

Collective attention and ranking methods

Gabrielle DEMANGE¹

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Abstract

Ranking systems are becoming increasingly important in many areas, in the Web environment and academic life for instance. Presumably a ranking helps individuals to make decisions by providing them with relevant information. In a world with a tremendous amount of choices, a ranking plays also the crucial role of influencing the attention that is devoted to the various alternatives. In recurrent situations, attention will, in turn, alter the new statements on which subsequent rankings will be based. The paper proposes an analysis of this feedback by studying some reasonable dynamics that a ranking method may induce. The feedback is shown to depend strongly on the used ranking method. Two main families of methods are investigated, one based on the notion of 'handicaps', the other one on the notion of peers' rankings.

Keywords ranking, scoring, invariant method, peers' method, attention, handicap, scaling matrix.

1 Introduction

The use of rankings is becoming pervasive in many areas including academia for ranking researchers, journals, universities, and the Web environment for ranking Internet pages. The public good aspect of information explains the use of rankings. Rankings are based on a costly process of gathering and summarizing some relevant information on the alternatives in a particular topic. When such information is relevant to anyone, the publication of rankings avoids each individual to pay the search and processing costs. For that very reason, rankings have some influence on the attention that is devoted to the various alternatives. In recurrent situations, attention will, in turn, alter the new statements on which subsequent rankings will be based. This paper proposes an analysis of the feedback between rankings, attention intensities, and statements by studying some reasonable dynamics.

A ranking problem is described by a set of items to be ranked and a set of 'experts' who provide some statements on which the ranking will be based. Rankings here are cardinal, meaning that scores are assigned to items. Let us describe some prominent problems. In a ranking of journals based on citations, journals are both the items to be ranked and the experts, and the statements are the number of citations by articles in a journal towards articles published in different journals. In a ranking of Web pages based on the link structure, the statements are given by the links from a page to the other pages. Here also the items to be ranked -the Web pages- coincide with the experts. This is not the case in our third example, an apportionment problem. The problem is to allocate the seats in an assembly to parties as a result of the votes of various electoral bodies, regional for example. Here the items to be ranked are the parties, the experts are the regions, and a region's statement is given by the number of votes gathered by each party in the region. Voting by committee in which

¹PSE-EHESS, address 48 bd Jourdan, 75014 Paris, France e-mail demange@pse.ens.fr.

voters are asked to weigh the candidates is also a ranking problem. However, such situations are plagued with plain strategic issues that are not the subject of this paper. The focus is on recurrent situations in which the influence of experts is 'diffuse', channelled through the impact of rankings on attention.

The analysis bears on ranking methods that satisfy two important properties. The first property is *intensity invariance*. The property has been introduced for dealing with the situations in which the 'intensity' of statements is not controlled, as in the two first examples above, where neither the number of citations per article nor the number of links from a page are restricted *a priori*. To avoid an expert to increase its impact on the final ranking by an inflation in its statements, experts' statements are adjusted so as to obtain an 'intensity invariant' ranking method (there are other justifications, as explained in the paper). In the journals' example, the cites of a journal are deflated by the average number of cites per article, so that the ranking depends on the proportions of the cites allocated by journals to the different journals. In the Web environment, one deflates a link from a site by the total number of links from that site. This is what is performed by one of the most well known methods, the 'invariant' method, which serves as a basis to PageRank of Google (Page et al. 2004). Factoring out the intensity of outward links avoid pages to increase their score by inflating the number of links.

The second property, that of 'supporting weights' views a method as simultaneously assigning scores to the items and weights to the experts. Given the experts' statements, the ranking writes as a weighted combination of the experts' statements in which furthermore the scores and the weights form some sort of an equilibrium relationship. The property is satisfied by most current methods although it has not been made explicit so far. The counting method, which ranks items according to their received totals, simply assigns equal constant weights to the experts. The invariant method, alluded to above, determines which pages are influential on the basis that a page is influential if it is heavily cited by other influential pages. This creates a loopback effect between the score as an item and the weight as an expert: by its very definition the invariant method assigns scores and weights that form an equilibrium, that is their equalization. This property is useful for various reasons. In particular, it helps us to define new methods through equilibrium relationships and to give a precise definition to what a peers' method is.

The first part of the paper considers static problems, in which the experts' statements are given. I introduce a new ranking method that is both intensity-invariant and supported by equilibrium weights. The equilibrium is based on the notion of handicaps. There are indeed strong relationships between rankings and handicaps. Since the purpose of handicaps is to adjust the marks received by items so as to equalize their 'strength', rankings and handicaps are inversely related to each other: the handicap of i is half that of ℓ if i can be said to be twice as good as ℓ , that is if i 's score is the double of that of ℓ . The method, called the *handicap-based method*, is characterized by simple properties. The computation of the handicap-based ranking relies on a well-known procedure of matrix scaling, called RAS method or iterative proportional fitting procedure.

The second part of the paper studies a recurrent framework to analyze the influence of rankings. This influence is driven by their impact on attention intensities. In a context in which the number of alternatives to consider is huge, experts cannot carefully assess each one and tend to pay more attention to those whose score is higher. For example, while working on a paper, a researcher who uses rankings tends to read more the journals whose ranks are higher. Voters tend to pay more attention to parties with large scores in the past because these parties have more chances of winning or simply because they have more resources to spend on the media. An 'influence function' describes how the current ranking modifies attention intensities. This generates a joint dynamics on rankings and statements because statements depend on both preferences and attention: the current ranking

modifies attention intensities, hence the next statements on which next ranking is based. An intuition is that, as past statements have an impact on future statements through rankings computation, we might expect ‘the rich to get richer’. For example journals with a lot of past citations are more likely to be cited again, which may result in an improvement in their scores. However, the impact of such self-enforcing mechanism may differ according to the ranking method. Our aim is to investigate more precisely this link between a ranking method and the dynamics.

The analysis is carried out on intensity-invariant methods but the main insights remain, and are most likely to be reinforced, when intensity is not factored out. The influence function is first assumed to take a simple linear form. Contrasted results are obtained for two different classes of methods. The first class, called the generalized handicap-based methods, is obtained from the handicap-based method by modifying the experts’ weights. The class includes both the handicap-based and the counting methods. These methods guarantee stability in the sense that, given preferences for the experts, the sequence of rankings converges towards a unique rest point (see more precisely Proposition 3).

The second class is the class of peers’ methods. The rationale behind a peers’ method is that the ability of an individual to perform (measured by his score) is correlated with his ability to judge others’ performance. In particular, for a method supported by weights, a minimal requirement is that an individual who receives a small score is also assigned a small expert’s weight. This defines a peers’ method. The invariant method is a peers’ method since scores and experts’ weights are equalized). In Demange (2009) I show that the dynamics for the invariant method may admit multiple limit points. According to Propositions 4 and 5, such multiplicity is bound to happen: Whatever peers’ method, the dynamics may admit multiple limit points for some preferences, each one corresponding to a different support (the support is the subset of items that keep a positive score). Furthermore, the supports of the limit points are independent of the peers’ method. Such result illustrates the self-sustaining aspect of a peers’ method. Self-sustainability here is not obtained through plain manipulation but through the coordination device induced by the influence of the ranking.

This paper is about the convergence of behaviors and statements. This is also the concern of the large literature that analyzes the influence of opinions channelled by ‘neighbors’ in a partially connected network (see e.g. the book of Goyal 2005, deMarzo et al. 2003 for instance). These papers analyze situations in which individuals receive private signals about a state of the world. One main question is whether (non-strategic) communication will lead opinions to converge to a common belief and, if it converges, how this belief relates to the initial opinions and the network structure. Instead here information -the ranking- is made public and influences all experts in an identical way. The impact however differs across experts because they differ in their preferences.² The analysis shows that the interplay of preferences and the ranking method may induce a variety of different outcomes.

Researchers in computer science have also concerns about the influence of the rankings provided by search engines. The main criticism is that rankings are biased towards already popular webpages, thus preventing the rise in popularity of recently created high quality pages. There has been some proposals to correct the bias, such as introducing some randomness in the rankings (Pandey et al. 2005), or to account of the date of creation of a page in the computation of the ranking (Cho, Roy, and Adams 2005).

Finally, the paper is also related to the social choice theory approach that aims to characterize methods through properties or ‘axioms’. The most relevant papers for our work are the axiomatizations of the invariant method (Palacios-Huerta and Volij 2004, Slutzki and Volij 2006, and Altman and Tennenholtz 2005). Here the handicap-based method is introduced and an axiomatization is provided. Finally, the computation of the handicap-based ranking relies on a well-known method of

²Recently, Golub and Jackson (2008) introduces some form of heterogeneity in preferences in a network framework.

matrix scaling, called RAS method or iterative proportional fitting procedure. The procedure has been used in various areas, in statistics for adjusting contingencies tables, in economics for balancing international trade accounts (Bacharach 1965).

The paper is organized as follows. Next section presents ranking methods, gives examples, and defines some properties. Section 3 introduces the handicap-based method and provides a characterization with some simple axioms. Section 4 is devoted to the dynamics under a linear influence function. Section 5 presents some extensions, and Section 6 discusses some related literature and concludes. Some technical proofs are given in Section 7.

2 Ranking methods

This section describes the framework, recalls some well known methods, and introduces standard properties.

2.1 The framework

Let N be the set of items to be ranked. Items can be individuals, journals, articles, political parties. Let M be the set of 'judges'. Judges can be experts, voters. In the judgment by 'peers', the two sets coincide. In the following, an element of N is called an *individual* and an element of M an *expert*, keeping in mind the different interpretations. The cardinality of N is denoted by n and that of M by m .

Experts provide some statements on which the ranking of the individuals will be based. Experts' statements are described by a $n \times m$ *statement matrix* $\Pi = (\pi_{i,j})$, in which j 's column represents j 's statement over N . Given these statements, one seeks for a ranking that assigns to each individual i a non-negative number r_i , called the *score* of i . The aim of the ranking is to provide the relative strength of the n individuals. This means that not only the order of the individual scores matters but also their values up to a multiplicative constant. Thus, normalizing the sum of the scores to 1, a ranking of N is given by a vector r in the simplex Δ_N : $\Delta_N = \{r = (r_i) \in \mathbb{R}^n, r_i \geq 0, \sum_i r_i = 1\}$.

A method assigns a ranking to each possible statement matrix. Statement matrices are first restricted to be positive, that is $\pi_{i,j}$ to be all positive. Formally, given N and M , a *ranking or scoring* method F assigns to each positive $n \times m$ matrix Π a positive ranking $r = F(\Pi)$ in Δ_N .

The three examples described in the introduction are cast into this framework.

For ranking journals based on citations, N and M are both given by the set of journals to be compared, statements are the number of citations by articles in journal j towards articles published in journal i . To be more precise, let $C_{i,j}$ be the total number of cites from j to i in a relevant period. Cites are normalized to account for the total number n_j of articles in j : this gives matrix Π in which the value $\pi_{i,j} = \frac{C_{i,j}}{n_j}$ is the average number of references of an article from j to i .

For ranking pages on Internet based on the link structure, the two sets of individuals and experts N and M coincide with the set of 'relevant' pages and the method defines a ranking of the pages that is based on the links within N . Hence the statement matrix Π is the adjacent matrix of the Web network: it has $\pi_{i,j}$ equal to 1 if page j points to i and 0 otherwise. The matrix has many zeros because many pages are not pointing to each other. A perturbation technic makes the matrix positive.

In an apportionment problem, N is the set of parties, M is the set of constituencies, say regions, and statements are the number of votes from region j to party i . One wants to allocate a given number of seats to the parties, while possibly 'over'-representing' a region with small population and 'under'-representing one with large population for instance (i.e. not simply counting the total

of the votes obtained by each party). A first task amounts to assign a ranking to parties based on Π (this leaves aside integer problems (seats are not dividible) and the allocation of the seats within parties; for a detail analysis of this problem, see Balinski and Demange 1989-b and 1989-a).

We describe natural properties that one may want any method to satisfy. Intensity invariance, uniformity, and exactness appear in the literature under various names. Homogeneity and equilibrium weights are new.

Basic properties Intensity invariance requires the ranking not to be affected by a scaling of a column.

Definition 1 *A method F is intensity invariant if $F(\Pi') = F(\Pi)$ for Π' the matrix obtained from Π by multiplying a column by any positive μ .*

An intensity invariant method can be defined on matrices whose intensity is 'factored out' and then extended to all positive matrices. Specifically, let \mathcal{M} be the set of matrices with positive elements and columns' sums equal to 1. Let us assign to each positive matrix Π the matrix $[\Pi]$ in \mathcal{M} obtained by normalizing the columns: $[\pi]_{i,j} = \pi_{i,j}/\pi_{*j}$ where π_{*j} denotes the sum $\sum_{i \in N} \pi_{i,j}$. The value $[\pi]_{i,j}$ is the *share* assigned by j to i (or received by i from j). Applying iteratively to each column the intensity invariance property, a method is intensity invariant if and only if $F(\Pi) = F([\Pi])$ for each positive Π . Thus, intensity invariant methods can be defined on \mathcal{M} and then extended to all positive matrices. In particular, the *intensity invariant version* G of a method F is defined by setting $G(\Pi) = F([\Pi])$ for each Π .

Intensity invariance basically assigns *a priori* a sum to each expert's statements, where *a priori* means independently of the statement matrix Π . The property can be justified differently depending on the context. Let us give two different justifications.

A first justification, as alluded to in the introduction, is that one does not want an expert to 'weigh' more because of an inflation in its statements. In the case of journals, factoring out reference intensity avoids to introduce a bias due to the fact that the average number of cites per article differs across journals.³ In the case of Internet, intensity invariance avoids that a page improves its score simply by multiplying the pages it points to. A second justification is that one wants *a priori* to 'correct' or adjust the experts' statements. In an apportionment problem for instance, in which an expert represents an electoral regional body, the primitives are the number of votes in each region. One may want to 'over'-represent a region with small population and 'under'-represent one with large population for instance, so that each one expects to have the same impact on the final result.⁴ This is performed by adjusting for the sizes of the population, so as to obtain the same total of adjusted votes in each region. More generally, in some problems, one may want to treat experts in a different way. This feature can easily be accommodated by assigning different experts' totals and by adjusting each column sum to its assigned total.

To state the next properties, we introduce balanced matrices. Balanced matrices constitute a benchmark in which there is no rationale for distinguishing between individuals if experts are not discriminated *a priori*: rows obtain equal scores when one simply computes the mean of the received

³Recall that $\pi_{i,j}$ is the average number of references of an article from j to i . Factoring out intensity implies that one works with $\pi_{i,j}/\pi_{*j}$, which describes in which proportion the cites made by an article in j are received by i on average. See Palacios-Huerta and Volij (2004) for an analysis of the impact of cite intensities on the ranking of economic journals.

⁴We implicitly assume here that the method is required to be anonymous, in which anonymity means that the ranking is independent on a permutation of the columns. Without such a condition, the imposed sum on an expert has not much meaning, or, put differently, requiring an equal sum for each expert's statements imposes no restriction.

shares, or put equivalently, rows obtain equal scores under the counting method. Specifically, a matrix is said to be (a, b) -balanced if each row receives the same total as well as each column: $\pi_{i*} = a$ each i , $\pi_{*j} = b$ each j (where a and b are linked by $na = mb$). When there is no need to specify the values for a and b , we simply say that the matrix is balanced. For N equal to M , the matrix is proportional to a bi-stochastic matrix. Let \mathbf{e}_N denotes the 'flat' ranking with all its components equal: $\mathbf{e}_N = (\frac{1}{n})$.

Definition 2 A method F is **uniform** if $F(\Pi) = \mathbf{e}_N$ for all balanced Π . A method is **exact** if the reverse is true for normalized matrices: $F([\Pi]) = \mathbf{e}_N$ implies that $[\Pi]$ is balanced.

A method is uniform if it assigns equal scores to each balanced statement matrix. Exactness asks conversely that, once the experts' statements have been normalized, individuals obtain equal scores only if they receive the same total. Requiring exactness for all matrices, i.e., that equal scores are assigned only to balanced matrices, is too strong because no intensity invariant and uniform method satisfies such a requirement. To see this, start with a balanced matrix. Its ranking is e_N by uniformity. Multiply the columns by distinct numbers. The obtained matrix is not balanced but intensity invariance requires its ranking to be e_N : the method is not exact.

Homogeneity The homogeneity property that we introduce now is very natural but has not yet been considered in the literature, as far as we know.⁵ For cardinal rankings, the values taken by the scores, and not only the order, matter. In fact, due to the normalization, a ranking is characterized by the relative scores. Saying that the score of i is twice the score of k should mean that i is 'twice as good' as k . In this interpretation, if we start with a matrix and all the shares of i are multiplied by a factor, i 's relative position should be multiplied by the same factor. This is the homogeneity property.

Definition 3 A method is **homogeneous** if multiplying row i by a positive scalar λ multiplies its rank relative to other rows by the same λ .

Uniformity and exactness are inherited by a method when factoring out intensities. This is not true for the homogeneity property, as illustrated by the counting method.

Three methods: The counting, invariant, and Hits methods We define the methods for any matrix Π . The intensity invariant version is obtained by applying the method to the normalized matrices $[\Pi]$.

The counting method is the simplest method. It assigns scores proportional to the total number of received statements:

$$r_i = \frac{\pi_{i*}}{\sum_{\ell} \pi_{\ell*}} \text{ where } \pi_{i*} = \sum_{j \in M} \pi_{i,j}. \quad (1)$$

The counting method defined by (1) is homogeneous. The intensity invariant version is not. Consider the following example.

$$\Pi = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \Pi' = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}$$

The intensity invariant version of the counting method assigns equal scores, $(1/2, 1/2)$ to Π . Matrix Π' is obtained by multiplying the first row by 2 but the method assigns $(13/20, 7/20)$ instead of

⁵The homogeneity axiom introduced in Palacios-Huerta and Volij (2004) differs: their axiom bears on a given matrix that has two proportional rows whereas ours bears on two distinct matrices.

(2/3, 1/3). The reason is that expert 1 likes relatively more individual 1 than expert 2. Hence the total of expert 1 increases more than that of expert 2 so that when normalizing Π' , the adjustment on 1's statements is larger than on 2's. This explains why the score of 1 is less than doubled relative to that of 2.

The counting method treats experts equally. Instead, the next methods - the Liebowitz-Palmer⁶ and the invariant method- treat experts differently. The sets of individuals and experts coincide ($N = M$). The methods are 'peers' methods based on the idea that the statements made by a peer (as an expert) should be weighed by his score (as an individual). This induces a loopback effect: a score of an individual is defined as proportional to the sum of the received shares weighted by the experts' scores. Specifically the method looks for r in Δ_N that satisfies

$$r_i = \lambda \sum_{j \in N} \pi_{i,j} r_j \text{ for each } i \text{ for some positive } \lambda. \quad (2)$$

These equations say that r is a positive eigenvector of matrix Π associated to the eigenvalue λ . By standard results on positive matrices, such an eigenvector exists, and is associated with the largest eigenvalue; it is called a 'principal' eigenvector. The vector r is unique because Π is irreducible and r is required to be in the simplex. The method is thus well defined.

The invariant method is the intensity invariant version of the Liebowitz-Palmer method. For a normalized matrix, the largest eigenvalue is one⁷ so that the invariant score of Π satisfies

$$r_i = \sum_{j \in N} [\pi_{i,j}] r_j \text{ for each } i. \quad (3)$$

The method is uniform, exact, but not homogeneous.

The Hits method, introduced by Kleinberg (1999), also ranks Web pages using the link structure between pages. Given a relevant set of pages,⁸ N , the Hits method defines a ranking of these pages, based on the links within N . Thus, as for the invariant method, the two sets of individuals and experts coincide. The method distinguishes two weights for each 'page', one associated with the relevance or authority of a page, the other with the adequacy of a page to point towards the relevant pages. The first set of weights defines the ranking, which should help users to find the relevant pages. The second set of weights identifies the pages -called 'hubs'- that are important because they point to relevant pages (but might be not useful to Internet users). Specifically the method assigns the ranking r and the experts weights q in Δ_N that satisfy

$$r_i = \sum_j \pi_{i,j} q_j \text{ each } i \text{ and } q_j = \lambda \sum_i \pi_{i,j} r_i \text{ each } j \quad (4)$$

for some positive λ . Quoting Kleinberg (1999), hubs and authorities exhibit a mutually relationship: a good authority is a page that is pointed to by many good hubs, a good hub is one that points to many good authorities.

In matrix form, (4) writes as $r = \Pi q$ and $q = \lambda \tilde{\Pi} r$ where $\tilde{\Pi}$ is the transpose of Π . Thus the 'authority' weights r and the 'hub' weights q are well defined as respectively the normalized

⁶This terminology is sometimes used because Liebowitz and Palmer (1984) used it (more precisely an approximation of it) for ranking economic journals. The methods have been (re)defined and used in various contexts: in sociology by Katz (1953) and Bonacich (1987), in academics for ranking journals by Pinski and Narin (1976), on Internet for ranking Web pages using the link structure between pages.

⁷Recall that the eigenvalues of a matrix and its transpose are identical. The set of equations $\sum_i \pi_{i,j} = 1$ for each j implies that $\mathbf{1}_N$, the n -vector with components equal to 1, is a positive eigenvector of the transpose of Π with eigenvalue 1.

⁸An important part of the paper deals with the determination of the relevant set of pages, following a query, that is our set N . This problem is left aside here.

principal eigenvectors of the positive matrices $\tilde{\Pi}\tilde{\Pi}$ and $\tilde{\Pi}\Pi$. The method is uniform, exact, but not homogeneous.

Although the two sets of individuals and experts coincide, we will not qualify the method as a peers' method because individual scores expert's weights may be uncorrelated (see the definition in section 4.3). The main insight of the Hits method is precisely to allow this.

2.2 Supporting Weights and peers' methods

The invariant and the Hits methods, by their very definition, assign weights to experts. This property is formally defined as follows.

Definition 4 *The method F is supported by $Q^F = (Q_j^F)$ where the Q_j^F are positive functions defined over the set of positive matrices if for each Π*

$$F_i(\Pi) = \sum_j \frac{\pi_{i,j}}{\sum_i \pi_{i,j}} Q_j^F(\Pi) \text{ each } i. \quad (5)$$

The vector $Q^F(\Pi)$ belongs to Δ_M . For an intensity invariant method, the weights are intensity invariant : $Q^F(\Pi) = Q^F([\Pi])$.

In the counting method, the weights are all equal to $1/m$. In the invariant method, the weight vector is the normalized principal eigenvector of Π , and in the Hits method, it is the normalized principal eigenvector of $\tilde{\Pi}\Pi$.

According to (5) the ranking $F(\Pi)$ is a convex combination of the normalized columns of Π with weights given by $Q^F(\Pi)$.⁹ Our interpretation is that the ranking and the supporting weights form an equilibrium relationship. This interpretation is the basis of the Hits method. Similarly, the invariant method seeks to equalize the ranking to the experts' weights. The rationale of the counting method is that no distinction should be made between experts. In line with this interpretation, even when there are multiple ways to write the ranking as a combination of the columns, the function Q^F is well specified. For the counting method for example, the columns are linearly dependent for m larger than n but the weights are well defined, set to $1/m$. Section 3 introduces a new method, called the handicap-based method, which is based on an alternative equilibrium relationship.

In some situations, the fact that a ranking writes as a convex combination of the experts' statements can be interpreted as an efficiency criterion. Let each expert represent a well defined person, and interpret j 's column as the bliss ranking of j with preferences decreasing in the euclidean norm to the bliss point. Then a ranking outside the convex hull of the bliss points is Pareto-dominated: its projection on the convex hull is preferred by every expert.

The property of supporting weights is useful for various reasons. Apart from defining new methods through equilibrium relationships, it helps us to transform a method, and to give a precise definition to what a peers' method is.

Transformed methods A method supported by weights can be transformed into another method by transforming the weights. Let F be supported by Q^F and g be a positive scalar function defined

⁹To show that the vector $Q^F(\Pi)$ must be in Δ_M , let $r = F(\Pi)$ be in Δ_N and $q = Q^F(\Pi)$ satisfy the relationships $r_i = \sum_j [\pi_{i,j}] q_j$ each i . Summing over i , exchanging sums, and using that columns' sum are equal to 1 yields:

$$\sum_i r_i = \sum_i \sum_j [\pi_{i,j}] q_j = \sum_j \left(\sum_i [\pi_{i,j}] \right) q_j = \sum_j q_j.$$

Since $\sum_i r_i = 1$, we obtain $\sum_{j \in M} q_j = 1$.

over $[0, 1]$. F is transformed by g into method G by adjusting the weights as follows. Given Π experts' weights are proportional to $g(Q_j^F(\Pi))$. This gives

$$G_i(\Pi) = \sum_{j \in M} \pi_{i,j} \frac{g(q_j)}{\sum_{k \in M} g(q_k)} \text{ for each } i \in N, q = Q^F(\Pi). \quad (6)$$

One may want to put restrictions on g , as we will see in next section. A family of methods is obtained from F by letting g to be homogeneous and decreasing: $g(x) = x^{1-\gamma}$ for some γ positive. This family contains the method F ($\gamma = 0$) and the counting method ($\gamma = 1$).

Peers' methods A peers' method should not be defined simply by the fact that the two sets of individuals and experts coincide. A precise definition to what a peers' method means can be easily given for methods supported by weights. The main idea underlying a peers' method is that the ability to provide correct expertise is positively related with the performance as an individual. This makes sense in a setting in which individuals are ordered by a single 'ability' parameter that drives their capacity both to perform and to judge others. In such a situation, the experts' weights should be correlated with individuals' scores. A minimal requirement is that an individual who receives a small score is also assigned a small expert's weight and vice-versa. We define a peers' method as one for which the weight as an expert is bounded relative to the score as an individual.

Definition 5 Let $N = M$. A method F supported by Q^F is a peers' method if there are positive k and k' such that $k' \leq Q_i(\Pi) \leq kF_i(\Pi)$ for each Π in \mathcal{M} .

Since the weights are bounded above by 1, requiring the ratio $(Q_i/F_i)(\Pi)$ to be bounded bears when we consider matrices Π with arbitrarily small scores i . The invariant method is a peers' method since F and Q coincide. The counting method (when applied to $N = M$) is clearly not a peers' method since an individual score can be arbitrarily small while his expert's weight is constant. The Hits method is not a peers' method. A simple example illustrates this. Consider the normalized matrix

$$\Pi(\epsilon) = \begin{pmatrix} 2\epsilon & \epsilon & \epsilon \\ 1/2 - \epsilon & \epsilon & 1 - 2\epsilon \\ 1/2 - \epsilon & 1 - 2\epsilon & \epsilon \end{pmatrix}.$$

As ϵ tends to 0, simple computation yields that the ranking assigned by the Hits method converges to $(0, 1/2, 1/2)$ and that experts' weights converge to $(1/3, 1/3, 1/3)$.¹⁰ For sake of comparison, the same limit ranking $(0, 1/2, 1/2)$ is obtained for the invariant method, which is also, by definition, the limit of the experts' weights.

Notation For a vector x , $x \gg 0$ means that all components are strictly positive, $x > 0$ all are nonnegative and one at least is positive. The support of x , $x > 0$ is the set of indices that are strictly positive.

Given a finite set I , $\mathbf{1}_I$ denotes the vector in \mathfrak{R}^I whose components are equal to 1, $\Delta_I = \{x = (x_i) \in \mathfrak{R}^I, x_i \geq 0, \sum_{i \in I} x_i = 1\}$ the simplex in \mathfrak{R}^I , i.e. the set of possible rankings of I , and $\mathbf{e}_I = \frac{1}{|I|} \mathbf{1}_I$ denotes the ranking with equal components.

¹⁰

$$\tilde{\Pi}(\epsilon)\Pi(\epsilon) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ \epsilon & \epsilon & 1 - 2\epsilon \\ \epsilon & 1 - 2\epsilon & \epsilon \end{pmatrix} \begin{pmatrix} 0 & \epsilon & \epsilon \\ 1/2 & \epsilon & 1 - 2\epsilon \\ 1/2 & 1 - 2\epsilon & \epsilon \end{pmatrix} \text{ converges to } \begin{pmatrix} 12 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}.$$

The principal eigenvector of $\tilde{\Pi}(\epsilon)\Pi(\epsilon)$ puts equal weights on 2 and 3 by symmetry. It converges to the eigenvector of the limit matrix that has this property, $(1/3, 1/3, 1/3)$, which is associated with eigenvalue $3/2$. There are two other positive eigenvectors, $(0, 1, 0)$ $(0, 0, 1)$ associated with eigenvalue 1.

Given a vector x in \mathcal{R}^I , $dg(x)$ denotes the diagonal $I \times I$ matrix with x_i as the i -th element on the diagonal.

Given I and J two non empty subsets of N and M respectively, let $\Pi_{I,J}$ denote the matrix obtained from Π by keeping the rows indexed by i in I and the columns indexed by j in J .

3 The handicap-based method

This section introduces a new method based on the notion of handicaps. The purpose of handicaps is to equalize the strengths between individuals. Handicaps and scores may be seen as inversely related: saying that the handicap of i is twice that of ℓ means that the score of i is half that of ℓ . The handicap-based method is based on this idea: it looks for individuals' handicaps and experts' weights that form an equilibrium relationship and assigns rankings inversely related to the equilibrium handicaps.

To motivate the equilibrium relationships between handicaps and experts' weights, let us consider an iterative process.¹¹ Starting with equal weights for the experts, compute handicaps so as to equalize the scores between individuals. Now, the advice of an expert who assigns large shares to individuals with high handicaps may be considered as unduly represented. This leads to a reassessment of experts' weights so as to equalize the sum of their statements weighted by handicaps. Then the handicaps have to be recomputed and so on. The process of adjustment in experts' weights and individuals' handicaps is shown to converge. The precise algorithm is the following.

Let us start by assigning handicaps to individuals that equalize their total adjusted count to given value, say 1: i 's handicap h_i^0 is the value by which i 's total count must be multiplied so as to obtain 1

$$\left(\sum_j \pi_{i,j}\right)h_i^0 = 1.$$

Let us assign experts' weights x^1 so as to equalize the distributed handicaps across the experts. The total number of handicaps distributed by expert j is evaluated to $\sum_i \pi_{i,j}h_i^0$, so that weights are given by

$$x_j^1 = \frac{n}{m} \frac{1}{\sum_i \pi_{i,j}h_i^0}.$$

(The factor $\frac{n}{m}$ is chosen so as to keep the overall sum of the terms in the matrix $(\pi_{i,j}h_i^0x_j^1)$ constant, here equal to n , so the weights may not sum to 1.) Now, the individuals' counts are adjusted to the weighted sums of their shares computed with the just defined experts' weights, which, in turn, give new values h^1 for the handicaps: $(\sum_j \pi_{i,j}x_j^1)h_i^1 = 1$, each i . Iterating the operations, the process defines two sequences h^τ , x^τ , $\tau = 1, \dots$ by the following equations

$$\left(\sum_j \pi_{i,j}x_j^\tau\right)h_i^\tau = 1 \text{ each } i \text{ and } \left(\sum_i \pi_{i,j}h_i^\tau\right)x_j^{\tau+1} = \frac{n}{m} \text{ each } j. \quad (7)$$

The process (7) can be seen as alternatively scaling the rows and the columns of matrix Π so as to have rows' sums equal to 1 and columns' sums equal to n/m . This process is known as the 'RAS'

¹¹Similarly, the rankings and experts' weights given by the invariant and the Hits methods can be seen as the equilibrium values of an iterative process (see Amir 2002, Kleinberg 1999, Liebowitz and Palmer 1984). Note that the process entirely differs from the one considered in Section 4. Here the matrix is kept fixed whereas in Section 4 it is modified along the process.

method¹² and the sequences (h^τ, x^τ) can be shown to converge to a positive vector (h, x) . Taking the limit in (7), (h, x) satisfies

$$\left(\sum_j \pi_{i,j} x_j\right) h_i = 1 \text{ each } i \text{ and } \left(\sum_i \pi_{i,j} h_i\right) x_j = \frac{n}{m} \text{ each } j. \quad (8)$$

Thus handicaps and experts' weights exhibit a mutual relationship: handicaps equalize individual weighted counts and experts' weights equalize the distributed handicaps.

The handicap-based method assigns a ranking inversely related to the handicap vector. This is a well defined method, as stated in the following proposition.

Proposition 1 *Given a positive matrix Π , there is a unique vector $r = (r_i)$ in Δ_N such that*

$$r_i = \sum_j \pi_{i,j} q_j \text{ for each } i \text{ where } \frac{1}{q_j} = \frac{m}{n} \sum_i \frac{\pi_{i,j}}{r_i} \text{ for each } j \quad (9)$$

or, equivalently, there is a unique pair $r = (r_i), q = (q_j)$, r in Δ_N such that the matrix P of general element $(p_{i,j} = \pi_{i,j} \frac{q_j}{r_i})$ is balanced with rows' sums equal to 1. The **handicap-based method** H assigns to each matrix the unique ranking r defined by (9). It is supported by the weights Q^H : $Q_j^H(\Pi) = \frac{q_j}{q_{*,j}}$.

H is intensity invariant, exact, and homogeneous.

The ranking and the experts' weights assigned by the handicap-based method can be seen as the vectors of adjustments of the statements matrix that are necessary to obtain a balanced matrix, one in which each individual receives equal scores.

Proof of Proposition 1. First denote by P the matrix with general element $p_{i,j} = \pi_{i,j} h_i q_j$ where h and x satisfies (8). The matrix P has its rows and columns sums equalized to 1 and n/m so it is $(1, n/m)$ -balanced. One can show that P is the unique $(1, n/m)$ -balanced matrix that is obtained from Π by multiplication of its rows and its columns by some numbers (see for example Balinski and Demange 1989-b). But the values of h and x for which (8) are satisfied are not unique: given a solution h, x , multiplying h by a positive λ and dividing x by the same λ yields another solution. Hence, we can choose the pair so that the ranking r defined by $r_i = 1/h_i$ is in Δ_N . Furthermore, denoting by q the corresponding value of x , we have $(p_{i,j} = \pi_{i,j} \frac{q_j}{r_i})$, hence equations (9) are satisfied.

To show that r is unique, taking the log of the expression $p_{i,j} = \pi_{i,j} \frac{q_j}{r_i}$ yields

$$\log(r_i) - \log(q_j) = \log(\pi_{i,j}) - \log(p_{i,j}) \text{ for each } i, j.$$

This is a linear system in $a_i = \log(r_i), b_j = \log(q_j)$ (the right hand side is uniquely defined since P is unique). The kernel is defined by $a_i - b_j = 0$ for each i, j , hence the a_i and b_j are all equal to some constant. This implies that the r and the q that satisfy (9) are all obtained by multiplication by a constant, hence there is a unique r that belongs to Δ_N : This proves that H is a well defined method.

Let us show that H satisfies all properties.

H is intensity invariant. Let Π' be obtained from Π by multiplying column j of Π by μ . Letting q' be the vector obtained from q by dividing q_j by μ , the vectors r and q' satisfy (9) for Π' . Hence, by the uniqueness result, $H(\Pi')$ is equal to r .

¹²The procedure has been used in various areas, in statistics for adjusting contingencies tables, in economics for balancing international trade accounts as developed by Bacharach (1965). The aim of the RAS method is to adjust a given matrix so as to satisfy constraints on the rows and columns sums, here respectively 1 and n/m . Hence the object of interest is the final adjusted matrix P of general element $p_{i,j} = \pi_{i,j} h_i x_j$. Instead we are interested in the vectors of adjustment, the handicaps and experts weights.

H is exact. Let $H(\Pi) = \mathbf{e}_N$ for Π a matrix in \mathcal{M} . Since its columns' sums are equal to 1, matrix Π is balanced, if its row's totals are equal. The second equation in (9) yields that the weight vector q satisfies $1/q_j = m(\sum_i \pi_{i,j})$ for each j , which gives $q_j = 1/m$ since Π is in \mathcal{M} . Plugging $r_i = 1/n$ and $q_j = 1/m$ for each i, j into the first equations of (9) we obtain that each row's total is equal to m/n . Hence the matrix Π is balanced.

H is homogeneous. For $\Pi' = dg(\rho)\Pi$ let $s_j = \sum_i \pi_{i,j}\rho_i$ denote the columns' sums of Π' . We have $[\pi'_{i,j}] = \rho_i\pi_{i,j} = \frac{\rho_i r_i \pi_{i,j}}{x_j} s_j$. Hence, by the uniqueness property, the handicap-based ranking r' for Π' is the vector in Δ_N proportional to $(\rho_i r_i)$: this proves homogeneity. This also implies that the weights associated to $\Pi' = dg(\rho)\pi$ are proportional to $(q_j s_j)$, which gives

$$\begin{aligned} H_i(dg(\rho)\pi) &= \frac{\rho_i r_i}{\sum_i \rho_i r_i} \text{ each } i, Q_j^H(dg(\rho)\pi) = \frac{q_j s_j}{\sum_i \rho_i r_i} \text{ each } j \\ \text{where } r &= H(\Pi), q = Q^H(\Pi), s_j = \sum_i \pi_{i,j}\rho_i. \end{aligned} \quad (10)$$

■

The next proposition provides a characterization of the handicap-based method.

Proposition 2 *The handicap-based method is the unique ranking method that is intensity invariant, exact, and homogeneous.*

Proof of Proposition 2. Let method F satisfy the properties listed in Proposition 2.

Given $r = F(\Pi)$, divide each row i by r_i so as to obtain matrix $\Pi' = dg(1/r_1, \dots, 1/r_n)\Pi$. We show that the ratio of the score of i over that of j is multiplied by r_j/r_i so that the scores are equalized: $F(\Pi') = \mathbf{e}_N$. This is proved by applying iteratively the homogeneity property of F . Start by dividing row 1 by r_1 so as to obtain matrix $\Pi_1 = dg(1/r_1, 1, \dots, 1)\Pi$. Homogeneity implies that $F(\Pi_1)$ is the ranking in Δ_N proportional to $(1, r_2, \dots, r_n)$. Then divide row 2 of Π_1 by r_2 so as to obtain matrix $\Pi_2 = dg(1/r_1, 1/r_2, 1, \dots, 1)\Pi$. Homogeneity implies that $F(\Pi_2)$ proportional to $(1, 1, r_3, \dots, r_n)$. Iterating up to n we finally obtain the matrix $\Pi' = dg(1/r_1, \dots, 1/r_n)\Pi$ and that its ranking has all its components equal: $F(\Pi') = \mathbf{e}_N$.

By intensity invariance of F we have $F([\Pi']) = F(\Pi') = \mathbf{e}_N$. Exactness implies that $[\Pi']$ is balanced, i.e., that its rows' sums are equal to m/n . Since $[\pi'_{i,j}] = \frac{\pi_{i,j}}{r_i \sum_\ell \pi_{\ell,j}/r_\ell}$, this writes

$$\sum_j \frac{\pi_{i,j}}{r_i \sum_\ell \pi_{\ell,j}/r_\ell} = \frac{m}{n} \text{ each } i, \text{ or } r_i = \sum_j \pi_{i,j} q_j \text{ each } i \text{ where } \frac{1}{q_j} = \frac{m}{n} \sum_\ell \pi_{\ell,j}/r_\ell \text{ each } j.$$

Thus r and q satisfy (9). Since r is in Δ_N , r is equal to $H(\Pi)$. ■

4 Dynamics

The statement matrix so far has been taken as given. It implicitly reflects preferences. Indeed, the premise of ranking methods is that statements are related to preferences. In particular, citations or links are considered as a positive vote. Even so, the absence of citation to an article is not necessarily a negative vote simply because the paper might not have been read. In a context with many alternatives (potentially many relevant papers to read, many sites to visit) individuals are not considering each alternative. Or they are not devoting the same amount of attention to each one. As a result, statements depend on both preferences and attention. This induces a channel through which rankings have some influence because they modify attention intensities hence the statements. We examine this influence by specifying how rankings modify attention.

4.1 The influence model

Attention intensities are described by a positive n -vector (b_i) , where b_i represents the intensity spent on i . In the context of journals for example, attention represents the selection of the journals that are read in statistical terms. When attention differs across two journals, the articles in the journal with the higher value for attention intensity have more chances to be read, everything equal. In the case of voting, attention represents the time spent by voters on listening to the parties.

Attention intensity modifies experts' statements. Let us interpret $\pi_{.j} = (\pi_{i,j})_{i \in N}$ as j 's 'true' preferences, that is the probability for j to state a positive vote on i when j evaluates each i with equal attention. Bias in attention b results in statements proportional to $(\pi_{i,j} b_i)$.

The influence of a ranking is described by an 'influence function' that assigns attention intensities to a ranking in Δ_N , where attention to i is increasing in the score of i . Natural influence functions are build on a scalar function B defined on $[0, 1]$, positive, and increasing: Given a public ranking r , attention intensities are proportional to $(B(r_i))$. This yields the statement matrix

$$\pi'_{i,j} = \pi_{i,j} B(r_i)$$

(since we consider intensity invariant methods we do not need to normalize the matrix at this stage). Examples of functions B are $B(x) = x^\alpha$ or $B(x) = e^{\alpha x}$ with $\alpha > 0$ (the lower α , the less an individual is influenced by the ranking).

The statement matrix Π' leads to an adjustment in the ranking, which, when made public, will modify the attention intensities, hence the statements. This generates a joint dynamics on statements and rankings in which current statements determine the ranking which will influence subsequent statements through the impact on attention intensities. We are interested in this dynamics, and how it relates to the ranking method and the influence functions. We investigate first the case where the influence function is linear, $B(x) = x$.

At the beginning of period $t + 1$, the ranking $r^{(t+1)}$ is published on the basis of the matrix $\Pi^{(t)}$ given by

$$\pi_{i,j}^{(t)} = \pi_{i,j} r_i^{(t)} \text{ or in matrix notation } \Pi^{(t)} = dg(r^{(t)})\Pi.$$

Thus, the joint dynamics followed by the statement matrix and the ranking are defined by

$$\Pi^{(t)} = dg(r^{(t)})\Pi, r^{(t+1)} = F(\Pi^{(t)}). \quad (11)$$

Matrix $\Pi^{(t)}$ is obtained from Π by multiplying row i by $r_i^{(t)}$ for each i . Given Π , the statement matrices along the process are all of the form $dg(\rho)\Pi$ for some positive ρ . For a method supported by weights Q^F , to simplify notation, we use $\mathbf{q}_j(r) = Q_j^F(dg(r)\Pi)$. The dynamics followed by $r^{(t)}$ writes

$$r_i^{(t+1)} = \sum_j \frac{\pi_{i,j} r_i^{(t)}}{\sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)}} \mathbf{q}_j(r^{(t)}) \text{ each } i \text{ in } N. \quad (12)$$

To study the dynamics, we make some continuity assumptions. First, the scores and the weights are continuous functions over the set of positive matrices. This is a natural requirement and all methods introduced so far satisfy it. Second, we need also to consider what happens when some scores become arbitrarily small. Suppose that the sequence of rankings and supporting experts' weights converge and denote with $*$ the limit values. Taking the limit in (12), we have:

$$r_i^* = \sum_j \frac{\pi_{i,j} r_i^*}{\sum_{\ell} \pi_{\ell,j} r_\ell^*} q_j^* \text{ for each } i \text{ where } q_j^* = \lim_{r^{(t)} \rightarrow r^*} \mathbf{q}_j(r^{(t)}). \quad (13)$$

(Note that the sum $\sum_{\ell} \pi_{\ell,j} r_{\ell}^*$ is strictly positive for any ranking, even with null components since each $\pi_{\ell,j}$ is strictly positive.) The limit conditions (13) are met for i with a null r_i^* . This is a self-enforcing mechanism: an individual whose score is null is not assessed, hence not cited, which in turn justifies a null score. But such a mechanism may not be robust unless a limit point enjoys a minimum of stability. The stability of a limit point requires to consider the behavior of the ranking and the weights associated to statement matrices $dg(\rho)\Pi$ for ρ in a neighborhood of r^* . For a point r^* that has some null components, the limit matrix $dg(r^*)\Pi$ has null rows so that the method may not be defined. This leads to the continuity assumption (b).

Continuity assumption (C)

(a) F and Q^F are continuous functions over the set of positive matrices.

(b) $\mathbf{q}(\rho) = Q^F(dg(\rho)\Pi)$ has a well defined limit when ρ tends to r^* where r^* is any vector in Δ_N . The limit is denoted by $\mathbf{q}(r^*)$.

Condition (a) does not deserve any comment. Condition (b) bears on the boundary of the set of positive matrices. It is clearly satisfied by the counting method since experts' weights are constant.

The invariant method satisfies Condition (b) as well. Given r^* with some null components, the limit $\mathbf{q}(r^*)$ is derived as follows. Denoting by I the support of r^* , consider $dg(r^*)\Pi_{vert I \times I}$ the $I \times I$ matrix formed by deleting the rows and the columns of $dg(r^*)\Pi$ indexed by I . As ρ converges to r^* , the ranking and the weight vector converge on I towards the normalized principal eigenvector of that restricted matrix and to zero outside I . More generally, for a peers' method, the weights' experts converge to zero for any i whose score converges to zero. Condition (b) is thus satisfied if the weights on I converge, which is a mild assumption.

The handicap-based method satisfies Condition (b), as can be seen from (10): We have

$$\lim_{\rho \rightarrow r^*} H_i(dg(\rho)\Pi) = \frac{r_i^* H_i(\Pi)}{\sum_{\ell} r_{\ell}^* H_{\ell}(\Pi)}, \lim_{\rho \rightarrow r^*} Q_j^H(dg(\rho)\Pi) = \frac{Q_j^H(\Pi) s_j}{\sum_{\ell} r_{\ell}^* H_{\ell}(\Pi)} \text{ where } s_j = \sum_i \pi_{i,j} r_i^*.$$

In contrast to the invariant method, the limits of the scores and the weights are not obtained by applying the handicap-based method to the restricted matrix $(dg(r^*)\Pi)_{I \times I}$ (still denoting by I the support of r^*). In particular, let two matrices Π and Π' have identical rows on I . Although the matrices $dg(r^*)\Pi$ and $dg(r^*)\Pi'$ coincide, the limits $\lim_{\rho \rightarrow r^*} Q^H(dg(\rho)\Pi)$ and $\lim_{\rho \rightarrow r^*} Q^H(dg(\rho)\Pi')$ may differ: They depend on the statements of experts outside I through the values taken by H and Q^H on Π . The reason is that as the scores of some individuals tend to zero, their handicaps grow in the same proportion, and, as a result, they continue to affect the experts' weights.

The continuity assumption (C) requires the continuity over a subset of nonnegative matrices, those with some null rows and the other strictly positive. Of course, it would be simpler to consider methods that are continuous over the whole set of all nonnegative normalized matrices (these matrices have at least one positive element in each column). But very few methods are continuous over this larger set. The invariant method is not because a non-negative matrix may admit distinct principal eigenvectors in Δ_N , and continuity would require to deal with a multi-valued function. Similar difficulties also arise for the handicap-based method: as we have just seen, continuity would require considering multi-valued function.

From now on, we assume the continuity assumption (C) to be satisfied without repeating it.

Rest points and their support A fixed point of the dynamics may have null components, as seen before from (13). We are however interested in robust results, as described by stability. Recall that a limit point r^* is locally asymptotically stable if the process converges to r^* for an open set of initial values for r^0 around r^* . Dividing (12) by $r_i^{(t)}$ gives an expression for the growth rate of i 's

score. If the sequence converges to a limit point with null i component, the limit growth rate must be less than 1. This gives the following necessary conditions for a point to be stable.

Definition 6 *Necessary conditions for r^* to be stable for the dynamics (12) are*

$$\sum_j \frac{\pi_{i,j}}{\sum_{\ell \in N} \pi_{\ell,j} r_\ell^*} \mathbf{q}_j(r^*) \leq 1 \text{ for each } i \text{ with an equality if } r_i^* > 0. \quad (14)$$

A point that satisfies (14) is called a rest point.

We display a family of methods under which convergence is guaranteed: given a matrix Π the sequence converges towards a unique rest point whatever the initial r^0 . In contrast, we show that no peers' method guarantees the uniqueness of a rest point nor convergence.

4.2 On the convergence of generalized handicap-based methods

This section shows that convergence towards a unique point is guaranteed for a family of methods based on the handicap-based methods. A generalized handicap-based method is defined as a transform of the handicap-based method by an homogeneous and concave function: $g(x) = x^{1-\gamma}$ for γ positive. From (6) such a method G writes as

$$G_i(\Pi) = \sum_{j \in M} [\pi_{i,j}] \frac{q_j^{1-\gamma}}{\sum_{k \in M} q_k^{1-\gamma}} \text{ for each } i \in N \text{ where } q = Q^H(\Pi). \quad (15)$$

The handicap-based method obtains for $\gamma = 0$ and the counting method for $\gamma = 1$. We prove a general convergence result for any method in the family obtained for positive γ . (When γ is larger than 1, there is an important change in the transformation since the weights are in the reverse order of the handicap weights. One may want to exclude these values. The next proposition is however valid.)

Proposition 3 *Consider a generalized handicap-based method (15) with γ strictly positive. There is a unique rest point, which is furthermore globally stable: the dynamics (12) converges to it for any initial value of r .*

Proposition 3 is proved by showing that the following function L is a Lyapounov function for the dynamics:

$$\begin{aligned} L(\rho) &= \frac{1}{1-\gamma} \sum_j Q_j^H(\Pi) (\sum_i \pi_{i,j} \rho_i)^{1-\gamma} & \gamma > 0, \gamma \neq 1 \\ &= \sum_j Q_j^H(\Pi) \ln(\sum_i \pi_{i,j} \rho_i) & \gamma = 1. \end{aligned}$$

The case $\gamma = 1$ corresponds to the counting method. The handicap-based method, which obtains for γ equal to zero, is not covered by Proposition 3. As shown below, multiple rest points are possible but in degenerate situations. When γ is negative, multiplicity is more severe, as will be illustrated with the case $\gamma = 1$.

A technical lemma Starting with a matrix Π , the iterate of the matrices during the process are all of the form $dg(\rho)\Pi$. The next lemma states conditions on the behavior of the experts' weights along the process under which there is convergence.

Lemma 1 *Let a method be supported by Q . Given a column-normalized matrix Π , assume that there are some functions ψ_k , k in M , defined from the set of positive scalar numbers, \mathbb{R}_+ , to itself, continuous and decreasing such that*

$$Q_j(dg(\rho)\Pi) = \frac{s_j\psi_j(s_j)}{\sum_{k \in M} s_k\psi_k(s_k)} \text{ for each } j \in M \text{ where } s_j = \sum_{i \in N} \pi_{i,j}\rho_i. \quad (16)$$

Then there is a unique rest point, which is globally stable.

Expression (16) states how the modification of the statement matrix by ρ affects the experts' weights. As each row i is multiplied by ρ_i , the columns' sums are modified into the s_j . Conditions (16) require the experts' weights to depend only on these sums and in a separable way (up to the normalization).

The proof of Lemma 1 is given in the appendix. It goes as follows. Let function L be defined by $L(\rho) = \sum_j \Psi_j(\sum_i \pi_{i,j}\rho_i)$ where Ψ_j is a primitive of ψ_j . If expression (16) holds, then the conditions (14) on a rest point exactly coincide with the first order conditions of the maximization of L over r in Δ_N . This immediately implies that there is a unique rest point when L is strictly concave, that is ¹³ when the ψ_j are strictly decreasing. Furthermore, in that case, L is shown to be a Lyapounov function for the process, which ensures the convergence to the rest point. When the ψ_j are not decreasing, L is not concave, and the points satisfying the first order conditions associated to the program still coincide with the rest points but several rest points are possible. Furthermore L is no longer a Lyapounov function (see the two examples at the end of this section).

Proof of Proposition 3. For the handicapped-based method, it follows from (10) that the weight vector $Q^H(dg(\rho)\Pi)$ is the normalized vector proportional to $(Q_j^H(\Pi)s_j)$. For a method transformed by $x^{1-\gamma}$, the weights Q are proportional to $[Q^H(dg(\rho)\Pi)]^{1-\gamma}$, hence they satisfy

$$Q_j(dg(\rho)\Pi) \propto Q_j^H(\Pi)^{1-\gamma} s_j^{1-\gamma}. \quad (17)$$

Thus (16) holds for functions ψ_j given by

$$\psi_j(s_j) = Q_j^H(\Pi)^{1-\gamma} s_j^{-\gamma}. \quad (18)$$

For γ positive, the functions ψ_j are decreasing. ■

The handicapped-based method For the handicapped-based method, the function L is linear so that uniqueness of a rest point is not guaranteed. In the case in which there are several maximizers to L , each one is a rest point. This case arises when there are several i that achieve the maximal handicap-based score for Π . Let $r = H(\Pi)$ and $q = Q^H(\Pi)$, the values assigned by the handicapped-based method to Π . Denote by I the set of individuals for which the handicap-based score is maximal. We show that the rest points are all those with null components outside I . In the general case, I is a singleton hence there is a unique rest point.

Recall that $r^{(t+1)}$ and $q^{(t+1)}$ are the rankings and the weights assigned by the handicapped-based method to $\Pi^{(t)} = dg(r^{(t)})\Pi$. Using (10) we have for each $t \geq 0$

$$r_i^{(t+1)} = \lambda^{(t+1)} r_i^{(t)} r_i \text{ and } q_j^{(t+1)} = \lambda^{(t+1)} \left(\sum_{\ell} \pi_{\ell,j} r_{\ell}^{(t)} \right) q_j \quad (19)$$

¹³Denoting the function on the left-hand side of (14) by $\phi_i(r)$, the conditions (14) write as $\phi_i(r) \leq 1$, with an equality for $r_i = 0$. Since we consider the maximization of L over the closed set Δ_N , the ϕ_i are not the partial derivatives of L with respect to r_i , hence they do not have to satisfy the integrability conditions $\partial\phi_i/\partial r_{\ell} = \partial\phi_{\ell}/\partial r_i$, for each i, ℓ .

where $\lambda^{(t+1)} = 1/ \langle r^{(t)}, r \rangle$ ensures that $r^{(t+1)}$ is in Δ_N . Thus, the growth rates, $\gamma_i^{(t+1)} = r_i^{(t+1)}/r_i^{(t)}$, are proportional to the handicap-based scores (r_i). This implies that the ratio of the growth rate of i over that of ℓ , $\gamma_i^{(t+1)}/\gamma_\ell^{(t+1)}$, stays constant equal to the ratio of their handicap-based scores r_i/r_ℓ .

Let i be not in I , that is i 's handicap-based score is not maximal: $r_i < r_\ell$ for some ℓ . Since rankings are in the simplex and i 's growth's rate is less than ℓ 's, i 's score $r_i^{(t+1)}$ must converge to 0. Thus any rest point has null components on $N - I$.

For i and ℓ both in I , the handicap score ratio r_i/r_ℓ is equal to one, thus their growth rates are equal. this implies that the ratio $r_i^{(t)}/r_\ell^{(t)}$ stays constant equal to their initial values $r_i^{(0)}/r_\ell^{(0)}$. Thus $r^{(t)}$ converges towards the ranking whose components are proportional on I to their initial values and are null outside. Thus any ranking with null components on $N - I$ is a rest point. ■

Example of multiplicity Consider a generalized handicap-based method G with $g(x) = x^2$ (which corresponds to $\gamma = -1$). Expression (18) is still valid so that (16) is satisfied for $\psi_j(s_j) = Q_j^H(\Pi)s_j$. Since these functions are increasing, Lemma 1 does not apply. We show that multiple rest points are possible. Let the 2×2 matrix

$$\Pi = \begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}.$$

Easy computation gives that the handicap based rankings and weights are

$$r \propto (\sqrt{(1-a)b}, \sqrt{a(1-b)}), \quad q \propto (\sqrt{(1-b)b}, \sqrt{a(1-a)}) \quad (20)$$

where \propto indicates 'proportional to'. From (10) the weights associated to matrix $dg(\rho)\Pi$ are 'proportional to' (q_1s_1, q_2s_2) , and by definition the weights (w_1, w_2) associated by G are proportional to the square values $((q_1s_1)^2, (q_2s_2)^2)$. We look for conditions under which the ranking $(1,0)$ is a rest point, i.e. satisfies (14). The conditions write

$$(1-a)\frac{w_1}{s_1} + b\frac{w_2}{s_2} = 1 \quad \text{and} \quad \frac{a}{(1-a)}\frac{w_1}{s_1} + \frac{(1-b)}{b}\frac{w_2}{s_2} \leq 1$$

where $s_1 = 1 - a$ and $s_2 = b$. The first condition writes $w_1 + w_2 = 1$ so it is surely satisfied. The second condition writes

$$\frac{a(1-a)(q_1)^2}{(q_1(1-a))^2 + (q_2b)^2} + \frac{(1-b)b(q_2)^2}{(q_1(1-a))^2 + (q_2b)^2} \leq 1.$$

Plugging the expression of q given in (20) and rearranging,¹⁴ the inequality can be rewritten as $3a + b \leq 1 + 4ab$. Exchanging the role of a and b , the ranking $(0,1)$ is a rest point if $a + 3b \leq 1 + 4ab$. Clearly, the set of values for which both rankings $(1,0)$ and $(0,1)$ are rest points is an open set: multiplicity is a robust phenomena.

4.3 Peers' methods

This section analyzes the dynamics under a peers' method. Recall that the ratio of an expert's weight to his individual score admit a positive lower bound and a finite upper bound. By continuity (C), this

¹⁴Since (q_1, q_2) is proportional to $(\sqrt{(1-b)b}, \sqrt{a(1-a)})$ the inequality is equivalent to

$$2a(1-a)(1-b)b \leq (1-b)b(1-a)^2 + (a(1-a)b)^2,$$

that is $2a(1-b) \leq (1-b)(1-a) + ab$ and finally $0 \leq 1 - b - 3a + 4ab$.

implies that the weight $\mathbf{q}_j(r)$ is null if and only if r_j is null. Quite strong results are obtained using only this property. These results extend the analysis on the invariant method (Demange (2009)). There I show that the dynamics may have multiple rest points, or even multiple locally stable rest points. It turns out that this multiplicity is bound to occur with peers' methods.

Let us first study rest points. The next proposition provides a characterization of their supports.

Proposition 4 *Consider a peers' method. Given Π , subset I of N is the support of a rest point if and only if*

$$\text{there is } x \text{ in } \mathfrak{R}^I, x \gg 0, \Pi_{I \times I} x = \mathbf{1}_I, \Pi_{N-I \times I} x \leq \mathbf{1}_{N-I}. \quad (21)$$

The fact that (21) is necessary for I to be the support of a rest point is straightforward. Recall the conditions on a rest point (14): $\sum_j \frac{\pi_{i,j}}{\sum_{\ell \in N} \pi_{\ell,j} r_\ell^*} \mathbf{q}_j(r^*) \leq 1$ for each i with an equality if $r_i^* > 0$. With a peers' method, $\mathbf{q}_i(r^*)$ is null whenever r_i^* is null. Hence the vector x in \mathfrak{R}^I is defined by

$$x_j = \frac{\mathbf{q}_j(r^*)}{\sum_i \pi_{i,j} r_i^*}, \text{ each } j \in I \quad (22)$$

is positive and satisfies $\sum_{j \in I} \pi_{i,j} x_j = 1$ for each $i \in I$ and $\sum_{j \in I} \pi_{i,j} x_j \leq 1$ for each $i \notin I$, that is the system of linear inequalities (21).¹⁵

To show that conversely (21) guarantees the existence of a rest point with support in I , we need to find r^* in Δ_I that satisfies (22). We build a correspondence whose fixed points are solutions to (22) (see the details in the section proof). We want to find r that equalizes the ratios $\frac{\mathbf{q}_j(r)}{x_j \sum_i \pi_{i,j} r_i}$. Given r with a j 's ratio $\frac{\mathbf{q}_j(r)}{x_j \sum_i \pi_{i,j} r_i}$ that is not minimum, we assign to j a null score. The behavior of a peers' system ensures that a fixed point is strictly positive, hence that the ratios are equalized on I . Condition (21) then implies that the ratios are equalized to 1.

Proposition 4 can be interpreted as follows. Consider first the whole set N . Conditions (21) state the existence of a positive vector x for which $\Pi x = \mathbf{1}_N$, or $\sum_j \pi_{i,j} x_j = 1$, each i . In words, there is a set of experts' weights that equalize the expected weighted totals across individuals for the true preferences. Thus, N is the support of a rest point when experts' preferences are not in a clear way in favor of some individuals. (When the matrix Π is invertible, the condition simply writes as $\Pi^{-1} \mathbf{1}_N \gg 0$.) Consider now a subset I of N and interpret similarly x as a weight vector on I . Conditions (21) require the existence of a weight vector on I that equalizes the weighted score of each individual in I , and in addition, by the inequality condition, that gives a lower weighted score to individuals not in I .

From Proposition 4, the supports of the rest points are independent of the peers' method. The rest points, that is the precise values assumed by the scores on such a support, are not. We now study the convergence.

Proposition 5 *Consider a peers' method. There are matrices Π for which the dynamics admit several locally stable points.*

It is easy to understand why a problem may admit multiple stable rest points. Let preferences be sufficiently antagonistic in the following way. Take I a subset and choose preferences from I to $N - I$ small enough and the same for $N - I$ to I , that is the $\pi_{i,j}$ small enough for i in I and j not in I or the reverse. First I or a subset of I is the support of a rest point, and similarly for $N - I$. The key point is that the stability of a rest point null on $N - I$ is independent of the values of $\pi_{i,j}$ for j not in I , that is of the preferences of $N - I$.

¹⁵Observe that the same argument can be used for any method and gives $\Pi_{I \times M} x = \mathbf{1}_I$. For $N = M$ and a method that is not a peers' method, the vector x is not required to be null outside the support of r^* . As a result, for I strict subset of N the system $\Pi_{I \times N} x = \mathbf{e}_I$ has more unknown than equations, hence is not much informative.

5 Non linear influence functions

So far we have assumed a specific form, linear, for the influence of rankings. This section considers more general influence functions B , $B(r) = r^\alpha$, as described in Section 4.1. Specifically, given an announced ranking, j 's statements are now proportional to $\pi_{i,j}B(r_i)$. Thus given $r^{(t)}$, the adjusted citation matrix $\Pi^{(t)}$ is

$$\pi_{i,j}^{(t)} = \pi_{i,j}B(r_i^{(t)}) = \pi_{i,j}r_i^{(t)\alpha}.$$

With a slight abuse of notation let r^α denote the vector (r_k^α) . Using as usual $\mathbf{q}_j(\rho) = Q_j^F(dg(\rho)\Pi)$, the dynamics write

$$r_i^{(t+1)} = \sum_j \frac{\pi_{i,j}r_i^{(t)\alpha}}{\sum_\ell \pi_{\ell,j}r_\ell^{(t)\alpha}} \mathbf{q}_j(r^{(t)\alpha}) \text{ each } i. \quad (23)$$

A rest point r^* satisfies

$$\sum_j \frac{\pi_{i,j}r_i^{*1-\alpha}}{\sum_\ell \pi_{\ell,j}r_\ell^{*\alpha}} \mathbf{q}_j(r^{*\alpha}) \leq 1 \text{ for each } i \text{ with an equality if } r_i^* > 0. \quad (24)$$

Intuitively, as α increases, the influence becomes more discriminating. The analysis differs sensibly depending on the value of α with respect to 1, the boundary case $\alpha = 1$ being the case studied in the previous sections.

Diminishing marginal impact: $\alpha < 1$ For $\alpha < 1$, (23) implies that the growth rate of i 's score is strictly larger than 1 for r_i small enough. As a result, the possible rest points are necessarily strictly positive and satisfy

$$\sum_j \frac{\pi_{i,j}r_i^{*1-\alpha}}{\sum_\ell \pi_{\ell,j}r_\ell^{*\alpha}} \mathbf{q}_j(r^{*\alpha}) = 1 \text{ for each } i. \quad (25)$$

More generally, even without convergence, no individual's score becomes arbitrarily small because the growth rate of an individual score is superior to 1 for a low enough score. We can say more for generalized handicap-based methods. Lemma 1 extends as follows. Let a method satisfy the conditions (16), which guarantee the dynamics to converge to a unique rest point for a linear influence function. Then the dynamics converges for influence functions $B(x) = x^\alpha$ any $\alpha < 1$. The proof follows the same lines as that of Lemma 1 by considering the function L defined by $L(\rho) = \sum_j \Psi_j(s_j)$ where Ψ_j is a primitive of ψ_j but with the sum s_j given by $s_j = \sum_i \pi_{i,j}\rho_i^\alpha$. For Ψ_j concave (ψ_j decreasing) and $\alpha > 1$, L is strictly concave. The rest point coincides the unique maximizer of L over the rankings, and in addition L is a Lyapounov function (see the details at the end of the proof of Lemma 1). This result implies that the process (23) converges for any generalized handicap-based method with $g(x) = x^{1-\gamma}$ and influence $B(r) = r^\alpha$ with $\alpha \leq 1$, $\gamma \leq 0$ with at least one strict inequality.

Increasing marginal impact: $\alpha > 1$ For $\alpha > 1$, the limit rankings may be expected to be more discriminative than in the linear case as there are increasing marginal rewards to have high scores. We show here that this is indeed true for the invariant method : any ranking concentrated on a single point is a stable point.

The proof is as follows. Starting with a score for individual i low enough, we show that i 's score will decrease exponentially to 0. Thus the sequence $r^{(t)}$ converges to $\mathbf{1}_{\{\ell\}}$ for initial rankings in a

neighborhood of $\mathbf{1}_{\{\ell\}}$, that is rankings for which all scores except that of ℓ are small enough. The proof relies on the following inequality :

$$\forall t > 0, r_i^{(t+1)} \leq C r_i^{(t)\alpha}. \quad (26)$$

Assuming (26) for the moment, iteration from 0 up to t implies

$$r_i^{(t)} \leq \alpha^{1+\alpha+\dots+\alpha^{t-1}} r_i^{(0)\alpha^t}, \text{ or } r_i^{(t)} \leq \alpha^{\frac{1}{1-\alpha}} \left[\alpha^{\frac{1}{\alpha-1}} r_i^{(0)} \right]^{\alpha^t}.$$

Hence if $\alpha^{\frac{1}{\alpha-1}} r_i^{(0)} < 1$ then $r_i^{(t)}$ converges to 0 as t tends to ∞ . As a consequence, if all $r_i^{(0)}$ are small enough for $i \neq \ell$, then the rankings will converge to $\mathbf{1}_{\{\ell\}}$.

It remains to show (26). For the invariant method, the dynamics is given by

$$\frac{r_i^{(t+1)}}{r_i^{(t)\alpha}} = \sum_j \pi_{i,j} \frac{r_j^{(t+1)}}{\sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)\alpha}}.$$

We have to show that the right hand side is bounded above. Let us denote by π_{max} and π_{min} respectively the maximum and minimum of the elements in matrix Π .

We first provide a lower bound to $\sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)\alpha}$. Observe that

$$\pi_{min} \sum_{\ell \in N} r_\ell^{(t)\alpha} \leq \sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)\alpha}. \quad (27)$$

To bound $\sum_{\ell \in N} r_\ell^{(t)\alpha}$ we apply Holder inequality¹⁶ to the vectors $r^{(t)}$ and $\mathbf{1}_N$ with the parameters $p = \alpha$ and $q = \alpha/(\alpha - 1)$ (q is positive since $\alpha > 1$). This yields

$$\sum_{\ell \in N} r_\ell^{(t)} \leq \left(\sum_{\ell \in N} r_\ell^{(t)\alpha} \right)^{\frac{1}{\alpha}} n^{\frac{\alpha-1}{\alpha}}$$

Since $\sum_{\ell \in N} r_\ell^{(t)} = 1$, this writes $1 \leq \left(\sum_{\ell \in N} r_\ell^{(t)\alpha} \right)^{\frac{1}{\alpha}} n^{\frac{\alpha-1}{\alpha}}$ and we obtain $n^{(1-\alpha)} \leq \sum_{\ell \in N} r_\ell^{(t)\alpha}$. Using (27) gives

$$\pi_{min} n^{(1-\alpha)} \leq \pi_{min} \sum_{\ell \in N} r_\ell^{(t)\alpha} \leq \sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)\alpha}.$$

This inequality together with $\pi_{i,j} \leq \pi_{max}$ and (27) gives

$$\frac{r_i^{(t+1)}}{r_i^{(t)\alpha}} \leq \sum_j \pi_{max} \frac{r_j^{(t+1)}}{\pi_{min} n^{(1-\alpha)}} = \frac{\pi_{max}}{\pi_{min} n^{(1-\alpha)}}$$

Thus inequality (26) holds for C the value on the right hand side. ■

6 Concluding remarks

TO BE COMPLETED

¹⁶Holder inequality is $\sum_{\ell} x_{\ell} y_{\ell} \leq (\sum_{\ell} x_{\ell}^p)^{1/p} (\sum_{\ell} y_{\ell}^q)^{1/q}$, for p and q positive related by $1/p + 1/q = 1$.

7 Proofs

Proof of Lemma 1. Let us assume (16) for some functions ψ_j . Let Ψ_j be a primitive of ψ_j and L the function $L(\rho) = \sum_j \Psi_j(\sum_i \pi_{i,j} \rho_i)$, or for short $\sum_j \Psi_j(s_j)$. Consider the program (\mathcal{P})

$$(\mathcal{P}) : \max_{\rho} L(\rho) \text{ over } \rho \geq 0, \sum_i \rho_i \leq 1$$

Let μ denote the multiplier associated to the constraint $\sum_i \rho_i \leq 1$. The first order conditions satisfied by a solution r of (\mathcal{P}) are

$$\frac{\partial L}{\partial \rho_i} = \sum_j \psi_j(s_j) \pi_{i,j} \leq \mu \text{ each } i \text{ with } = \text{ for } r_i > 0. \quad (28)$$

Multiplying by r_i the inequality for each i and summing over i yields $\sum_i \sum_j \psi_j(s_j) \pi_{i,j} r_i = \mu(\sum_i r_i)$. Exchanging the sums yields $\sum_i \sum_j \psi_j(s_j) \pi_{i,j} r_i = \sum_j \psi_j(s_j) \sum_i \pi_{i,j} r_i = \sum_j \psi_j(s_j) s_j$. Hence using $\sum_i r_i = 1$ gives the value of μ : $\mu = \sum_j \psi_j(s_j) s_j$ and the first order conditions (28) are equivalent to

$$\sum_j \pi_{i,j} \psi_j(s_j) \leq \sum_k s_k \psi_k(s_k) \text{ each } i \text{ with } = \text{ for } r_i > 0. \quad (29)$$

Now given that $(\mathbf{q}_j(\rho))$ satisfy (16), i.e. are proportional to $(\psi_j(s_j))$ we show that the conditions (14) on a rest point

$$\sum_j \frac{\pi_{i,j}}{\sum_{\ell \in N} \pi_{\ell,j} r_{\ell}^*} \mathbf{q}_j(r^*) \leq 1 \text{ for each } i \text{ with an equality if } r_i^* > 0$$

coincide with the first order conditions (29), of program (\mathcal{P}) . This is immediate since $\mathbf{q}_j(\rho) = \frac{s_j \psi_j(s_j)}{\sum_k s_k \psi_k(s_k)}$ gives

$$\sum_j \frac{\pi_{i,j}}{\sum_{\ell \in N} \pi_{\ell,j} r_{\ell}} \mathbf{q}_j(\rho) = \sum_j \pi_{i,j} \frac{\psi_j(s_j)}{\sum_k s_k \psi_k(s_k)}. \quad (30)$$

Let us assume in addition the ψ_j to be decreasing. Then the functions Ψ_j are strictly concave and there is a unique solution to program (\mathcal{P}) characterized by (29). This implies that there is a unique rest point.

To prove convergence, we show that L is a Lyapounov function: the sequence $L(r^{(t)})$ strictly increases with t as long as $r^{(t)}$ differs from $r^{(t-1)}$. We have $\frac{\partial L}{\partial r_i}(r) = \sum_j \psi_j(s_j) \pi_{i,j}$. Multiplying by r_i , summing over i , and exchanging the sums yields $\sum_{\ell \in N} r_{\ell} \frac{\partial L}{\partial r_{\ell}}(r) = \sum_{j \in M} s_j \psi_j(s_j)$. Hence from (30), the dynamics (12) followed by $r^{(t+1)}$: $r_i^{(t+1)} = \sum_j \frac{\pi_{i,j} r_i^{(t)}}{\sum_{\ell \in N} \pi_{\ell,j} r_{\ell}^{(t)}} \mathbf{q}_j(r^{(t)})$ each i , can be written as

$$r_i^{(t)} = \frac{\frac{\partial L}{\partial r_i}(r^{(t-1)})}{S^{(t-1)}} r_i^{(t-1)} \text{ each } i \text{ where } S^{(t-1)} = \sum_{\ell \in N} r_{\ell}^{(t-1)} \frac{\partial L}{\partial r_{\ell}}(r^{(t-1)}). \quad (31)$$

These equations mean that the growth rates of the components of r are proportional to the gradient of L . Consider the difference $L(r^{(t)}) - L(r^{(t-1)})$. The concavity of L implies

$$L(r^{(t)}) - L(r^{(t-1)}) \geq \sum_i \frac{\partial L}{\partial r_i}(r^{(t-1)}) [r_i^{(t)} - r_i^{(t-1)}] \text{ with } > \text{ if } r^{(t)} \neq r^{(t-1)}. \quad (32)$$

We first show that

$$\frac{\partial L}{\partial r_i}(r^{(t-1)}) [r_i^{(t)} - r_i^{(t-1)}] \geq r_i^{(t-1)} \left[\frac{\partial L}{\partial r_i}(r^{(t-1)}) - S^{(t-1)} \right] \quad (33)$$

Rewriting (31) as

$$r_i^{(t)} - r_i^{(t-1)} = \frac{r_i^{(t-1)}}{S^{t-1}} \left[\frac{\partial L}{\partial r_i} (r^{(t-1)} - S^{(t-1)}) \right]$$

The term on the right inside the brackets is positive (resp. negative) if the partial derivative $\frac{\partial L}{\partial r_i}$ is larger (resp. smaller) than $S^{(t-1)}$: inequality (33) follows by multiplying by $\frac{\partial L}{\partial r_i}$.

Summing the inequalities (33) over i yields

$$\sum_i \frac{\partial L}{\partial r_i} (r^{(t-1)}) [r_i^{(t)} - r_i^{(t-1)}] \geq \sum_i r_i^{(t-1)} \frac{\partial L}{\partial r_i} (r^{(t-1)}) - S^{t-1} = 0$$

Hence from (32), the value of L strictly increases as long as $r^{(t)}$ differs from $r^{(t-1)}$, that is as long as the rest point is not reached: L is a Lyapounov function and the sequence converges to the rest point.

Extension to influence functions $B(x) = x^\alpha$, $\alpha < 1$. Let L be defined by $L(\rho) = \sum_j \Psi_j(\sum_i \pi_{i,j} \rho_i^\alpha)$ where Ψ_j is a primitive of ψ_j . Consider as above the program \mathcal{P} of maximization of L over Δ_N . The program is strictly concave and the unique rest point is interior. By similar computations, one checks that the first order conditions are

$$\sum_j \pi_{i,j} r_i^{\alpha-1} \frac{\psi_j(s_j)}{\sum_k s_k \psi_k(s_k)} = 1 \text{ each } i \text{ with } s_j = \sum_i \pi_{i,j} r_i^\alpha. \quad (34)$$

Applying $\mathbf{q}_j(\rho) = \frac{s_j \psi_j(s_j)}{\sum_k s_k \psi_k(s_k)}$ to $\rho_i = r_i^\alpha$ each i , these conditions coincide with the conditions (25) on a rest point. The proof that L is a Lyapounov function for the process with $\alpha < 1$ is identical to that for $\alpha = 1$. ■

Proof of Proposition 4. We proved in the text that conditions (21) are necessary for the existence of a rest point with support I . Let us show the converse. Recall the notation $\mathbf{q}(r) = Q^F(dg(r)\Pi)$. Under Assumption (C), function \mathbf{q} is continuous over the whole set Δ_N and furthermore, r_i null implies $\mathbf{q}_i(r)$ null by the peers' property. Given x that satisfies (21) we need to prove the existence of r^* positive in Δ_I that satisfies (22):

$$x_j = \frac{\mathbf{q}_j(r^*)}{\sum_i \pi_{i,j} r_i^*}, \text{ each } j \in I.$$

Consider the correspondence from Δ_I to itself defined by

$$\Phi(r) = \{\rho \in \Delta_I \text{ s.t. } \rho_k = 0 \text{ for each } k \text{ that does not minimize } \frac{\mathbf{q}_j(r)}{x_j(\sum_i \pi_{i,j} r_i)} \text{ over } j \in I\}. \quad (35)$$

It is easy to check that the continuity of the function \mathbf{q} implies that the correspondence Φ is upper hemi-continuous. Since Φ is convex-valued, it has a fixed point by Kakutani theorem, $r^* \in \Phi(r^*)$. We show that r^* is positive and satisfies (22).

By contradiction, assume $r_i^* = 0$ for some i . The peers' property implies that $\mathbf{q}_i(r^*)$ is null. Since \mathbf{q} takes values in Δ_I , there is k in I with $\mathbf{q}_k(r^*) > 0$. Thus the minimum of the $\frac{\mathbf{q}_j(r^*)}{x_j(\sum_i \pi_{i,j} r_i^*)}$ over j in I is zero (achieved at $j = i$) and is not achieved at $j = k$. Hence any ρ in $\Phi(r^*)$ has $\rho_k = 0$. However, applying the peers' property again, r_k^* must be positive because $\mathbf{q}_k(r^*) > 0$. Hence $r^* \in \Phi(r^*)$ cannot hold, which gives the contradiction.

Thus a fixed point r^* of Φ is a vector in Δ_I with all its components positive. Since $r^* \in \Phi(r^*)$, $\Phi(r^*)$ contains a strictly positive vector. By the definition (35), this implies that the ratios

$\mathbf{q}_j(r^*)/(x_j(\sum_i \pi_{i,j} r_i^*))$ are equalized across j : there is some λ such that $\mathbf{q}_j(r^*) = \lambda x_j(\sum_i \pi_{i,j} r_i^*)$ for each j . Summing these equations over j yields that λ is equal to 1: the sum on the left hand side is $\sum_j \mathbf{q}_j(r^*)$ which is equal to 1, and the sum on the right hand side is $\lambda \sum_j x_j(\sum_i \pi_{i,j} r_i^*)$ which is equal to $\lambda \sum_i r_i^*(\sum_j \pi_{i,j} x_j) = \lambda$ since x satisfies (21). Hence $\lambda = 1$, which proves (22). ■

Proof of Proposition 5. Let a peers' method.

We first show that for matrices with low enough values in the sub-matrix $\Pi_{N-I,I}$, the scores on $N - I$ converge to zero if their initial values are low enough, whatever dynamics on I , that is whether or not the scores on I converge. The growth rate of i 's score satisfies

$$\frac{r_i^{(t+1)}}{r_i^{(t)}} = \sum_j \pi_{i,j} \frac{\mathbf{q}_j(r^{(t)})}{\sum_\ell \pi_{\ell,j} r_\ell^{(t)}}. \quad (36)$$

Let a matrix Π be such that for some $k < 1$ we have

$$\sum_j \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} \leq k \text{ for each } i \notin I \text{ and for each } r \text{ with null components on } N - I. \quad (37)$$

Such matrices exist, as checked below. We will show that, starting from rankings with small enough values outside I , the growth rate of the scores on $N - I$ is strictly smaller than 1, hence the scores converge to zero.

By continuity of the \mathbf{q} , similar inequalities to (37) hold for r with small enough components on $N - I$. Formally, let $\hat{\Delta}_I$ be the subset in Δ_N formed with the rankings with null components on $N - I$, and given $\epsilon > 0$ let $\mathcal{V}(\epsilon)$ be the neighborhood of $\hat{\Delta}_I$ where the components on $N - I$ are smaller than ϵ . For k' with $k < k' < 1$, there is $\epsilon > 0$ such that

$$\sum_j \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} \leq k', i \notin I, \text{ for each } r \in \mathcal{V}(\epsilon)$$

Assume the ranking belongs to $\mathcal{V}(\epsilon)$ at some date t . By (36), the growth rates of all components on $N - I$ are strictly smaller than 1, hence the ranking at date $t + 1$ also belongs to $\mathcal{V}(\epsilon)$. By induction, the sequence stays in $\mathcal{V}(\epsilon)$ at any further date and furthermore the components on $N - I$ converge to zero because their growth rates is smaller than k' .

It remains to show that there are indeed matrices for which (37) hold. The simplest example will do but it should be clear that many matrices could be used. Let a matrix in which $\pi_{\ell,j} = \alpha$ for i in I and j in I , and $\pi_{\ell,j} = \beta$ for i not in I , j in I :

$$\begin{pmatrix} \alpha & \alpha & \times, \times \\ \dots & \dots & \dots \times, \times \\ \alpha & \dots & \alpha \times, \times \\ \beta & \dots & \beta \times, \times \\ \beta & \dots & \beta \times, \times \end{pmatrix}$$

Take $\alpha|I| + \beta|N - I| = 1$ so that the sums of the columns indexed by I are equal to 1.

For r with null components on $N - I$, we have $\mathbf{q}_j(r) = 0$ for any j not in I by definition of a peers' method, and $\sum_\ell \pi_{\ell,j} r_\ell = \alpha$ for any j in I . Hence we obtain successively

$$\sum_j \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} = \sum_{j \in I} \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} = \sum_{j \in I} \frac{\mathbf{q}_j(r)}{\alpha}$$

Now using $\pi_{i,j} = \beta$ for i not in I and j in I , we obtain

$$\sum_j \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} = \frac{\beta}{\alpha} \quad i \notin I$$

Since $\alpha|I| + \beta|N - I| = 1$, $\beta/\alpha < 1$ for β small enough hence (37) is met.

To conclude the proof, we need to find matrices with several locally stable points.

First observe that conditions (37) on a matrix Π only bear on the values of the matrix on $N - I \times I$, i.e., the $\pi_{i,j}$ for i not in I and j in I (because $\mathbf{q}_j(r)$ is null for j not in the support of r). Hence the values of the matrix on $I \times I$ can be chosen so that there is convergence to a point with support I : this gives a locally stable point with support I . Furthermore the conditions on Π bear only on the columns indexed by I . Thus the values of the matrix on $N - I$ can also be chosen so that the same result hold on $N - I$: this ensures the existence of another stable point with support included in $N - I$ (the construction can clearly be extended so as to get more than 2 stable points). ■

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