Group Rewards and Voting in a Poisson Voting Game *

Alastair Smith[†] Tom LaGatta [‡]
Bruce Bueno de Mesquita [§]

March 2013

Abstract

Using a Poisson games framework of Myerson (1998, 2000), we model elections in which parties offer contingent prizes to those identifiable groups of voters that offer the highest level of political support. Equilibrium behavior is driven by voters competing to win preferential treatment for their group rather than by policy concerns. In the spirit of Duverger's Law, we show that two prize-seeking groups actively support each party in each prize competition, with not more than four groups in two-party contests. The results address variance in turnout in elections, political rewards, the persistence of dominant parties and the dynamics of group identity.

1 Introduction

We present a rational choice model of voting to examine the consequences for turnout and other aspects of electoral politics when parties target selective re-

*Earlier versions of this paper have been presented at the Midwest Political Science Association meeting (April 2012) and the W. Allen Wallis Institute of Political Economy Conference (September 2012). We gratefully acknowledge the useful comments of the audiences at these meetings and others. We are particularly grateful to several anonymous referees, Srinivasa Varadhan, Avidit Acharya, Ethan Bueno de Mesquita, David Myatt, John Morgan and Scott Tyson.

[†]Alastair.Smith@nyu.edu

[‡]lagatta@cims.nyu.edu

[§]bruce.buenodemesquita@nyu.edu

wards to those groups of voters that offer them the highest level of electoral support. The model explains voting behavior in competitive electoral systems, such as those in established democracies, and in patronage systems in which a dominant party persists in the presence of free and fair elections even if the dominant party is widely regarded by citizens as inferior to other parties. The model resolves turnout problems seen in standard accounts of rational voting. In particular, we explain variance in voter turnout, including high turnout, in large electorates both in competitive and non-competitive elections. The model also resolves credibility issues in vote buying accounts within patronage democracies and explains the persistence of dominant parties. It identifies the incentives parties have to break large electorates, such as Congressional districts, into many sub-competitions, such as individual precincts and wards, as a means to influence turnout so as to appear to have a large electoral mandate.

As in other voting models, in our model voters individually compare the benefits and costs of voting for each of two parties or from abstaining (Downs 1957; Riker and Ordeshook 1968). A classic criticism of such an approach, and one of Green and Shapiro's (1996) main arguments against rational choice modeling, is that in a large electorate voters are extremely unlikely to be pivotal. Hence, if voting is costly and their vote is extremely unlikely to matter, then critics question why voters turn out at all. In our approach voters are unlikely to be pivotal in altering who wins and yet still may turn out to vote. Rather than being concerned only over which party wins, at least some voters in our model are motivated by the influence they have over the distribution of contingent prizes such as pork or other benefits targeted at specific, identifiable blocs of voters. As will be shown, prize pivotality creates competition for prizes that helps coordinate the votes of group members. Each individual in a group has incentives to vote similarly to her compatriots. Under the right circumstances, those incentives can outweigh partisan preference and the cost of voting. The model helps as well to address the potential for the polarization of groups while providing a theory of group formation and group dynamics. That is to say, the model predicts when social cleavages are likely to be politically active or to be activated by entrepreneurial politicians. When electoral competition is dominated by a single party, group identities are shown to be fixed and rigid, with little reason for individuals to migrate between groups. However, competitive elections make group identities dynamic and create incentives for individuals to move between groups. Competitive elections also create incentives for political entrepreneurs to redefine the boundaries between groups, creating the incentive for multiple competitions over prizes even in a single, winner-take-all electoral setting.

The model draws attention to the symmetry of group size, the competitiveness of elections in terms of which party wins and to how the nature of prizes affect equilibrium levels of turnout, the stability of voter groups, and the incentives for migration between groups. In winner-take-all political environments, competition for prizes results in only two groups actively supporting each political party, suggesting Duvergerian-style competition between groups seeking the prizes that parties dole out. Although the model reinforces two group competition at the party level, in other respects the results qualify and add nuance to Duverger's Law (Duverger 1963; Riker 1982). It shows, for instance, that with two party competition there can be anywhere between 2 and 4 active and stable groups in each prize competition. When prizes are rival in nature, politicians are incentivized to break the political competition into a series of lower level prize competitions, such as rewards at the precinct level in a US Congressional race.

The basic setup integrates Smith and Bueno de Mesquita's (2012) concept of Contingent Prize Allocation Rules (CPAR) and Myerson's (1998, 2000) Poisson random population of voters model. In the CPAR examined here, parties provide selective rewards (a.k.a. prizes) to identifiable groups of voters based on their level of electoral support. One simple illustration of such a grouping is geographical districts. In established democracies parties rarely observe the vote choice of individuals, but they readily observe the level of electoral support at the ward or precinct level. Under the CPAR applied here, parties reward the most supportive group. For instance, in the geographical context they might allocate new infrastructure projects such as a new school or hospital to the most supportive precinct within a larger electoral district. Alternatively they could reward a particular ward or precinct through the provision of superior services, such as better trash pickup or more reliable street plowing. Parties might also disproportionately hire public employees from the most supportive group. This was a standard practice within the party machines which dominated many large US cities (Allen 1993). Richard J. Daley, the long term mayor of Chicago, was notorious in this respect (Rakove 1975). This is perhaps unsurprising since the internal rules of the Democratic Party of Cook County (which contains Chicago) specifies that on committees, ward representatives are given voting rights in proportion to the level of democratic votes their ward delivered in previous elections (Gosnell 1937). US national parties also structure rules to reward their loyalists. For instance, both parties skew Presidential nomination procedures in favor of states that gave them high levels of support in previous elections (see for instance, Democratic Party Headquarters 2007).

Although the model is applicable to any set of groups whose electoral support is observable, for ease of exposition we often focus on geographical groupings. Others, in a different context, have shown that non-geographic group voting can also be observable (Chandra 2004). Groups might, therefore, be based on religion, ethnicity or profession instead of geographical voting districts. What matters is that parties can, in the aggregate, observe who supports them and can selectively reward groups.

The paper proceeds as follows. Section 2 places our investigation in the context of the relevant literature. The basic structure of the model is introduced in section 3. Section 4 then develops the logic of outcome and prize pivotality, methods for approximating pivot values and examines limiting cases. Equilibrium voting behavior for the symmetric case is discussed in section 5. Section 6 explores equilibria in the presence of asymmetry. That section focuses attention on how competitive elections induce a fluidity into group membership that is lacking when one party is almost certain to win the election. In addition, to identifying stable arrangements of groups, section 6 explores how parties benefit from creating numerous lower level prize competitions between pairs of groups when prizes are rival. Finally, section 7 concludes with a discussion of the main theoretical findings.

2 Literature Review

Pivotality lies at the heart of rational choice models of voting behavior (Downs, 1957; Aldrich 1993; Riker and Ordeshook 1968; Ferejohn and Fiorina 1974). Voters not only assess their expected rewards under each party, they also factor in the likelihood that their vote matters. Formally, a vote only matters if it breaks a tie or turns defeat into a tie. Under all other circumstances, an additional vote is immaterial in determining which party wins. In a large electorate, even if the outcome is expected to be close, the probability that a voter's vote matters is extremely small (Myerson 1998, 2000). Although Myatt (2011; see also Krishna and Morgan 2012), building on a result by Good and Mayer (1975), suggests that in the presence of uncertainty about the relative popularity of the parties, pivot probabilities do not go to zero as quickly as in perfect information models, the likelihood of influencing an electoral outcome nevertheless is small. Others argue that electoral influence is best assessed using statistical predictions based on forecasts of vote shares (Gelman, King and Boscardin 1998). For instance, Gelman, Silver and Edlin (2010) suggest that in the 2008 US Presidential election the average voter had about a one in 60 million chance of influencing the outcome although this figure varied greatly by state. However pivotality is calculated, voters are expected to have only a minuscule influence on outcomes. Voters might therefore be expected to abstain, producing the puzzle of why so many people choose to vote.

That electoral turnout is much higher than anticipated by pivotal-voting models is a central critique of rational choice (Green and Shapiro 1996; see Feddersen 2004 and Geys 2006 for surveys of this literature). The critique seems especially pertinent in electoral systems with a dominant party since such a party, with near certainty, continually wins. In the case of a single-party system, for instance, it seems surprising that anyone votes unless they are compelled to. Yet we will show that politicians can use contingent prizes to determine the size of their mandate in non-competitive, dominant party settings and that they can do so in a manner consistent with the decision calculus of citizens in competitive elections. We propose a solution to the puzzle of variation in rational voter turnout in competitive and in non-competitive elections.

Many branches of the voting literature consider factors beyond pure policy comparisons of parties. Castanheira (2003), Razin (2003) and Meirowitz and Shotts (2009), for instance, suggest that the signaling value of voting is important because the margin of victory influences policy implementation. Others point to voters being motivated by personal or local benefits, such as patronage and pork (Ferejohn 1974; Fenno 1978; Schwartz 1987; Stokes 2005, 2007). Most relevant to the discussion here is Schwartz's (1987) expected utility model in which he argues that voters care about how their precinct votes in terms of potentially courting favor from the victorious party. He shifts the focus of turnout from the global policy difference between parties to the selective provision of local public goods or club goods to sub-electorates. Although he maintains a focus on pivotality with regards to which party wins, his decision-theoretic approach concentrates on a smaller, local level of analysis where voters are likely to be more influential than at the macro level. Our strategic analysis focuses on pivotality with regard to the distribution of prizes.

The vote buying literature assesses where parties can most effectively buy electoral support (Ansolabehere and Synder. 2006; Myerson 1993, Dekel et al 2008, Kovenock and Roberson 2009, Cox and McCubbins 1986, Lindbeck and Weibull 1987, and Dixit and Londregan 1995, 1996). One common question is whether parties increase their vote share more by offering turnout-inducing rewards to party loyalists or to marginal voters, a swing in whose vote might be critical. Such approaches treat the parties as strategic competitors while the voters respond to rewards in a non-strategic manner. Consistent with the critique of pivotal voting, although these vote buying tactics increase the attractiveness of one party relative to another, they do not mitigate the problem that individual voters have little influence over electoral outcomes. Given the low probability of influencing the outcome, making a party more attractive only minimally increases the incentive to vote for it. Further, such approaches fail to explain voter turnout in non-competitive elections in which one party is virtually certain to win.

Pork or patronage rewards are often proffered for voter support (Ferejohn 1974; Fenno 1978; Kitschelt and Wilkinson 2007; Stokes 2005). By offering upfront bribes and the prospects of rewards, such as jobs or better services after the election, patronage based parties directly influence voters. Even if such targeted rewards might be economically inefficient relative to public goods (Lizzeri and Persico 2001), vote buying is politically valuable because it obfuscates the pivotality issue. Since the quid pro quo of benefits for votes is carried out at the individual level, pivotality is not relevant for patronage models. Parties buy individual votes rather than make themselves electorally more attractive. As such, voters do not discount the value of the party by the likelihood that their vote is influential. Yet a number of credibility issues surround patronage vote buying (Stokes 2005, 2007).

At least in established democracies, all of which have a secret ballot, once a voter enters the voting booth parties can not observe whether the voter delivers the promised vote (although see Gerber et al 2009). Neither can the voter be certain that a party will deliver its promised rewards after the election. Norms and reciprocity are often offered as solutions to these credibility issues (see Kitschelt and Wilson 2007 for a review), Stokes (2005) developed a repeated play model that explicitly addresses these concerns. However, other problematic issues remain. For example, relatively few voters receive goods from the party. Stokes (2005 p. 315) illustrates the problem with the example of an Argentinean party worker given ten tiny bags of food with which to buy the 40 voters in her neighborhood. Further, survey evidence by Brusco, Nazareno and Stokes (2004) suggests that the receipt of bribes does not guarantee that voters support the party. Similarly, Guterbock (1980) found that Chicago residents who received party service were no more likely to vote Democratic than those receiving no favors. The contingent prize allocation perspective resolves these issues. It provides an equilibrium mechanism for credibly rewarding voters. Even though individual votes cannot be observed, voters turnout to support a party in the hope of winning prizes for their group.

3 Basic Setup

Consistent with standard models, we examine a voter's influence on which party wins the election. We refer to this as the outcome pivot and distinguish it from what we refer to as prize pivotality. Smith and Bueno de Mesquita (2012) illustrate how voters retain their prize pivotality even in large electorates with a stylized example of three villages each with n voters. The victorious party offers to build a hospital, or other project, in the village that gives it the most votes. There is an equilibrium in which all voters support one party and the pivotality of the vote choice is 1/3 even if the electorate is large. In this illustration, no single voter is influential in affecting which party wins. Yet, by voting for the dominant party each voter gives their village a 1/3 chance of receiving the prize. If they abstain or vote for another party, then their village has one fewer votes than the other villages and so their village has no chance of receiving the prize. Provided that the cost of voting is less than 1/3 of the value of the prize, all voters strictly want to support the dominant party.

Smith and Bueno de Mesquita's (2012) example is highly stylized and their model is of limited generality. It considers only three groups and everyone is assumed to turn out to vote. Here we develop a more general model with broader implications. Rather than assume perfectly informed voters and politicians, we model uncertainty about bloc sizes and voter preferences. We treat the population size (and therefore the population of each group) as a Poisson random variable. Given this assumption, no one is quite certain how many voters there are in each group. Additionally, each voter has a personal -and private- evaluation of one party relative to the other, which we also model as a random variable. We impose minimal assumptions on the probability distribution describing voters' preferences.

Myerson (1998, 2000) shows that treating population size as a Poisson random variable creates a flexible framework within which it is straightforward to analyze pivotal voting decisions. For instance, he shows how the Poisson approach avoids the messy combinatoric calculations involved in large fixed population voting models (Palfrey and Rosenthal 1983, 1985; Ledyard 1984; Krishna and Morgan 2012). Consistent with these models he finds that in large electorates, voters are only outcome pivotal in very close elections and even then their influence is small; so even a small cost of voting discourages significant turnout. The key to the analyses presented here is that while outcome pivotality quickly approaches zero as the electorate grows, prize pivotality goes to zero much more slowly. In particular, outcome pivot is proportional to $\frac{1}{\sqrt{n_T}}e^{-n_T(\sqrt{p}-\sqrt{q})^2}$, where n_T is expected total population size and p and q are the probabilities that voters support parties $\mathcal{A}$ and $\mathcal{B}$ respectively. In contrast, with n_T made up of numerous groups of expected size n, prize pivotality is proportional to $\frac{1}{\sqrt{n}}$. Except in the case of perfect electoral balance ($p = q$), outcome pivotality declines at an exponential rate as the expected total population size increases. Therefore, except in extremely close elections or very small electorates, prize pivots dominate outcome pivot. The larger the electorate the greater the relative importance of the prize pivot.

We formally develop the concept of prize pivotality and derive approximations for the extent to which voters influence electoral outcomes and the distribution of prizes in large electorates of uncertain size. As a contrast to the standard policy-only model, we characterize equilibria in the context of prize competition. We illustrate how prizes and groups effect electoral outcomes and turnout in competitive elections and in non-competitive elections where one party dominates. Contingent prizes increase turnout in both settings. We solve the model first for situations involving symmetrically sized groups and then for asymmetrically sized groups. In both settings stable equilibria involve two groups actively supporting each party. The final section examines group dynamics and how politicians organize groups into political competitions.

We assume an election takes place between two parties, $\mathcal{A}$ and $\mathcal{B}$, for a single office. All voters have the option of voting for party $\mathcal{A}$, voting for party $\mathcal{B}$ or abstaining. Each voter pays a cost c to vote; abstention is free. The outcome of the election is determined by a plurality of the votes cast, with ties decided by a coin flip. To begin, we assume there is a large number of voters, divided into K roughly equally sized groups. The assumption of equally sized groups is especially pertinent for democratic elections subject to the constraint of one person, one vote. The groups are indexed $1, 2, .., K$. Although these groups might be based on any underlying societal cleavage, they can simply be thought of as geographically based wards within an electoral district.

Group k has size N_k which we treat as an unknown Poisson random variable with mean n_k. Therefore, $\Pr(N_k = x) = f_{n_k}(x) = \frac{n_k^x}{x!}e^{-n_k}$ and $\Pr(N_K \leq x) = F_{n_k}(x) = \frac{\Gamma(x+1,n_k)}{x!} = \sum_{z=0}^{x} f_{n_k}(z)$ where Γ is the incomplete gamma function. The total number of voters is $N_T = \sum_{k=1}^{K} N_k$, which, by the aggregation property of the Poisson distribution (Johnson and Kotz 1993), is also a Poisson random variable with mean $n_T = \sum_{k=1}^{K} n_k$. Note that to avoid confusion, we denote the expected size of the total population with a subscript T.

Let p_k represent the average probability that members of group k vote for $\mathcal{A}$, and let q_k represent the probability that members of k vote for party $\mathcal{B}$. By the decomposition property of the Poisson distribution (Johnson and Kotz 1993), A_k, the number of votes for party $\mathcal{A}$ in group k, is a Poisson random variable with mean $\lambda_k = p_k n_k$. We use the notation $(p, q) = ((p_1, q_1), ..., (p_K, q_K))$ as the profile of vote probabilities, $(\lambda, \gamma) = ((\lambda_1, \gamma_1), ..., (\lambda_K, \gamma_K))$ as the profile of expected votes for parties $\mathcal{A}$ and $\mathcal{B}$ and $(A, B) = ((A_1, B_1), ..., (A_K, B_K))$ as the profile of actual votes. Party $\mathcal{A}$ wins the election if it receives more votes than party $\mathcal{B}$ ($\sum_{k=1}^{K} A_k > \sum_{k=1}^{K} B_k$); ties are resolved by a coin flip. The goal of this paper is to characterize profiles of vote probabilities that can be supported in Nash equilibria, show how these equilibria vary within a winner-take all environment and examine the incentives this creates for group formation/maintenance.[1]

Voters care both about policy benefits and any potential prizes the parties distribute. With regard to policy benefits, voter i receives a policy reward of

[1]Elsewhere we explore the properties of other CPAR, (Smith 2011).

ε_i if party $\mathcal{A}$ wins the election and a policy payoff of 0 if $\mathcal{B}$ wins. The random variable ε_i represents individual i's private evaluation of party $\mathcal{A}$ relative to party $\mathcal{B}$. We assume the individual evaluations are independently identically distributed with distribution $\Pr(\varepsilon_i < r) = G(r)$.

In addition to policy benefits, individuals care about the benefits and rewards that parties might provide to specific groups of voters. The nature of these benefits can vary widely and often depend on the nature of groups. For instance, if groups are defined across occupational categories, then a party can reward one group relative to another with favorable regulatory or trade policies. If groups are based on religion, then a party can privilege a particular group with legislation that favors its particular faith or by grants to organizations associated with that faith. Similarly, a party could adopted preferential hiring practices to reward a particular ethnic group. When groups are geographically defined, parties can reward the people in one locale relative to people in another by basing pork projects in one area or by providing superior services there.

The concern here is with allocation mechanisms rather than on what is being allocated. Hence rather than work with the litany of titles for benefits we simply refer to all preferential rewards and benefits as *prizes*, ζ. When necessary, we label the value of these prize as ζ_A and ζ_B according to which party hands them out. What is essential for our basic model is that parties can observe the level of political support from each group and that there exists a means of preferentially rewarding groups.

We will also explore the non-rival versus rival nature of prizes later in this analysis. As it turns out, this factor influences the optimal division of society into groups from the perspective of political parties and citizens. Although in practice all policies have private and public goods components, we contrast the limiting cases. We treat a prize as a non-rival local public good or a pure club good if its provision is unrelated to the size of the group that benefits from it. An example of such a prize might be the granting of primacy to a specific language. At the other extreme, prizes can be completely rival in nature and so the more people who need to receive the benefit, the more expensive its provision becomes. For instance, if a party simply gave money to each member of a group, then the cost of providing the prize increases linearly in the size of the group. While in reality there is great variation in the nature of prizes, we focus on the limiting cases. We refer to the first case, where the marginal cost of increasing group size is zero, as a non-rival prize. Prizes based on private goods are rival in nature and have a constant marginal cost of providing the prize as group size increases. However, until we examine the relative cost of prize provisions under different arrangements of groups, the essential point is that the members of the group to which the prize is allocated get benefits worth ζ. With this setup in mind, we explore how parties can condition their distribution of prizes on the vote outcome (A, B).

3.1 Contingent Prize Allocation Rules (CPAR)

Parties can allocate prizes to the various groups in many ways. Let $GA_k(A, B)$ be the expected value of the prize that party $\mathcal{A}$ provides to group k if the vote profile is (A, B). Although we develop the logic of our arguments with respect to party $\mathcal{A}$, throughout there are parallel considerations with respect to party $\mathcal{B}$. That is, each party allocates prizes.

Although there are many plausible CPARs, we focus here on a specific Winner-Takes-All Rule, while Smith (2011) examines other prize-distribution mechanisms. Under this rule party $\mathcal{A}$ rewards the most supportive group (or groups). Other groups receive nothing.

$$GA_k(A, B) = \begin{cases} \zeta \quad if \quad A_k = \max\{A_1, ..., A_K\} > 0 \\ 0 \qquad\qquad otherwise \end{cases}$$

Under the WTA rule, party $\mathcal{A}$ gives a prize to the most supportive group (or groups). This rule creates a race between the groups to get the prize. Members of each group, therefore, have an impetus to turnout to support party $\mathcal{A}$ in the hope that their additional vote tips the balance and wins the prize for their group.

4 Pivotality and Voting

The concept of pivotality lies at the heart of rational choice analyses of voting. Voters weigh the costs and benefits of voting: they vote for the alternative they prefer (at least in two-party competition), but they only vote when their expected influence on the outcome outweighs the cost of voting. The standard concept of this influence is the likelihood of shifting the outcome from one party to another. We refer to this as the outcome pivot, OP_A, which is defined formally below.

Voters can also be pivotal in terms of the distribution of the prize. That is, by voting for party $\mathcal{A}$, a voter not only increases the likelihood that party $\mathcal{A}$ wins, she also increases the probability that her group will be the most supportive group and so receive selective benefits from party $\mathcal{A}$. We refer to the likelihood of being pivotal in terms of prize allocation as the Prize Pivot, PP_A. In all cases we define analogous terms with respect to party $\mathcal{B}$.

As Myerson (1998) demonstrates, the Poisson model provides a convenient framework for modeling pivotality. The approach assumes that the size of a group is a Poisson distributed random variable. In the case of the current model, group k has N_k members, where N_k is an unknown random variable that is Poisson distributed with mean n_k. Given this Poisson assumption, from the perspective of each member of group k, the votes of the other $N_k - 1$ members of k (excluding themselves) can also be assumed to be Poisson distributed with mean n_k. This result, which Myerson (1998, Theorem 2 p. 384) refers to as *environmental equivalence*, means that each voter's calculation about the other members of the group is mathematically equivalent to an external analyst's perspective of the whole group.

Environmental equivalence results from two factors that perfectly offset each other. The first factor is a signal about group size. Given that an individual is known to be a member of the group provides the signal that the expected group size is larger than the prior mean. The second factor is that when formulating her optimal actions, a voter considers only the $N_k - 1$ other members of the group. These two factors result in each voter's perception of the other members of her group being identical to the analyst's perception of the whole group. This feature makes the Poisson framework especially attractive for modeling pivotality.

The proposition below provides a definition and calculation of outcome pivot. For a voter in group k, the probability that by voting for $\mathcal{A}$ rather than abstaining she alters the electoral outcome is referred to as the Outcome Pivot, OP_A. Given vote probability profile (p, q), the number of votes for party $\mathcal{A}$ in district k is a Poisson random variable with mean $\lambda_k = p_k n_k$ and the total number of votes for $\mathcal{A}$ is also a Poisson random variable A with mean $n_T p = \sum_{k=1}^{K} p_k n_k$, where $n_T = \sum_{k=1}^{K} n_k$ and p is the weighted average probability of voting for $\mathcal{A}$.

Analogously, the total number of votes for party $\mathcal{B}$ is B, a Poisson random variable with mean $n_T q = \sum_{k=1}^{K} q_k n_k$. Given the well-known result that an individual's vote only influences who wins if it breaks a tie or turns a loss into a draw (see Riker and Ordeshook 1968 for instance), the proposition below defines and characterizes OP_A.

Proposition 1 *Given the vote probability profile* (p, q)*,*

$$\begin{aligned} OP_A &= \Pr(\mathcal{A} \textit{ wins} \mid \textit{voter } i \textit{ votes } \mathcal{A}) - \Pr(\mathcal{A} \textit{ wins} \mid \textit{voter } i \textit{ abstains}) \quad (1) \\ &= \frac{1}{2}\Pr(A = B) + \frac{1}{2}\Pr(A = B - 1) \\ &= e^{-n_T(p+q)}\frac{1}{2}(I_0(2n_T\sqrt{pq}) + (\frac{q}{p})^{\frac{1}{2}} I_1(2n_T\sqrt{pq})) \quad (2) \end{aligned}$$

where $A = \sum_{k=1}^{K} A_k$, $B = \sum_{k=1}^{K} B_k$ *and* $I_m(x)$ *is the modified Bessel function of the first kind.*

Proof. From Skellam (1946), if A and B are Poisson random variables with means $n_T p$ and $n_T q$, respectively, then $Sk(n_T p, n_T q, m) = \Pr(A - B = m) = e^{-(n_T p + n_T q)}(\frac{n_T p}{n_T q})^{\frac{m}{2}} I_{|m|}(2n_T\sqrt{pq})$, where I_m is the modified Bessel function of the first kind. The function Sk is called the Skellam distribution with parameters $n_T p$ and $n_T q$. Therefore OP_A is simply the average of the Skellam distribution evaluated at $m = 0$ and $m = -1$. So $OP_A = e^{-n_T(p+q)}((\frac{p}{q})^{\frac{0}{2}}\frac{1}{2}(I_0(2n_T\sqrt{pq}) + (\frac{q}{p})^{\frac{1}{2}} I_1(2n_T\sqrt{pq}))$. The Outcome Pivot with respect to voting for party $\mathcal{B}$ is analogously defined, OP_B. ■

Voters not only affect which party wins but also the distribution of prizes. Prize Pivot, $PP_{A,k}(p, q)$, refers to the expected change in the prize distribution from party $\mathcal{A}$ for group k if a member of k votes for $\mathcal{A}$ rather than abstains and the vote probability profile is (p, q). To simplify notation we generally omit the profile (p, q).

Proposition 2 *The prize pivot,* $PP_{A,k}$*, for any individual voter in group* k *is as follows:*

$$\begin{aligned} PP_{A,k} &= E[Prize|vote\mathcal{A}] - E[Prize|abstain] \quad (3) \\ &= \zeta \sum_{a=0}^{\infty} f_{n_k p_k}(a) \left(\prod_{j \neq k} F_{n_j p_j}(a+1) - \prod_{j \neq k} F_{n_j p_j}(a) \right) \end{aligned}$$

Proof. Suppose $A_k = a_k$. If a voter in group k abstains then her group receives the prize ζ if $a_k \geq \max\{A_{j\neq k}\}$. Since A_j is Poisson distributed with mean $n_j p_j$, $\Pr(a_k \leq A_j) = F_{n_j p_j}(a_k)$ and the probability that a_k is the maximum of all groups' support for $\mathcal{A}$ is $\prod\limits_{j\neq k} \Pr(A_j \leq a_k) = \prod\limits_{j\neq k} F_{n_j p_j}(a_k)$. Since A_k is Poisson distributed, group k's expected prize if the voter abstains is $\sum_{a=0}^{\infty} f_{n_k p_k}(a) \prod\limits_{j\neq k} F_{n_j p_j}(a)$.

If the voters votes for $\mathcal{A}$, then $\Pr(a_k + 1 \geq \max\{A_{j\neq k}\}) = \prod\limits_{j\neq k} \Pr(A_j \leq 1 + a_k) = \prod\limits_{j\neq k} F_{n_j p_j}(a_k+1)$ and the expected prize for k is $\zeta \sum_{a=0}^{\infty} f_{n_k p_k}(a) \prod\limits_{j\neq k} F_{n_j p_j}(a+1)$. Therefore $PP_{A,k} = \zeta \sum_{a=0}^{\infty} f_{n_k p_k}(a)(\prod\limits_{j\neq k} F_{n_j p_j}(a+1) - \prod\limits_{j\neq k} F_{n_j p_j}(a))$. Further, $PP_{A,k}$ is continuous in all components of (p, q) because the underlying Poisson distributions are continuous. ∎

As the number of voters grows large, there are simple approximations for the pivots to which we now turn. These approximations are subsequently applied to investigate the substantive properties of equilibria.

4.1 Asymptotic Approximations of Pivots

Outcome and prize pivots both arise and are of interest in settings with very large numbers of prospective voters. Therefore we use asymptotic approximations of the Bessel function to generate reliable estimates of pivot probabilities. To derive these approximations we assume the expected number of voters, $n_k p_k$, is relatively large and the number of groups K is relatively small.

As the expected number of voters converges towards infinity, the approximations converge to their true value. We indicate the accuracy of these approximation in finite populations.

Proposition 3 *Outcome Pivot: OP_A: If $n_T = \sum_{k=1}^{K} n_k$, $p = \frac{1}{n_T} \sum_{k=1}^{K} p_k n_k$ and $q = \frac{1}{n_T} \sum_{k=1}^{K} q_k n_k$ then*

$$OP_A \sim \widetilde{OP}_A = \frac{1}{2\sqrt[4]{pq}\sqrt{\pi n_T}} \frac{\sqrt{p}+\sqrt{q}}{2\sqrt{p}} \cdot e^{-n_T\left(\sqrt{p}-\sqrt{q}\right)^2} \quad (4)$$

This is the same approximation used by Myerson (1998), so we provide only a brief sketch. As derived above the difference between the vote for $\mathcal{A}$ and $\mathcal{B}$ is Skellam distributed: $OP_A = e^{-n_T(p+q)}((\frac{p}{q})^{\frac{m}{2}} \frac{1}{2}(I_0(2n\sqrt{pq}) + (\frac{q}{p})^{\frac{1}{2}} I_1(2n_T\sqrt{pq}))$. The modified Bessel function of the first kind, $I_m(x)$ is a well known mathematical function that for fixed m and large x is well approximated (Abramowitz and Stegun 1965, p. 377):

$$I_{|m|}(x) \sim \frac{e^x}{\sqrt{2\pi x}}(1 - \frac{4m^2-1}{8x} + \frac{(4m^2-1)(4m^2-9)}{2!(8x)^2} - \frac{(4m^2-1)(4m^2-9)(4m^2-25)}{3!(8x)^3} + ...)$$

We use the first term of this approximation $I_{|m|}(x) \sim \frac{e^x}{\sqrt{2\pi x}}$ and equation 4 follows directly. To check the accuracy we evaluate $(I_m(x) - \frac{e^x}{\sqrt{2\pi x}})/I_m(x)$ for $m = 0, 1$. Ninety nine percent accuracy is attained when $x > 38.2$. The approximation become better as x increases.

For large populations the outcome pivot estimates are accurate. For instance, if the population mean is $n_T = 100,000$ and voters support parties $\mathcal{A}$ and $\mathcal{B}$ with probability $p = .5$ and $q = .5$, then the approximation error for the outcome pivot is around .0001%.

Asymptotic approximations of prize pivots exist. We define the set of groups whose members actively support party $\mathcal{A}$, that is with positive probability, as $\mathcal{A}$-*active* groups, $W_A = \{k \in \{1,..,K\} : p_k > 0\}$. The proposition below characterizes an approximation of $PP_{A,1}$ when there are two such groups. The appendix provides a series of approximations of prize pivots for three or more $\mathcal{A}$ -active groups, the simplest of which relies upon the expansion of the products in equation 3, making a Gaussian approximation of the Poisson distribution and Laplace's method of integration.

Proposition 4 *Winner-Take-All Prize Pivot Approximation:*

If there are 2 $\mathcal{A}$-active groups with expected votes for $\mathcal{A}$ of λ_1 and λ_2, then as $\lambda_i \to \infty$, $PP_{A,1} \approx \frac{1}{2\sqrt{\pi}\sqrt[4]{\lambda_1\lambda_2}}\frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}}e^{-(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2}$.

Proof. $PP_{A,1} = \zeta \Pr(A_1 = A_2 + 1)$. A_1 and A_2 are Poisson random variables with means λ_1 and λ_2, the difference $A_1 - A_2$ is Skellam distributed: $\Pr(A_1 - A_2 = -1) = Sk(\lambda_1, \lambda_1, -1) = e^{-(\lambda_1+\lambda_2)}\sqrt{\frac{\lambda_2}{\lambda_1}}I_{|1|}(2\sqrt{\lambda_1\lambda_2})$, where I_m is the modified Bessel function of the first kind. Using the approximation (discussed above) that $I(x) = e^x/\sqrt{2\pi x}$, $PP_{A,1} \approx e^{-(\lambda_1+\lambda_2)}\sqrt{\frac{\lambda_2}{\lambda_1}}\frac{e^{2\sqrt{\lambda_1\lambda_2}}}{\sqrt{2\pi 2\sqrt{\lambda_1\lambda_2}}} = \frac{1}{2\sqrt{\pi}\sqrt[4]{\lambda_1\lambda_2}}\frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}}e^{-(\sqrt{\lambda_1}-\sqrt{\lambda_2})^2}$. ■

Figure 1 illustrates several important properties of outcome and prize pivots and helps build intuition as to how pivotality affects equilibrium behavior with a symmetric example. The figure assumes 10,000 expected voters who support party $\mathcal{B}$ with probability $q = .3$. The x-axis plots p, the rate at which voters support party $\mathcal{A}$. The number of voters affects the outcome pivot through two terms: $\frac{1}{\sqrt{n_T}}$ and $e^{-n_T(\sqrt{p}-\sqrt{q})^2}$. The first of these terms suggests outcome pivot probabilities decay at the relatively slow rate of $1/\sqrt{n_T}$. However, unless the election is expected to be extremely close, $p = q$, then the rate of decay of the outcome pivot is dominated by the exponential term $e^{-n_T(\sqrt{p}-\sqrt{q})^2}$. When support for the parties is not perfectly evenly matched, then as the number of voters gets large the outcome pivot collapses towards a point mass (although see Myatt 2012 which shows that aggregate uncertainty about the relative popularity of parties smooths this point mass somewhat). Unless voters are virtually certain that the election will be a statistical dead-heat ($p = q$), the desire to alter which party wins generates little incentive to vote. In contrast, in the symmetric case, the competition for prizes provides a stable incentive for individuals to vote. As seen in figure 1, whether voters are divided into 2 or 5 groups, voters remain pivotal with respect to the allocation of prizes. This provides an incentive for individual voters to turnout even if their vote has no effect on who wins. Moving from left to right in figure 1, prize pivots decline as the expected turnout of voters increases. In particular, if the expected turnout from each group is λ, then the prize pivot declines at the rate of $\frac{1}{\sqrt{\lambda}}$. The origin of this $1/\sqrt{\lambda}$ can most easily be see from the prize pivot approximation for two groups. As seen in the prize only limiting case, equilibrium behavior induces similar levels for support from all $\mathcal{A}$-active groups. We now examine how prize pivots affect equilibrium voting behavior.

Figure 1 about here

The above results allow us to succinctly express voting decisions.

4.2 Voting Calculus

Suppose we consider any fixed vote profile $(p, q) = ((p_1, q_1), (p_2, q_2), ..., (p_K, q_K))$ that describes the probability with which members of each group support $\mathcal{A}$ and $\mathcal{B}$ respectively. Given this profile, the following equations characterize individual private evaluations of party $\mathcal{A}$ relative to $\mathcal{B}$ (that is value of ε_i) such that an individual in group k is indifferent between her vote choices.

$$U_k(vote\mathcal{A}) - U_k(abstain) = (\tau_{Ak})OP_A + PP_{A,k} - c = 0 \quad (5)$$

$$U_k(vote\mathcal{B}) - U_k(abstain) = (\tau_{Bk})OP_B + PP_{B,k} - c = 0 \quad (6)$$

$$U_k(voteA) - U_k(voteB) = (\tau_{ABk})(OP_A - OP_B) + PP_{A,k} - PP_{B,k} = 0 \tag{7}$$

The thresholds, τ_{Ak}, τ_{Bk} and τ_{ABk} that solve these equations characterize Nash equilibria.

Theorem 5 *There exist vote probability profiles (p, q) supported by Nash equilibrium voting behavior: voter i in group k votes for party $\mathcal{A}$ if $\varepsilon_i > \max\{\tau_{Ak}, \tau_{ABk}\}$; votes for $\mathcal{B}$ if $\varepsilon_i < \min\{\tau_{Bk}, \tau_{ABk}\}$ and abstains if $\min\{\tau_{Bk}, \tau_{ABk}\} < \varepsilon_i < \max\{\tau_{Ak}, \tau_{ABk}\}$. The thresholds, τ_{Ak}, τ_{Bk} and τ_{ABk}, solve equations 5, 6, and 7 for each group and $p_k = 1 - G(\max\{\tau_{Ak}, \tau_{ABk}\})$ and $q_k = G(\min\{\tau_{Bk}, \tau_{ABk}\})$.*

Proof. Given the Poisson population assumption, there is always some, albeit very small, probability that i is the only voter. In such a setting, her vote would determine the outcome. This ensures that $OP_A > 0$ and $OP_B < 0$. Therefore equation 5 is an increasing linear function of τ_{Ak}. Therefore for any given vote profile (p, q), there is a unique threshold that solves the equations (and the same for equations 6, and 7). As annotated, these three equations correspond to differences in expected value from each of the voter's actions. If $\varepsilon_i > \max\{\tau_{Ak}, \tau_{ABk}\}$ then, i votes for $\mathcal{A}$, since $U_k(vote\mathcal{A}) > U_k(abstain)$ and $U_k(vote\mathcal{A}) > U_k(vote\mathcal{B})$. Similarly if $\varepsilon_i < \min\{\tau_{Bk}, \tau_{ABk}\}$, then i votes for $\mathcal{B}$. Given the thresholds, an individual in group k votes for $\mathcal{A}$ with probability $\widetilde{p}_k(p, q) = 1 - G(\max\{\tau_{Ak}, \tau_{ABk}\})$ and votes for $\mathcal{B}$ with probability $\widetilde{q}_k(p, q) = G(\min\{\tau_{Bk}, \tau_{ABk}\})$. Since both outcome and prize pivots are continuous in all components of the vote profile (p, q), the τ thresholds, and hence $\widetilde{p}_k(p, q)$ and $\widetilde{q}_k(p, q)$, are continuous in all components of the vote profile. Let $M : [0, 1]^{2K} \to [0, 1]^{2K}$ be this best response function for all the groups. That is to say, M maps (p, q) into simultaneous best responses for all groups $(\widetilde{p}, \widetilde{q}) = ((\widetilde{p}_1(p, q), \widetilde{q}_1(p, q)), ..., (\widetilde{p}_K(p, q), \widetilde{q}_K(p, q)))$. As M is continuous and maps a compact set back into itself, by Brouwer's fixed point theorem, a fixed point exists. If $G()$ is discontinuous then $\widetilde{p}_k(p, q)$ is an upper hemi-continuous mapping, voters randomize at the indifference points and a fixed point exists by Kakutani's fixed point theorem. ■

4.3 Competition for prizes

The literature focuses on the case where there are no prizes and voters are pivotal only in terms of which party wins (Ledyard 1984; Myerson 1998; Krishna and

Morgan 2012). In that policy-only case, turnout is relatively low and elections are close as expected vote shares are similar. We contrast that case with the other limiting case in which there are no policy differences between the parties and groups of voters compete solely for the prizes. We focus initially on the case in which the election is non-competitive and so competition is only for prizes offered by party $\mathcal{A}$. In this setting, while the election is lopsided in outcome it is close in terms of the distribution of prizes.

As introduced earlier, W_A is the set of groups whose members actively support party $\mathcal{A}$, that is with positive probability. The proposition below establishes that there are at least two groups that actively support $\mathcal{A}$. Further, in general, all the groups in W_A provide the same expected number of votes for $\mathcal{A}$. The exception is when some of the supportive groups are sufficiently small that they have fewer prospective voters than the expected number of votes from other groups. In this case all the members of the smaller groups vote for $\mathcal{A}$ in an effort to boost their chance of winning the prize in the event the actual votes from other groups fall below expectations. Our exploration of the prize-only limiting case in a non-competitive election starts with the following proposition, which is similar to Krishna and Morgan (2012).

Proposition 6 *If* $\lambda_j = p_j n_j$ *and* $\lambda_k = p_k n_k$ *then*

$$\begin{aligned} \Delta &= PP_{A,j} - PP_{A,k} \qquad (8) \\ &= \zeta \sum_{a=0}^{\infty}([f_{\lambda_j}(a) f_{\lambda_k}(a+1) - f_{\lambda_k}(a) f_{\lambda_j}(a+1)] \prod_{i \neq j,k} F_{\lambda_i}(a)) \\ &\quad +\zeta \sum_{a=0}^{\infty}((f_{\lambda_j}(a) f_{\lambda_k}(a+1) - f_{\lambda_k}(a) f_{\lambda_j}(a+1)) \prod_{i \neq j,k} f_{\lambda_i}(a+1)) \\ &\quad +\zeta \sum_{a=0}^{\infty}([f_{\lambda_j}(a) F_{\lambda_k}(a) - f_{\lambda_k}(a) F_{\lambda_j}(a)] \prod_{i \neq j,k} f_{\lambda_i}(a+1)) \end{aligned}$$

Further, if $p_i n_i = 0$ *for all* $i \neq j, k$ *and* $\lambda_j > \lambda_k$*, then* $PP_{A,j} < PP_{A,k}$

Proof. Let $M(a)$ represent the distribution of the greatest number of votes for $\mathcal{A}$ by any groups other than j or k: $M(a) = \Pr(Max_{i \neq j,k}\{A_i\} \leq a) = \prod_{i \neq j,k} F_{\lambda_i}(a)$. Let $m(a)$ be the associated probability mass function. Note that, if $n_i p_i = 0$, then $M(a) = 1$ and $m(a) = 0$ for all $a > 0$.

Noting that $M(a+1) = M(a) + m(a+1)$ and $F(a+1) = F(a) + F(a+1)$, the prize pivot for group j can be written as,

$$
\begin{aligned}
PP_{A,j} &= \zeta \sum_{a=0}^{\infty} f_{\lambda_j}(a)[M(a+1)F_{\lambda_k}(a+1) - M(a)F_{\lambda_k}(a)] \qquad (9) \\
&= \zeta \sum_{a=0}^{\infty} f_{\lambda_j}(a)[(M(a)+m(a+1))(F_{\lambda_k}(a)+f_{\lambda_k}(a+1)) - M(a)F_{\lambda_k}(a)] \\
&= \zeta \sum_{a=0}^{\infty} f_{\lambda_j}(a)\left(M(a)f_{\lambda_k}(a+1) + m(a+1)F_{\lambda_k}(a) + f_{\lambda_K}(a+1)m(a+1)\right)
\end{aligned}
$$

The difference between $PP_{A,j}$ and $PP_{A,k}$ is:

$$
\begin{aligned}
\Delta &= PP_{A,j} - PP_{A,k} \qquad (10) \\
&= \zeta \sum_{a=0}^{\infty} M(a)[f_{\lambda_j}(a)f_{\lambda_k}(a+1) - f_{\lambda_k}(a)f_{\lambda_j}(a+1)] \\
&+ \zeta \sum_{a=0}^{\infty} m(a+1)(f_{\lambda_j}(a)f_{\lambda_k}(a+1) - f_{\lambda_k}(a)f_{\lambda_j}(a+1)) \\
&+ \zeta \sum_{a=0}^{\infty} m(a+1)[f_{\lambda_j}(a)F_{\lambda_k}(a) - f_{\lambda_k}(a)F_{\lambda_j}(a)]
\end{aligned}
$$

The first term and second terms contain

$$
[f_{\lambda_j}(a)f_{\lambda_k}(a+1) - f_{\lambda_k}(a)f_{\lambda_j}(a+1)] = \frac{\lambda_j^a \lambda_k^a e^{-\lambda_k} e^{-\lambda_j}}{a!a!}(\lambda_k - \lambda_j) < 0
$$

Hence if $M(0) = 1$ then $\Delta < 0$.

The third term of Δ contains the term $[f_{\lambda_j}(a)(F_{\lambda_k}(a)) - f_{\lambda_k}(a)(F_{\lambda_j}(a))]$ which can be written as

$$
\begin{aligned}
&\frac{\lambda_j^a e^{-\lambda_j}}{a!} \sum_{x=0}^{a} \frac{\lambda_k^x e^{-\lambda_k}}{x!} - \frac{\lambda_k^a e^{-\lambda_k}}{a!} \sum_{x=0}^{a} \frac{\lambda_j^x e^{-\lambda_j}}{x!} \\
= \; &\frac{e^{-\lambda_j} e^{-\lambda_k}}{a!} \sum_{x=0}^{a} \frac{\lambda_j^x \lambda_k^x}{x!}(\lambda_j^{a-x} - \lambda_k^{a-x}) \qquad (11)
\end{aligned}
$$

Since $(\lambda_j^{a-x} - \lambda_k^{a-x}) > 0$, the third term of Δ is positive, so Δ cannot be definitively signed if $M(0) < 1$. Substitution of $M(a)$ and $m(a)$ into equation 10 produces 8. ■

As we will see later when we are ready to explore asymmetric group size, there is a fundamental tradeoff in voting behavior. When there are 2 $\mathcal{A}$-active voting groups, then increasing the size of a group decreases any member's probability of voting. If there are more than 2 $\mathcal{A}$-active voting groups, then individual members of the group expected to have the largest turnout themselves have an increased incentive to vote, leading to instability in the number of groups. Before we can explore that however, we return to the argument for symmetric group sizes and prize-only competition. The following proposition characterizes equilibria in prize-only competition in a limiting case. Later we relax some of the limitations but it is useful first to establish the expected behavior in this situation so that we can see cleanly how each modification in conditions; that is, whether the election is non-competitive or competitive, when groups sizes are symmetric or asymmetric, and when prizes are non-rival or rival, alters equilibrium behavior.

Proposition 7 *If party $\mathcal{A}$ offers a large prize ($\zeta_A > c$ and $\zeta_B = 0$), voting is costly ($c > 0$) and there are no policy differences ($G(x) = 0$ for $x < 0$ and $G(x) = 1$ for $x \geq 0$), then there exist Nash equilibria in which at least two groups actively support $\mathcal{A}$. Further, if $j \in W_A$, then,either $p_j n_j = Max_{i \in W_A} p_i n_i$ or $p_j = 1$. For non-$\mathcal{A}$-active groups, $i \notin W_A$, $p_i = 0$ and $\prod_{j \in W_A} F_{n_j p_j}(1) \leq c$.*

Proof. From theorem 5, equilibria exist. Further, from Myerson (1998), we know all equilibria in random population games are type symmetric so we restrict attention to such strategies. From equation 5 and with no policy differences, equilibria require $PP_{A,k} = c$. We now examine a series of cases:

1) Suppose no-one votes: $p_j = 0$ for all groups ($j = 1, ..., K$). Then by voting any voter could ensure that her group wins the prize: Therefore, $PP_{A,j} = \zeta_A > c$. Someone wants to vote, so provided the prize is larger than the cost of voting we can rule out $p_j = 0$ for all groups.

2) Suppose $p_j > 0$ for only one group. A member of this group is pivotal if and only if no one else in her group votes. For group j, $PP_{A,j} = \Pr(A_j = 0) = e^{-(n_j p_j)} \geq c$. But for a voter in group i, where $p_i = 0$, a vote for party $\mathcal{A}$ wins the prize if $A_j \leq 1$. Therefore for group i, $PP_{A,i} = \Pr(A_j = 0) + \Pr(A_j = 1) > e^{-(n_j p_j)}$ which contradicts $p_i = 0$.

3) We now consider the case where at least two groups vote with positive probability: $p_j, p_k > 0$. If $p_j \in (0,1)$, then $PP_{A,j} = c$. If $p_k \in (0,1)$, then $PP_A(k) = c$ and proposition 6 implies $n_j p_j = n_k p_k$. If $p_k = 1$ then $PP_{A,k} \geq c$.

Therefore, in groups that provide party $\mathcal{A}$ support, either all $\mathcal{A}$-active groups generate the same number of expected votes or all members of a group vote for $\mathcal{A}$.

4) Suppose no one in group i votes ($p_i = 0$), then the chance of being prize pivotal is less than the cost of voting. Specifically, if $p_j, p_k > 0$ then $p_i = 0$ implies $PP_{A,i} = \prod_{j \in K} F_{n_j p_j}(1) \leq c$. This last condition ensure that no one in group i wants to vote. ■

In non-competitive elections with prize-only equilibria in which only party $\mathcal{A}$ offers a prize, we are essentially in a one-party environment and, therefore, only party $\mathcal{A}$ receives any votes. Despite there being only one credible party in this limiting case, the extant groups divide into two sets: those that actively support $\mathcal{A}$ (W_A) and those that provide no support ($K \backslash W_A$). In general, the $\mathcal{A}$-*active* groups generate the same expected number of voters for $\mathcal{A}$ and it is straightforward to see that $n_i p_i$ increases as the size of the prize ζ increases and decreases in the cost of voting, c. So, broadly speaking, politicians in this prize-only setting can shape the turnout rate – or their mandate – by varying the size of the prize. Put differently,a budget constraint imposed on the prize's size places a limit on turnout. The exception to this group-turnout symmetry arises when the expected size of an active group is less than the expected number of votes from other $\mathcal{A}$-active groups. This situation can arise in equilibrium and when it does every voter in the smaller group votes for $\mathcal{A}$. Contingent prize allocation aligns the incentives of voters within groups and coordinates their actions. Either many voters in a group vote for $\mathcal{A}$ or none of them vote. The CPAR creates a competition between the groups that is supported by individually rational voting. It also creates an incentive for a dominant party to sustain more than one group or faction within its ranks.

4.4 Asymmetry and instability

Proposition 6 derives the difference between the prize pivots of $\mathcal{A}$-*active* groups. When there are only two $\mathcal{A}$-active groups, then the prize pivot of the smaller group is larger. The next lemma tackles the question of the relative size of prize pivots in the presence of three or more $\mathcal{A}$-active groups. Although in full generality the relative sizes of prize pivots are complicated, the following lemma characterizes approximations that are closely related to prize pivots for substantively interesting cases. From equation 8 in proposition 6 the difference in prize pivots between groups 1 and 2 is

$$\begin{aligned}
\Delta &= PP_{A,1} - PP_{A,2} \\
&= \zeta \sum_{a=0}^{\infty} ([f_{\lambda_1}(a) f_{\lambda_2}(a+1) - f_{\lambda_2}(a) f_{\lambda 1}(a+1)] \prod_{i \neq 1,2} F_{\lambda_i}(a)) \\
&\quad + \zeta \sum_{a=0}^{\infty} ((f_{\lambda_1}(a) f_{\lambda_2}(a+1) - f_{\lambda_2}(a) f_{\lambda 1}(a+1)) \prod_{i \neq 1,2} f_{\lambda_i}(a+1)) \\
&\quad + \zeta \sum_{a=0}^{\infty} ([f_{\lambda_1}(a) F_{\lambda_2}(a) - f_{\lambda_2}(a) F_{\lambda_1}(a)] \prod_{i \neq 1,2} f_{\lambda_i}(a+1))
\end{aligned}$$

Δ is a combination of summations over the products of CDFs and PMFs. As expected votes (the λ's) increase the PMF terms become relatively small. Thus for large elections the second term of Δ which involves products of three PMFs becomes small relative to the first and third terms which involve the product of two PMFs and a CDF. Hence, $\widehat{\Delta}$, defined as the first and third terms of Δ, approximates Δ for large λ.

In contrast to the cases with 2 $\mathcal{A}$-*active* groups, equilibria involving 3 or more $\mathcal{A}$-active groups are knife-edged and breakdown with the introduction of asymmetry. The following lemma states that, if there are $w > 2$ $\mathcal{A}$-active groups and group 1 slighly increases its expected number of votes above that of the other groups, then the prize pivot is greater for group 1 than the other groups. In such a setting, the number of $\mathcal{A}$-active groups is unstable because the increase in prize pivot induces still further support for A from the members of group 1 and erodes the incentive to vote in the other groups.

Lemma 8 *Suppose there are w $\mathcal{A}$-active groups with the following expected number of votes for party $\mathcal{A}$, $\lambda_1 = \rho\lambda$, $\lambda_2 = \lambda_3 = ... = \lambda_w = \lambda$, and*

$$\begin{aligned}
\widehat{\Delta} &= \zeta \sum_{a=0}^{\infty} (f_{\lambda_1}(a) f_{\lambda_2}(a+1) - f_{\lambda_2}(a) f_{\lambda 1}(a+1)) F_{\lambda_2}(a)^{w-2} \qquad (12) \\
&\quad + \zeta \sum_{a=0}^{\infty} (f_{\lambda_1}(a) F_{\lambda_2}(a) - f_{\lambda_2}(a) F_{\lambda_1}(a))(w-2) f_{\lambda_2}(a+1)
\end{aligned}$$

Evaluated at $\rho = 1$, $\frac{d\widehat{\Delta}}{d\rho} > 1$.

Proof. Let $\widehat{\Delta} = \zeta \sum_{a=0}^{\infty} S(a)$ where $S(a)$ represents the terms in the summation of equation 12. The proof involves showing every $S(a)$ term is positive. From the Poisson distribution, $F_\lambda(a) = e^{-\lambda} \sum_{x=0}^{a} \frac{\lambda^x}{x!}$ and $f_\lambda(a) = e^{-\lambda} \frac{\lambda^{a-x} \lambda^x}{a!}$.

Therefore we rewrite $S(a)$ and substitute $\lambda_1 = \rho\lambda$ and $\lambda_2 = \lambda$. Hence for the $w = 3$ case,

$$\begin{aligned} S(a) &= \frac{\lambda^a}{a!}\frac{\lambda^a}{a!}\frac{\lambda}{a+1}e^{-\rho\lambda-\lambda-\lambda}\{\sum_{x=0}^{a}\frac{\lambda^x}{x!}\rho^a(1-\rho)+\sum_{x=0}^{a}\frac{\lambda^x}{x!}\rho^x(\rho^{a-x}-1)\} \\ &= \frac{\lambda^a}{a!}\frac{\lambda^a}{a!}\frac{\lambda}{a+1}e^{-\rho\lambda-\lambda}\{\sum_{x=0}^{a}e^{-\lambda}\frac{\lambda^x}{x!}\rho^x(\rho^{a-x}(1-\rho)+(\rho^{a-x}-1))\} \\ &= \frac{\lambda^a}{a!}\frac{\lambda^a}{a!}\frac{\lambda}{a+1}e^{-\rho\lambda-\lambda}\{\sum_{x=0}^{a}f_\lambda(x)(\rho^{a-x}(1-\rho)+(\rho^{a-x}-1))\} \end{aligned}$$

Let $X(a) = \sum_{x=0}^{a} f_\lambda(x)(\rho^a(1-\rho)+(\rho^a-\rho^x))$

The terms $x = a$ and $x = a-1$ in the summation are negative but all other terms are positive.

Now differentiate $X(a)$ with respect to ρ

$\frac{dX(a)}{d\rho} = \sum_{x=0}^{a} f_\lambda(x)\rho^{a-1}(2a - a\rho - \rho - x\rho^{x-a})$

Which evaluated at $\rho = 1$ is

$\frac{dX(a)}{d\rho} = \sum_{x=0}^{a} f_\lambda(x)(a-1-x) = (a-1)f_\lambda(0) + (a-2)f_\lambda(1) + ... + 2f(a-3) + f(a-2) + 0f(a-1) - f(a)$

If $a \geq \lambda + 1$ then $f(a-2) \geq f(a)$ (by single peakedness of Poisson distribution) and therefore $\frac{dX(a)}{d\rho} > 0$. Now we show the sum of terms $2f(a-3) + f(a-2) + 0f(a-1) - f(a)$ is positive by rewriting them as $\frac{e^{-\lambda}\lambda^{a-3}}{(a-3)!}(2 + \frac{\lambda}{(a-2)} - \frac{\lambda^3}{(a-2)(a-1)a})$. Simple differentiation of the term in parentheses shows it is decreasing in λ. By single-peakedness $2f(a-3)+f(a-2)+0f(a-1)-f(a) > 0$ for $\lambda > a-1$. By $(2+\frac{\lambda}{(a-2)} - \frac{\lambda^3}{(a-2)(a-1)a})$ decreasing in λ, $2f(a-3)+f(a-2)-f(a) > 0$ must also be positive for smaller λ. Thus for $a \geq 3$, $\frac{dX(a)}{d\rho} > 0$. Since λ is large, $X(0)$, $X(1)$ and $X(2)$ are vanishingly small compared to $X(a)$ evaluated around $a = \lambda$. Hence, for $w = 3$, $\frac{d\widehat{\Delta}}{d\rho} > 0$.

Next we show the result holds for $w > 3$.

$$\begin{aligned} \widehat{\Delta} = \zeta\sum_{a=0}^{\infty}&[(f_{\lambda_1}(a)f_\lambda(a+1) - f_\lambda(a)f_{\lambda 1}(a+1))F_\lambda(a)^{w-2} \\ &+ (f_{\lambda_1}(a)F_\lambda(a) - f_\lambda(a)F_{\lambda_1}(a))(w-2)f_\lambda(a+1)] > 0 \end{aligned}$$

since

$$|(f_{\lambda_1}(a)f_\lambda(a+1) - f_\lambda(a)f_{\lambda 1}(a+1))F_\lambda(a)^{(w-2)}| < |(f_{\lambda_1}(a)f_\lambda(a+1) - f_\lambda(a)f_{\lambda 1}(a+1))F_\lambda(a)|$$

$$(f_{\lambda_1}(a)f_\lambda(a+1) - f_\lambda(a)f_{\lambda 1}(a+1))F_\lambda(a) < (f_{\lambda_1}(a)F_\lambda(a) - f_\lambda(a)F_{\lambda_1}(a))f_\lambda(a+1)$$

and

$$(f_{\lambda_1}(a)F_\lambda(a)-f_\lambda(a)F_{\lambda_1}(a))f_\lambda(a+1) < (w-2)(f_{\lambda_1}(a)F_\lambda(a)-f_\lambda(a)F_{\lambda_1}(a))f_\lambda(a+1)$$

The first inequality follows from $F_\lambda(a)^{(w-2)} < F_\lambda(a) \in (0,1)$. The second inequality was proven above when $w = 3$. The final inequality follows from multiplication by $(w-2)$. ■

Lemma 8 indicates that equilibria involving three or more $\mathcal{A}$-active groups are knife-edged cases. If perfectly balanced in expected votes, then three or more groups can survive and be active within a party but the slightest deviation from perfect balance leads to a collapse down to two-active groups within a party (and within a party competition, as we explain later). Figure 2 illustrates why. The figure examines the case of three $\mathcal{A}$-active groups and plots the prize pivots for each group as asymmetry is introduced. Specifically the x-axis plots the relative difference in size between the expected turnouts. On the right hand side of figure 2 the larger group has 1% higher turnout than the medium group, which in turn has 1% higher turnout than the smaller group. Under symmetric turnout, all groups have identical prize pivots (left hand side of figure). Even a slight amount of asymmetry induces instability. Suppose initially that all three groups have the same expected level of support for $\mathcal{A}$, 1000 votes in this example. Now consider the thought experiment as to what happens as group 1 increases its relative support for $\mathcal{A}$. In the case of two groups, this higher expected number of $\mathcal{A}$ votes reduced the prize pivot for the larger group. This reduction in prize pivot disincentivizes members of group 1 from voting and so restores balance between the expected votes of each group. In contrast, with three groups there is positive feedback. The increased number of votes by group 1 increases the incentive for members of this larger group to vote for $\mathcal{A}$. This positive feedback results in even more people in group 1 wanting to vote for $\mathcal{A}$, a situation that is unstable. Hence, although equilibrium exist with three or more $\mathcal{A}$-active groups producing the same expected turnout, these equilibria are knife-edged and sensitive to any slight perturbations in relative turnouts. Analogous to Duverger's result that two party competition evolves under majoritarian rule (Duverger 1982, Riker 1963), WTA Contingent Prize Allocation Rules result in stable competition between two groups within any individual party to obtain prizes. As we will see later,this circumstance provides an inducement for parties to break competition for a single office down into many sub-competitions such as the competition within each US state to contribute to the electoral college majority for a given party's candidate in the national, presidential election.

Figure 2 about here

With these results in hand, we can now examine the model's implications regarding voting behavior.

5 Equilibrium Voting Behavior

The competitiveness of elections plays an important role in distinguishing expected voting behavior in the model that follows. Informally, by non-competitive elections we mean an election in which one party is virtually certain to win. In terms of the model, this implies that voters have virtually no prospects of influencing which party wins, which we formally treat as $OP_A \to 0$. In this section we illustrate symmetric voting behavior in competitive and non-competitive settings, after which we examine asymmetric voting behavior between groups that leads to polarization in which some groups predominantly back party $\mathcal{A}$, other back party $\mathcal{B}$ and other remain non-aligned.

Contingent Prize Allocation Rules induce higher turnout in both competitive and noncompetitive elections. An example serves as a convenient illustration. Figure 3 considers the fully symmetric case in which members of all groups support parties A and B at the same rate, the total expected number of voters is $n_T = 100,000$, valuations of party $\mathcal{A}$ relative to $\mathcal{B}$ (ε) are normally distributed and the cost of voting is $c = 0.05$. In this setting, voters influence both the outcome of the election and the allocation of prizes by both parties. The decision to support $\mathcal{A}$ or $\mathcal{B}$ rather than abstain solve equations 5 and 6. The size of the prize parties distribute is plotted on the x-axis, with the assumption that both parties have the same sized prize to offer. Absent any prize, turnout is low, with only about 0.5% of prospective voters turning out, with half voting for each party. As the value of the prizes increases then so too does support for each party. For instance, if the value of the prize is $\zeta = 8$, that is 400 times the cost of voting, then support for each party is about 27%; that is, 54% turnout. Inducing such turnout in the absence of prizes requires the variance in policy preferences of the parties to be approximately 23,500 times the cost of voting. Such calculations rely heavily on appropriate scaling so we should not read too much into the precise numbers. However, in relative terms, these figure illustrate how prizes induce more political support than policy differences alone.

Figure 3 about here

The relative importance of prizes compared to policy preferences, as just illustrated, has two significant implications for Black's (1958) and Downs's (1957) views of party competition. First, if candidates converge to the median voter position, as predicted in Downsian competition, then policy differences are of course irrelevant and prizes alone dictate voting behavior. Second, because the impact of prizes can be so much larger than the impact of policy differences, there may be little incentive for parties to converge on policy.

In contrast to the standard rational voter story, elections need not be close to induce turnout. Indeed support for party $\mathcal{A}$ would be relatively unchanged even if few people supported party $\mathcal{B}$. Suppose for instance that party $\mathcal{B}$ had a far smaller prize to offer than party $\mathcal{A}$, as might occur if $\mathcal{A}$ were a long term incumbent and $\mathcal{B}$ had never held office and was not expected to do so in the near future. Cases of dominant parties are the norm in many nations around the world (Kitschelt and Wilkinson 2007). Here we relax the limiting conditions in proposition 7 in which $\zeta_B = 0$ so that we induce votes for each party. If parties $\mathcal{A}$ and $\mathcal{B}$ offer prizes worth ζ_A and ζ_B, respectively, and all groups vote symmetrically, then A obtains approximately $(\zeta_A/\zeta_B)^2$ times the votes that B receives.[2] If $\zeta_A > \zeta_B$, then the assumption of non-competitive elections is well justified since the probability that party $\mathcal{B}$ wins is trivial. For instance, if $\zeta_A = 16$ and $\zeta_B = 4$, then $OP_A = \frac{1}{\sqrt{\pi}\sqrt{\lambda}} e^{-4\lambda^2}$ where λ is the expected number of supporters for party $\mathcal{B}$. In the case of 2 groups and a voting cost of $c = \frac{1}{5\sqrt{\pi}}$, then, with 1000 expected votes for party $\mathcal{B}$, the chance of any voter being outcome pivotal is about $\frac{1}{160}\frac{\sqrt{10}}{\sqrt{\pi}} e^{-9000}$. In such equilibria, elections are free and fair, but challengers can never expect to win. The incumbents use the resources of office to offer large prizes in the form of patronage goods. With little prospects of attaining office and hence the resources with which to distribute prizes, challengers garner relatively little support. Reform is hard. Even if all the voters want political change, they want such changes enacted by voters in other groups rather than risk their group's access to prizes by diminishing their group's vote share. Once incumbents are expected to continually win, such

[2] This is most easily seen by assuming two groups and substitution into the 2 group prize pivot approximation.

expectations become self fulfilling. Arriola (2012) suggests such a pattern is broken by political challengers backed by corporate wealth as such parties can offers the prizes needs to attract political support.

5.1 Polarization and Asymmetric Equilibrium Behavior

The equilibrium groups need not behave symmetrically, although as the examples above illustrated they may. Returning to the nomenclature introduced earlier, W_A referred to $\mathcal{A}$-*active* groups, those whose voters supported party $\mathcal{A}$ with positive probability. These groups compete for the prize offered by party $\mathcal{A}$. To avoid the need for a more general definition of active groups, for this section we assume the distribution of preferences over the parties, $G(x)$, is the uniform distribution over the interval -1/2 to 1/2. Let W_B refer to the set of groups that actively support party $\mathcal{B}$. Although these assumptions are relaxed in the next section, we assume groups are symmetric in size and preferences. The proposition below generalizes proposition 7.

Proposition 9 *For suitably sized prizes, there exist equilibria in which there are at least two groups in W_A, at least two groups in W_B and members of remaining groups vote for neither party. For interior solutions, the vote probabilities satisfy the following equations:*

$$\text{For } i \in W_A : OP_{A,i}(\frac{1}{2} - p_i) + PP_{A,i} = c \tag{13}$$

$$\text{For } j \in W_B : OP_{B,j}(q_j + \frac{1}{2}) + PP_{B,j} = c \tag{14}$$

$$\text{For } k \notin W_A, W_B : OP_{A,k}(1/2) + PP_{Ak} < c \text{ and } OP_{B,k}(1/2) + PP_{B,k} < c \tag{15}$$

Proof. Given the uniform distribution, if $p_i \in (0,1)$ then $\tau_i = \frac{1}{2} - p_i$. Similarly, if $q_i \in (0,1)$ then $\tau_i = q_i + \frac{1}{2}$. Equations 13 and 14 are the equilibrium equations 5 and 6. Ensuring $p_i \in (0,1)$ and $q_i \in (0,1)$ places bounds on prize size. The constraint that non-aligned groups do not vote also places a constraint on prize size. Since expected turnout for these groups is zero, a vote by a member of such groups is only pivotal in the allocation of the prize if the turnout from all other groups is 0 or 1: Hence $PP_{A,i \notin W_A} = \zeta_A \Pr(A_1, ..., A_K \leq 1) =$

$\zeta_A \prod_{k=1,\ldots,K} F_{\lambda_k}(1) = \zeta_A \prod_{i \in W_A} F_{\lambda_i}(1)$. Applied to equation 15 this places an upper bound on prize size.

The proof that there must be at least two active groups follows steps 1 and 2 in the proof of proposition 7. ■

Proposition 9 shows that even when outcome pivot considerations are taken into account equilibria retain many of the properties seen in the limiting prize only cases. In particular, provided prizes are bigger than the cost of voting, then voters from at least two groups actively vote for each party. The $\mathcal{A}$-active and $\mathcal{B}$-active groups can be either disjoint, such that certain groups support only party $\mathcal{A}$ while other groups support only party $\mathcal{B}$, or there could be overlap such that members of certain groups actively support both parties. More than two active groups for either party can only be supported under conditions of perfect balance, proposition 7. Thus, except for perfect balance, there are not less than 2 and not more than 4 stable groups within a party-office competition. Two groups per party supports the Duvergerian view of winner-take-all settings. The results, however, add nuance to the Duvergerian perspective because with two parties there can be up to four groups within any competition for prizes. Furthermore, we will show later that parties can break competition for office into many smaller prize competitions – as in precinct votes in a single Congressional district or electoral college votes across states in a single presidential election. Then we will see that while no more than four groups can be stable within any of the sub-competitions, many more than four groups can be supported in the overall party competition.

For convenience, the equilibria were stated for the uniform distribution and the definition of active groups involved any positive probability of voting for a party. However, we might imagine modifying these definitions. Intuitively, members of $\mathcal{A}$-active groups have the prospects of affecting both which party wins (OP) and the distribution of prizes ($PP_{A,i}$). In expectation, the aggregation of these individual incentives delivers λ_j expected votes for $\mathcal{A}$. In contrast non-aligned groups generate far fewer votes, zero in the case above. However, if the uniform distribution assumption is relaxed and there is full support over the preference types, then a preference extremist (ε very large) in non-aligned group i wants to vote in order to influence the outcome of the election. $\mathcal{A}$-active groups can be redefined as groups with levels of support for $\mathcal{A}$ above some substantial threshold to accommodate such a generalization. By voting for $\mathcal{A}$, an extremist in a non-aligned group could potentially win the prize for her group. However, since her group generates relatively few votes for $\mathcal{A}$, then her prospects of being prize pivotal become vanishingly small as the number of votes for $\mathcal{A}$ grows. Returning to the definition of prize pivot and supposing that the expected number of votes from $\mathcal{A}$-active groups, λ_i, vastly exceeds the expected number of votes in non-$\mathcal{A}$-active groups, λ_k, then

$$PP_{A,k} = \zeta \sum_{a=0}^{\infty} f_{\lambda_k}(a) \left(\prod_{i \in W_A} F_{\lambda_i}(a+1) - \prod_{i \in W_A} F_{\lambda_i}(a) \right) \approx 0$$

Since $\lambda_i > \lambda_k$, for those a where $f_{\lambda_k}(a)$ is of substantial magnitude $\prod_{i \in W_A} F_{\lambda_i}(a+1)$ and $\prod_{i \in W_A} F_{\lambda_i}(a)$ are approximately zero. And when $\prod_{i \in W_A} F_{\lambda_i}(a+1) - \prod_{i \in W_A} F_{\lambda_i}(a)$ is substantial, at values of a around λ_i, $f_{\lambda_j}(a)$ is tiny. Hence, for members of groups whose expected support for A is substantially less than the expected support from other groups, the prize pivot is very small. The incentive for members of such groups is primarily to influence the electoral outcome. The voting calculus in such groups is approximately $OP_A(\tau_{A,k}) \approx c$. In contrast voters in $\mathcal{A}$-active groups are motivated by both outcome and prize considerations: $OP_A(\tau_{A,i}) + PP_{A,i} \approx c$. Given expectations about whether or not their group has a realistic prospect of being awarded the prize, voter incentives fulfill such expectations.

Thus far we have explored symmetric and asymmetric equilibrium behavior under conditions in which group sizes are symmetric. Now we examine the implications when group sizes are asymmetric.

6 Asymmetry and Group Formation

When elections are competitive, asymmetries in group size and in preferences are important. Although in the symmetric case, knife edged equilibria involving more that two $\mathcal{A}$-active groups exist, in the presence of asymmetry in terms of group size or group party preferences, parties draw the bulk of their support from just two groups.

We start by examining how asymmetries between groups affects the stability of equilibria under competitive elections. Then we examine group formation and dynamics from the perspective of voters and political entrepreneurs and parties. How voters migrate between groups and how political entrepreneurs redefine group identity depends upon the nature of prizes and the competitiveness of elections.

6.1 Asymmetry and Instability

Above we examined equilibria with asymmetric voting behavior where voters behaved differently depending upon whether they were in $\mathcal{A}$-active groups, $\mathcal{B}$-active groups or neither. This section examines the consequences of structural asymmetries between groups rather than simply behavioral differences between groups. The results are phrased in terms of asymmetries with respect to group size, but shifts in group preferences for one party over the other have similar effects.

A useful starting point for exploring asymmetry is the equilibrium conditions for groups 1 and 2 that govern indifference between voting for A and abstaining. From equation 5

$$OP_A\tau_{A,1} + PP_{A,1} = c = OP_A\tau_{A,2} + PP_{A,2} \tag{16}$$

Rearranging this equation, substituting $\tau_{A,1} = G^{-1}(1-p_1)$ and noting that while the prize pivot varies by group, the outcome pivot does not, yields:

$$OP_A(G^{-1}(1-p_1) - G^{-1}(1-p_2)) = PP_{A,2} - PP_{A,1} \tag{17}$$

In the case of the uniform distribution analyzed above, the $G^{-1}(1-p_1) - G^{-1}(1-p_2)$ term is simply $p_2 - p_1$. For group 2 to support $\mathcal{A}$ at a higher rate than group 1 ($p_2 > p_1$) implies that $PP_{A,2} > PP_{A,1}$. Combining this with proposition 6, leads directly to the following result.

Proposition 10 *In competitive elections ($OP_A > 0$), if there are two $\mathcal{A}$-active groups and $n_1 > n_2$, then $p_2 > p_1$ and $n_1p_1 > n_2p_2$. However, as $OP_A \to 0$, $p_1 \to p_2$ and $n_1p_1 \to n_2p_2$.*

This result implies that with competitive elections the larger group has a higher expected turnout even though the smaller group supports $\mathcal{A}$ at a higher rate. Intuitively, for both groups to provide the same level of support for $\mathcal{A}$ requires that a higher proportion of the smaller group votes. However this implies that the indifferent type in the smaller group ($\tau_{A,2}$) likes $\mathcal{A}$ less than the indifferent type in the larger groups ($\tau_{A,1}$) and is therefore less motivated to turnout. In non-competitive elections this preference distinction between groups is of little consequence. When the electoral winner is not in doubt, policy preferences over parties do not enter voters' considerations. Yet, in the competitive election setting, the larger group generates a higher expected level of support for $\mathcal{A}$ and is therefore, on average, expected to win the prize that $\mathcal{A}$ offers more often. This differential expectation of winning prizes shapes voter incentives to migrate from one group to another, as we explore below.

It is worth revisiting the implications for asymmetry when there are three or more $\mathcal{A}$-active groups. Three or more groups can exist in equilibrium, albeit knife-edged equilibrium, in the symmetric setting. No such equilibria with three or more groups generically exist in the presence of asymmetry and competitive elections. Lemma 8 suggests that when one group has a larger expected turnout, then its members are incentivized to turnout at a yet higher rate. Under asymmetric conditions, only two groups actively support $\mathcal{A}$ and only two groups actively support $\mathcal{B}$. Of course these groups can, but need not be, the same groups. Thus, in a two-party winner-take-all setting with asymmetric group size and competitive elections, among electorally-active groups there are expected to be no fewer than two and no more than four per prize competition. There can, of course, be any number of non-electorally active groups. Further, as we shall explore shortly, when prizes are rival, politicians want to create many prize competitions so that overall there can be many active groups despite the constraint that there be only 2 $\mathcal{A}$-active groups and 2 $\mathcal{B}$-active groups per prize competition.

Given these general results, we now explore the incentives of voters to form groups and the incentives of politician to structure prize competitions.

6.2 Voter Organized Groups

The competition for prizes affects which social cleavages are active and the evolution of group identity. The competitiveness of elections and the extent to which prizes are rival or non-rival affects the incentives of voters to migrate between groups. If a voter were offered the choice to switch groups prior to playing the voting game, then her propensity to do so depends upon several factors. First, there is an innate personal cost to switching group identity. Such an emigration cost depends upon the nature of groups. If groups are geographically based then the cost is that of relocating. Other emigration costs might be less tangible, such as learning a new language or religious practice. Second, beyond the personal cost of emigration, groups differ in the extent to which they welcome members. Extant members can charge a high immigration cost for people wishing to join. Alternatively, they might actively seek to redefine their group's identity to be more inclusive. The willingness of people to pay the costs of migration and the barriers groups set to entry depend upon the level of political competition, the size of prizes and the rival/non-rival nature of the prizes.

When elections are non-competitive the rate of migration between groups is low and group identities are static. Since the outcome of the election is a foregone conclusion, policy preferences are irrelevant considerations and so, beyond needing enough members, precise group size has little effect on equilibrium turnout. Voters gain little from migration as each $\mathcal{A}$-active group has the same equilibrium probability of winning the prize and, if the prizes are rival, then extant members of groups want to restrict entry because additional members dilute their share of the prizes without increasing the likelihood that the group wins a prize. Thus, in the non-competitive electoral setting, with groups incentivized to raise the cost of immigration for potential migrants and migrants having little to gain from migration, group identities are relatively fixed, especially if prizes are rival.

Groups dynamics are more fluid under competitive electoral settings. Figure 4 plots the rate at which the larger and smaller group vote for party $\mathcal{A}$ as the difference in their relative size increases. Although group 2 votes for $\mathcal{A}$ at a higher rate, this does not offset the larger number of members in group 1. Therefore, the expected number of votes from group 1 is larger than that from group 2 and as a consequence group 1 wins the prize more often. In contrast, in the non-competitive setting, the rates at which groups 1 and 2 support party $\mathcal{A}$ completely offsets the size differences between the groups. The solid lines in figure 4 shows the likelihood that group 1 is allocated the prize. In the non-competitive setting, size differences have no impact and in equilibrium each group is equally likely to be awarded the prize. Yet as the difference in group sizes grows in the competitive electoral setting, group 1 becomes increasingly likely to be awarded the prize. This has important implications for group dynamics and electoral competitiveness.

In the competitive setting, both groups 1 and 2 have incentives to absorb additional members from non-$\mathcal{A}$-active groups (or from each other). By taking on additional members, each group increases the probability of attaining the prize. However, this benefit has to be contrasted against the number of members who share a rival prize. The desire to increase the likelihood of winning prizes induces an openness on the part of groups (lowering immigration costs) and the prospects of winning prizes creates an incentive for individuals to migrate. However, the incentive to welcome immigrants into the group evaporates when the dilution of the value of the prize exceeds the marginal improvement in the probability of winning the prize. Groups are more open to absorbing new members when prizes are non-rival and so added members do not dilute the value of the prize. This is an important implication that is worth fleshing out more fully.

The fluidity of group membership has the potential to undermine the competitiveness of elections when one group is more successful at recruiting members than another. The calculations in figure 4 assume elections remain maximally competitive (that is, $p = q$) as relative group size changes. However, absent parallel shifts in the $\mathcal{B}$-active groups such a supposition is untenable. As group 1 grows in size relative to group 2, both groups reduce their support for party $\mathcal{A}$ because prize pivotality declines as group 1 becomes more likely to be awarded the prize, as shown in proposition 10. As the size difference between groups 1 and 2 increases,the number of votes for party A declines in both groups and so $\mathcal{A}$ loses more elections. Such a reduction in electoral competitiveness reduces the incentive to migrate and dampens disparities between groups. As elections become non-competitive, the incentive for voters to form larger groups vanishes. The necessity that elections remain competitive limits the extent to which one $\mathcal{A}$-active group can be more successful than the other at recruiting new members.

We have seen that the equilibrium-induced coordination among group members imposes limits on how far apart groups drift in size when migration across groups is possible. Politicians also have incentives to influence group divisions. We now examine those incentives and how they may influence electoral competition and turnout.

6.3 How politicians organize groups

Although many group identities, such as race, religion or ethnicity, are primordial, others are artificial constructs created by politicians (Chandra 2004). For instance, the City of Chicago is divided into 50 wards. Why did politicians create 50 groups, instead of 2, 20 or 200, and how is political competitiveness structured between these groups. Given lemma 8, organizing direct prize competition between 50 wards is unlikely to engender high levels of political support because in the presence of asymmetry only two $\mathcal{A}$-active groups are part of a stable equilibrium. However, rather than organize a single competition between many groups, parties can create numerous competitions between smaller subsets of groups; and when prizes are rival it is in their interest to do so.

To illustrate how parties structure competition, suppose there are initially 16 roughly even sized groups and a dominant party wants to obtain some fixed level of support, say 60%, while minimizing its expenditure on prizes. Given 60% support, elections are non-competitive so we restrict attention to prize motivations. If party $\mathcal{A}$ offers a single prize to the most supportive of the 16 groups, then in equilibrium only two groups will be highly supportive and the 60% support goal can not be realized. However, parties can structure competition differently through various aggregation processes. For instance, the parties might aggregate the 16 initial groups into 2 larger groups, each of which contains 8 of the original 16 groups. In geographical terms, this might mean creating 2 counties each containing 8 wards and then awarding the prize to the most supportive county. Alternatively, the party might form 8 counties each containing 2 groups – a stable configuration – and have a competition for a prize in each of the eight counties. Which strategy is optimal depends upon the rivalness of prizes.

Table 6.3 calculates the relative cost of providing rival and non-rival prizes under different configurations. Three factors affect the overall cost of prize provision. First, as the number of competitions increases, then so too does the number of prizes that the party must provide. Second, the cost of providing a prize depends upon how many people share it, at least for rival prizes. For instance, if competition is between 2 counties each composed of 8 groups, then the party provides a single prize that benefits the members of 8 of the original groups. If the prize is non-rival, then the cost of providing it is independent of the size of the group receiving it. Third, group size affects pivotality at the rate of $\frac{1}{sqrt(n)}$. By breaking the competition for prizes into a series of competitions between smaller groups, a party reduces the value of prize it needs to offer to elicit 60% support.

Table 6.3 shows the product of the three factors affecting the relative cost of buying 60% support and highlights the optimal configuration for rival and non-rival prizes. With non-rival prizes, the cost of rewarding a group is the same whether the group is large or small. Therefore parties prefer to allocate non-rival prizes in a single large contest. In contrast, when prizes are rival, leaders prefer to split competitions into smaller sub-competitions. For instance, the relative cost of eliciting 60% support from a single large competition is $\sqrt{(8)}$ times more expensive that gaining the same overall level of support from 8 small competitions. In both these cases, the same number of people receive rewards, but small group size in the latter case increases each voter's prize pivot allowing the party to buy the same level of support with smaller prizes.

Table 6.3: Competitions and the Overall Cost of Prize Provision.

Number of Competitions	1	2	4	8
Size of Aggregated Group [a]	8	4	2	1
Non-Rival Prize: Relative Cost	$\mathbf{1 \cdot 1 \cdot \sqrt{8}}$	$2 \cdot 1 \cdot \sqrt{4}$	$4 \cdot 1 \cdot \sqrt{2}$	$8 \cdot 1 \cdot \sqrt{1}$
Rival Prize: Relative Cost	$1 \cdot 8 \cdot \sqrt{8}$	$2 \cdot 4 \cdot \sqrt{4}$	$4 \cdot 2 \cdot \sqrt{2}$	$\mathbf{8 \cdot 1 \cdot \sqrt{1}}$

[a] Aggregate size expressed in number of primordial groups.

Competition over non-rival prizes, such as language or religious supremacy, takes place at the national level. In contrast, when prizes are rival, such as traditional patronage goods, then parties create numerous smaller competitions. Rakove (1975 Ch 4) describes the distribution of rewards under Mayor Daley's control of Chicago's Democratic Party Machine which relied on a hierarchical version of the competition structure presented here. Although not completely winner-take-all, resources flowed disproportionately to those who delivered large Democratic vote shares. The Cook County Democratic party played ward bosses off against each other. Those who succeeded in obtaining large Democratic vote shares got greater access to patronage jobs and resources than other ward bosses. In turn, ward bosses played precinct bosses off against each other by funneling resources to those who turned out the Democratic vote. Voters in heavily Democratic precincts turned out to support the Democrats as this increased local access to resources.

7 Conclusions

Using a Poisson games framework of Myerson (1998, 2000), we have modeled elections in which parties offer contingent prizes to those identifiable groups of voters that offer the highest level of political support. The model demonstrates that even in large populations, in which voters have little influence on the outcome of elections, they retain significant influence over the distribution of prizes. This remains true even in lopsided elections, giving all political parties, whether in competitive or non-competitive environments, an incentive to encourage factions so as to manage turnout and achieve the appearance of a mandate whether they are popular or not. We have specified the conditions under which voter turnout fluctuates as a function of four considerations: the value of contingent prizes; the extent to which prizes are rival or non-rival; the degree to which elections are competitive; and the extent to which the size of voter groups are symmetric or asymmetric. Equilibrium behavior is more likely to be driven by voters competing to win preferential treatment for their group than by policy concerns. The model also provides a modifed and more nuanced understanding of the implications of winner-take-all settings beyonds the standard Duvergerian account.

The results also provide insights into group dynamics. In non-competitive electoral settings, voters have little reason to shift groups or alter their political identity and group members have little reason to welcome the entreaties of others to join them. In contrast, competitive elections induce fluidity in group membership, at least up to a limit. When prizes are rival, then extant groups of voters are happy to welcome new members as long as they improve the probability of winning the prize more than they dilute the value of the prize. In this way we can see both fluidity and self-sustaining features to prize-motivated groups and explanations for variance in turnout.

8 References

Abramowitz, Milton; Stegun, Irene A., eds. 1965. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, New York: Dover.

Aldrich, John. 1993. "Rational Choice and Turnout." American Journal of Political Science. 37(1): 246-78.

Allen, Oliver E. 1993. The Tiger: The Rise and Fall of Tammany Hall. Reading MA: Addison-Wesley Publishing Company.

Ansolabehere, Stephen and James Synder. 2006. Party Control of State Government and the Distribution of Public Expenditures. Scandinavian Journal of Economics, 108(4): 547-569.

Arriola, Leonardo R. 2012. Multi-Ethnic Coalitions in Africa: Business Financing of Opposition Election Campaigns (Cambridge Studies in Comparative Politics). Cambridge University Press.

Brusco,V., M. Nazareno and S. Stokes. 2004.Vote Buying In Argentina. Latin American Research Review 39(2):66-88.

Castanheira, Micael. 2003. Victory Margins and the Paradox of Voting, European Journal of Political Economy, 19(4): 817-841.

Chandra, Kanchan. 2004. Why Ethnic Parties Succeed. Cambridge University Press.

Cox, Gary W., and Matthew D. McCubbins. 1986. "Electoral Politics as a Redistributive Game." Journal of Politics 48 (May): 370–89.

Dekel, Eddir, Matthew O. Jackson, Asher Wolinsky. 2008. Vote Buying: General Elections. Journal of Political Economy, 116(2): 351-380.

Democratic Party Headquarters. 2007. "Call for the 2008 Democratic National Convention." February 2, 2007. Washington DC: Democratic Party Headquarters

Dixit, Avinash., and John Londregan. 1995. "Redistributive politics and economic efficiency," American Political Science Review 89:856-866.

Dixit, Avinash, and John Londregan. 1996. "The Determinants of Success of Special Interests in Redistributive Politics." Journal of Politics 58 (November): 1132–55.

Black, Duncan. 1948. "On the rationale of group decision-making". Journal of Political Economy 56: 2334

Downs, Anthony. 1957. An Economic Theory of Democracy. New York: Harper and Row Publishers.

Duverger, M. 1963. Political parties: their organization and activity in the modern state, North, B. and North R., tr. New York: Wiley, Science Ed.

Feddersen, Timothy J. 2004. Rational Choice Theory and the Paradox of Not Voting. Journal of Economic Perspectives. 18(1): 99-112.

Fenno, Richard F. Jr. 1978. Home Style: House Members in their Districts.

Little, Brown.

Ferejohn, John. Pork Barrel Politics: Rivers and Harbors Legislation, 1947-1968. Stanford, CA: Stanford University Press, 1974

Ferejohn, John and Morris P. Fiorina. 1974. "The Paradox of Not Voting: A Decision Theoretic Analysis," American Political Science Review 68(2): 525-536.

Gelman, Andrew, Gary King and W. John Boscardin. 1998. Estimating the probability of events that have never occurred: When is your vote decisive? Journal of the American Statistical Association 93:1-9.

Gelman, Andrew, Nate Silver and Aaron Edlin. 2010. What is the Probability your Vote Will Make a Difference. Economic Inquiry.

Gerber, Alan S., Gregory A. Huber, David Doherty and Conor M. Dowling. 2009. Is There a Secret Ballot?: Ballot Secrecy Perceptions and Their Implications for Voting Behavior. Working Paper, Yale University

Geys, Benny. 2006. 'Rational' Theories of Voter Turnout: A Review. Political Studies Review 4:16-35.

Good, I. J., and L. S. Mayer. 1975. "Estimating the Efficacy of a Vote," Behavioral Science, 20(1), 25–33.

Gosnell, Harold F. 1937. Machine Politics: Chicago Model. Chicago: University of Chicago Press

Green, Donald and Ian Shapiro. 1996. Pathologies of Rational Choice Theory: A Critique of Applications in Political Science. New Haven CT: Yale University Press.

Guterbock, Thomas M. 1980. Machine Politics in Transition: Party and Community in Chicago. Chicago and London: University of Chicago Press.

Johnson, N.L., Kotz, S., Kemp, A.W. (1993) Univariate Discrete distributions (2nd edition). Wiley.

Kitschelt, Herbert and Steven I. Wilkinson (ed). 2007. Patrons, Client and Policies: Patterns of Democratic Accountability and Political Competition. Cambridge: Cambridge University Press.

Kovenock, Dan, Brian Roberson. 2009. Inefficient redistribution and inefficient redistributive politics. Public Choice 139: 263—272

Krishna, Vijay and John Morgan. 2012. Majority Rule and Utilitarian Welfare. Working paper.

Ledyard, John. 1984. The pure theory of large two-candidate elections, Public Choice 44: 7-41.

Lindbeck, A. and J. Weibull. 1987. "Balanced budget redistribution and the outcome of political competition." Public Choice 52:273–97.

Lizzeri, Alessandro and Nicola Persico. 2001. The Provision of Public Goods under Alternative Electoral Incentives. American Economic Review, 91(1): 225-239

Meirowitz, Adam and Kenneth Shotts. 2009. "Pivots versus Signals in Elections" Journal of Economic Theory 144:744-771.

Myatt, David. 2012. A Theory of Protest Voting. Working paper, London Business School.

Myatt, David. 2011. A Rational Choice Theory of Voter Turnout. Working paper, London Business School.

Myerson, Roger B. 1993. "Incentives to Cultivate Favored Minorities Under Alternative Electoral Systems", American Political Science Review, 87(4): 856-69.

Myerson, Roger B. 1998. Population Uncertainty and Poisson Games. International Journal of Game Theory. 27: 375-392.

Myerson, Roger B. 2000. Large Poisson Games. Journal of Economic Theory 94: 7-45.

Palfrey, Thomas R. and Howard Rosenthal. 1983. A Strategic Calculus of Voting. Public Choice 41: 7-53.

Palfrey, Thomas R. and Howard Rosenthal. 1985. Voter Participation and Strategic Uncertainty. American Political Science Review 79: 62-78.

Rakove, Milton K. 1975. Don't Make No Waves, Don't Back No Losers: An Insider's Analysis of the Daley Machine. Bloomington IN: Indiana University Press.

Razin, Ronny. 2003. Signaling and election motivations in a voting model with common value and responsive candidates. Econometrica, 71(4): 1083–1119.

Riker, William H. 1982. The Two-Party System and Duverger's Law: An Essay on the History of Political Science American Political Science Review, 76(4):753-766.

Riker, William H. and Peter O. Ordeshook. 1968. A Theory of the Calculus of Voting. American Political Science Review 62: 25-42.

Schwartz, Thomas. 1987. Your Vote Counts on Account of the Way It

Is Counted: An Institutional Solution to the Paradox of Not Voting. Public Choice, 54(2): 101-121.

Skellam, J. G. (1946) "The frequency distribution of the difference between two Poisson variates belonging to different populations". Journal of the Royal Statistical Society: Series A, 109 (3), 296.

Smith, Alastair and Bruce Bueno de Mesquita. 2012. Contingent Prize Allocation and Pivotal Voting. British Journal of Political Science 42: 371-392.

Stokes, Susan C. 2007. Political Clientelism. In The Oxford Handbook of Comparative Politics. Edited by Carles Boix and Susan C. Stokes. Oxford, UK: Oxford University Press.

Stokes, Susan C. 2005. Perverse Accountability: A Formal Model of Machine Politics with Evidence from Argentina. American Political Science Review 99(3): 315-325.

9 Appendix

In this appendix we derive two asymptotic approximation for the prize pivot for the general case of K $\mathcal{A}$-active groups. The proposition below provides an approximation for $K = 3$ groups based upon the expansion of equation 3, hence we refer to it as the expansion approximation. The proof deals with the general case of K groups. As K increases, the expansion approximation involves $(K-1)+\frac{(K-1)(K-2)}{2}+\frac{(K-1)(K-2)(K-3)}{3!}+...$ terms. However as expected turnouts become large the latter terms become small relative to the first $(K-1)$ terms.[3] As K increases this results in many terms, although in symmetric cases many of these terms are the same. Therefore we also offer an alternative difference approximation, that for all sizes of K, involves only 2 terms. Both approaches yield accurate approximations as np increases. For instance, for $K = 3$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1000$, the error in the expansion approximation is about 1.5%. The comparable error for the difference approximation is 1.4%. As expected turnouts increase, the approximation become increasingly accurate. Let expected number of votes for party $\mathcal{A}$ from group i be $\lambda_i = n\omega_i p_i$ where n is average group size.

[3]The $(K-1)$ terms involve summations over $K-2$ CDF terms and 2 PMF terms. The $(K-1)(K-2)/2$ terms involve summations over $K-3$ CDF terms and 3 PMF terms. As the λ's become large the PMF become small relative to the CDF terms, so the later terms can be ignored.

Proposition 11 *Winner-Take-All Prize Pivot Approximation:*

If there are $K = 3$ groups, then as $\lambda_i \to \infty$,

$$PP_{A,1} \approx \widetilde{PP_{A,1}} = \zeta e^{ny_2(a_2^*)}\sqrt{\frac{2\pi}{n|y_2''(a_2^*)|}} + \zeta e^{ny_3(a_3^*)}\sqrt{\frac{2\pi}{n|y_3''(a_3^*)|}} + \zeta e^{ny_{23}(a_{23}^*)}\sqrt{\frac{2\pi}{n|y_{23}''(a_{23}^*)|}}$$

where

$$y_2(a) = -\omega_1 p_1 - \omega_2 p_2 + \frac{1}{n}\log(\frac{(n\omega_1 p_1)^a}{\Gamma(a+1)}\frac{(n\omega_2 p_2)^a}{\Gamma(a+1)}\frac{(n\omega_2 p_2)}{(a+1)}\prod_{j\neq 1,2}\Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}}))$$

$$y_3(a) = -\omega_1 p_1 - \omega_3 p_3 + \frac{1}{n}\log(\frac{(n\omega_1 p_1)^a}{\Gamma(a+1)}\frac{(n\omega_3 p_3)^a}{\Gamma(a+1)}\frac{(n\omega_3 p_3)}{(a+1)}\prod_{j\neq 1,3}\Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}}))$$

$$y_{23}(a) = -\omega_1 p_1 - \omega_2 p_2 - \omega_3 p_3 + \frac{1}{n}\log(\frac{(n\omega_1 p_1)^a}{\Gamma(a+1)}\frac{(n\omega_2 p_2)^a}{\Gamma(a+1)}\frac{(n\omega_2 p_2)}{(a+1)}\frac{(n\omega_3 p_3)^a}{\Gamma(a+1)}\frac{(n\omega_3 p_3)}{(a+1)})$$

and a_2^* is the unique solution to $y_2'(a) = 0$, a_3^* is the unique solution to $y_3'(a) = 0$ and a_{23}^* is the unique solution to $y_{23}'(a) = 0$. $\Phi()$ represents the standard normal distribution and $\Gamma()$ is the gamma function.

Proof. Now we derive the general K group expansion approximation. From the definition of prize pivot we expand the products:

$$\begin{aligned} PP_{A1} &= \zeta\sum_{a=0}^{\infty} f_{\lambda_1}(a)(\prod_{j\neq 1} F_{\lambda_j}(a+1) - \prod_{j\neq 1} F(a)) \\ &= \zeta\sum_{\alpha=0}^{\infty} f_{\lambda_1}(a)\prod_{j\neq 1}(F_{\lambda_j}(a) + \frac{\lambda_j}{(a+1)}f_{\lambda_j}(a)) - \zeta\sum_{\alpha=0}^{\infty} f_{\lambda_1}(a)\prod_{j\neq 1}F_{\lambda_j}(a) \\ &= \zeta\sum_{\alpha=0}^{\infty} f_{\lambda_1}(a)[\sum_{i\neq 1}\frac{\lambda_i}{(a+1)}f_{\lambda i}(a)\prod_{j\neq i,1}F_{\lambda_j}(a) + \\ &\quad \sum_i\sum_j \frac{\lambda_i}{(a+1)}f_{\lambda i}(a)\frac{\lambda_j}{(a+1)}f_{\lambda_j}(a)\prod_{k\neq i,j,1}F_{\lambda_k}(a) + ...] \end{aligned}$$

We index terms as follows: $s_i = \zeta\sum_{\alpha=0}^{\infty} f_{\lambda_1}(a)\frac{\lambda_i}{(a+1)}f_{\lambda i}(a)\prod_{j\neq i,1}F_{\lambda_j}(a)$ and $s_{ij} = \zeta\sum_{\alpha=0}^{\infty} f_{\lambda_1}(a)\frac{\lambda_i}{(a+1)}f_{\lambda i}(a)\frac{\lambda_j}{(a+1)}f_{\lambda_j}(a)\prod_{k\neq i,j,1}F_{\lambda_k}(a)$. If there are three groups, then $PP_{A,1} = s_1 + s_2 + s_{12}$. Note that as the λ's become large, the higher terms (s_{ij} and s_{ijk} ...) become small relative to s_i's. We estimate each term as an integral given by the Euler-Maclarin formula:

$$\sum_{a=0}^{w} h(a) = \int_1^w h(a)da - B_1(h(w) - h(1)) + \sum_{y=1}^{r}\frac{B_{2y}}{(2y)!}(h^{(2y-1)'}(w) - h^{(2y-1)'}(1)) + R$$

where $B_1 = -\frac{1}{2}$, $B_2 = 1/6$, $B_3 = 0, B_4 = -1/30, B_6 = 1/42$,... are the Bernoulli numbers and our interest is the case as $w \to \infty$. These estimates are well justified since once the λ's are greater than 100, the higher order derivatives evaluated at 0 and 3λ are less than 10^{-50}. We examine a single term,

Hence

$$\begin{aligned} s_2 &\approx \widetilde{s_2} = \zeta \int_{a=0}^{\infty} f_{\lambda_1}(a) \frac{\lambda_i}{(a+1)} f_{\lambda i}(a) \prod_{j \neq i,1} F_{\lambda_j}(a) da \\ &= \zeta \int_{a=0}^{\infty} e^{-\lambda_1} \frac{\lambda_1^a}{\Gamma(a+1)} e^{-\lambda_2} \frac{\lambda_2^a}{\Gamma(a+1)} \frac{\lambda_2}{(a+1)} \prod_{j \neq 1,2} F_{\lambda_j}(a) da \end{aligned}$$

Next we substitute $n_i = \omega_i n$ where n is the average group size and, by the central limit theorem, the Poisson distribution converges to the Gaussian distribution: $F_{n\omega_j p_j}(a) \approx \Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}})$

$$\begin{aligned} \widetilde{s_2} &= \zeta \int_{a=0}^{\infty} e^{-n\omega_1 p_1} e^{-n\omega_2 p_2} \frac{(n\omega_1 p_1)^a}{\Gamma(a+1)} \frac{(n\omega_2 p_2)^a}{\Gamma(a+1)} \frac{(n\omega_2 p_2)}{(a+1)} \prod_{j \neq 1,2} \Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}}) da \\ &= \zeta \int_{0}^{\infty} e^{n(y_2(a))} da \end{aligned}$$

where

$$\begin{aligned} y_2(a) &= -\omega_1 p_1 - \omega_2 p_2 + \frac{1}{n} \log(\frac{(n\omega_1 p_1)^a}{\Gamma(a+1)} \frac{(n\omega_2 p_2)^a}{\Gamma(a+1)} \frac{(n\omega_2 p_2)}{(a+1)} \prod_{j \neq 1,2} \Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}})) \\ &= -\omega_1 p_1 - \omega_2 p_2 + \frac{1}{n} (a \log(n\omega_1 p_1) + (a+1) \log(n\omega_2 p_2) - 2 \log(\Gamma(a+1) \\ &\quad - \log(a+1) + \sum_{j \neq 1,2} \log(\Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}}))) \end{aligned}$$

The derivatives of $y_2(a)$ with respect to a are[4]

$$n.y_2'(a) = \log(n\omega_1 p_1) + \log(n\omega_2 p_2) - 2\psi(a+1) - \frac{1}{a+1} + \sum_{j \neq 1,2} \frac{1}{\sqrt{n\omega_j p_j}} \frac{\phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}})}{\Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}})}$$

and

$$n.y_2''(a) = -2\psi^{(2)}(a+1) + \frac{1}{(a+1)^2} - \sum_{j \neq 1,2} \frac{1}{n\omega_j p_j} \left(\frac{(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}}) \phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}})}{\Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}})} + \frac{\phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}})^2}{\Phi(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}})^2} \right) < 0$$

[4]The derivatives of the Γ function are the digamma function $\psi(x) = \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ and the polygamma function $\psi^{(m)}(x) = \frac{d^m}{dx^m} \log \Gamma(x)$.

Let a_2^* be the unique solution to $n.y_2'(a_2^*) = 0$.

The integral $\widetilde{s_2} = \zeta \int_0^\infty e^{n(y_2(a))} da$ is of the standard form for a Laplace integral. By Taylor series about a_2^*,

$$\begin{aligned} \widetilde{s_2} &= \zeta \int_0^\infty e^{n(y_2(a))} da = \zeta \int_0^\infty e^{n(y_2(a_2^*)+(a-a_2^*)y_2'(a_2^*)+\frac{1}{2}(a-a_2^*)^2 y_2''(a_2^*)+\ldots)} da \\ &\sim \zeta \int_0^\infty e^{n(y_2(a_2^*)+(a-a_2^*)y_2'(a_2^*)+\frac{1}{2}(a-a_2^*)^2 y_2''(a_2^*))} da \end{aligned}$$

The approximation is justified since y_1 has a unique maximum at $a = a_2^*$, and the terms decay rapidly in n.

At a_2^*, $y_2'(a_2^*)$, the second term in the integrand, is 0 and the third term $(e^{\frac{1}{2}(a-a_2^*)^2 y_2''(a_2^*)})$ has the form of a Gaussian distribution. Therefore, as $np \to \infty$,

$$\begin{aligned} \widetilde{s_2} &\approx \zeta e^{n y_2(a_2^*)} \int_0^\infty e^{\frac{1}{2}(a-a_2^*)^2 y_2''(a_2^*)} da \\ &\approx \zeta e^{n y_2(a_2^*)} \sqrt{\frac{2\pi}{n|y_2''(a_2^*)|}} \end{aligned}$$

The other terms in the integral are similarly defined so for the 3 group cases $PP_{A,1} \approx \widetilde{PP_{A,1}} = \zeta e^{n y_2(a_2^*)} \sqrt{\frac{2\pi}{n|y_2''(a_2^*)|}} + \zeta e^{n y_3(a_3^*)} \sqrt{\frac{2\pi}{n|y_3''(a_3^*)|}} + \zeta e^{n y_{23}(a_{23}^*)} \sqrt{\frac{2\pi}{n|y_{23}''(a_{23}^*)|}}$ where

$$\begin{aligned} y_3(a) &= -\omega_1 p_1 - \omega_3 p_3 + \frac{1}{n} \log\left(\frac{(n\omega_1 p_1)^a}{\Gamma(a+1)} \frac{(n\omega_3 p_3)^a}{\Gamma(a+1)} \frac{(n\omega_3 p_3)}{(a+1)} \prod_{j \neq 1,3} \Phi\left(\frac{a - n\omega_j p_j}{\sqrt{n\omega_j p_j}}\right)\right) \\ y_{23}(a) &= -\omega_1 p_1 - \omega_2 p_2 - \omega_3 p_3 + \frac{1}{n} \log\left(\frac{(n\omega_1 p_1)^a}{\Gamma(a+1)} \frac{(n\omega_2 p_2)^a}{\Gamma(a+1)} \frac{(n\omega_2 p_2)}{(a+1)} \frac{(n\omega_3 p_3)^a}{\Gamma(a+1)} \frac{(n\omega_3 p_3)}{(a+1)}\right) \end{aligned}$$

a_3^* solves $y_3'(a_3^*) = 0$ and a_{23}^* solves $y_{23}'(a_{23}^*) = 0$. ■

The approximation in proposition 11 involves three terms, although the higher order term (s_{23}) becomes vanishing small and unimportant as λ increases. However for case of K groups there are $K-1$ first order terms and $(K-1)(K-2)/2$ second order terms. In the symmetric case ($\lambda_i = \lambda_j$) these $K-1$ terms are identical. However, when groups differ this approach involves many terms. Therefore we offer an alternative difference approximation.

Proposition 12 *As* $\lambda_i \to \infty$ *then* $PP_{A,1} \approx \widetilde{\widetilde{PP}}_{A1} = \zeta e^{n z_1(a_1^*)} \sqrt{\frac{2\pi}{n|z_1''(a_1^*)|}} -$

$\zeta e^{nz_2(a_2^*)}\sqrt{\frac{2\pi}{n|z_2''(a_2^*)|}}$ *where*

$$
\begin{aligned}
z_1(a) &= -\omega_1 p_{1+}\frac{1}{n}((a-1)\log n + (a-1)\log(\omega_1 p_1) - \log\Gamma(a) + \sum_{j\neq 1}\log\Phi(\frac{a - n\omega_i p_i}{\sqrt{n\omega_i p_i}})) \\
z_2(a) &= -\omega_1 p_1 + \frac{1}{n}(a\log n + a\log(\omega_1 p_1) - \log\Gamma(a+1) + \sum_{j\neq 1}\log\Phi(\frac{a - n\omega_1 p_1}{\sqrt{n\omega_1 p_1}}))
\end{aligned}
$$

where a_1^* *and* a_2^* *are the unique solutions to* $\frac{d}{da}z_1'(a) = 0$ *and* $\frac{d}{da}z_2'(a) = 0$.

Proof.

$$
\begin{aligned}
PP_{A1} &= \zeta\sum_{a=0}^{\infty} f_{\lambda_1}(a)(\prod_{j\neq 1} F_{\lambda_j}(a+1) - \prod_{j\neq 1} F(a)) \qquad (18) \\
&= \zeta\sum_{a=0}^{\infty} f_{\lambda_1}(a)\prod_{j\neq 1} F_{\lambda_j}(a+1) - \zeta\sum_{a=0}^{\infty} f_{\lambda_1}(a)\prod_{j\neq 1} F_{\lambda_j}(a)
\end{aligned}
$$

Reindexing the first summation in $PP_{A,1}$ by $a' = a+1$ and extracting the $a = 0$ term from the second sum yields

$$
\begin{aligned}
PP_{A1} &= \zeta\sum_{a'=1}^{\infty} f_{\lambda_1}(a'-1)\prod_{j\neq 1} F_{\lambda_j}(a') - \zeta\sum_{\alpha=1}^{\infty} f_{\lambda_1}(a)\prod_{j\neq 1} F_{\lambda_j}(a) - \zeta f_{\lambda_1}(0)\prod_{j\neq 1} F_{\lambda_j}(0) \\
&= \zeta\sum_{\alpha=1}^{\infty} f_{\lambda_1}(a-1)\prod_{j\neq 1} F_{\lambda_j}(a) - \zeta\sum_{\alpha=1}^{\infty} -f_{\lambda_1}(a)\prod_{j\neq 1} F_{\lambda_j}(a) + \zeta\prod_{j} f_{\lambda_j}(0)
\end{aligned}
$$

where we combine the two sums since the a' index is just a dummy variable, and we simplify the last term using the fact that $F_{\lambda_j}(0) = f_{\lambda_j}(0)$.

The term $\zeta\prod_j f_{\lambda_j}(0)$ is effectively zero since $\lambda_j \to \infty$ by assumption. We now adopt an analogous approach to that in proposition 11. First, we use the Euler-Maclaurin formula to write this sum as an integral. Second we approximate $\prod_{j\neq 1} F_{\lambda_j}(a)$ with a Gaussian. We then construction two Laplace integrals.

$$
\begin{aligned}
PP_{A1} &\approx \zeta\int_1^{\infty} e^{-\lambda_1}\frac{\lambda_1^{a-1}}{\Gamma(a)}\prod_{j\neq 1}\Phi(\frac{a-\lambda_j}{\sqrt{\lambda_j}})da - \zeta\int_1^{\infty} e^{-\lambda_1}\frac{\lambda_1^{a}}{\Gamma(a+1)}\prod_{j\neq 1}\Phi(\frac{a-\lambda_j}{\sqrt{\lambda_j}})da \\
&\approx \zeta\int_1^{\infty} e^{nz_1(a)}da - \zeta\int_1^{\infty} e^{nz_2(a)}da = I_1 - I_2
\end{aligned}
$$

where

$$
\begin{aligned}
z_1(a) &= -\omega_1 p_{1+}\frac{1}{n}((a-1)\log n + (a-1)\log(\omega_1 p_1) - \log\Gamma(a) + \sum_{j\neq 1}\log\Phi(\frac{a - n\omega_i p_i}{\sqrt{n\omega_i p_i}})) \\
z_2(a) &= -\omega_1 p_1 + \frac{1}{n}(a\log n + a\log(\omega_1 p_1) - \log\Gamma(a+1) + \sum_{j\neq 1}\log\Phi(\frac{a - n\omega_1 p_1}{\sqrt{n\omega_1 p_1}}))
\end{aligned}
$$

Let a_1^* and a_2^* be the unique solutions to $\frac{d}{da}z_1'(a) = 0$ and $\frac{d}{da}z_2'(a) = 0$. As $np \to \infty$, the Laplace integrals are

$$\begin{aligned} I_1 &\approx \zeta e^{nz_1(a_1^*)} \int_0^\infty e^{\frac{1}{2}(a-a_1^*)^2 z_1''(a_1^*)} da \\ &\approx \zeta e^{nz_1(a_1^*)} \sqrt{\frac{2\pi}{n|z_1''(a_1^*)|}} \end{aligned}$$

and

$$I_2 \approx \zeta e^{nz_2(a_2^*)} \sqrt{\frac{2\pi}{n|z_2''(a_2^*)|}}$$

Thus as $np \to \infty$, $\widetilde{\widetilde{\widetilde{PP}}}_{A1} = \zeta e^{nz_1(a_1^*)} \sqrt{\frac{2\pi}{n|z_1''(a_1^*)|}} - \zeta e^{nz_2(a_2^*)} \sqrt{\frac{2\pi}{n|z_2''(a_2^*)|}}$ approximates PP_{A1}. ■

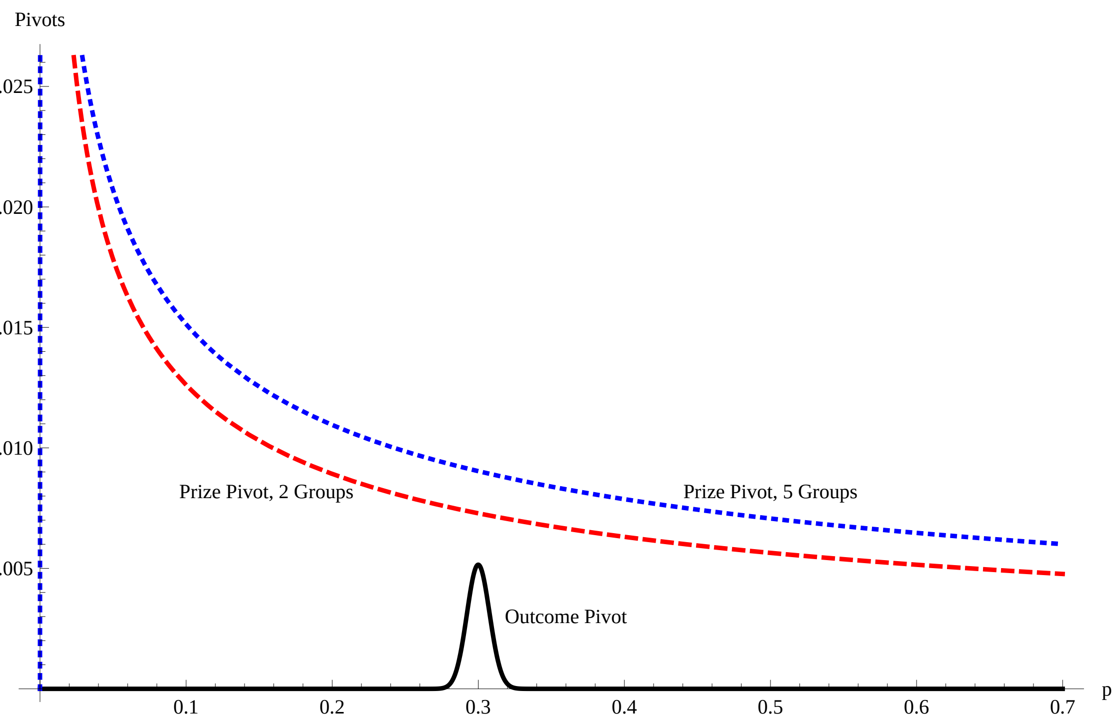

Figure 1: Outcome and Prize Pivots

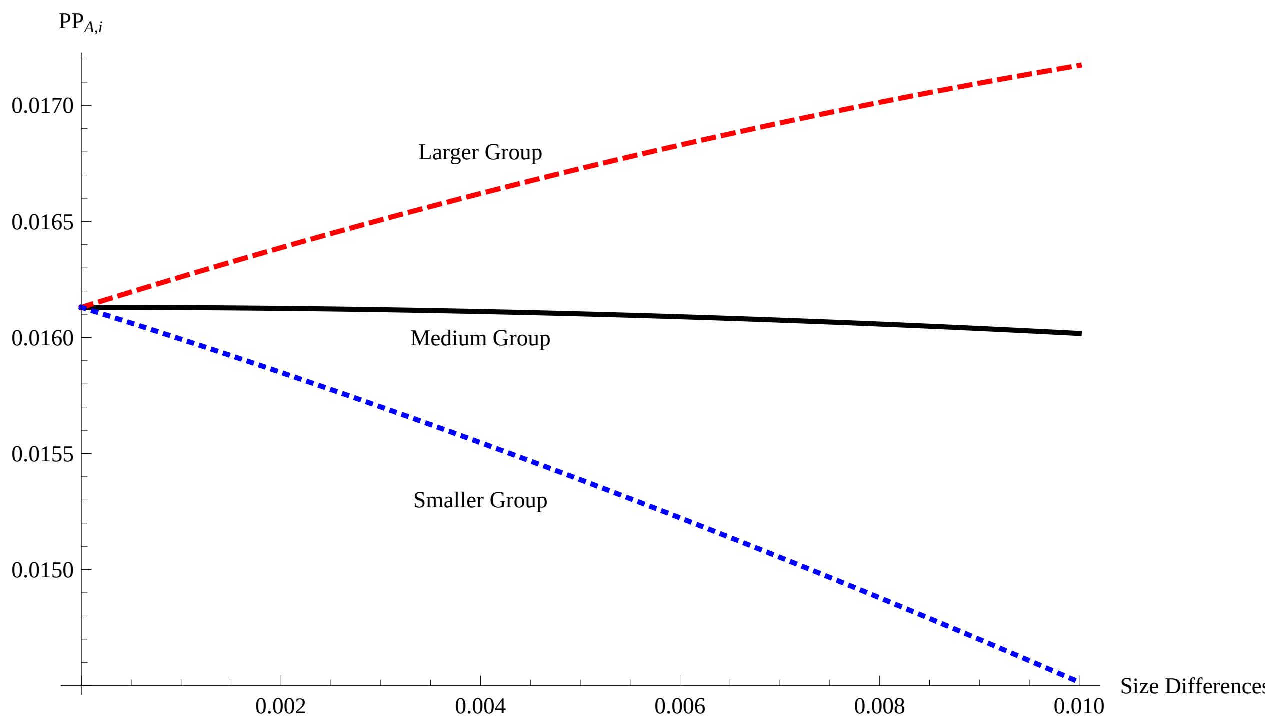

Figure 2: Prize Pivots and Asymmetric Group Size

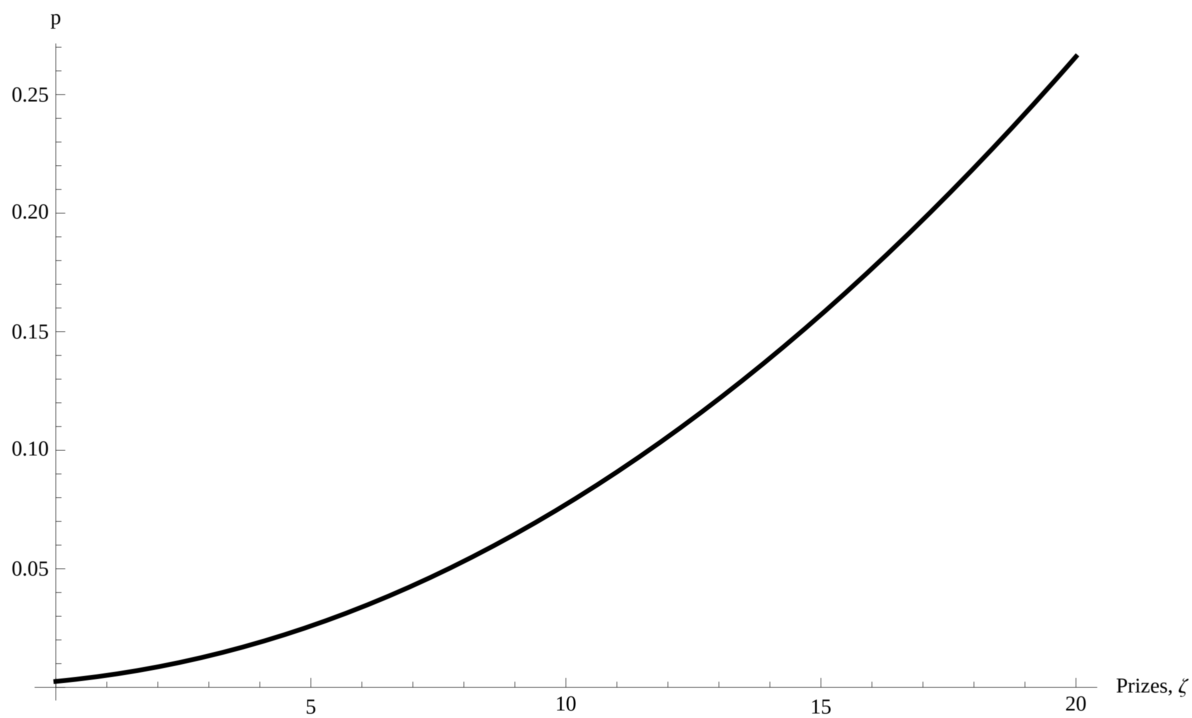

Figure 3: Turnout in a Fully Symmetric Competitive Election

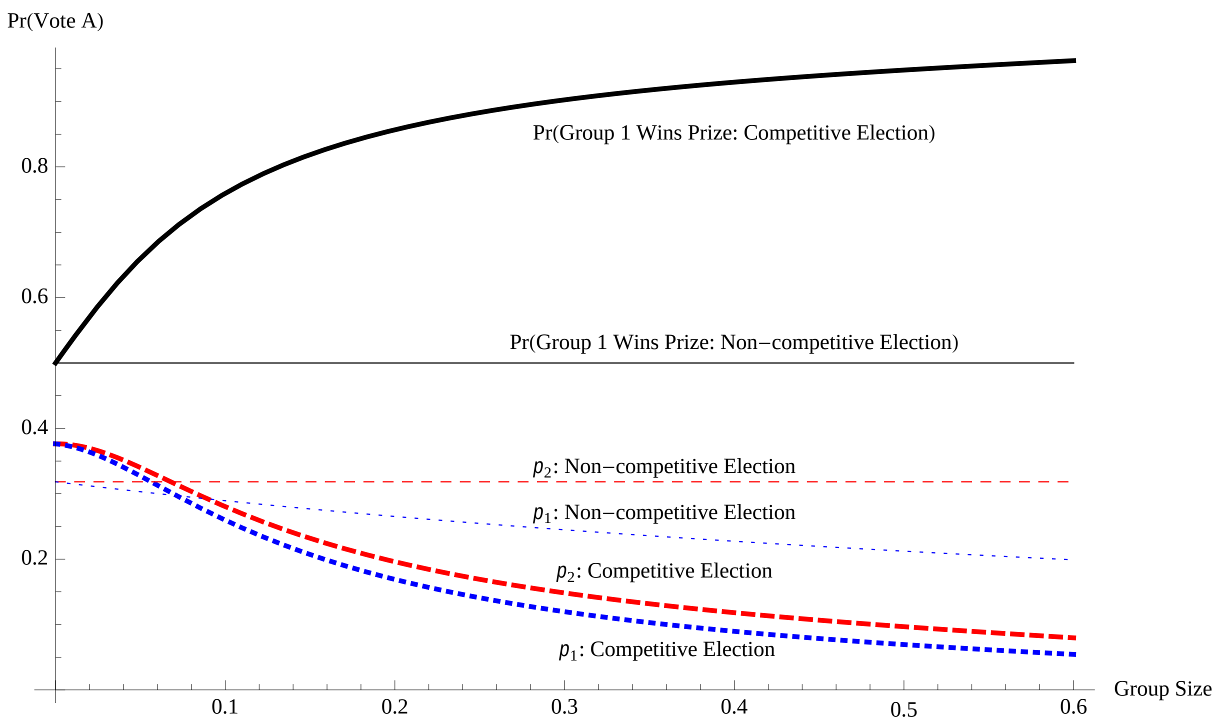

Figure 4: How Differences in Group Size Influence Turnout and Which Party Wins in Competitive and Non-Competitive Elections