Less than exponential growth with non-constant discounting

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Warwick Growth Workshop, 7-11 July

Diminishing returns (DR) in Neoclassical growth models lead to stagnation if households discount at a positive constant rate.

Endogenous growth literature solution

Constant returns (CR) + positive constant discount rate $\Longrightarrow$ exponential long-run growth transitional dynamics $\nexists$

Our proposal and the research question

Diminishing returns (DR) + positive diminishing discount rate $\nRightarrow$ exponential long-run growth

Is it possible a **less than exponential unbounded growth?**

Three different concepts of equilibria

Endogenous growth literature in the 90s

Balanced growth path: The **growth rate is constant** ⇒ output and consumption grow exponentially

Regular growth or quasi-arithmetic growth literature

The **growth rate declines** at a rate which is proportional to itself

- Mitra (1983)
- Pezzey (2004)
- Asheim et al (2007)
- Groth et al. (2010)
- Bazhanov (2013)

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Less than exponential unbounded growth (LEUG)

Definition

An economy exhibits less than exponential unbounded growth (LEUG) if the growth rate of consumption being strictly positive, converges to zero and ensures an unlimited amount of consumption as time goes to infinity.

$$\gamma_c(t) = \frac{\dot{c}}{c} > 0,\ \forall t \geq 0, \quad \lim_{t\to\infty} \gamma_c(t) = 0, \quad \int_t^{\infty} \gamma_c(h)dh = +\infty.$$

Outline

1. Introduction ✓
2. Less than exponential unbounded growth in the standard Ramsey Model
3. The Ramsey model with non-constant discounting

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2. Less than exponential unbounded growth in the standard Ramsey Model
   - DR $\Rightarrow$ ever decreasing rate of return
   - Requirements: Constant population, net-of-capital-depreciation production function and consumers do not discount the future
   - Result: The rate of return decreases so slow as to guarantee that consumption grows unboundedly (LEUG)
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3. The Ramsey model with non-constant discounting
   - DR $\Rightarrow$ ever decreasing rate of return
   - Requirements: Constant population, net-of-capital-depreciation production function and consumers discount the future at a non-constant rate
   - Result: The rate of return decreases slower than a weighted mean of the instantaneous discount rates $\Rightarrow$ consumption grows unboundedly (LEUG)

2. LEUG in the standard Ramsey Model

The production function and utility function

$$y(t) = f(k(t)) = Ak^{\alpha}(t), \qquad A > 0, \; \alpha \in (0,1)$$

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \text{ for } \sigma \neq 1$$

Dynamics in the standard Ramsey model

$$\dot{k} = f(k) - c - (n+\delta)k,$$

▸ Phase Diagram

$$\gamma_c(t) \equiv \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left[r(t) - (\delta + \rho)\right], \quad r(t) = f'(k(t)) = \alpha A k(t)^{\alpha-1}$$

Constant parameters: $\delta > 0$, $n > 0$, $\sigma > 0$.

2. LEUG in the Ramsey Model when $n = \delta = \rho = 0$

The rate of return, r and the saving rate, s

$$c = (1-s)f(k) = (1-s)Ak^{\alpha}, \qquad r = f'(k) = \alpha A k^{\alpha-1}$$

$$\begin{aligned} \dot{r} &= -\frac{1-\alpha}{\alpha} s r^2, \qquad r(0) = r_0 = \alpha A k_0^{\alpha-1}, \\ \dot{s} &= -(1-s)\left(\frac{1}{\sigma} - s\right) r \qquad s \in [0,1]. \end{aligned}$$

▸ Phase Diagram

$\rho = \delta = n = 0$

There exits a set of initial conditions $s_0 \in [1/\sigma, 1)$ for every given $r_0 > 0$ such that the pattern of LEUG arises.

3. The Ramsey model with non-constant discounting

- Individuals are highly impatient in the near future, but more patient when they are confronted with choices in the distant future (Laibson, 1997)
- Time-varying discount rates lead to time-inconsistency problem (Ramsey, 1928; Strotz, 1956; Pollak, 1968; Goldman, 1980)
- A sophisticated agent is aware that he cannot precommit his future behavior, but adopts a strategy of optimal planning against its future self (Karp, 2007; Marín-Solano and Navas, 2009)
- Barro (1999) deduced a modified Ramsey rule for a Neoclassical growth model with non-constant discounting and sophisticated agents

Instantaneous time discount rate	
Barro (1999)	Our proposal
$\underbrace{\lim_{j\to+\infty}\rho(j)=\bar{\rho}>0}_{\text{equivalence with constant discounting}}$	$\underbrace{\lim_{j\to+\infty}\rho(j)=0}_{\text{a new pattern of growth arises}}$

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Households

$$\max_{c_t(h)} \int_t^{\infty} u\left[c_t(h)\right] \theta(h-t)\, dh$$

s.t.: $\dot{k}_t(h) = f(k_t(h)) - c_t(h), \quad k_t(t) = \bar{k}_t.$

The discount function and the instantaneous discount rate

$$\theta(j) \geq 0, \quad \dot{\theta}(j) < 0, \quad \theta(0) = 1$$

$$\rho(j) = -\frac{\dot{\theta}(j)}{\theta(j)} > 0, \quad \dot{\rho}(j) < 0, \quad \lim_{j \to +\infty} \rho(j) = 0$$

3. The Ramsey model with non-constant discounting

Households

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s.t.: $\dot{k}_t(h) = f(k_t(h)) - c_t(h), \quad k_t(t) = \bar{k}_t.$

Modified Ramsey Rule (Barro, 1999). Sophisticated consumers

$$\gamma_c(t) \equiv \frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left(r(t) - \lambda(t)\right).$$

$\lambda(t)$ weighted mean of instantaneous discount rates

$$\lambda(t) = \frac{\int_0^{\infty} \rho(j) w(t,j) dj}{\int_0^{\infty} w(t,j) dj} > 0, \qquad w(t,j) = \theta(j) e^{(1-\sigma)\int_t^{t+j} \gamma_c(\tau) d\tau}.$$

3. The Ramsey model with non-constant discounting

Despite DR, there is a chance for unbounded growth

$$\gamma_c(t) = \frac{1}{\sigma}\left[r(t) - \overbrace{\rho}^{\text{constant}}\right] \qquad \text{Standard Ramsey rule}$$

$$\gamma_c(t) = \frac{1}{\sigma}\left[r(t) - \overbrace{\lambda(t)}^{non-\text{constant}}\right] \qquad \text{Modified Ramsey rule}$$

Temporal evolution of $\lambda(t)$

$$\dot{\lambda} = -\rho_0 B - \lambda\left(\lambda - B\right) - B\int_0^{\infty}\left[\dot{\rho}(j) - \rho^2(j)\right] w(t,j)dj.$$

where $\rho(0) = \rho_0$ and $B(t) = \left[\int_0^{\infty} w(t,j)dj\right]^{-1}$.

A specific discount function

Temporal evolution of $\lambda(t)$

$$\dot{\lambda} = -\rho_0 B - \lambda(\lambda - B) - B\int_0^{\infty}\left[\dot{\rho}(j) - \rho^2(j)\right] w(t,j)\,dj.$$

$$\dot{\rho}(j) - \rho^2(j) = -\phi\rho(j), \quad \phi > 0.$$

Temporal evolution of $\lambda(t)$

$$\dot{\lambda} = -\rho_0 B - \lambda\left(\lambda - B\right) + \phi\lambda.$$

$$\dot{\rho}(j) - \rho^2(j) = -\phi\rho(j), \quad \phi > 0.$$

$$\theta(j) = 1 - \frac{\rho_0}{\phi}\left(1 - e^{-\phi j}\right), \quad \rho(j) = \frac{\rho_0\phi}{\rho_0 + (\phi - \rho_0)e^{\phi j}}.$$

with $0 < \rho_0 < \phi$, $\quad \lim_{j\to\infty} \theta(j) = 1 - \frac{\rho_0}{\phi} \in (0,1).$

Delays are not discounted in the very long run

$$\lim_{j\to+\infty} \rho(j) = 0.$$

A four-differential equation system

$$\dot{r} = -\frac{1-\alpha}{\alpha} s r^2, \quad r(0) = r_0,$$

$$\dot{s} = -(1-s)\left[\left(\frac{1}{\sigma} - s\right) r - \frac{\lambda}{\sigma}\right],$$

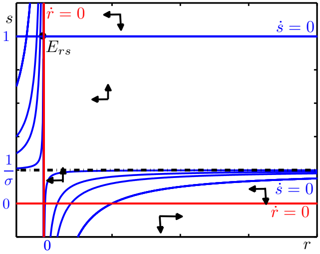
$$\dot{B} = \frac{1-\sigma}{\sigma}(r-\lambda) B + B(B-\lambda),$$

$$\dot{\lambda} = -\rho_0 B - \lambda(\lambda - B) + \phi\lambda$$

Final condition for a LEUG path

$$\lim_{t\to\infty} \frac{\lambda(t)}{B(t)} = \frac{\rho_0}{\phi}$$

▸ Proof



Proposition

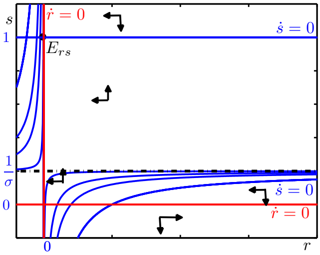
For any initial point with $s_0 \in (1/\sigma, 1)$, $r_0 > 0$, $B_0 > 0$ and $\lambda_0 > 0$, the saving rate increases towards 1 and the rate of return decreases towards 0.

Phase diagram

Corollary

Among the set of steady-state equilibria of the form $(0, s^*, 0, 0)$, $s^* \in [0,1]$, only the equilibrium with $s^* = 1$ can be asymptotically stable.

Saddle-path stability for $s^* = 1/\sigma$?



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Saddle-path stability for $s^* = 1/\sigma$?

Lemma

The steady states $(0, s^*, 0, 0)$ are non-hyperbolic equilibria, characterized by a one-dimensional unstable manifold and a three-dimensional center manifold.

2^{nd}-order approximation of the center manifold

$$\lambda = \frac{\rho_0}{\phi} B \left[1 + \frac{\sigma - 1}{\phi\sigma} \left(r - \frac{\rho_0}{\phi} B \right) \right]$$

- Those trajectories starting on the center manifold will never leave it.
- If initial point does not lay on the center manifold, trajectory will diverge from the equilibrium (**saddle-path stability**)

- Is there a set of initial points on the center manifold from which the trajectories asymptotically converge to the steady state?
- And if so, are these convergent trajectories LEUG paths?

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The flow in the center manifold near an equilibrium $(0, s^*, 0, 0)$

$$\begin{aligned}
\dot{r} &= -\frac{(1-\alpha)r^2}{\alpha}s, \quad r(0) = r_0, \\
\dot{s} &= -(1-s)\left\{\frac{1}{\sigma}\left(r - \frac{\rho_0}{\phi}B\right)\left[1 + \frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right] - sr\right\}, \\
\dot{B} &= B\left\{\frac{1}{\sigma}\left(r - \frac{\rho_0}{\phi}B\right)\left[1 + \frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right] - (r - B)\right\}
\end{aligned}$$

Convergence on the center manifold

New variable

$z = B/r$ which must be $z < \phi/\rho_0$ along a LEUG path

Time-elimination

$\tau = -\ln(r/r_0)$, with r_0 the initial value of the rate of return

$$
\begin{aligned}
\frac{dz}{d\tau} &= \frac{dz/dt}{d\tau/dt} = \frac{\alpha}{1-\alpha}\frac{z}{s}\left\{\frac{1}{\sigma}\left(1-\frac{\rho_0}{\phi}z\right)\left(1+\frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right)-(1-z)\right\}+z,\\
\frac{ds}{d\tau} &= \frac{ds/dt}{d\tau/dt} = -\frac{\alpha}{1-\alpha}\frac{1-s}{s}\left\{\frac{1}{\sigma}\left(1-\frac{\rho_0}{\phi}z\right)\left(1+\frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right)-s\right\},\\
\frac{dB}{d\tau} &= \frac{dB/dt}{d\tau/dt} = \frac{\alpha}{1-\alpha}\frac{B}{s}\left\{\frac{1}{\sigma}\left(1-\frac{\rho_0}{\phi}z\right)\left(1+\frac{\sigma-1}{\sigma\phi}\frac{\rho_0}{\phi}B\right)-(1-z)\right\}
\end{aligned}
$$

Convergence on the center manifold

Four steady states (z^*, s^*, B^*)

$$(0, 1, 0), \quad (0, 1/\sigma, 0)$$

$$\left(\frac{\phi\,(\alpha(\sigma-1)-(1-\alpha)\sigma)}{(\phi\sigma-\rho_0)\alpha}, 1, 0\right), \quad \left(\frac{\phi\,(\sigma\alpha-1)}{\phi\sigma\alpha-\rho_0}, \alpha\frac{\phi-\rho_0}{\phi\sigma\alpha-\rho_0}, 0\right)$$

Proposition (saddle-path stability)

Assuming $\sigma > 1/\alpha$, for any given values of r_0 and $\gamma_c(0)$, there exists a unique value s_0 (lower than $1/\sigma$) from which variables (z, s, B) will converge to

$$(z^*, s^*, B^*) = (0, 1/\sigma, 0)$$

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$$(z^*, s^*, B^*) = (0, 1/\sigma, 0)$$

Convergence on the center manifold

Proposition

Assuming $\alpha > 1/2$ *and* $\sigma > \alpha/(2\alpha - 1)$, *for any given values of* r_0 *and* $\gamma_c(0)$,

- $i)$ *from any* $s_0 \geq 1/\sigma$, *the saving rate will increase and variables* (z, s, B) *will converge to* $(0, 1, 0)$. $\Leftarrow$ **asymptotically local stability**.
- $ii)$ *there exists a unique value* s_0 *(lower than* $1/\sigma$*) from which the trajectories will converge to* $\left(\frac{\phi(\alpha(\sigma-1)-(1-\alpha)\sigma)}{(\phi\sigma-\rho_0)\alpha}, 1, 0\right)$. $\Leftarrow$ **saddle-path stability**

Additionally, assuming $\sigma > 1/\alpha$, *for any given value of* r_0, *there exist values for* $\gamma_c(0)$ *and* s_0, *from which the variables will converge to* $\left(\frac{\phi(\sigma\alpha-1)}{\phi\sigma\alpha-\rho_0}, \alpha\frac{\phi-\rho_0}{\phi\sigma\alpha-\rho_0}, 0\right)$. $\Leftarrow$ **saddle-path stability**

Speed of convergence

Proposition

For any of the convergent trajectories to any of the four steady states the rate of return converges towards zero as an $\mathcal{O}(t^{-1})$ function.

$$-\frac{(1-\alpha)}{\alpha} r \leq \frac{\dot{r}}{r} \leq -\frac{(1-\alpha)}{\alpha} s_{\min} r \text{ for all } t \geq 0$$

Speed of convergence

The rate of return converges towards zero as an $\mathcal{O}(t^{-1})$ function

$$-\frac{1-\alpha}{\alpha}r < \frac{\dot{r}}{r} < -\frac{1-\alpha}{\alpha}s_{\min}r.$$

$$\overbrace{k_0\left[1+\widetilde{\gamma}_0(1-\alpha)t\right]^{\frac{1}{1-\alpha}} < k(t) < k_0\left[1+\gamma_0(1-\alpha)t\right]^{\frac{1}{1-\alpha}}}^{\text{LEUG}}$$

$$c_0\left[1+\widetilde{\gamma}_0\frac{1-\alpha}{\alpha}t\right]^{\frac{\alpha}{1-\alpha}} < c(t) < c_0\left[1+\gamma_0\frac{1-\alpha}{\alpha}t\right]^{\frac{\alpha}{1-\alpha}}$$

Conclusions

- A Neoclassical Ramsey growth model does not necessarily lead to stagnation if $\rho = \delta = n = 0$
  - $\gamma_c(t) > 0 \;\; \forall t \geq 0, \lim_{t \to +\infty} \gamma_c(t) = 0,$
  - $\int_t^{\infty} \gamma_c(h)dh = +\infty \Rightarrow \lim_{t \to +\infty} c(t) = +\infty$
- Our proposal: A positive but declining instantaneous discount rate ($\rho(t) > 0$; $\dot{\rho}(t) < 0$)
  - Consumers are highly impatient about immediate consumption, but much more patient when they are confronted with choices in the far future (Laibson (1997)).
- Our discount function meets the "impatience" requirement of Laibson (1997) and, additionally

$$\lim_{t \to +\infty} \rho(t) = 0$$

Conclusions

- A family of time-varying discount functions, for which the asymptotic equilibrium (cannot be reached in finite time), and is characterized by a three-dimensional center manifold.
$$\theta(j) = 1 - \frac{\rho_0}{\phi}\left(1 - e^{-\phi j}\right)$$
- A set of convergent trajectories for which the rate of return declines towards zero but consumption will grow forever (positive $\gamma_c(t)$ for ever), at a declining rate ($\lim_{t \to +\infty} \gamma_c(t) = 0$).
- The growth rate of consumption declines so slowly that
$$\int_t^{\infty} \gamma_c(h)dh = +\infty \Rightarrow \lim_{t \to +\infty} c(t) = +\infty$$
- A non-constant but declining discount rate leads to LEUG.

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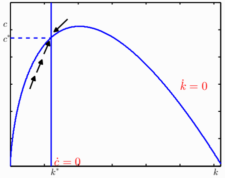
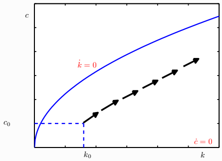
$n,\ \delta,\ \rho > 0$



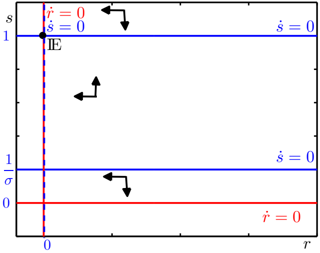
- A saddle-path stable steady-state
- Transitional dynamics: growing consumption and capital per capita converging to the steady-state
- Long-run stagnation

$n = \delta = \rho = 0$



- Might the economy exhibit an infinite period of growth in consumption and capital?
- Will consumption increase unboundedly?

Back



$\sigma > 1$ existence of stable equilibria

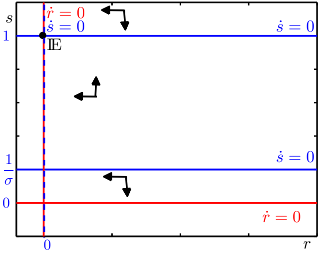
Return 1 Return 2

- If $s_0 = 1/\sigma$, it remains constant and $r(t)$ is the solution of a Bernoulli differential equation

$$\dot{r} = -\frac{1-\alpha}{\alpha\sigma} r^2 \Rightarrow \begin{cases} k(t) = k_0 \left[1 + \gamma_0 (1-\alpha) t\right]^{\frac{1}{1-\alpha}} \\ c(t) = c_0 \left[1 + \gamma_0 \frac{1-\alpha}{\alpha} t\right]^{\frac{\alpha}{1-\alpha}} \end{cases}$$

- If $s_0 \in (1/\sigma, 1]$ the integral curves will converge to the point $\mathbb{E} = (0, 1)$

$$-\frac{1-\alpha}{\alpha} r < \frac{\dot{r}}{r} < -\frac{1-\alpha}{\alpha\sigma} r.$$



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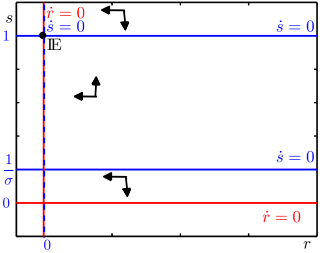

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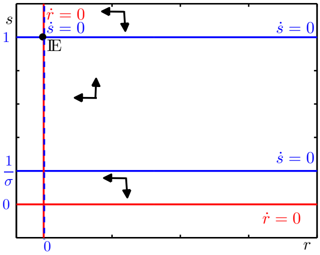


$\sigma > 1$ existence of stable equilibria

Return 1 Return 2

- If $s_0 = 1/\sigma$, it remains constant and $r(t)$ is the solution of a Bernoulli differential equation

$$\dot r = -\frac{1-\alpha}{\alpha\sigma} r^2 \Rightarrow \left\{ \begin{array}{c} k(t) = k_0 \left[1 + \gamma_0 (1-\alpha) t\right]^{\frac{1}{1-\alpha}} \\ c(t) = c_0 \left[1 + \gamma_0 \frac{1-\alpha}{\alpha} t\right]^{\frac{\alpha}{1-\alpha}} \end{array} \right\}$$

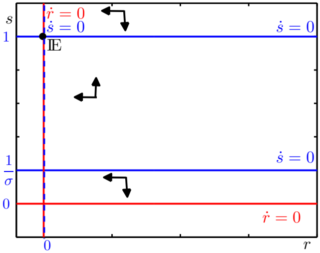
- If $s_0 \in (1/\sigma, 1]$ the integral curves will converge to the point $\mathbb{E} = (0, 1)$

$$-\frac{1-\alpha}{\alpha} r < \frac{\dot r}{r} < -\frac{1-\alpha}{\alpha\sigma} r.$$

$$k_0 \left[1 + \widetilde{\gamma}_0 (1-\alpha) t\right]^{\frac{1}{1-\alpha}} < k(t) < k_0 \left[1 + \gamma_0 (1-\alpha) t\right]^{\frac{1}{1-\alpha}}$$

$$c_0 \left[1 + \widetilde{\gamma}_0 \frac{1-\alpha}{\alpha} t\right]^{\frac{\alpha}{1-\alpha}} < c(t) < c_0 \left[1 + \gamma_0 \frac{1-\alpha}{\alpha} t\right]^{\frac{\alpha}{1-\alpha}}$$

2. LEUG in the Ramsey Model when $n = \delta = \rho = 0$



$\sigma > 1$ existence of stable equilibria

◂ Return 1 ◂ Return 2

- If $s_0 = 1/\sigma$, it remains constant and $r(t)$ is the solution of a Bernoulli differential equation

$$\dot r = -\frac{1-\alpha}{\alpha\sigma} r^2 \Rightarrow \left\{ \begin{array}{c} k(t) = k_0 \left[1 + \gamma_0 (1-\alpha) t\right]^{\frac{1}{1-\alpha}} \\ c(t) = c_0 \left[1 + \gamma_0 \frac{1-\alpha}{\alpha} t\right]^{\frac{\alpha}{1-\alpha}} \end{array} \right\}$$

- If $s_0 \in (1/\sigma, 1]$ the integral curves will converge to the point $\mathbb{E} = (0, 1)$

$$-\frac{1-\alpha}{\alpha} r < \frac{\dot r}{r} < -\frac{1-\alpha}{\alpha\sigma} r.$$

$$\overbrace{k_0 \left[1 + \widetilde{\gamma}_0 (1-\alpha) t\right]^{\frac{1}{1-\alpha}} < k(t) < k_0 \left[1 + \gamma_0 (1-\alpha) t\right]^{\frac{1}{1-\alpha}}}^{\text{LEUG}}$$

$$c_0 \left[1 + \widetilde{\gamma}_0 \frac{1-\alpha}{\alpha} t\right]^{\frac{\alpha}{1-\alpha}} < c(t) < c_0 \left[1 + \gamma_0 \frac{1-\alpha}{\alpha} t\right]^{\frac{\alpha}{1-\alpha}}$$

Proof

$$\frac{\lambda(t)}{B(t)} = N(t) = \int_0^\infty -\dot{\theta}(j) e^{(1-\sigma)\int_t^{t+j}\gamma_c(\tau)d\tau} dj.$$

Because function $\dot{\theta}(j)$ is Lebesgue integrable by the Lebesgue's dominated convergence theorem (see, for example, Apostol (1991)), it follows that

$$\lim_{t\to\infty}\int_0^\infty -\dot{\theta}(j) e^{(1-\sigma)\int_t^{t+j}\gamma_c(\tau)d\tau} dj = \int_0^\infty \lim_{t\to\infty}\left[-\dot{\theta}(j) e^{(1-\sigma)\int_t^{t+j}\gamma_c(\tau)d\tau}\right] dj$$

From the mean value theorem there exists an intermediate $\omega \in [t, t+j]$ such that

$$\int_t^{t+j}\gamma_c(\tau)d\tau = \gamma_c(\omega)j.$$

Then, along a LEUG path

$$\lim_{t\to\infty}\int_t^{t+j}\gamma_c(\tau)d\tau = \lim_{\omega\to+\infty}\gamma_c(\omega)j = 0.$$

As a consequence, the result follows.