alpha}(B^{|N|}, \bar{D}_{k>1,\alpha}(A^{|N|}))$$

$$= R_{k>1,\alpha}(B^{|N|}) = CR^*_{k>1,\alpha}(A^{|N|})$$

□

Proof of lemma 3. First the 'only if' part of the statement is addressed. Let $x \in \mathcal{P}_A$ and $S \subseteq \{i \in N : \ x_i = A\}$ be parochial. $I_0(S)$ must be nonempty. For any α, $i \in I_0(S)$, $N \supseteq T \supseteq \{i\}$, as

$$u_i(x) = \alpha|N_i| > |N_i| \geq u_i(\tilde{x}_i = B, \hat{x}_{-i}) \quad \text{for any } \hat{x}_{-i},$$

we have:

$$\tilde{x}_T \in A_T(x) \quad \implies \quad \tilde{x}_i = A.$$

Then, by induction, for any $i \in I_m(S)$, $m \geq 1$, $N \supseteq T \supseteq \{i\}$, as

$$u_i(x) \geq \alpha(|N_i| - |N_i \setminus S|) > |N_i| - |N_i \setminus S| \geq |N_i| - |N_i \cap I_{m-1}(S)|$$

$$\geq u_i(\tilde{x}_i = B, \tilde{x}_{I_{m-1}(S)} = A^{|I_{m-1}(S)|}, \hat{x}_{-(I_{m-1}(S) \cup \{i\})}) \quad \text{for any } \hat{x}_{-(I_{m-1}(S) \cup \{i\})},$$

we have:

$$\tilde{x}_T \in A_T(x) \quad \implies \quad \tilde{x}_i = A.$$

So, for any $x^t \in \mathcal{P}_A$: $x_S^{t+1} = A^{|S|}$ and therefore $x^{t+1} \in \mathcal{P}_A \not\supseteq B^{|N|}$. So $x \in \mathcal{P}_A$ implies $x \notin \bar{D}_{k,\alpha}(B^{|N|})$.

Now the 'if' part of the statement is addressed. Assume that $x \notin \bar{D}_{k,\alpha}(B^{|N|})$ for any k, α. If there exists feasible $S \subseteq N$ such that:

$$(*) \quad x_S \neq B^{|S|} \quad \text{and}$$
$$\exists\, \underline{\alpha} : \forall\, i \in S : \quad u_i(B^{|S|}, x_{-S}) \geq u_i(x),$$

then, assuming $k \geq |S|$, we have $S \in \text{supp}(F_k)$. Assuming $\alpha < \underline{\alpha}$:

$$B^{|S|} \in A_S(x) \quad \text{so} \quad P_{k,\alpha,0}(x, (B^{|S|}, x_{-S})) > 0.$$

Starting from $x^t \notin \bar{D}_{k,\alpha}(B^{|N|})$, let $x^{t+1} = (B^{|S|}, x_{-S})$ for such an S, and iterate until a state, $\tilde{x}$, is reached such that there does not exist feasible S which satisfies $(*)$. Let $T = \{i \in N : \tilde{x}_i = A\}$. Then $I_0(T) \neq \emptyset$ or T would satisfy $(*)$ as there would exist α such that:

$$\forall\, i \in T : \quad u_i(B^{|T|}, \tilde{x}_{-T}) = |N_i| \geq \alpha(|N_i| - 1) \geq u_i(\tilde{x}).$$

If $\nexists\, m$ such that $I_m(T) = T$, then choose m such that $I_m(T) = I_{m-1}(T)$. Then:

$$\forall\, i \in T \setminus I_m(T) : \quad |N_i \setminus T| > |N_i \cap I_{m-1}(T)| = |N_i \cap I_m(T)|.$$

But then, for all $i \in T \setminus I_m(T)$, there exists α such that:

$$u_i(B^{|T \setminus I_m(T)|}, \tilde{x}_{-(T \setminus I_m(T))}) \geq |N_i \setminus T| + |N_i \cap (T \setminus I_m(T))|$$

$$> \alpha(|N_i \cap I_{m-1}(T)| + |N_i \cap (T \setminus I_m(T))|) = \alpha|N_i \cap T| \geq u_i(\tilde{x})$$

so $T \setminus I_m(T)$ satisfies $(*)$ and we have a contradiction. Therefore $\exists\, m$ such that $I_m(T) = T$. T is a parochial set such that $x_T = A^{|T|}$. Therefore $x \in \mathcal{P}_A$.

□

Proof of proposition 5. SKETCH - to be expanded.

- Any convergent path must eventually lead to a cycle of length at least $4a$ of players who play A. Take a path that gives the fastest way to get to a configuration in which a cycle of length over $4a$, say C_{4a}, of players plays A. The time to attain this will bound waiting time from below. Consider only paths that have switches from B to A (this is fine due to monotonicity of the better responses).

- Take players who 'influence' C_{4a} by switching to A. For these to speed the switch to A these sets of players must comprise cycles, and these cycles must be of size less than or equal to $4a$.

- The path cannot involve A moving between these sets of influencing players as if it did, a new cycle would be created by the path between the influencing players, the path from the influencing players to C_{4a}, and a section of C_{4a}. This new cycle cannot be of size more than $4a$ as this would contradict the path we are analyzing being the fastest way to attain a cycle of A players of length more than $4a$. It cannot be of size less than or equal to $4a$ as it shares edges with the cycles of influential players which are of size $\leq 4A$, an event with asymptotically zero probability as $|N|$ becomes large (McKay et al., 2004).

- No two players in an influencing cycle can be co-influenced by a player outside of the cycle, or else either a cycle of $> 4a$ already exists or a cycle of $\leq 4a$ contradicts no edge sharing by small cycles.

- An influencing cycle of length b, C_b, requires $\lceil b/2 \rceil$ errors and influences at most $\lceil b+1/2 \rceil$ players in C_{4a}.

- At any configuration in which all players in C_{4a} do not play A, at most a single error is required to leave said configuration. This is because there is always a vulnerable cycle of influencing players at the edge of the set of A players on C_{4a}. This means that the formation of each influencing cycle adds at least $\lceil b/2 \rceil - 1$ to the order of the waiting time for every $\lceil b+1/2 \rceil$ players in C_{4a} which are influenced.

- So at least 1 is added to the order for every 2 influenced. Each few (say every 2) influenced could combine with other influenced players to influence a further player. The order of the waiting time until all players on C_{4a} play A is then at least ε^{-a}.

□

Proof of proposition 6. For a clique S, let $W_A = \{x \in X : \ x_S = A^{|S|}\}$, $W_B = \{x \in X : \ x_S = B^{|S|}\}$. Note that $x_S \notin \{A^{|S|}, B^{|S|}\}$ implies $x \notin \Lambda_{k,\alpha}$. Now, from any state in W_A, to escape the basin of attraction of W_A (and enter the basin of attraction of W_B) requires that some player in S play B without making an error. This will be easier when the external neighbor of such a player plays B, and harder when his external neighbor plays A. Note that each player in S has $|S|$ neighbors. This gives:

$$|S| \geq \alpha(|S| - 1 - R(W_A)) \quad \Longrightarrow \quad R(W_A) \geq \frac{(\alpha-1)}{\alpha}|S| - 1$$

$$\Longrightarrow \quad R(W_A) = \left\lceil \frac{(\alpha-1)}{\alpha}|S| - 1 \right\rceil$$

and

$$|S| \geq \alpha(|S| - CR^*(W_B)) \quad \Longrightarrow \quad CR(W_B) \geq \frac{(\alpha-1)}{\alpha}|S|$$

$$\Longrightarrow \quad CR(W_B) = \left\lceil \frac{(\alpha-1)}{\alpha}|S| \right\rceil .$$

For moves from W_B to W_A, coalitions of players can coordinate their move to playing A. For each such player this gives an extra $(k-1)$ neighbors who play A, thus reducing resistances for such moves. Such moves are easier when external neighbors play A, and harder when they play B.

$$\alpha(1 + (k-1) + R(W_B)) \geq |S| \quad \Longrightarrow \quad R(W_B) \geq \frac{|S|}{\alpha} - k$$

$$\Longrightarrow \quad R(W_B) = \left\lceil \frac{|S|}{\alpha} - k \right\rceil$$

and

$$\alpha((k-1) + R(W_B)) \geq |S| \quad \Longrightarrow \quad CR(W_A) \geq \frac{|S|}{\alpha} - k + 1$$

$$\Longrightarrow \quad CR(W_A) = \left\lceil \frac{|S|}{\alpha} - k + 1 \right\rceil .$$

Simple manipulation then shows that:

$$|S| \geq \frac{(k+1)\alpha}{2-\alpha} \quad \Longrightarrow \quad R(W_B) > CR(W_B)$$
$$|S| \leq \frac{(k-3)\alpha}{2-\alpha} \quad \Longrightarrow \quad R(W_A) > CR(W_A)$$

The result follows from Theorem 2 of Ellison (2000). □

Proof of proposition 7. For given x, if $N_S = A^{|N_S|}$ then $A^{|S|} \in A_S(x)$. Furthermore, if $N_S = A^{|N_S|}$ and $x_S = A^{|S|}$ then $A_S(x) = \{A^{|S|}\}$. Therefore $N_S = A^{|N_S|}$, $x \in \Lambda_{k,\alpha}$ implies $x_S = A^{|S|}$. If $N_S \neq A^{|N_S|}$ and $|S| > \frac{\alpha}{\alpha-1}$ then $\alpha(|S|-1) > |S|$ and the same argument applies. If $N_S \neq A^{|N_S|}$ and $|S| < \frac{\alpha}{\alpha-1}$ then for some player in S with an external neighbor who plays B, his payoff $|S|$ from playing B is higher than his maximum payoff from playing A of $\alpha(|S|-1)$, so he will play B. But then all of the players in S will have a neighbor who plays B, and the same argument will apply to them. □

Appendix B. Simulation methodology

Appendix B.1. Introduction

We describe below the methodology behind the simulation results presented in the paper. Full, commented, code for the simulations is available upon request from the authors.

Appendix B.2. The Main Algorithm

In Algorithm 1 we present pseudo-code to run a single replicate of the model under a given set of parameters as required. After random seed initialisation (see section Appendix B.4), the underlying network is either generated or imported from the emperical network library. Strategies are initialised in all simulations to the inefficient coordination outcome (All-B) and payoffs calculated as per the model presented in the paper. The main loop is iterated until either the maximum number of steps (T) is reached or the convergence condition has been reached. We note that in the simulation of the model, one iteration is defined by the addressing of a single coalition, S, defined by the coalition procedure at the beginning of the iteration (see section Appendix B.3 below). Convergence in all simulations required the fraction of 'B' players in the population to be less than γ, the convergence limit.

Algorithm 1 One simulation replicate for given parameter values

Require: T, the maximum number of iterations in a simulation; G_{type} or G_{id}, the network type to generate or real network ID to import; $|N|$, the number of agents in the population; α, the efficient coordination payoff; k, the maximum size of a coalition; ε, the tremble probability; p_b coalition branching parameter; γ convergence limit.

{*initialise:*}
seed ← Set and record random seed
$G \leftarrow$ Create or import network($G_{type}|G_{id}$)
$x \leftarrow$ initialise strategy vector(n)
$\pi \leftarrow$ Update payoffs(G, x, α)
$t \leftarrow 1$
converged? ← FALSE
{*main loop:*}
while ($t < T$) AND (*converged*? = FALSE) **do**
  $S \leftarrow$ Get coalition(G, k, p_b)
  $x \leftarrow$ Obtain better response(S, G, x, π, α)
  $x \leftarrow$ Apply tremble(S, x, ε)
  $\pi \leftarrow$ Update payoffs(G, x, α)
  converged? ← Test convergence(x, γ)
  $t \leftarrow t + 1$
end while
return (*converged*?, t)

Appendix B.3. Coalition formation

The problem we address is to identify a subset of up to k agents, $S \subseteq N$, who form a connected sub-graph of G such that a path exists from any agent in S to any other agent in S. In terms of combinatorics, the order of complexity of the problem, $\Omega_G(k)$ depends strongly on the topology of the substrate graph G. For example, if G is the complete graph G^C, then $\Omega_{G^C}(k) \sim \binom{|N|}{k}$, which for the generated networks in this paper ($|N| = 256, k \in \{4, 8\}$) gives rise to over 174 million and over 4×10^{14} ways of forming S respectively. On the other hand, for a 2−regular ring network G^R, $\Omega_{G^R}(k)$ will be equal to just $|N|$ since the topology constrains the composition of S to $|N|$ consecutive index sets, each with a different starting vertex. However, in most cases, even a small amount of density at the local level complicates the picture dramatically.

For this reason, rather than enumerating every possible set S for a given graph G prior to running each simulation (and so, choosing some S from the pre-defined library each iteration) we instead opt to perform a *run-time* algorithm to define S each iteration. The algorithm is as follows: 1) Choose l from a uniform distribution on $\{1, \dots k\}$. 2) choose $i \in N$ and initiate the current *buds* b, of the algorithm to be the set of adjacent vertices to i, and initate the coalition $S = \{i\}$; 3) randomly (uniform) add vertices from b to S until $|S| = l$ or b is 1; 4) *if* $|S| = l$, *return* S, *else if* $|S| < l$, for fraction p_b of vertices in b obtain their adjacent vertices (not in S) and create a fresh set b with these; 5) repeat steps 2 to 4 until the algorithm returns. There is one further detail to the algorithm, in step 4, if no new *bud* vertices can be found, for the current vertex being considered from b, then the algorithm moves back one step to the previous bud point to the current vertex and looks for new vertices. The algorithm keeps rolling-back like this until a new source of vertices can be found. This detail prevents the algorithm from getting stuck with a single bud (for $p_b \to 0$, say) which is in fact a leaf-node in G. In effect, the algorithm moves out through the local neighbourhood of i, layer by layer, until S is filled. We define $p_b \in (0, 1]$ to be the *branching parameter* since for $p_b \to 0$ coalition sets will take an approximately *linear* topology, whilst for $p_b \to 1$, the algorithm is 'greedy' and returns coalitions with density (connectivity) approximating the local density of the substrate graph G. The role of p_b is exemplified by Fig. B.16. In order to investigate the impact of p_b on the results, we ran a set of simulations over p_b for various coalition sizes relevant to the present study (results not reported, but available on request). Since no significant differences in our results were found, we have opted to use $p_b = 0.5$ in all simulations reported in the paper.

Appendix B.4. The random stream

The random stream used in the simulations is the MATLAB stream method `mt19937ar`, which is described by the MATLAB documentation as 'Mersenne

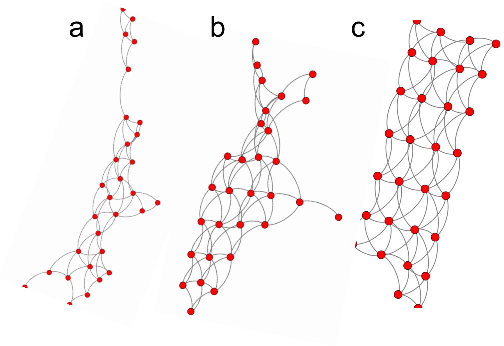


Figure B.16: The outcome of the coaltion formation algorithm is exemplified by coalitions formed from a 100 agent, unwrapped Moore lattice, building coalition sizes of 30 with p_b equal to 0.0 (a); 0.5 (b); and 1.0 (c). The transition from a linear to dense topology is evident.

Twister with Mersenne prime $2^{19937} - 1$'. In each experiment (unless indicated), 20 replicates were used with this stream, each with an individual seed defined by the number of the replicate (e.g. replicate 1 of 20 passed '1' to the initialisation method) to create the stream[23]. Thus, each replicate in a given experiment was conducted with identical random number stream conditions to the corresponding replicate of another experiment. Such a setup ensures that any differences amongst experiments can be attributed wholly to the conditions of the experiment and not the random number streams in play.

Appendix B.5. Networks

The simulation study considered two general categories of networks: generated (or, artificial) networks; and emperical (or, real) networks. In the first category, we constructed square lattices with Von-Neumann (VN) neighborhoods and ring networks via simple linear algebra methods written by one of the authors (available on request). Next, we adopted the algorithm of Watts and Strogatz (1998) to randomly 'rewire' 10% of the edges of these networks to form so-called 'small-world' (SW) networks, and then 100% of the edges to form random (rand) networks. The figure of 10% rewiring was settled on after inspection of the characteristic path length ($L(G)$) and clustering coefficient ($C(G)$) measures of each network type after systematic rewiring from 0% to 100% (results not shown). The object for a 'small-world' graph, $G(N, K)$ having $|N|$ vertices and K average degree, is to obtain $L(G, N, K)' \sim 1$ yet $C(G, N, K)' \gg 1$ where, L' and C' are

[23] Actual Matlab code used:
`s1 = RandStream.create('mt19937ar','seed',r);`
`RandStream.setDefaultStream(s1);`, where 'r' is the replicate number.

Table B.6: Summary statistics for the networks considered in the simulations.

Network Type	$\|N\|$	E	K	$C(G)$	$L(G)$	$\frac{C(G)}{C(G_{R(N,K)})}$	$\frac{L(G)}{L(G_{R(N,K)})}$
Artificial networks							
Square lattice (VN)	256	512	4.0	0.0000	8.0	0.0	2.0
				(na)	(na)		
VN (small world)	256	512	4.0	0.0032	5.3	0.2	1.3
				(0.0026)	(0.1)		
VN (random)	256	512	4.0	0.0126	4.2	0.8	1.1
				(0.0047)	(0.0)		
Ring	256	512	4.0	0.4981	32.4	31.9	8.1
				(na)	(na)		
Ring (small world)	256	512	4.0	0.3701	6.5	23.7	1.6
				(0.0166)	(0.4)		
Ring (random)	256	511	4.0	0.0093	4.2	0.6	1.1
				(0.0041)	(0.0)		
Scale-free	256	510	4.0	0.0809	3.4	5.2	0.9
				(0.0259)	(0.1)		
Real networks							
Zachary's Karate Club (Zachary, 1977)	34	78	4.6	0.571	2.41	4.2	1.04
Network Th. Co-authorship (Newman, 2006b,a)	379	914	4.8	0.741	6.042	58.3	1.6
Schwimmer's Taro Exchange (Schwimmer, 1973)	22	39	3.5	0.339	2.494	2.1	1.02
Kapferer's Tailor Shop (Kapferer, 1972)	39	158	8.1	0.458	1.772	2.2	1.01
Facebook sub-graph (McAuley and Leskovec, 2012)	95	325	6.8	0.559	3.509	7.8	1.5

the ratio of the characteristic path length and clustering coefficient of the network G to the equivalent limit values for a random network of similar $|N|$ and K, $G_{R(N,K)}$. In other words, a 'small world' network will have an equivalently low average path length to a counterpart random network but much higher local edge density as given by the relatively high clustering coefficient. That is,

$$L(G, N, K)' = \frac{L(G)}{L(G_{R(N,K)})} \tag{B.1}$$

and,

$$C(G, N, K)' = \frac{C(G)}{C(G_{R(N,K)})}. \tag{B.2}$$

As in the reference, in the random limit, $L(G_{R(N,K)}) = ln(|N|)/ln(K)$ and $C(G_{R(N,K)}) = K/|N|$.

In Table B.6 and Fig. B.17 we present summary statistics for the networks used in the simulations where $|N|$, E and K are the number of vertices, edges and the average degree of the network respectively, whilst $C(G)$ and $L(G)$ respectively give the clustering coefficiency and characteristic path length of the network. The values of C' and L' are given for convenience in the last two columns of the table, whilst the same values are plotted in the figure. As can be seen, the networks we study cover a diverse region within L'—C' space.

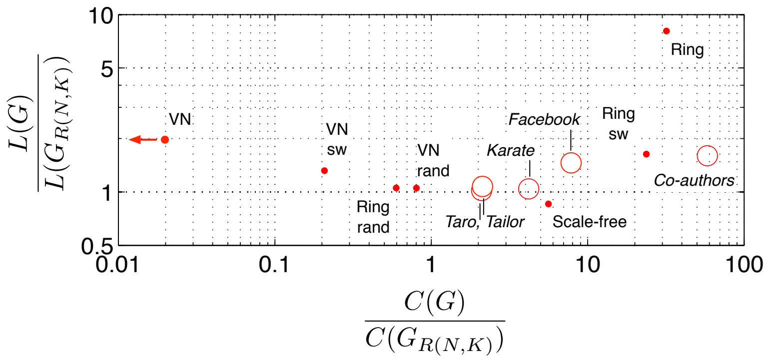


Figure B.17: The networks considered in the simulations displayed in L'—C' space. Refer Table B.6 for details.

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