ONE-SIDED UNCERTAINTY AND DELAY IN REPUTATIONAL BARGAINING

Dilip Abreu
Princeton University

David Pearce
New York University

and

Ennio Stacchetti
New York University

Preliminary and Incomplete; Not for Circulation

May 20, 2011

This research was supported in part by the National Science Foundation.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Introduction

Rubinstein (1982) delighted economists by establishing uniqueness of perfect equilibrium in an infinite horizon bargaining model. Once the surprise wore off, attention moved to another intriguing feature of the model: in the unique equilibrium, agreement is reached immediately. While this did not square well with some real-world phenomena (protracted haggling over prices, strikes in labor negotiations and so on), it was expected that introducing asymmetric information into the model would easily produce delay to agreement. If the purpose of holding out for a better deal is to signal the strength of one's bargaining position, then the existence of asymmetric information (without which there would be nothing to signal) might naturally be expected to go hand in hand with delay to agreement.

The asymmetric information bargaining literature did not unfold exactly as hoped. The early papers revealed a vast multiplicity of perfect Bayesian equilibria, even for one-sided asymmetric information (Rubinstein (1985)) or for only two periods in the case of bilateral informational asymmetry (Fudenberg and Tirole (1983)). More specific results relied on severely limited strategy spaces (Chatterjee and Samuelson (1987)), appeals to "reasonable" selections from the equilibrium correspondences (Sobel and Takahashi (1983), Cramton (1984), Chatterjee and Samuelson (1988)) or axiomatic restrictions of equilibrium (Rubinstein (1985) and Gul and Sonnenschein (1988)). The latter two papers study one-sided asymmetric information and produce solutions with a "Coasean" flavor:[1] the uninformed player, facing an opponent drawn from a distribution of payoff types, does as badly as she would if she instead faced, with certainty, the strongest possible opponent from that distribution. Moreover, there is virtually *no delay to agreement.* (Both these results apply to situations where offers can be made frequently.)

This paper investigates the effects of introducing behavioral perturbations[2] into a bargaining model with one-sided asymmetric information. Will this achieve equilibrium selection, as in Fudenberg and Levine (1989), Abreu and Gul (2000) and Abreu and Pearce (2007)? And if so, will the predictions agree with the axiomatic treatment of Rubinstein (1985)? The model we perturb is the leading example of the class covered by Rubinstein (1985). Player A is of known preferences, but she is unsure which of two discount rates player B uses to discount his payoff stream.

---

[1] Coase (1971) conjectured that when a durable goods monopolist faces buyers with a distribution of valuations, most sales will occur almost immediately, at a price near the infimum of that valuation distribution. Coase assumed that the monopolist was free to adjust prices frequently. In the 1980's a series of papers verified the conjecture with increasing conclusiveness and generality. See especially Stokey (1981), Bulow (1982), Fudenberg, Levine and Tirole (1985) and Gul, Sonnenschein and Wilson (1986), and for a critique of some of the axioms imposed, Ausubel and Deneckere (1989).

[2] A trio of papers (Kreps and Wilson (1982), Kreps, Milgrom, Roberts and Wilson (1982) and Milgrom and Roberts (1982)) opened the reputational literature by demonstrating the power that even small reputational perturbations could have in games with long or infinite horizons. Fudenberg and Levine (1989) show the success a patient long-run player can have against short-run opponents by choosing which of many reputational types to imitate. Much of the ensuing literature is surveyed authoritatively by Mailath and Samuelson (2006).

We begin in Section 2 by considering stationary behavioral types, who never waver from the demands they make at the beginning of the game. This is the class of types used by Myerson (1991) and Abreu and Gul (2000). We show that equilibrium is essentially unique: with high probability, play ends almost immediately, and the uninformed player's expected payoff is virtually what she would have received in a full information Rubinstein (1982) solution if her opponent were known to be the stronger (more patient) of the two possible rational types. This is in agreement with Rubinstein (1985), and reinforces the message of Inderst (2005), who showed that in a durable goods monopoly problem, endowing the monopolist with an ex ante reputation for (possibly) being a behavioral type, does *not* overturn Coase's predictions unless that ex ante probability is substantial. See Kim (2009) for extensions of that work.

Abreu and Pearce (2007) established that the stationary behavioral types are a "sufficiently rich" class[3] of perturbations to consider in stationary bargaining games (or more generally, in repeated games with contracts). There are reasons to doubt that this is true with asymmetric information. Cramton (1984) emphasizes the importance for an informed player of delaying his first offer, to signal strength (his Introduction opens with a dramatic illustration from military history). Accordingly, in Section 3 we expand the set of behavioral types for the informed player so that player B can use the tactic of delaying making an offer, without losing his reputation for being behavioral. This innovation turns out to be crucial. While equilibrium is still unique, it takes an entirely different form from Section 2. A hybrid equilibrium results from the patient player B trying to use delay to separate himself from the impatient version of B. For many parameter values, the uninformed player A does *better* than in the Coasean solution, and there is considerable expected *delay to agreement*.

Notably, the payoffs of the unique solution of Section 3 are continuous in the prior probability (call it α_2) that the informed player B is the more patient of his two rational types. Contrast this to Rubinstein (1985), where players' payoffs are discontinuous at $\alpha_2 = 0$ (the uninformed player's payoff is given by the full information Rubinstein solution with impatient player B when $\alpha_2 = 0$, whereas it is given roughly by the full information solution with the *patient* player B whenever α_2 is positive). The same troubling discontinuity is familiar from the durable goods monopoly problem, where only the lower support of the price distribution matters to the seller.

Atemporal Types

Two players bargain over the division of a surplus. Player A opens the game by demanding a share $a \in [0, 1]$ of the surplus. Having observed this demand, B can accept it or make a counteroffer $b \in (1 - a, 1]$ (interpreted as the share of the surplus that B demands). It is convenient to adopt a continuous/discrete time model in which a player can change his demand at any time in $\{1, 2, 3, \dots\}$, but can concede to an outstanding demand at any $t \in [0, \infty)$. This modeling device, introduce by Abreu and Pearce (2007),

[3]More precisely, there is no advantage to being able to imitate any type outside this class, even if your opponent can.

allows us to do war of attrition calculations in continuous time, while avoiding the usual pathologies that arise in continuous time games. If at any time $t \in \{1, 2, 3, \dots\}$ the players make (simultaneously) compatible demands (a, b) (that is, $a + b \leq 1$), one of the two divisions of surplus $(a, 1-a)$ or $(1-b, b)$ is implemented with probability 1/2 each, and the game ends.

The players can be "rational" or "behavioral" independently. Rational player A's discount rate is $r^A > 0$ and rational player B's discount rate is either r_1^B or r_2^B, where $r_1^B > r_2^B > 0$ and $\alpha_k = \mathbb{P}[r_k^B \mid \text{B is rational}]$, $k = 1, 2$. Each player's type (whether he is rational or behavioral and the value of his discount rate) is private information. Rational players are assumed to maximize the expected discounted value of their shares. If agreement is never reached, they both receive 0 payoffs. Behavioral types for A and B are represented by two finite sets $\mathbb{A}, \mathbb{B} \subset (0, 1)$. A behavioral player A of type $a \in \mathbb{A}$ makes the initial demand a, never changes her initial demand, and accepts (immediately) a counteroffer b if and only if $1 - b \geq a$. Behavioral types $\mathbb{B}$ are similarly defined. Player $i \in \{A, B\}$ is behavioral with probability z^i and for each $a \in \mathbb{A}$ and $b \in \mathbb{B}$,

$$\pi^A(a) = \mathbb{P}[a \mid \text{A is behavioral}] \quad \text{and} \quad \pi^B(b) = \mathbb{P}[b \mid \text{B is behavioral}].$$

We will denote this incomplete information game by $\Gamma(r, \alpha, z^A, z^B)$, where $r = (r^A, r_1^B, r_2^B)$ and $\alpha = (\alpha_1, \alpha_2)$. The parameters $\mathbb{A}$, $\mathbb{B}$, π^A and π^B are held fixed throughout.

Let $\bar{a} = \max \mathbb{A}$ and $\bar{b} = \max \mathbb{B}$. We assume that $[\min \mathbb{A}] + \bar{b} > 1$, $\bar{a} + [\min \mathbb{B}] > 1$, and that $\pi^A(a) > 0$ and $\pi^B(b) > 0$ for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$.

Hereafter, we find it convenient to call $\mathcal{A}$ the rational player A and $\mathcal{B}_k$ the rational player B with discount rate r_k^B, $k = 1, 2$. We will require that $\mathcal{A}$'s initial demand belongs to $\mathbb{A}$, and similarly that $\mathcal{B}_k$'s initial demand belongs to $\mathbb{B}$. There is no restriction on either player's subsequent demands.[4] In equilibrium, $\mathcal{A}$ chooses an initial posture $a \in \mathbb{A}$ with probability $\varphi^A(a)$, and after observing a, $\mathcal{B}_k$ chooses an initial posture $b \in \mathbb{B}$ with probability $\varphi_k^B(b|a)$. A pair of choices $(a, b) \in \mathbb{A} \times \mathbb{B}$ with $a + b > 1$ leads to the subgame $\Gamma(r, \hat{\alpha}_1(a, b), \hat{\alpha}_2(a, b), \hat{z}^A(a), \hat{z}^B(a, b), a, b)$, where

$$\begin{aligned}
\hat{z}^A(a) &= \frac{z^A \pi^A(a)}{z^A \pi^A(a) + (1 - z^A)\varphi^A(a)} \\
\hat{z}^B(a, b) &= \frac{z^B \pi^B(b)}{z^B \pi^B(b) + (1 - z^B)[\alpha_1 \varphi_1^B(b|a) + \alpha_2 \varphi_2^B(b|a)]} \\
\hat{\alpha}_k(a, b) &= \frac{(1 - z^B)\alpha_k \varphi_k^B(b|a)}{z^B \pi^B(b) + (1 - z^B)[\alpha_1 \varphi_1^B(b|a) + \alpha_2 \varphi_2^B(b|a)]} \qquad k = 1, 2.
\end{aligned}$$

are the posterior probabilities that player A is behavioral, and that player B is behavioral or $\mathcal{B}_k$, respectively. For simplicity, we will often omit the arguments (a, b) and simply write, for example, $\hat{z}^A$ and $\hat{z}^B$ instead of $\hat{z}^A(a)$ and $\hat{z}^B(a, b)$.

---

[4]This is without loss of generality. The choice of $\bar{a} \in \mathbb{A}$ for $\mathcal{A}$ weakly dominates any $a \notin \mathbb{A}$, and when A demands $a \in \mathbb{A}$, a counteroffer $b \notin \mathbb{B}$ yields a unique equilibrium in which $\mathcal{B}_k$ concedes to a right away with probability 1, $k = 1, 2$. This is another expression of Coasean dynamics (see Section 8.8 of Myerson (1991), Proposition 4 of Abreu and Gul (2000) and Lemma 1 of Abreu and Pearce (2007)).

For any $(a,b) \in \mathbb{A} \times \mathbb{B}$ with $a+b>1$, the subgame $\Gamma(r, \hat{\alpha}_1, \hat{\alpha}_2, \hat{z}^A, \hat{z}^B, a, b)$ is played as a war of attrition (WOA) and has a unique equilibrium, similar to the equilibrium obtained by Abreu and Gul (2000). After adopting the initial postures (a,b), each rational player randomly chooses a time to accept the opponent's demand (if the opponent has not accepted already). A rational player has the option of changing his initial demand at any $t \in \{1, 2, \dots\}$. But doing so would reveal that he is rational and in the unique equilibrium of the continuation game he should concede to the opponent's demand immediately.[5] Let $\mu^i = (1-\hat{z}^i) \times$ (probability that i concedes at time 0) be the (total) probability that $i \in \{A, B\}$ concedes at time 0. Then $\mu^A \cdot \mu^B = 0$ since when $\mu^B > 0$, for example, $\mathcal{A}$ strictly prefers to wait at time 0. If $0 < \mu^B \leq \hat{\alpha}_1$, then only $\mathcal{B}_1$ concedes immediately with positive probability, but if $\mu^B > \hat{\alpha}_1$, then $\mathcal{B}_1$ concedes immediately with probability 1 and $\mathcal{B}_2$ concedes immediately with positive probability. Thus, if the players do not concede immediately, the relevant posteriors become

$$\hat{z}^i(0) = \frac{\hat{z}^i}{1-\mu^i} \quad i = A, B, \quad \hat{\alpha}_2(0) = \min\left\{\frac{\hat{\alpha}_2}{1-\mu^B}, 1-\hat{z}^B(0)\right\},$$

and $\hat{\alpha}_1(0) = 1 - \hat{z}^B(0) - \hat{\alpha}_2(0)$. After time 0, the WOA is divided into two intervals $(0, \tau_1]$ and $(\tau_1, \tau_2]$. In $(0, \tau_1]$, A concedes at a constant Poisson rate λ_1^A, and in $(\tau_1, \tau_2]$, A concedes at constant Poisson rate λ_2^A, while B concedes at constant Poisson rate λ^B in the whole interval $(0, \tau_2]$, where

$$\lambda_k^A(a,b) = \frac{r_k^B(1-a)}{a+b-1} \quad k = 1, 2, \qquad \lambda^B(a,b) = \frac{r^A(1-b)}{a+b-1}.$$

(Again, we will often omit the arguments (a,b) for λ^B and λ_k^A, $k = 1, 2$, when the relevant (a,b) are clear from context.) If $\hat{\alpha}_1(0) = 0$ then $\tau_1 = 0$, and if $\hat{\alpha}_2 = 0$ then $\tau_2 = \tau_1$. If $\hat{\alpha}_1(0) = 0$ or $\hat{\alpha}_2(0) = 0$, the WOA is exactly that studied by Abreu and Gul (2000). When $\hat{\alpha}_1 > 0$, λ_1^A keeps $\mathcal{B}_1$ indifferent between conceding and waiting, but at τ_1 A becomes convinced that she is not dealing with $\mathcal{B}_1$, and switches to the concession rate λ_2^A that keeps $\mathcal{B}_2$ indifferent between conceding and waiting. The concession rate λ^B keeps $\mathcal{A}$ indifferent in $(0, \tau_2]$. The players' reputations (that is, the posteriors that they are behavioral) grow exponentially over time:

$$\hat{z}^B(\tau) = \hat{z}^B(0)e^{\lambda^B \tau} \quad \text{and} \quad \hat{z}^A(\tau) = \begin{cases} \hat{z}^A(0)e^{\lambda_1^A \tau} & \tau \in (0, \tau_1] \\ \hat{z}^A(\tau_1)e^{\lambda_2^A \tau} & \tau \in (\tau_1, \tau_2]. \end{cases}$$

At time τ_2 both players' reputations reach 1 simultaneously. Given μ^A and μ^B, τ_1 and τ_2 are defined by

$$[\hat{z}^B(0) + \hat{\alpha}_2(0)]e^{\lambda^B \tau_1} = 1 \quad \text{and} \quad \hat{z}^B(0)e^{\lambda^B \tau_2} = 1.$$

[5]The argument is as in Footnote 4. This argument presumes that when a player reveals rationality, his opponent does not do so at the same time. It is easy to show that in equilibrium both players cannot reveal rationality with positive probability at the same time. See Lemma 5 for an explicit treatment in the (novel) context of the temporal model.

If at least one player is rational, the subgame ends in agreement randomly in the interval $[0, \tau_2]$ with probability 1; if both players are behavioral, the subgame never ends.

The next four lemmas analyze the game by backward induction. Lemma 1 summarizes the solution of the subgame we have been discussing, taking the offers made and the updated beliefs as parameters. The expression L in the lemma captures the relative "strengths" of the two players' positions: unless the situation is perfectly balanced, where $L = 1$, one of the players is "weak", and needs to concede with positive probability at time zero. Player A's position is weakened by any of the following changes: an increase in her demand or a decrease in B's demand, an increase in her rate of interest or a decrease in either r_1^B or r_2^B, or a decrease in her reputation $\hat{z}^A$ or an increase in $\hat{z}^B$.

In Lemma 1 below recall the definitions of μ^A, μ^B, λ_k^A, $k = 1, 2$, and λ^B from above.

Lemma 1. *$\Gamma(r, \hat{\alpha}_1, \hat{\alpha}_2, \hat{z}^A, \hat{z}^B, a, b)$ with $(a, b) \in \mathbb{A} \times \mathbb{B}$ and $a + b > 1$ has a unique equilibrium. Let*

$$L = \frac{[\hat{z}^A]^{\lambda^B}}{[\hat{z}^B + \hat{\alpha}_2]^{\lambda_1^A - \lambda_2^A} [\hat{z}^B]^{\lambda_2^A}}.$$

When $L \leq 1$, $\mu^B = 0$ and

$$\mu^A = 1 - L^{1/\lambda^B} = 1 - \frac{\hat{z}^A}{[\hat{z}^B + \hat{\alpha}_2]^{(\lambda_1^A - \lambda_2^A)/\lambda^B} [\hat{z}^B]^{\lambda_2^A/\lambda^B}}.$$

When $L \geq 1$, $\mu^A = 0$ and

$$\mu^B = \begin{cases} 1 - \hat{z}^B/[\hat{z}^A]^{\lambda^B/\lambda_2^A} & \text{if } \ 1 - \hat{z}^B/[\hat{z}^A]^{\lambda^B/\lambda_2^A} \geq \hat{\alpha}_1 \\ 1 - 1/L^{1/\lambda_1^A} & \text{otherwise.} \end{cases}$$

At time τ_2, $\hat{z}^A(\tau_2) = \hat{z}^B(\tau_2) = 1$.

Remark: Note that when $\hat{\alpha}_2 = 0$ or $\hat{\alpha}_1 = 0$, the WOA reduces to that studied by Abreu and Gul (2000) where there is only one type of rational player B. In this case,

$$L = \frac{[\hat{z}^A]^{\lambda^B}}{[\hat{z}^B]^{\lambda_1^A}} \quad \text{if } \hat{\alpha}_2 = 0, \quad \text{and} \quad L = \frac{[\hat{z}^A]^{\lambda^B}}{[\hat{z}^B]^{\lambda_2^A}} \quad \text{if } \hat{\alpha}_1 = 0.$$

Proof. Most of the results follow directly from the analysis in Abreu and Gul (2000). Thus, here we only deduce the value of μ^B when $L > 1$. When $\mu^B \geq \hat{\alpha}_1$, $\mathcal{B}_1$ concedes immediately with probability 1 (and $\mathcal{B}_2$ concedes immediately with nonnegative probability). Thus, if B does not concede immediately, A concludes that she is dealing with $\mathcal{B}_2$ or a behavioral type. Consequently, $\tau_1 = 0$ and A concedes to b at a constant Poisson rate λ_2^A in the interval $(0, \tau_2]$. Thus

$$\frac{\hat{z}^B}{1 - \mu^B} e^{\lambda^B \tau_2} = 1 \quad \text{and} \quad \hat{z}^A e^{\lambda_2^A \tau_2} = 1.$$

These equations imply that

$$1-\mu^B = \hat{z}^B/[\hat{z}^A]^{\lambda^B/\lambda_2^A}.$$

When $\mu^B < \hat{\alpha}_1$, $\mathcal{B}_1$ concedes immediately with probability less than 1, and $\tau_1 > 0$. In this case,

$$\left[\frac{\hat{\alpha}_2+\hat{z}^B}{1-\mu^B}\right]e^{\lambda^B\tau_1}=1, \quad \frac{\hat{z}^B}{1-\mu^B}e^{\lambda^B\tau_2}=1 \quad \text{and} \quad \hat{z}^A e^{\lambda_1^A\tau_1+\lambda_2^A(\tau_2-\tau_1)}=1.$$

These equations imply that

$$1-\mu^B = \left[\frac{1}{L}\right]^{1/\lambda_1^A}$$

□

Lemma 2 concerns the limiting properties of equilibrium after offers have been made, as the initial reputations approach zero (in any manner not violating an arbitrarily loose bound). The striking result here is that player B's strength or weakness is affected neither by the interest rate r_1^B of his more impatient rational type, nor by the probability $\hat{\alpha}_1$ of that type. Only r_2^B and $\hat{\alpha}_2$ contribute to his strength (along with the impatience of player A). This is explained by the fact that for $\hat{z}^A$ and $\hat{z}^B$ very small, almost 100% of the war of attrition will be spent in the second phase (see the paragraphs preceding Lemma 1 above), in which A faces the more patient type of B (if B is rational). To understand why, consider the following example. Fix $\hat{\alpha}_1$, the probability that B is the impatient rational type, at .9; the residual probability is divided between the probability (bounded above by .1) that B is the patient rational type, and $\hat{z}^B$, the probability B is behavioral. Absent any concessions at time 0, it takes a fixed amount of time τ_1 (dependent on $\hat{\alpha}_1$, which we will not change) to finish the first stage of the war of attrition (given the rate λ^B at which B needs to concede to A, Bayes' Rule determines the time τ_1 at which nine tenths of the B population, that is, all the impatient ones, will have conceded). Now let $\hat{z}^B$ approach zero. The length of the entire war of attrition grows without bound, but the first stage is not increasing in length. Even though the impatient type was more abundant than the patient type at time zero, A spends almost 100% of the war of attrition fighting the patient type, when $\hat{z}^B$ is negligible. For this reason, λ_2^A appears in the statement of Lemma 2, whereas λ_1^A does not.

For any $R > 1$ and $\bar{z} > 0$, define the cone and truncated cone

$$K(R) = \{(z^A, z^B) \mid z^A > 0, z^B > 0 \text{ and } \max\{z^A/z^B, z^B/z^A\} \le R\},$$
$$K(R,\bar{z}) = \{(z^A, z^B) \in K(R) \mid z^A \le \bar{z} \text{ and } z^B \le \bar{z}\}.$$

Lemma 2. *Let $R > 1$ and $\{z^\ell\} \subset K(R)$ be a sequence such that $z^\ell = (z^{A\ell}, z^{B\ell}) \downarrow (0,0)$. For each ℓ, let φ^ℓ be a PBE of $\Gamma(r,\alpha,z^\ell)$. Assume that $\varphi^\ell \to \varphi^\infty$ in $\mathbb{R}^M \times \mathbb{R}^{N\times M}$. For a given $(a,b) \in \mathbb{A}\times\mathbb{B}$ with $a+b>1$, consider the corresponding subgames $\Gamma(r,\hat{\alpha}^\ell,\hat{z}^\ell,a,b)$. Let $(\hat{z}^{B\infty},\hat{\alpha}_1^\infty,\hat{\alpha}_2^\infty)$ be the limit of $\{(\hat{z}^{B\ell},\hat{\alpha}_1^\ell,\hat{\alpha}_2^\ell)\}$.*

(i) *If* $\varphi_1^{B\infty}(b|a) + \varphi_2^{B\infty}(b|a) > 0$ *and* $\lambda_2^A > \lambda^B$, *then* $\mu^{B\ell} \to 1$.
(ii) *If* $\varphi^{A\infty}(a) > 0$ *and* $\hat{z}^{B\infty} > 0$, *then* $\mu^{A\ell} \to 1$.
(iii) *If* $\varphi^{A\infty}(a) > 0$, $\hat{\alpha}_2^{B\infty} > 0$ *and* $\lambda^B > \lambda_2^A$, *then* $\mu^{A\ell} \to 1$.

Proof. See Appendix.

For each $a \in (0,1)$ and $k = 1, 2$, let

$$b_k^*(a) = \max\left\{1 - a, 1 - \frac{r_k^B}{r^A}(1-a)\right\}.$$

Assume A offers $a \in (0,1)$. When $b_k^*(a) > 1 - a$, $b_k^*(a)$ is the "balanced counter-demand" that equalizes the Poisson rates of concessions when A only faces $\mathcal{B}_k$ (and behavioral types):

$$\lambda_k^A(a, b_k^*(a)) = \frac{r_k^B(1-a)}{a + b_k^*(a) - 1} = \frac{r^A(1 - b_k^*(a))}{a + b_k^*(a) - 1} = \lambda^B(a, b_k^*(a)).$$

When the demand a is too modest, any counter-demand $b > 1 - a$ is excessive, that is, yields $\lambda_k^A > \lambda^B$. In this case we define $b_k^*(a) = 1 - a$. For $k = 1, 2$, we assume that

$$b_k^*(a) \notin \mathbb{B} \quad \text{and} \quad \min \mathbb{B} < b_k^*(a) \quad \text{for all } a \in \mathbb{A}.$$

Let

$$\lfloor b_k^*(a) \rfloor = \max\left\{1 - a, \max\{b \in \mathbb{B} \mid b < b_k^*(a)\}\right\} \quad \text{and} \quad \tilde{a}^* \in \text{argmin}_{a \in \mathbb{A}} \lfloor b_2^*(a) \rfloor.$$

For simplicity,[6] we assume that the argmin is a singleton. Observe that when $\lfloor b_k^*(a) \rfloor > 1 - a$, it is the largest behavioral demand $b \in \mathbb{B}$ such that $\lambda^B(a,b) > \lambda_k^A(a,b)$. Furthermore, if $r_1^B > r_2^B$ (as assumed), there clearly exists $\underline{\Delta} > 0$ such that $\lambda_1^A(a, b_2^*(a) - \Delta) > \lambda^B(a, b_2^*(a) - \Delta)$ for all $\Delta \leq \underline{\Delta}$ and $a \in \mathbb{A}$. We will assume throughout that the grid of types $\mathbb{B}$ is fine enough that $\lfloor b_2^*(a) \rfloor > b_2^*(a) - \underline{\Delta}$ for all $a \in \mathbb{A}$.

Suppose that the reputational perturbations z^A and z^B are very slight. Once A has made an equilibrium demand $a \in \mathbb{A}$, with high probability player B responds by demanding the highest amount $b \in \mathbb{B}$ such that b is less greedy than the balanced demand $b_2^*(a)$. Any demand higher than this, if offered with noticeable probability in equilibrium, would leave B in a weak position, from which he would need to concede with probability near 1. Lemma 3 establishes the payoff consequences for each player.

[6] Our results can be rephrased throughout – at the cost of some clumsiness in the statements and proofs – for the general case.

Lemma 3. *For any $R > 1$ and $\epsilon > 0$, there exists $\bar{z} > 0$ such that for all $z \in K(R, \bar{z})$, for any Perfect Bayesian equilibrium (φ^A, φ^B) of $\Gamma(r, \alpha, z)$, and for any $a \in \mathbb{A}$, the corresponding expected payoff for $\mathcal{A}$ satisfies*

$$v^A(a, z) \geq 1 - \lfloor b_2^*(a) \rfloor - \epsilon.$$

Moreover, if $\varphi^A(a) \geq \epsilon$, then

$$v_k^B(a, z) \geq \lfloor b_2^*(a) \rfloor - \epsilon \quad k = 1, 2.$$

Proof. **Step 1**: To simplify notation, let $\hat{b}_2 = \lfloor b_2^*(a) \rfloor$. For the first part, assume by way of contradiction that there exist a sequence $\{z^\ell\}$, $z^\ell = (z^{A\ell}, z^{B\ell}) \to 0$, $a \in \mathbb{A}$ and a corresponding sequence of PBE $\{\varphi^\ell\}$ such that $v^{A\ell}(a, z^\ell) < 1 - \hat{b}_2 - \epsilon$ for all ℓ. Without loss of generality, we can also assume that $\varphi^\ell \to \varphi^\infty$ in $\mathbb{R}^M \times \mathbb{R}^{N \times M}$. For each ℓ and $b \in \mathbb{B}$, consider the corresponding subgame $\Gamma(r, \hat{\alpha}^\ell(a, b), \hat{z}^{A\ell}(a), \hat{z}^{B\ell}(a, b), a, b)$. Clearly $\mathcal{A}$ is guaranteed a payoff of at least $1 - b$ in this subgame (since $\mathcal{A}$ can always concede to b). Therefore, $\mathcal{A}$'s payoff is at least $1 - \hat{b}_2$ whenever $b \leq \hat{b}_2$. Suppose now that $b \in \mathbb{B}$ is such that $b > \hat{b}_2$ and $\varphi_k^{B\infty}(b|a) > 0$ for $k = 1$ or $k = 2$. Since $b > \hat{b}_2$, $\lambda_2^A(a, b) > \lambda^B(a, b)$. Then, Lemma 2 *(i)* implies that lim $\mu^{B\ell}(a, b) = 1$ and $\mathcal{A}$'s total expected payoff in the subgame after the demands (a, b) are made is bounded below by $(1 - z^{B\ell})a + z^{B\ell}(1 - b) - \epsilon/4 \geq 1 - \hat{b}_2 - \epsilon/2$ for all ℓ sufficiently large since $a \geq 1 - \hat{b}_2$. Finally, if $b \in \mathbb{B}$ is such that $b > \hat{b}_2$ and $\varphi_k^{B\infty}(b|a) = 0$ for $k = 1, 2$, then $\mathcal{A}$'s expected payoff is only bounded below by 0, but the probability of reaching the subgame with offers (a, b) is $z^{B\ell}\pi^B(b)$. Thus, $\mathcal{A}$'s total expected payoff after making the demand a is bounded below by $1 - \hat{b}_2 - \epsilon$ for all ℓ sufficiently large, a contradiction.

Step 2: For the second part, assume again by contradiction that there exist a sequence $\{z^\ell\}$, $z^\ell = (z^{A\ell}, z^{B\ell}) \to 0$, $a \in \mathbb{A}$ and a corresponding sequence of PBE $\{\varphi^\ell\}$ such that $\varphi^{A\ell}(a) \geq \epsilon$ and for either $k = 1$ or $k = 2$, $v_k^B(a, z^\ell) < \hat{b}_2 - \epsilon$ for all ℓ. Without loss of generality, assume that $\varphi^\ell \to \varphi^\infty$.

For each $b \in \mathbb{B}$ consider the corresponding subgame $\Gamma(r, \hat{\alpha}^\ell(a, b), \hat{z}^\ell(a, b), a, b)$. Assume that $\varphi_2^{B\infty}(b|a) > 0$. Then $\hat{\alpha}_2^\ell(a, b) \to \alpha_2\varphi_2^{B\infty}(b|a)/[\alpha_1\varphi_1^{B\infty}(b|a) + \alpha_2\varphi_2^{B\infty}(b|a)] > 0$. Furthermore, if $b = \hat{b}_2$, then $\lambda^B(a, b) > \lambda_2^A(a, b)$. Then, Lemma 2 *(iii)* implies that $\mu^{A\infty}(a, b) = 1$ and consequently $v_2^B(a, z^\ell) \to \lfloor b_2^*(a) \rfloor$. As $\mathcal{B}_1$ could also choose to counteroffer $\hat{b}_2$ (possibly out of equilibrium), $\lim v_1^B(a, z^\ell) \geq \hat{b}_2$. But, since $\mathcal{B}_2$'s payoff must weakly exceed $\mathcal{B}_1$'s, $v_1^B(a, z^\ell) \leq v_2^B(a, z^\ell)$ for all ℓ, and it follows that $v_1^B(a, z^\ell) \to \hat{b}_2$ as well. To complete the proof, we establish that $\varphi_2^{B\infty}(b|a) = 0$ for all $b \neq \hat{b}_2$.

Step 3: Consider any $b \in \mathbb{B}$ with $b < \hat{b}_2$ and suppose that $\varphi_2^{B\infty}(b|a) > 0$. For the corresponding subgames $\Gamma(r, \hat{\alpha}^\ell(a, b), \hat{z}^\ell(a, b), a, b)$, without loss of generality, assume that $(\hat{z}^{B\ell}(a, b), \hat{\alpha}_1^\ell(a, b), \hat{\alpha}_2^\ell(a, b)) \to (\hat{z}^{B\infty}(a, b), \hat{\alpha}_1^\infty(a, b), \hat{\alpha}_2^\infty(a, b))$. Then $\hat{\alpha}_2^\infty(a, b) > 0$. Furthermore, $\lambda^B(a, b) > \lambda_2^A(a, b)$. Then, by Lemma 2(iii), $\mu^{A\infty}(a, b) = 1$ and $v_2^B(a, z^\ell) \to b$. As in Step 2, we may also conclude that $v_1^B(a, z^\ell) \to b$. Now consider $\hat{b}_2$. If $\hat{z}^{B\infty}(a, \hat{b}_2) >$

0, then by Lemma 2(ii), $\mu^{A\infty}(a,\hat{b}_2) = 1$, which contradicts $v_2^B(a,\hat{z}^\ell) \to b < \hat{b}_2$. Hence $\hat{z}^{B\infty}(a,\hat{b}_2) = 0$. If $\varphi_1^{B\infty}(\hat{b}_2|a) = 0$ then $\hat{\alpha}_2^\infty(a,\hat{b}_2) = 1$. Note that $\hat{z}^{B\infty}(a,\hat{b}_2) + \hat{\alpha}_1^\infty(a,\hat{b}_2) + \hat{\alpha}_2^\infty(a,\hat{b}_2) = 1$. Then, By Lemma 2 *(ii)*, $\mu^{A\infty}(a,\hat{b}_2) = 1$, which yields a contradiction as before. Hence $\varphi_1^{B\infty}(\hat{b}_2|a) > 0$ and $\varphi_1^{B\ell}(\hat{b}_2|a) > 0$ for large ℓ. Therefore

$$\mu^{A\ell}(a,\hat{b}_2)\hat{b}_2 + (1-\mu^{A\ell}(a,\hat{b}_2))(1-a) \approx b,$$

which implies that $\mu^{A\infty}(a,\hat{b}_2) < 1$. Let τ_1^ℓ be the time until which A concedes at rate $\lambda_1^A(a,\hat{b}_2)$ in equilibrium φ^ℓ (see Lemma 1 for a definition), and let $E^\ell(\rho) = e^{-(\rho+\lambda_1^A(a,\hat{b}_2))\tau_1^\ell}$. Then

$$\int_0^{\tau_1^\ell} e^{-\rho s}\lambda_1^A(a,\hat{b}_2)e^{-\lambda_1^A(a,\hat{b}_2)s}ds = \frac{\lambda_1^A(a,\hat{b}_2)}{\rho+\lambda_1^A(a,\hat{b}_2)}(1-E^\ell(\rho)).$$

If $\mathcal{B}_2$ mimics $\hat{b}_2$, he obtains a payoff of

$$\tilde{v}_2^{B\ell} = \mu^{A\ell}(a,\hat{b}_2)\hat{b}_2 + (1-\mu^{A\ell}(a,\hat{b}_2))\left[\frac{\lambda_1^A(a,\hat{b}_2)}{r_2^B+\lambda_1^A(a,\hat{b}_2)}(1-E^\ell(r_2^B))\hat{b}_2 + E^\ell(r_2^B)(1-a)\right]$$

Recall that $[\hat{\alpha}_2^{B\ell} + \hat{z}^{B\ell}]e^{\lambda^B\tau_1^\ell} = 1$, so $E^\ell(\rho) = [\hat{\alpha}_2^{B\ell} + \hat{z}^{B\ell}]^{(\rho+\lambda_1^A(a,\hat{b}_2))/\lambda^B(a,\hat{b}_2)}$. Then, $\varphi_1^{B\infty}(\hat{b}_2|a) > 0$ implies that $\lim E^\ell(r_2^B) < 1$. Since

$$\frac{\lambda_1^A(a,\hat{b}_2)}{r_1^B+\lambda_1^A(a,\hat{b}_2)}\hat{b}_2 = 1-a$$

and $\lambda_1^A/(r_1^B+\lambda_1^A) < \lambda_1^A/(r_2^B+\lambda_1^A)$, we have that $\tilde{v}_2^{B\infty} > b$, a contradiction.

Step 4: Finally, consider any $b \in \mathbb{B}$ with $b > \hat{b}_2$, and suppose that $\varphi_2^{B\infty}(b|a) > 0$. Now $\lambda_2^A(a,b) > \lambda^A(a,b)$ and by Lemma 2(i), $\mu^{B\infty}(a,b) = 1$. Thus, $v_2^B(a,\hat{z}^\ell) \to 1-a$, and hence $v_1^B(a,\hat{z}^\ell) \to 1-a$ also. Now consider $\hat{b}_2$. As in Step 3 we conclude that $\varphi_1^{B\infty}(\hat{b}_2|a) > 0$. Now consider $\mathcal{B}_2$'s payoff from mimicking $\hat{b}_2$. If $\hat{z}^{B\infty}(a,\hat{b}_2) + \hat{\alpha}_2^\infty(a,\hat{b}_2) > 0$, then by Lemma 2 *(ii)* or *(iii)*, $\mu^{A\infty}(a,\hat{b}_2) = 1$, which contradicts $v_2^B(a,\hat{z}^\ell) \to 1-a$. Hence $\hat{\alpha}_1^\infty(a,\hat{b}_2) = 1$. Furthermore, $\mu^{A\infty}(a,\hat{b}_2) = 0$. Now we can simply repeat the end of Step 3 (which merely uses $\mu^{A\infty}(a,\hat{b}_2) < 1$) to conclude that $\tilde{v}_2^{B\infty} > 1-a = \lim v_2^B(a,\hat{z}^\ell)$, a contradiction. □

Recall that $\tilde{a}* = \arg\max \lfloor b_2^*(a) \rfloor$.

Corollary. *For any $R \in (0,\infty)$ and $\epsilon > 0$, there exists $\bar{z} > 0$ such that for all $z \in K(R,\bar{z})$, and for any Perfect Bayesian equilibrium (φ^A,φ^B) of $\Gamma(r,\alpha,z)$, $\varphi^A(a) < \epsilon$ for all $a \neq \tilde{a}^*$.*

Theorem 1. *For any $R \in (0,\infty)$ and $\epsilon > 0$, there exists $\bar{z} > 0$ such that for all $z \in K(R,\bar{z})$, and for any Perfect Bayesian equilibrium (φ^A,φ^B) of $\Gamma(r,\alpha,z)$,*

$$v^A(z) \geq 1 - \lfloor b_2^*(\tilde{a}^*) \rfloor - \epsilon \qquad \textit{and} \qquad v_k^B(z) \geq \lfloor b_2^*(\tilde{a}^*) \rfloor - \epsilon, \quad k = 1,2.$$

Let a^* be such that $r^A a^* = r_2^B(1-a^*)$. The demand a^* is such that its balanced counter-demand proposes the same partition: $b_2^*(a^*) = 1-a^*$. Note that if the grids of types $\mathbb{A}$ and $\mathbb{B}$ are fine then $\tilde{a}^* \approx a^*$ and $\lfloor b_2^*(\tilde{a}^*) \rfloor \approx 1-a^*$, and more generally, $\lfloor b_2^*(a) \rfloor \approx b_2^*(a)$ for all $a \in \mathbb{A}$.

Temporal Types

In the model of Section 2, there was no scope for a rational player B to signal his patience: delaying making a demand would reveal his rationality, giving A the decisive reputational advantage. Here, we remedy the situation in the simplest possible way, allowing for reputational types of the informed player that wait a variety of lengths of time before making a demand. It turns out that in equilibrium, B can now signal that he is either behavioral, or the patient rational type. This can have dramatic implications for the payoffs achieved by the two players, showing that in this asymmetric information environment, simple atemporal types are not canonical in the sense of Footnote 3 (whereas they are in the symmetric information settings of Abreu and Pearce (2007)).

The temporal model differs from the atemporal model in that player B is now allowed to make his initial counteroffer with delay. Once player A makes her initial demand a, player B can accept it or wait until some time $t \in [0, \infty)$ to make a counter-demand $b \in (1-a, 1]$. Similarly to the previous model, once the counter-demand b is made, the players can change their (counter)demands only at times $\{t+1, t+2, \dots\}$, but can concede to an outstanding demand at any time $\tau \in [t, \infty)$. Rational players $\mathcal{A}$ and $\mathcal{B}_k$, $k = 1, 2$, have discount rates r^A and r^B_k, $k = 1, 2$, respectively, where $r^B_1 > r^B_2$. Behavioral players A are represented by $\mathbb{A}$ as before (with the same interpretation), but behavioral types for B are now represented by the set $\mathbb{B} \times [0, \bar{T}]$, where $\bar{T}$ is a sufficiently long time (as we discuss later). A behavioral player B of type (b, t) makes his initial counter-demand b at time t, never changes his demand, and concedes (immediately) to a demand a if and only if $1 - a \geq b$. We also assume that behavioral types for B are "reactive" in the sense that if player A puts herself in a vulnerable position by changing her initial demand before player B has made a counter-demand, then a behavioral type (b, t) becomes more aggressive and immediately demands $\bar{b}$ (and only accepts a demand $a \leq 1 - \bar{b}$).

Let z^i be the probability that player $i \in \{A, B\}$ is behavioral, $\pi^A(a)$ be the conditional probability that A is type $a \in \mathbb{A}$ given that she is behavioral, and $\pi^B(b, t)$ be the conditional probability density that B is type $(b, t) \in \mathbb{B} \times [0, \bar{T}]$ given that he is behavioral. We assume that $\pi^B(b, t)$ is continuous in t for each $b \in \mathbb{B}$, and that there exists $\underline{\pi} > 0$ such that

$$\pi^A(a) \geq \underline{\pi} \quad \text{and} \quad \pi^B(b, t) \geq \underline{\pi} \quad \text{for all} \quad a \in \mathbb{A} \text{ and } (b, t) \in \mathbb{B} \times [0, \bar{T}].$$

We denote this game by $\Gamma(r, \alpha, z)$.

As before, we assume that $[\min \mathbb{A}] + \bar{b} > 1$ and conversely. Furthermore, we assume that

$$\bar{b} e^{-r^B_2 \bar{T}} < 1 - \bar{a},$$

so that the more patient player $\mathcal{B}_2$ (and therefore $\mathcal{B}_1$ as well) would prefer to accept the demand $\bar{a}$ immediately to waiting until after $\bar{T}$ to make the counterdemand $\bar{b}$, even if $\bar{b}$ were then immediately accepted by A. This is what we meant earlier by $\bar{T}$ being sufficiently long.

As in the previous section and for the same reasons, we assume that the initial demands of rational players are compatible with them being behavioral. In equilibrium, $\mathcal{A}$

chooses $a \in \mathbb{A}$ with probability $\varphi^A(a)$. For each demand $a \in \mathbb{A}$, player $\mathcal{B}_k$ either chooses to accept a immediately or to mimic a behavioral type $(b,t) \in \mathbb{B} \times [0,\bar{T}]$ with probability density $\varphi^B_k(b,t|a)$, $k=1,2$. Consequently, after the demand $a \in \mathbb{A}$ is made, A is behavioral with posterior probability

$$\hat{z}^A(a) = \frac{z^A \pi^A(a)}{z^A \pi^A(a) + (1-z^A)\varphi^A(a)}, \tag{1}$$

and after the counter-demand $(b,t) \in \mathbb{B} \times [0,\bar{T}]$, B is behavioral or $\mathcal{B}_k$, $k=1,2$, with posterior probabilities

$$\hat{z}^B(a,b,t) = \frac{z^B \pi^B(b,t)}{z^B \pi^B(b,t) + (1-z^B)[\alpha_1 \varphi^B_1(b,t|a) + \alpha_2 \varphi^B_2(b,t|a)]} \quad \text{and} \tag{2}$$

$$\hat{\alpha}_k(a,b,t) = \frac{(1-z^B)\alpha_k \varphi^B_k(b,t|a)}{z^B \pi^B(b,t) + (1-z^B)[\alpha_1 \varphi^B_1(b,t|a) + \alpha_2 \varphi^B_2(b,t|a)]} \quad k=1,2, \tag{3}$$

The analysis in the preceding Section leads one to suspect, correctly, that each side will eventually imitate a behavioral type, and a war of attrition (or an immediate probabilistic concession) ensues. Lemma 5 establishes the payoff consequences of being the first to reveal rationality. Lemma 4 points out that before player B has spoken, A is in a particularly delicate situation: if she reveals rationality, she expects B to act like the most aggressive behavioral type (because he is in a winning position no matter what types he imitates). The only exception to this expectation is if matters are even worse for A, because the equilibrium expectation if B responds to As revealing rationality by revealing rationality himself, gives A less than $1-\bar{b}$ (that is, the equilibrium expectation is a particularly adverse selection for A from the set of equilibria of the full information subgame.[7])

Lemma 4. *If player $\mathcal{A}$ (who chooses $a \in \mathbb{A}$ at $t=0$) reveals rationality before B makes a counterdemand, then $\mathcal{A}$'s continuation payoff is at most $1-\bar{b}$.*

Lemma 5. *Suppose A demands $a \in \mathbb{A}$ at time $t=0$ and B counterdemands $(b,t) \in \mathbb{B} \times [0,\bar{T}]$ where $a+b>1$. Suppose neither player has revealed rationality prior to time $s \geq t$ and that revealing rationality at s is in the support of $\mathcal{A}$'s equilibrium strategy. Then, if $\mathcal{A}$ reveals rationality at s and B has not, $\mathcal{A}$'s resulting equilibrium continuation payoff is at most $1-b$ and B's is at least b. An analogous conclusion holds when B is the first to reveal rationality at s. Moreover, in equilibrium, if A reveals rationality with positive probability at s, then B does not, and conversely.*

Consider a particular equilibrium of the game, and the subgame after A has made some demand $a \in \mathbb{A}$. In the subgame, there are expected discounted equilibrium payoffs

[7] Notice that offers in that subgame are simultaneous, so the uniqueness result of Rubinstein (1982) does not apply.

v_1 and v_2 for the impatient and patient rational types of B, respectively, discounted to time zero. Because $\mathcal{B}_2$ could adopt $\mathcal{B}_1$'s equilibrium strategy if he wanted to, $v_2 \geq v_1$. If $\mathcal{B}_k$ waits until some positive time t to make a demand $b \in \mathbb{B}$ in response to a, he must expect, if (b, t) is in the support of his equilibrium strategy in this subgame, to receive a payoff of $v_k e^{r_k^B t}$ (discounted to t) by doing so. Figure 1 shows the "indifference curves" for the respective types of player B, and some particular demand $b \in \mathbb{B}$ they might consider making. Note that if neither rational type of B ever demands b at time t, then if A observes (b, t), she concludes that she faces a behavioral type and concedes immediately. In this situation, the payoff $\mathcal{B}_k$ would receive if he deviated to making the demand (b, t), would be

$$b_z = (1 - \hat{z}^A)b + \hat{z}^A(1 - a)$$

(because if he is not conceded to, he waits an instant and concedes himself). Figure 1 also shows b_z.

Figure 1

In Figure 1, t^* labels the time at which $\mathcal{B}_1$'s indifference curve cuts b_z, and t_2 the time at which the two indifference curves intersect. (Eventually we shall introduce a time $t_1 < t_2$.) Consider times such as t' after t^* but before $\mathcal{B}_2$'s indifference curve cuts b_z. $\mathcal{B}_1$ will never demand b at time t' (even immediate acceptance by rational A would be insufficient to give him his equilibrium payoff). But (b, t') must be in the support of $\mathcal{B}_2$'s equilibrium strategy: if it were not, deviating to it would yield $\mathcal{B}_2$ a payoff of b_z since it would be taken by A as evidence that B was behavioral.

We see that to the right of t^*, the equilibrium must provide a payoff ramp that keeps $\mathcal{B}_2$ on his level set. How is this accomplished? When $\mathcal{B}_2$ asks for b at t', there may be a concession by one side, followed by a WOA constructed so that each side is indifferent about conceding at any time. If $\mathcal{B}_2$'s payoff is ramping up to the right of t^*, it must be that A is conceding to B with increasing probability as B waits longer before speaking. Given the single-crossing nature of the level sets, a natural question is: Is there a fully separating equilibrium in the subgame, in which to the left of t_2, instantaneous concession probabilities by A rise at the rate that keeps type $\mathcal{B}_1$ indifferent, and to the right, at the rate that keeps type $\mathcal{B}_2$ indifferent? Unfortunately, things are not that simple. $\mathcal{B}_2$ would have a profitable deviation: ask for b at t'' slightly to the left of t_2, and get not only the concession payoff that $\mathcal{B}_1$ would receive there, but also the advantage of playing a WOA in which A's concession rate is calculated to keep the less patient $\mathcal{B}_1$ indifferent ($\mathcal{B}_2$ therefore receives surplus by playing this WOA, and this is a bonus to $\mathcal{B}_2$ beyond the payoff that $\mathcal{B}_1$ gets). Here we say that $\mathcal{B}_2$ is "sneaking in" and playing the WOA (against an unsuspecting player A). We remark that $\mathcal{B}_1$ has no incentive to sneak in to the right of t_2: the slow WOA that $\mathcal{A}$ fights with $\mathcal{B}_2$ does not interest him.

Notice that (b, t'') must be in the equilibrium support of both types. (If it were in the support of $\mathcal{B}_2$ only, then $\mathcal{B}_2$ would get no bonus from the WOA, and his entire payoff would come from the concession A gives him, which type $\mathcal{B}_1$ could collect just as well as $\mathcal{B}_2$; then $\mathcal{B}_1$ and $\mathcal{B}_2$ would have the same payoffs at t'', a contradiction. We have already seen why it cannot be in the support of $\mathcal{B}_1$ only.) If A, upon hearing the demand b at t'', thinks it much more likely to have been said by $\mathcal{B}_1$ than type $\mathcal{B}_2$, then in the event that a WOA ensues (rather than a concession by A), it will have a long initial phase in which A concedes at a rate that would keep $\mathcal{B}_1$ indifferent; this is a major bonus for $\mathcal{B}_2$, and leaves him above his equilibrium indifference curve. On the other hand, if A thinks (b, t'') is sufficiently more likely to have been said by $\mathcal{B}_2$, the WOA will have a very brief first stage, which gives $\mathcal{B}_2$ such a small bonus that his expected payoff is below his equilibrium indifference curve. We show in the Densities section of the Appendix that there exist unique densities for $\mathcal{B}_1$ and $\mathcal{B}_2$ such that each type's payoff lies on the respective indifference curve.

Return for a moment to the fiction that player A believes, upon hearing the demand b any time before t^*, that there is no chance it was made by $\mathcal{B}_2$. We can plot the curve showing the expected utility $\mathcal{B}_2$ would get if he were to sneak in under this circumstance; we call this the *sneaking-in curve.* Denote the sneaking-in value at t by $\underline{v}_2(t)$. (We do not make explicit the dependence of the sneaking in function on b; the relevant b will be clear from context.) Lemma 6 establishes that it is steeper than $\mathcal{B}_2$'s equilibrium indifference curve; Figures 4 and 5 illustrate the two possible cases (in case 1, the sneaking-in curve intersects $\mathcal{B}_2$'s indifference curve to the right of the vertical axis, at the time we shall call t_1; in case 2, the sneaking-in curve cuts the vertical axis above v_2, in which case we say $t_1 = 0$).

Lemma 6. *(i) $v_1 e^{r_1^B t} < \underline{v}_2(t)$ for $t < t^*$ and $v_1 e^{r_1^B t^*} = \underline{v}_2(t^*)$; and (ii) $\underline{v}_2(t)$ is steeper than $v_2 e^{r_2^B t}$ for all t such that $v_2 e^{r_2^B t} \leq \underline{v}_2(t)$.*

Proof. See Appendix.

The analysis thus far has focused on the "horizontal" aspects of behavior in (t, b) space, that is, for a given demand b, at what times will each type of player B be active? Much more detailed distributional analysis will follow. But we turn at the moment to the complementary "vertical" question: which demands b will be used by $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively? As z approaches 0, strong results emerge here.

Recall that in the previous Section with atemporal types, both $\mathcal{B}_1$ and $\mathcal{B}_2$ put almost all their weight (asymptotically) on $\lfloor b_2^*(a) \rfloor$, which would be "slightly generous" in response to a in a game where there was no impatient type (that is, $\alpha_1 = 0$). "Slightly generous" means "the least generous response that is more generous than balanced". This remains true for $\mathcal{B}_2$ in the temporal setting. But even asymptotically, $\mathcal{B}_1$ may put substantial weight on both $\lfloor b_2^*(a) \rfloor$ and $\lfloor b_1^*(a) \rfloor$. The latter occurs only very near $t = 0$. All of this is established in Lemma 8. Lemma 7 prepares the ground by proving that if $b \leq \lfloor b_k^*(a) \rfloor$, and if $\mathcal{B}_k$ has positive density on b over some interval and $\mathcal{B}_j$, $j \neq k$, does not, then that interval has to be very short. The idea is that if $\mathcal{B}_k$ is making a more-reasonable-than-balanced demand, and if he were getting a payoff ramp over a long interval, $\mathcal{A}$'s conditional concession probability to him could not be close to 1 for most of that interval. If $\mathcal{B}_k$ is being generous, why isn't he getting conceded to almost for sure? It can only be because he is present there with extremely high density. But that can't occur over more than a brief interval (vanishing with z) or else the density would integrate to more than 1.

Lemma 7. *For any $R > 1$ and $\epsilon > 0$, there exists $\bar{z} > 0$ such that for all $z \in K(R, \bar{z})$, for any Perfect Bayesian Equilibrium (φ^A, φ^B) of $\Gamma(r, \alpha, z)$, and for any $a \in \mathbb{A}$ with $\varphi^A(a|z) \geq \epsilon$, if for a given k and for $j \neq k$, $\varphi_j^B(\lfloor b_k^*(a) \rfloor, s|a, z) = 0$ for all $s \in [s_0, s_0 + \epsilon]$, then $v_k e^{r_k^B s_0} \geq \lfloor b_k^*(a) \rfloor - \epsilon$.*

Proof. As in the proof of Lemma 3, suppose by way of contradiction that the Lemma is false. Then there exists a sequence $\{z^\ell\} \subset K(R)$ such that $z^\ell = (z^{A\ell}, z^{B\ell}) \downarrow (0, 0)$, a corresponding sequence of PBE $\{\varphi^\ell\}$, and $a \in \mathbb{A}$ such that $\varphi_j^B(\lfloor b_k^*(a) \rfloor, s|a, z^\ell) = 0$ for all $s \in [s_0, s_0 + \epsilon]$ and $v_k e^{r_k^B s_0} < \lfloor b_k^*(a) \rfloor - \epsilon$. Moreover, for some $t^\ell \in [s_0, s_0 + \epsilon]$ it must be the case that $\varphi_k^{B\ell}(\lfloor b_k^*(a) \rfloor, t^\ell|a, z^\ell) \leq 1/\epsilon$ (since $\int \varphi_k^{B\ell} \leq 1$). The argument for $k = 2$ is virtually identical to Lemma 2 *(ii)*. We present instead the very similar argument for $k = 1$. Now,

$$\hat{z}^{B\ell} \geq \frac{z^{B\ell}\pi^B(b, t^\ell)}{z^{B\ell}\pi^B(b, t^\ell) + (1 - z^{B\ell})\alpha_1/\epsilon}$$

since $\varphi_1^{B\ell}(\lfloor b_1^*(a) \rfloor, t^\ell|a, z^\ell) \leq 1/\epsilon$ (and $\varphi_2^{B\ell}(\lfloor b_1^*(a) \rfloor, t^\ell|a, z^\ell) = 0$ by assumption). Furthermore, since $\hat{\alpha}_2^\ell = 0$, $L^\ell = [\hat{z}^{A\ell}]^{\lambda^B}/[\hat{z}^{B\ell}]^{\lambda_1^A}$, and

$$L^\ell \leq \left[\frac{z^{A\ell}\pi^A(a)}{z^{B\ell}\pi^B(b, t^\ell)} \times \frac{z^{B\ell}\pi^B(b, t^\ell) + (1 - z^{B\ell})\alpha_1/\epsilon}{z^{A\ell}\pi^A(a) + (1 - z^{A\ell})\varphi^{A\ell}(a)}\right]^{\lambda_1^A} [\hat{z}^{A\ell}]^{\lambda^B - \lambda_1^A}.$$

Since $\pi^B(b, t^\ell) \geq \underline{\pi}$,

$$\lim_{\ell \to \infty} L^\ell \leq \left[R\,\frac{\pi^A(a)}{\underline{\pi}}\,\frac{\alpha_1/\epsilon}{\varphi^{A\infty}(a)}\right]^{\lambda_1^A} \times \lim_{\ell \to \infty} [\hat{z}^{A\ell}]^{\lambda^B - \lambda_1^A}.$$

Now, $\lambda^B > \lambda_1^A$ and

$$\hat{z}^{A\ell} \leq \frac{z^{A\ell}\pi^A(a)}{z^{A\ell}\pi^A(a) + (1 - z^{A\ell})\epsilon}$$

since $\varphi^{A\ell}(a) \geq \epsilon$. It follows that $\lim\, [\hat{z}^{A\ell}]^{\lambda^B - \lambda_1^A} = 0$. Consequently, by Lemma 1, $\mu^{A\ell} \to 1$. But $\mu^{A\ell} \to 1$ implies $v_1^\ell e^{r^{B_1}(s_0 + \epsilon)} \geq v_1^\ell e^{r_1^B t^\ell} \to \lfloor b_1^*(a) \rfloor$, a contradiction □

We seek to characterize equilibrium payoffs in the reduced game when $z = (z^A, z^B) \downarrow 0$. Sometimes we'll write $\varphi^A(a|z)$ and $\varphi_k^B(b, t|a, z)$ to make explicit the dependence on z.

Lemma 8. *For any $R > 1$ and $\epsilon > 0$, there exists $\bar{z} > 0$ such that for all $z \in K(R, \bar{z})$, for any Perfect Bayesian Equilibrium (φ^A, φ^B) of $\Gamma(r, \alpha, z)$, and for any $a \in \mathbb{A}$ with $\varphi^A(a|z) \geq \epsilon$,*

(i) if $b > \lfloor b_2^*(a) \rfloor$ then $\varphi_k^B(b, t|a, z) \leq \epsilon$ for all $t \in (0, \bar{T}]$ and $k = 1, 2$.

(ii) if $b < \lfloor b_k^*(a) \rfloor$ then $\varphi_k^B(b, t|a, z) = 0$ for all $t \in (0, \bar{T}]$, $k = 1, 2$.

(iii) *if* $\lfloor b_1^*(a) \rfloor < b < \lfloor b_2^*(a) \rfloor$ *then* $\varphi_1^B(b, t|a, z) \leq \epsilon$ *for all* $t \in (0, \bar{T}]$.
(iv) $\varphi_1^B(\lfloor b_1^*(a) \rfloor, t|a, z) = 0$ *for all* $t \in [\epsilon, \bar{T}]$.

Proof. *(i)* As in the proof of Lemma 3, suppose by way of contradiction that *(i)* is false. Then there exist $a \in \mathbb{A}$, $b \in \mathbb{B}$ with $b > \lfloor b_2^*(a) \rfloor$, a sequence $\{z^\ell\} \subset K(R)$ such that $z^\ell = (z^{A\ell}, z^{B\ell}) \downarrow (0,0)$, a sequence $\{t^\ell\} \subset (0, \bar{T}]$, and a corresponding sequence of PBE $\{\varphi^\ell\}$ such that $\varphi_1^{B\ell}(b, t^\ell|a) + \varphi_2^{B\ell}(b, t^\ell|a) \geq \epsilon$ for all ℓ. Without loss of generality (taking a subsequence if necessary), we can assume that $t^\ell \to t$ and $\varphi^{A\ell}(a) \to \varphi^{A\infty}(a) \geq \epsilon$. For each $k = 1, 2$, if $\{\varphi_k^{B\ell}(b, t^\ell|a, z^\ell)\}$ contains a bounded subsequence, we define $\varphi_k^{B\infty}(b, t|a)$ to be the limit of that subsequence, otherwise we define $\varphi_k^{B\infty}(b, t|a) = \infty$. But now the analysis of Lemma 2 *(i)* applies exactly and we conclude that for large enough ℓ (along a subsequence), B concedes with strictly positive probability $\mu^{B\ell}$ at the start of the WOA in the subgame $\Gamma(r, \hat{\alpha}^\ell, \hat{z}^{A\ell}, \hat{z}^{B\ell}, a, b)$ at time t^ℓ. Indeed $\mu^{B\ell} \to 1$.

On the other hand, equilibrium payoffs for $\mathcal{B}_1$ and $\mathcal{B}_2$ in the subgame are bounded below by $1 - a$ and hence must be strictly greater than $1 - a$ after delay $t > 0$. It follows that in the above subgame, A must concede to B with strictly positive probability at the start of the WOA. This yields a contradiction since by Lemma 1, $\mu^{A\ell}\mu^{B\ell} = 0$. Notice that the proof of *(i)* does *not* require $\varphi^{A\infty}(a) > 0$.

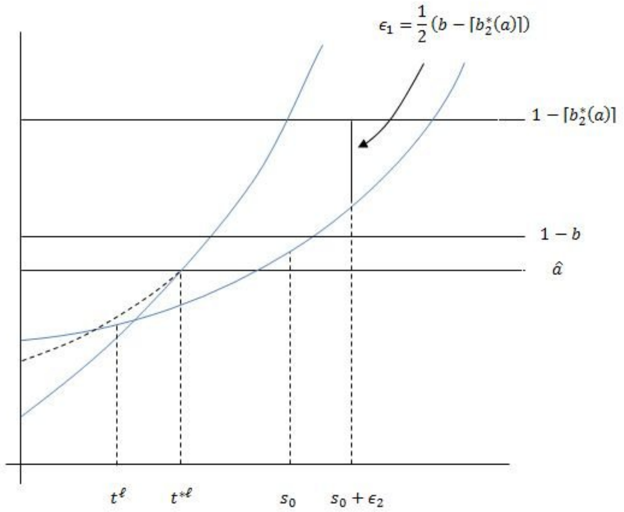

Figure 2

(ii) This builds on the derivation of the sneaking in function $\underline{v}_2$ of Lemma 6. Consider $b < \lfloor b_2^*(a) \rfloor$. We first argue that $\varphi_2^B(b, t|a, z) = 0$ for all $t \in [0, \bar{T}]$. Suppose not. Then there exist sequences as before with $\varphi_2^{B\ell}(b, t^\ell|a, z^\ell) > 0$. Since $\mathcal{B}_2$ can concede to a at time zero, it must be that $b > 1 - a$. Since the payoff from mimicking b at t^ℓ is strictly less than $b_z^\ell = \hat{z}^{A\ell}(1 - a) + (1 - \hat{z}^{A\ell})b$, it must be that $v_2^\ell e^{r_2^B t^\ell} < b_z^\ell$. Let $t^{*\ell}$ satisfy $v_1^\ell e^{r_1^B t^{*\ell}} = b_z^\ell$. We wish to first argue that $v_2^\ell e^{r_2^B s} = b_z^\ell$ for some $s > t^{*\ell}$. If $t^\ell \geq t^{*\ell}$ this is obvious (since $v_2^\ell e^{r_2^B t^\ell} < b_z^\ell$). Now suppose that $t^\ell < t^{*\ell}$.

If $\varphi_1^{B\ell}(b, t^\ell|a, z^\ell) = 0$, then $v_1^\ell e^{r_1^B t^\ell} \geq v_2^\ell e^{r_2^B t^\ell}$, since $\mathcal{B}_1$ always has the option of first counterdemanding b at t^ℓ. On the other hand, if $\varphi_1^{B\ell}(b, t^\ell|a, z^\ell) > 0$, then $v_2^\ell e^{r_2^B t^\ell} \leq \underline{v}_2(t^\ell)$. In either case, $v_2^\ell e^{r_2^B t^\ell} \leq \underline{v}_2(t^\ell)$. Since $v_2^\ell e^{r_2^B s}$ is flatter than $v_1^\ell e^{r_1^B s}$ and $\underline{v}_2(s)$ (see Lemma 6), $v_2^\ell e^{r_2^B s} = b$ for some $s > t^{*\ell}$. For all $t' \geq t^{*\ell}$, $v_1 e^{r_1^B t'} \geq b_z$, and consequently $\varphi_1^{B\ell}(\lfloor b_2^*(a) \rfloor, t'|a, z^\ell) = 0$. But this contradicts Lemma 7, as depicted in Figure 2 above, where $\epsilon_1 = [\lfloor b_2^*(a) \rfloor - b]/2$, ϵ_2 is defined by $b e^{r_2^B \epsilon_2} = \lfloor b_2^*(a) \rfloor - \epsilon_1$, and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

Now consider $b < b_1^*(a)$. We wish to argue that $\varphi_1^B(b, t|a, z) = 0$. Suppose not. Then, there exist sequences (analogous to the earlier sequences) such that $\varphi_1^{B\ell}(b, t^\ell|a, z^\ell) > 0$ for all ℓ. But this implies $v_1^\ell \leq b$. Consequently, there exists an interval $[0, s_1]$ such that, $b_1^*(a) - v_1^\ell e^{r_1^B s} > 2\epsilon > 0$ for all $s \in [0, s_1]$. Since $b < b_2^*(a)$ (because $b_1^*(a) < b_2^*(a)$), for ℓ large enough $\varphi_2^{B\ell}(b, t^\ell|a, z^\ell) = 0$ by our earlier conclusion. Let ϵ_1 and ϵ_2 be defined analogously to the above. Again we have contradicted Lemma 7 for $s_0 = 0$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

(iii) We know from *(ii)* that $\varphi_2^B(b, t|a, z) = 0$. If the result is not true there exist sequences as in *(i)* such that $\varphi_1^{B\infty}(b, t|a) > 0$ and $\varphi_2^{B\ell}(b, t^\ell|a, z^\ell) = 0 = \hat{\alpha}_2^\ell$ along the sequence. Observe that $\lambda_1^A > \lambda^B$. Now an argument almost identical to the proof of Lemma 2 *(i)* [replace $\hat{z}_2^\ell + \hat{\alpha}_2^\ell$ by $\hat{z}_2^\ell$ and subsequently λ_2^A by λ_1^A] implies $\mu^{B\ell} \to 1$. As in *(i)* above this yields a contradiction with the requirement that $\mu^{B\ell} > 0$.

(iv) Let $\bar{z}_0$ be chosen to satisfy *(i)*-*(iii)* for the given ϵ. If for all equilibria $v_1 e^{r_1^B \epsilon} \geq \bar{b}_1^*(a)$ then the result is clearly true. Let ϵ_1 and ϵ_2 satisfy

$$\lfloor b_1^*(a) \rfloor - 2\epsilon_1 = \lfloor b_1^*(a) \rfloor e^{-r_1^B \epsilon}$$
$$(\lfloor b_1^*(a) \rfloor - 2\epsilon_1) e^{r_1^B \epsilon_2} = \lfloor b_1^*(a) \rfloor - \epsilon_1$$

and define $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$. Applying Lemma 7 to the interval $s_0 = 0$ and ϵ_3 yields an upper bound $\bar{z}_1$. Choosing $\bar{z} = \min\{\bar{z}_0, \bar{z}_1\}$ gives us our desired conclusion. ☐

Corollary. *Under the conditions of the previous lemma, for any $a \in \mathbb{A}$, if $\hat{b}_2 \equiv \lfloor b_2^*(a) \rfloor > 1 - a$ then*

$$\int_0^{\bar{T}} \varphi_2^B(\hat{b}_2, t|a, z)dt \geq 1 - \bar{T}|\mathbb{B}|\epsilon.$$

Corollary. *Under the conditions of the previous lemma, for any $a \in \mathbb{A}$ such that $\lfloor b_2^*(a) \rfloor = 1 - a$,*

$$|v^A(a|z) - a| \leq \epsilon \quad \text{and} \quad |v_k^B(a|z) - (1 - a)| \leq \epsilon \quad k = 1, 2.$$

The fact that almost all of $\mathcal{B}_2$'s weight is devoted to $\lfloor b_2^*(a) \rfloor$, and that any of $\mathcal{B}_1$'s that is not devoted to $\lfloor b_2^*(a) \rfloor$ almost always goes on $\lfloor b_1^*(a) \rfloor$ near the origin, suggests that we can work with a reduced game where both players have severely restricted strategy spaces, in the vertical dimension. We define and study that game now. We will show that as z approaches 0, equilibria in the reduced game and the true game are essentially unique, and coincide.

The Reduced Game.

Denote by $\tilde{\Gamma}(r,\alpha,z)$ the "reduced game" in which after A makes an initial demand $a \in \mathbb{A}$, she cannot move until B makes a demand, and the set of B's behavioral types (depends on the choice a and) is $\{(\lfloor b_2^*(a)\rfloor, t) \mid t \in [0,\bar{T}]\}$. As part of the specification of the reduced game, if $\lfloor b_2^*(a)\rfloor = 1-a$, then the game ends with B accepting the demand a immediately. When $\lfloor b_2^*(a)\rfloor > 1-a$, the prior probability of type $(\lfloor b_2^*(a)\rfloor, t)$ is $z^B\pi^B(\lfloor b_2^*(a)\rfloor, t)$. Finally, if B counterdemands $\hat{b}_1 \equiv \lfloor b_1^*(a)\rfloor$ at time 0, this demand is accepted with probability 1 immediately by A.

We proceed to analyze $\tilde{\Gamma}(r,\alpha,z)$. The first step is to analyze the subgame $\tilde{\Gamma}(r,z,a)$ which arises after A chooses some $a \in \mathbb{A}$ with probability $\varphi^A(a) > 0$ (in equilibrium). By definition, when $\lfloor b_2^*(a)\rfloor = 1-a$, we have immediate agreement. Hereafter we assume $\lfloor b_2^*(a)\rfloor > 1-a$.

Fix an equilibrium of this subgame and suppose $\mathcal{B}_k$'s payoff in the subgame is v_k. As $\mathcal{B}_k$ is limited in the reduced game (for $t>0$) to the counterdemand $\hat{b}_2 = \lfloor b_2^*(a)\rfloor$, he only chooses the time at which he makes the counterdemand. Accordingly, we will simply write $\varphi_k^B(t)$ instead of $\varphi_k^B(\hat{b}_2, t|a)$. Since $r_1^B > r_2^B$, it follows immediately that $v_2 \geq v_1 \geq \hat{b}_1$. Since $\mathcal{B}_k$ must be indifferent among all times t for which $\varphi_k(t) > 0$, his continuation value at any such t (discounted to t) must be $v_k e^{r_k t}$.

If B's counterdemand is accepted immediately by $\mathcal{A}$ with probability 1, then B's expected value is b_z, and strictly less otherwise. Let t^* solve $v_1 e^{r_1^B t} = b_z$, t_2 be the time when the two payoff curves $v_1 e^{r_1^B t}$ and $v_2 e^{r_2^B t}$ intersect, and t_3 solve $v_2 e^{r_2^B t} = b_z$. By assumption, $t^* < \bar{T}$ and $t_3 < \bar{T}$. Furthermore, $\varphi_1^B(t) = 0$ for $t \geq t^*$ and $\varphi_2^B(t) = 0$ for $t \geq t_3$. Also, if $\varphi_1^B(t) = \varphi_2^B(t) = 0$ for some $t \in [0,\bar{T}]$, $\mathcal{B}_k$'s payoff from counteroffering $\hat{b}_2$ at time t (out of equilibrium) is b_z.

Fact 1: In any equilibrium, we must have

$$\Phi_k^B = \int_0^{\bar{T}} \varphi_k^B(t)dt = 1 \quad k = 1,2,$$

unless $v_k = \hat{b}_1$, in which case we may have $\Phi_k^B < 1$ combined with $\mathcal{B}_k$ making the demand $\hat{b}_1$ at time $t=0$ with probability $1-\Phi_k^B$.

Fact 2: Assume $v_i e^{r_i^B t} < v_j e^{r_j^B t}$ and $v_i e^{r_i^B t} < b_z$ for $\{i,j\} = \{1,2\}$. Then $\varphi_i^B(t) > 0$. Moreover, if $i=2$, then $\varphi_1^B(t) = 0$.

When $v_1 = v_2$ we obtained the configuration depicted in Figure 3 below. In this case, by Fact 2 $\varphi_1^B(t) = 0$ for all $t \in (0, t^*)$. Hence, $\Phi_1^B = 0$, so $\mathcal{B}_1$ makes the demand $\hat{b}_1$ at time 0 with probability 1, and this offer is accepted by A immediately (as per the rules of the reduced game). Hence $v_1 = v_2 = \hat{b}_1$.

Hereafter we assume that $v_2 > v_1$. Clearly $v_2 < b_z$, for if $v_2 = b_z$, then $v_1 = b_z$ also. (Since $v_2 = b_z$ is only possible if the counterdemand $\hat{b}_2$ at $t=0$ is accepted immediately by

$\mathcal{A}$ with probability 1.) But then we are in the case dealt with above where we concluded that $v_1 = v_2 = \hat{b}_1 < b_z$, a contradiction.

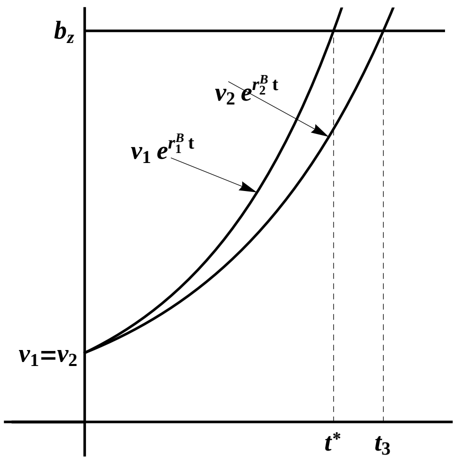

Figure 3: $v_1 = v_2$.

Fact 3: If $\varphi_k^B(t) > 0$ for $k = 1, 2$, then $v_1 e^{r_1^B t} < v_2 e^{r_2^B t} < \underline{v}_2(t)$.

Proof of Fact 3: Consider t such that in equilibrium $\varphi_k^B(t) > 0$ $k = 1, 2$. Then, in the WOA following B's counteroffer at t, A will concede at t with probability $\mu_1(t)$ as defined above, and thereafter at a rate of concession λ_1^A in $(t, t + \tau_1]$ and at rate λ_2^A in $(t + \tau_1, t + \tau_2)$. Player B concedes at rate λ^B. The times τ_1 and τ_2 satisfy:

$$(\hat{z}^B + \hat{\alpha}_2) e^{\lambda^B \tau_1} = 1, \quad \hat{z}^B e^{\lambda^B \tau_2} = 1 \quad \text{and} \quad \frac{\hat{z}^A}{1 - \mu_1(t)} e^{\lambda_1^A \tau_1} e^{\lambda_2^A (\tau_2 - \tau_1)} = 1.$$

Recall that

$$\frac{\hat{z}^A}{1 - \mu_1(t)} e^{\lambda_1^A \tau_1^S} = 1.$$

Clearly, $\tau_2 > \tau_1$, so $\tau_1^S > \tau_1$. Note that $\mathcal{B}_2$'s payoff (in equilibrium) is given by $v_2(t) = \mu_1 \hat{b}_2 + (1 - \mu_1) C$, where we simply replace τ_1^S by τ_1 in E. It follows that $v_1 e^{r_1^B t} < v_2 e^{r_2^B t} < \underline{v}_2(t)$. □

Fact 4: $t_2 < t^* < t_3$.

Proof of Fact 4: Since $v_1 < v_2$, $v_2 > \hat{b}_1$. Hence, by Fact 1, $\Phi_2^B = 1$ and $\varphi_2^B(t) > 0$ for some $t > 0$; at such a t, $v_2 e^{r_2^B t} < b_z$. If $t \geq t^*$ then $v_2 e^{r_2^B t} < b_z \leq v_1 e^{r_1^B t}$. Since $v_2 > v_1$, all this implies that $t_2 < t^*$. If $t < t^*$ and $v_1 e^{r_1^B t} > v_2 e^{r_2^B t}$, we may similarly conclude that $t_2 < t^*$. If $t < t^*$ and $v_1 e^{r_1^B t} \leq v_2 e^{r_2^B t}$, then by Fact 3 $v_1 e^{r_1^B t} \leq v_2 e^{r_2^B t} \leq \underline{v}_2(t)$. Since $\mathcal{B}_2$'s indifference curve cannot intersect $\underline{v}_2(\cdot)$ from below (Lemma 6)

$$v_2 e^{r_2^B s} - v_1 e^{r_1^B s} \leq \underline{v}_2(s) - v_1 e^{r_1^B s} \qquad \text{for all } s \in [t, t^*],$$

and since $\underline{v}_2(t^*) = v_1 e^{r_1^B t^*}$, we must have that $v_1 e^{r_1^B t_2} = v_2 e^{r_2^B t_2}$ for some $t_2 < t^*$. Finally, $t_2 < t^*$ implies that $t^* < t_3$ since $\mathcal{B}_2$'s indifference curve is flatter than $\mathcal{B}_1$'s. □

By Lemma 6, $v_2 e^{r_2^B t}$ can only intersect $\underline{v}_2(t)$ from above, and consequently at most once. Hence we can only have one of the two configurations below. In the first configuration $v_2 e^{r_2^B t}$ intersects $\underline{v}_2(t)$ at some $t_1 \in (0, t_2)$. The second configuration corresponds to the case when $v_2 < \underline{v}_2(0)$ and $v_2 e^{r_2^B t}$ does not intersect $\underline{v}_2(t)$ in $(0, t_2)$. In this case we define $t_1 = 0$. In both configurations $v_2 > \hat{b}_1$, and Fact 1 then implies that $\Phi_2^B = 1$.

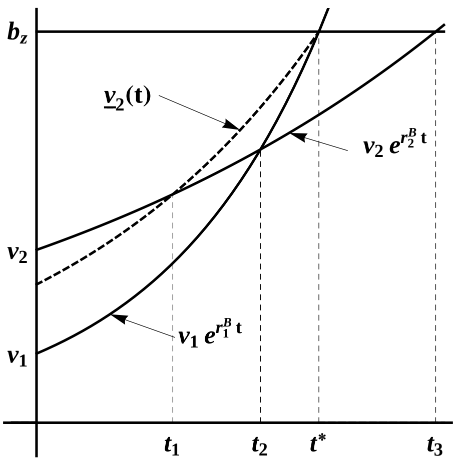

Figure 4: $v_1 < v_2$ and $v_2 > \underline{v}_2(0)$.

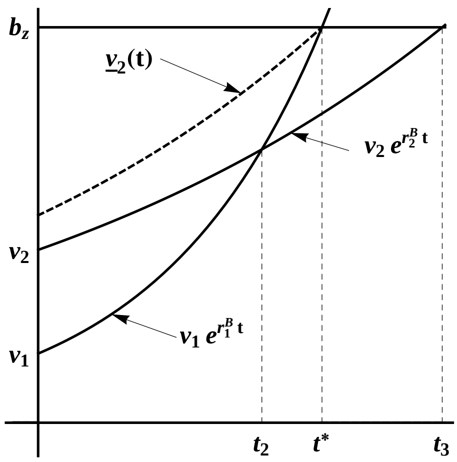

Figure 5: $v_1 < v_2$ and $v_2 \leq \underline{v}_2(0)$.

Lemma 9. *$\varphi_1^B(t) > 0$ and $\varphi_2^B(t) = 0$ for $t \in (0, t_1)$, $\varphi_1^B(t) > 0$ and $\varphi_2^B(t) > 0$ for $t \in (t_1, t_2)$, and $\varphi_1^B(t) = 0$ and $\varphi_2^B(t) > 0$ for $t \in (t_2, t_3)$.*

Proof. If $\mathcal{B}_1$ counteroffers $\hat{b}_2$ at any $t > t^*$, then his payoff (discounted to 0) is less than $b_z e^{-r_1 t} < v_1$. Hence $\varphi_1(t) = 0$ for all $t > t^*$. For the same reason, $\varphi_2(t) = 0$ for all $t > t_3$ Let $t < t_2$, so $v_1 e^{r_1 t} < v_2 e^{r_2 t} \leq b_z$ It follows that either $\varphi_1(t) > 0$ or $\varphi_2(t) > 0$. Assume that $\varphi_1(t) = 0$ and $\varphi_2(t) > 0$. After making the counteroffer $\hat{b}_2$ at time t, $\mathcal{B}_2$ can get his expected value $v_2 e^{r_2 t}$ in the ensuing WOA by waiting to see if $\mathcal{A}$ concedes to $\hat{b}_2$ right away, and conceding to a immediately if $\mathcal{A}$ does not. But $\mathcal{B}_1$ can obtain the same expected payoff by mimicking $\mathcal{B}_2$: counter-offering $\hat{b}_2$ at time t and following the same strategy after that. This is a contradiction. Hence, $\varphi_1(t) > 0$ for all $t \in (0, t_2)$. Finally suppose that $t \in (t_2, t_3)$. Again, either $\varphi_1(t) > 0$ or $\varphi_2(t) > 0$. Assume now that $\varphi_1(t) > 0$. $\mathcal{B}_1$ can get his expected value $v_1 e^{r_1 t}$ in the ensuing WOA by waiting until some time $t + \tau > t$ to concede if $\mathcal{A}$ does not concede first. But, if $\mathcal{B}_2$ mimicks $\mathcal{B}_1$, he expects a strictly higher payoff as $r_1 > r_2$, which is a contradiction since by assumption $v_1 e^{r_1 t} > v_2 e^{r_2 t}$. Hence $\varphi_1(t) = 0$ and $\varphi_2(t) > 0$ for all $t \in (t_2, t_3)$. □

For a given z, suppose that $\mathcal{A}$ chooses a with some probability $\varphi^A(a|z) > 0$, and let $\hat{z}^A$ be defined as in (1). Then, for any (v_1, v_2) in the relevant range, not necessarily corresponding to equilibrium values in the subgame following the offer a, we may compute the densities with which $(\hat{b}_2, t)$ is chosen by $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively so that the continuation values $v_k e^{r_k t}$, $k = 1, 2$ are achieved in the subgame following the offer of $\hat{b}_2$ at time t. These densities are denoted $\varphi_k^A(t|\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)$ and the corresponding integrals by $\Phi_k^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)$.[8]

Lemma 10. *For all $a \in A$ and $\epsilon > 0$, there exists $\delta > 0$ such that for any $v_1 \in [1-a, b_z)$ and v_2 with $v_2 e^{r_2^B t^*} \in [b_z - \delta, b_z)$, $|t_i - t^*| \leq \epsilon$ for $i = 1, 2, 3$. Furthermore, if $v_2 \leq \underline{v_2}(0)$ then $v_1 \geq b_z - \epsilon$.*

The function θ introduced in Lemma 11 is key and is central to our characterization results.

Lemma 11. *There exists a function $\theta : \mathbb{A} \to \mathbb{R}_+$ such that for all $a \in \mathbb{A}$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $v_1 \in [1-a, b_z)$ and v_2 with $v_2 e^{r_2^B t^*} \in [b_z - \delta, b_z)$ and $v_2 \geq \underline{v}_2(0)$,*

$$\left| \frac{\int_{t_1}^{t_2} \varphi_1^B(t|\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)dt}{\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)} - \frac{\alpha_2}{\alpha_1}\theta(a) \right| < \epsilon.$$

Lemma 12. *Assume $a \in \mathbb{A}$ and $\alpha_2\theta(a)/\alpha_1 < 1$. Then, there exists $\delta > 0$ such that for all $v_1 \in [1-a, b_z)$ and v_2 with $v_2 e^{r_2^B t^*} \in [b_z - \delta, b_z)$ and $v_2 < \underline{v}_2(0)$,*

$$\frac{\Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)}{\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)} < 1.$$

Lemma 13. *For any $a \in \mathbb{A}$, $R > 1$ and $\epsilon > 0$, there exists $\bar{z} > 0$ such that if $z \in K(R, \bar{z})$, and $\varphi^A(a|z) \geq \epsilon$, then for all $v_1 \in [\hat{b}_1, b_z)$ and $v_2 \in [v_1, b_z)$,*

$$\int_0^{t_1} \varphi_1^B(t|\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)dt < \epsilon.$$

Furthermore, if $\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2) = 1$, then $v_2 e^{r_2^B t^*} \geq b_z - \epsilon$.

The overall equilibrium involves the strategic choice of a by $\mathcal{A}$ and in this analysis it is convenient to make explicit the dependence of variables such as $\hat{b}_2$ on a. As noted earlier, we focus on the case of a small z. Recall the conditions of Lemma 8. We will argue that when $\varphi^A(a|z) \geq \epsilon$, then equilibrium in the subgame is "essentially" unique and consequently falls in one of the two cases described above. Furthermore, in case (i), $v_1 = \lfloor b_1^*(a) \rfloor$, $v_2 e^{r_2^B t^*} \approx \lfloor b_2^*(a) \rfloor$ and $\mathcal{B}_1$ counteroffers $\lfloor b_1^*(a) \rfloor$ at time 0 with probability close to $1 - \alpha_2\theta(a)/\alpha_1$, where $\alpha_2\theta(a)/\alpha_1$ is the limit value of $\Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)$ as

[8] In the reduced game the set of behavioral types in the subgame following the offer a is constrained. But, in the full game any counteroffer $b \in \mathbb{B}$ with $a + b > 1$ is possible, so in general, the densities will depend both on a and b.

$z \downarrow 0$. In case (ii) $v_1 \approx v_2 \approx \lfloor b_2^*(a) \rfloor$ and $\mathcal{A}$ concedes to $\lfloor b_2^*(a) \rfloor$ with probability close to 1. Let $D = [\lfloor b_1^*(a) \rfloor / \lfloor b_2^*(a) \rfloor]^{r^A/r_1^B}$. Player $\mathcal{A}$'s asymptotic payoff is

$$v^A(a) = \begin{cases} (\alpha_1 - \alpha_2\theta(a))(1 - \lfloor b_1^*(a) \rfloor) + \alpha_2(1+\theta(a))D(1 - \lfloor b_2^*(a) \rfloor) & \text{if (i)} \\ 1 - \lfloor b_2^*(a) \rfloor & \text{if (ii)} \end{cases}$$

Let $\hat{a} \in \mathbb{A}$ maximize $v^A(a)$ and suppose for simplicity that $\hat{a}$ is unique, then for small z, $\mathcal{A}$ mimics $\hat{a}$ with probability close to 1 and the analysis of the subgame following the choice of $\hat{a}$ is as outlined above. We state these results more formally below.

The payoff from efficient IC separation is $\alpha_1(1 - \lfloor b_1^*(a) \rfloor) + \alpha_2 D(1 - \lfloor b_2^*(a) \rfloor)$. When $\alpha_2\theta(a)/\alpha_1 < 1$, the difference between this payoff and $v^A(a)$ is

$$\frac{1-\alpha_1}{\alpha_1}\theta(a)[(1 - \lfloor b_1^*(a) \rfloor) - D(1 - \lfloor b_2^*(a) \rfloor)].$$

This difference is small when α_1 is close to 1 and/or $\theta(a)$ is small.

In Theorem 2 below we revert to the true game $\Gamma(r, \alpha, z)$. The main differences between $\Gamma(r, \alpha, z)$ and $\tilde{\Gamma}(r, \alpha, z)$ are that in $\Gamma(r, \alpha, z)$ many types (b, t) with $b \neq \hat{b}_2$ will also be mimicked with positive density, but by Lemma 5 these densities will go to zero with z. Furthermore, $\hat{b}_1$ will only be mimicked (if at all) in a small initial interval. In particular:

(1) $\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2) \geq 1 - \epsilon$ (and does not exactly equal 1),
(2) if $\alpha_2\theta(a)/\alpha_1 < 1$ then $|\Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2) - \alpha_2\theta(a)/\alpha_1| \leq \epsilon$.
(3) if $\alpha_2\theta(a)/\alpha_1 > 1$ then $\Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2) \geq 1 - \epsilon$,

where $\epsilon \downarrow 0$ as $z \to (0, 0)$. It is clear that the formulae for density functions do not depend upon whether we are in $\Gamma(r, \alpha, z)$ or $\tilde{\Gamma}(r, \alpha, z)$ and neither do the results regarding the ratios $[\Phi_1^B/\Phi_2^B]$.

Theorem 2. *Fix any $R > 1$ and $\epsilon > 0$. Consider any sequence $\{z^\ell\} \subset K(R)$ such that $z^\ell \to (0, 0)$, and a corresponding sequence of equilibria $\{\varphi^\ell\}$ for $\Gamma(r, \alpha, z^\ell)$. For a given $a \in \mathbb{A}$, assume $\varphi^{A\ell}(a) \geq \epsilon$ for all ℓ.*

(i) If $\alpha_2\theta(a)/\alpha_1 < 1$, then $|\lfloor b_1^*(a) \rfloor - v_1^\ell| < \epsilon$ for large ℓ, $\Phi_1^{B\ell}(\hat{z}^A, z^B, a, \lfloor b_2^* \rfloor, v_1, v_2) \to \alpha_2\theta(a)/\alpha_1$, and $\mathcal{B}_1$ counteroffers $\lfloor b_1^*(a) \rfloor$ with probability (close to) $1 - \alpha_2\theta(a)/\alpha_1$. Furthermore, if $\lfloor b_1^*(a) \rfloor e^{r_1^B t^*} = \lfloor b_2^*(a) \rfloor$, then $v_2^\ell \to \bar{v}_2^*$, where $\bar{v}_2^* e^{r_2^B t^*} = \lfloor b_2^*(a) \rfloor$.
(ii) If $\alpha_2\theta(a)/\alpha_1 > 1$, then $v_1^\ell, v_2^\ell \to \lfloor b_2^*(a) \rfloor$, and $\mathcal{A}$ concedes to $\lfloor b_2^*(a) \rfloor$ with probability (close to) 1 by time converging to 0.
(iii) Finally, if $\varphi^{A\ell}(a) \geq \epsilon$ for all ℓ, then a maximizes $v^A(\cdot)$.

Proof. (i) As usual, let $\hat{b}_k = \lfloor b_k^*(a) \rfloor$, $k = 1, 2$. By Lemma 5, $\Phi_2^{B\ell}(\hat{z}^A, z^B, a, \hat{b}_2, v_1^\ell, v_2^\ell) \geq 1 - \epsilon$ for large ℓ. By Lemma 9, $\Phi_1^{B\ell}(\hat{z}^A, z^B, a, \hat{b}_2, v_1^\ell, v_2^\ell) \leq \alpha_2\theta(a)/\alpha_1 + \epsilon$ for large ℓ. If $v_2^\ell \leq \underline{v}_2(0)$, then by Lemma 8 (which applies to Γ) $v_1^\ell, v_2^\ell \approx \hat{b}_2$; hence $v_1^\ell > \hat{b}_1$ and

$\Phi_1^{B\ell}(\hat{z}^A, z^B, a, \hat{b}_1, v_1^\ell, v_2^\ell) = 0$, and by Lemma 5, $\Phi_1^{B\ell}(\hat{z}^A, z^B, a, \hat{b}_2, v_1^\ell, v_2^\ell) \approx 1$. Consequently if $\alpha_2\theta(a)/\alpha_1 < 1$, $v_2^\ell \leq \underline{v}_2(0)$ yields a contradiction. Hence $v_2^\ell > \underline{v}_2(0)$. It follows that $\Phi_1^{B\ell}(\hat{z}^A, z^B, a, \hat{b}_2, v_1^\ell, v_2^\ell) \approx \alpha_2\theta(a)/\alpha_1 < 1$. Parts *(i)* – *(iii)* of Lemma 5 then imply that

$$\int_0^\epsilon \varphi_1^{B\ell}(t|\hat{z}^{A\ell}, z^{B\ell}, a, \hat{b}_1, v_1^\ell, v_2^\ell) \approx 1 - \frac{\alpha_2}{\alpha_1}\theta(a) > 0.$$

Consequently $v_1^\ell \approx \hat{b}_1$. Together with Lemma 8 *(ii)*, this completes the proof of *(i)*.

(ii) If $\alpha_2\theta(a)/\alpha_1 > 1$ then by the above discussion, for large ℓ, we must have $v_2^\ell < \underline{v}_2^\ell(0)$ and the rest of the characterization follows.

(iii) The preceding discussion clarifies that if $\varphi^{A\ell}(a) \geq \epsilon$ for large ℓ the payoffs to A in the corresponding subgame of Γ and $\tilde{\Gamma}$ converge. The conclusion follows directly. □

Existence

The proof of existence may be useful in other related environments; it uses a novel mix of constructive and non-constructive elements.

Let Σ be the unit simplex in $\mathbb{R}^{|\mathbb{A}|}$. That is,

$$\Sigma = \{\varphi^A \mid \varphi^A \geq 0 \quad \text{and} \quad \sum_{a\in\mathbb{A}} \varphi^A(a) = 1\}$$

For each $a \in \mathbb{A}$, let

$$V(a) = \{(v_{a1}, v_{a2}) \mid 1 - a \leq v_{a1} \leq v_{a2} \leq \bar{b}\} \quad \text{and} \quad V = \Pi_{a\in\mathbb{A}} V(a).$$

V is a compact and convex subset of $\mathbb{R}^{2|\mathbb{A}|}$.

To prove existence, we construct an upper hemicontinuous correspondence $\Psi : \Sigma \times V \to \Sigma \times V$ such that each of its fixed points agrees with an equilibrium of $\Gamma(r, \alpha, z)$.

For each $\varphi^A \in \Sigma$ and $a \in \mathbb{A}$, let the posterior $\hat{z}^A(a)$ be computed as in (1), and for each $b \in \mathbb{B}$, let $b_z(a) = (1 - \hat{z}^A(a))b + \hat{z}^A(a)(1 - a)$.

Fix $(\hat{z}^A(a), a, v) \in (0, 1) \times \mathbb{A} \times V$. Recall the definition of the intersection times (t^*, t_1, t_2, t_3) in Figures 3–5. These figures implicitly assumed that $a \leq v_{a1} \leq v_{a2}$, that $v_{a1} < b_z(a)$, and that $v_{a2}e^{r_2^B t^*} < b_z(a)$, where $v_{a1}e^{r_1^B t^*} = b_z(a)$. We now define the densities $\varphi^B(t|\hat{z}^A(a), z^B, a, b, v_a)$ for all $(b, t) \in \mathbb{B} \times [0, \bar{T}]$ using Equations (4)–(5) and (10)–(11) as follows:

$$\varphi^B(t|\hat{z}^A(a), z^B, a, b, v_a) = \begin{cases} ((4), 0) & \text{if } v_{a1}e^{r_1^B t} < b_z(a),\ v_{a2}e^{r_2^B t} \geq \underline{v}_2(t|\hat{z}^A(a), v_{a1}) \\ ((10), (11)) & \text{if } v_{a1}e^{r_1^B t} < b_z(a) \text{ and} \\ & \quad v_{a1}e^{r_1^B t} < v_{a2}e^{r_2^B t} < \underline{v}_2(t|\hat{z}^A(a), v_{a1}) \\ (0, (5)) & \text{if } v_{a2}e^{r_2^B t} \leq \min\{b_z(a), v_{a1}e^{r_1^B t}\} \\ (0, 0) & \text{if } v_{ak}e^{r_k^B t} \geq b_z(a),\ k = 1, 2. \end{cases}$$

Then, for $k = 1, 2$, define

$$\Phi_k^B(\hat{z}^A(a), z^B, a, b, v_a) = \int_0^{\bar{T}} \varphi_k^B(t|\hat{z}^A(a), z^B, a, b, v_a)dt$$
$$\bar{\Phi}_k^B(\varphi^A(a), v_a, a) = \sum_{b\in\mathbb{B}} \Phi_k^B(\hat{z}^A(a), z^B, a, b, v_a).$$

Let $\Gamma(r, \alpha, \hat{z}^A(a), z^B, a)$ be the subgame where player A has demanded $a \in \mathbb{A}$ and player B believes that A is behavioral with probability $\hat{z}^A(a)$. An equilibrium for $\Gamma(r, \alpha, \hat{z}^A(a), z^B, a)$ is a vector $v_a \in V(a)$ such that $\bar{\Phi}_k^B(\varphi^A(a), v_a, a) \leq 1$ and $[1 - \bar{\Phi}_k^B(\varphi^A(a), v_a, a)][v_{ak} - (1 - a)] = 0$ for $k = 1, 2$.

Define $\Psi_{(\varphi^A,a)} : V(a) \to V(a)$ by

$$\Psi_{(\varphi^A,a)}(v_a) = P_{V(a)}(v_{a1}\bar{\Phi}_1^B(\varphi^A(a), v_a, a), v_{a2}\bar{\Phi}_2^B(\varphi^A(a), v_a, a)),$$

where $P_{V(a)}(w)$ denotes the projection of $w \in \mathbb{R}^2$ into $V(a)$. $\Psi_{(\varphi^A,a)}(v_a)$ is continuous in (φ^A, v_a).

Theorem 3. *For each $z \in (0, 1)^2$, the game $\Gamma(r, \alpha, z)$ has an equilibrium.*

Proof. We first show that $v_a \in V(a)$ is an equilibrium of $\Gamma(r, \alpha, \hat{z}^A(a), z^B)$ if and only if v_a is a fixed point of $\Psi_{(\varphi^A,a)}$.

A vector v_a in the interior of $V(a)$ is a fixed point of $\Psi_{(\varphi^A,a)}$ if and only if $\bar{\Phi}_1^B(\varphi^A(a), v_a, a) = \bar{\Phi}_2^B(\varphi^A(a), v_a, a) = 1$. In this case, v_a is clearly an equilibrium of $\Gamma(r, \alpha, \hat{z}^A(a), z^B)$. The boundary of $V(a)$ is made up of three line segments. We now argue that $\Psi_{(\varphi^A,a)}$ has no fixed point in the upper or lower boundary of $V(a)$, and that a fixed point in the left boundary would correspond to an equilibrium where $\mathcal{B}_1$ concedes immediately to a with nonnegative probability. The left boundary is defined by $v_{a1} = 1 - a$ and $1 - a \leq v_{a2} < \bar{b}$ (we include here one of the endpoints that is at the intersection of the left boundary and the lower boundary, but not the other). A vector v_a in the left boundary is a fixed point of $\Psi_{(\varphi^A,a)}$ if and only if $\bar{\Phi}_k^B(\varphi^A(a), v_a, a) \leq 1$for $k = 1, 2$, with equality for $k = 2$ unless $v_{a2} = 1 - a$. But then, since $v_{a1} = 1 - a$, $[1 - \bar{\Phi}_k^B(\varphi^A(a), v_a, a)][v_{ak} - (1 - a)] = 0$ for $k = 1, 2$, and v_a is an equilibrium. Now, observe that when $v_a = (\bar{b}, \bar{b})$, $v_{ak}e^{r_k^B t} \geq b$ for all $(b, t) \in \mathbb{B}\times[0, \bar{T}]$ so $\bar{\Phi}^B(\varphi^A(a), v_a, a) = (0, 0)$. In this case, $\Psi_{(\varphi^A,a)}(v_a) = P_{V(a)}(0) = (1 - a, 1 - a) \neq v_a$, so v_a is not a fixed point. The upper boundary (excluding end points) is defined by $v_{a2} = 1 - \underline{b}$ and $a < v_{a1} < 1 - \underline{b}$, and the lower boundary is defined by $a < v_{a1} = v_{a2} < 1 - \underline{b}$. Assume v_a is in the upper boundary. Then for all $(b, t) \in \mathbb{B} \times [0, \bar{T}]$, $v_{a2}e^{r_2^B t} > b_z(a)$, and $\bar{\Phi}_2^B(\varphi^A(a), v_a, a) = 0$. Hence, $w = (v_{a1}\bar{\Phi}_1^B(\varphi^A(a), v_a, a), v_{a2}\bar{\Phi}_2^B(\varphi^A(a), v_a, a)) = (w_1, 0)$, where $w_1 \geq 0$, so $P_{V(a)}(w)$ lies in the lower boundary of $V(a)$ and $v_a \neq P_{V(a)}(w)$. Finally, assume v_a is in the lower boundary of $V(a)$. Then, for each $t \geq 0$, $v_{a2}e^{r_2^B t} \leq v_{a1}e^{r_1^B t}$ and $\bar{\Phi}_1^B(\varphi^A(a), v_a, a) = 0$. Hence, $w = (v_1\bar{\Phi}_1^B(\varphi^A(a), v_a, a), v_2\bar{\Phi}_2^B(\varphi^A(a), v_a, a)) = (0, w_2)$, where $w_2 \geq 0$, so $P_{V(a)}(w)$ lies in the left boundary of $V(a)$ and $v_a \neq P_{V(a)}(w)$. This establishes our claim.

For a given $(\varphi^A, v) \in \Sigma \times V$, we define $\mathcal{A}$'s payoff functions as follows

$$v_W^A(a, b|\varphi^A, v) = (1-b)\sum_{k=1}^{2} \alpha_k \int_0^{\bar{T}} \varphi_k^B(t|\hat{z}^A(a), z^B, a, b, v_a)e^{-r^A t}dt$$

$$v^A(a|\varphi^A, v) = a\sum_{k=1}^{2} \alpha_k [1 - \bar{\Phi}_k^B(\varphi^A(a), v_a, a)]^+ + \sum_{b\in\mathbb{B}} v_W^A(a, b|\varphi^A, v),$$

where $[\xi]^+$ is ξ if $\xi > 0$ and 0 otherwise. Let

$$\Sigma^*(\varphi^A, v) = \{\phi \in \Sigma \mid \phi(a) = 0 \text{ for all } a \notin \operatorname{argmax} v^A(\cdot|\varphi^A, v)\}.$$

The correspondence $\Psi : \Sigma \times V \to \Sigma \times V$ is then defined by

$$\Psi(\varphi^A, v) = (\Sigma^*(\varphi^A, v), (\Psi_{(\varphi^A, a)}(v_a) : a \in \mathbb{A})).$$

We now argue that (φ^A, v) is a fixed point of Ψ if and only if it corresponds to an equilibrium of $\Gamma(r, \alpha, z)$. Suppose that (φ^A, v) is a fixed point of Ψ. Then v_a is an equilibrium of $\Gamma(r, \alpha, \hat{z}^A(a), z^B, a)$ for each $a \in \mathbb{A}$. Also, by definition, $\varphi^A(a) > 0$ implies that a is an optimal offer for $\mathcal{A}$ given that $\mathcal{B}_k$ counteroffers (b, t) with density $\varphi_k^B(t|\hat{z}^A(a), z^B, a, b, v_a)$ and accepts a immediately with probability $1 - \bar{\Phi}_k^B(\varphi^A(a), v_a, a)$. Thus, (φ^A, φ^B) is an equilibrium for $\Gamma(r, \alpha, z)$. The converse is analogous.

Finally, since Ψ is upper hemicontinuous and convex-valued, by Kakutani's fixed point theorem, Ψ has a fixed point. □

Appendix

Atemporal Model.

Proof of Lemma 2

(i) We show that under the assumed conditions (and using the notation of Lemma 1),

$$L^\ell \to \infty \quad \text{and} \quad \frac{\hat{z}^{B\ell}}{[\hat{z}^{A\ell}]^{\lambda^B/\lambda^A_2}} \to 0.$$

The conclusion then follows from Lemma 1. Note that

$$\hat{z}^{A\ell} \geq \frac{z^{A\ell}\pi^A(a)}{z^{A\ell}\pi^A(a) + 1 - z^{A\ell}},$$

where the lower bound is attained when $\varphi^{A\ell}(a) = 1$. Therefore

$$\frac{\hat{z}^{A\ell}}{\hat{z}^{B\ell}} \geq \frac{z^{A\ell}}{z^{B\ell}} \frac{\pi^A(a)}{\pi^B(b)} \frac{z^{B\ell}\pi^B(b) + (1 - z^{B\ell})[\alpha_1\varphi_1^{B\ell}(b) + \alpha_2\varphi_2^{B\ell}(b)]}{z^{A\ell}\pi^A(a) + 1 - z^{A\ell}},$$

and

$$\liminf_{\ell\to\infty} \frac{\hat{z}^{A\ell}}{\hat{z}^{B\ell}} \geq \frac{1}{R} \frac{\pi^A(a)}{\pi^B(b)} [\alpha_1\varphi_1^{B\infty}(b) + \alpha_2\varphi_2^{B\infty}(b)] \equiv c > 0.$$

Note that $\varphi_1^{B\infty}(b) + \varphi_2^{B\infty}(b) > 0$ also implies that $\hat{z}^{B\infty} = 0$. Now,

$$L^\ell = \left[\frac{\hat{z}^{A\ell}}{\hat{z}^{B\ell}}\right]^{\lambda^B} \frac{1}{[\hat{z}^{B\ell}]^{\lambda_2^A - \lambda^B}} \frac{1}{[\hat{z}^{B\ell} + \hat{\alpha}_2^\ell]^{\lambda_1^A - \lambda_2^A}}$$

Since $\lambda_1^A > \lambda_2^A > \lambda^B$, $\lim\ L^\ell = \infty$. Furthermore,

$$\lim_{\ell\to\infty} \frac{\hat{z}^{B\ell}}{[\hat{z}^{A\ell}]^{\lambda^B/\lambda_2^A}} \leq \frac{1}{c} \lim_{\ell\to\infty} [\hat{z}^{A\ell}]^{1-\lambda^B/\lambda_2^A} = 0.$$

(ii) As remarked above, $\hat{z}^{B\infty} > 0$ implies that $\varphi_1^{B\infty}(b) = \varphi_2^{B\infty}(b) = 0$. Since $\varphi^{A\infty}(a) > 0$ and $\hat{z}^{B\infty} > 0$,

$$\lim_{\ell\to\infty} \hat{z}^{A\ell} = \lim_{\ell\to\infty} \frac{z^{A\ell}\pi^A(a)}{z^{A\ell}\pi^A(a) + (1 - z^{A\ell})\varphi^{A\ell}(a)} = 0$$

$$\lim_{\ell\to\infty} L^\ell = \lim_{\ell\to\infty} \frac{[\hat{z}^{A\ell}]^{\lambda^B}}{[\hat{z}^{B\ell} + \hat{\alpha}_2^\ell]^{\lambda_1^A - \lambda_2^A} [\hat{z}^{B\ell}]^{\lambda_2^A}} = 0.$$

Consequently, $\lim\ L^\ell = 0$ and by Lemma 1, $\mu^{A\ell} \to 1$.

(iii) As in (ii), $\hat{z}^{A\ell} \to 0$. Also

$$\hat{z}^{B\ell} \geq \frac{z^{B\ell}\pi^B(b)}{z^{B\ell}\pi^B(b) + 1 - z^{B\ell}}$$

and $\hat{z}^{B\ell} + \hat{\alpha}_2^\ell \leq 1$ for each ℓ. Hence

$$\begin{aligned} L^\ell &= \left[\frac{\hat{z}^{A\ell}}{\hat{z}^{B\ell}}\right]^{\lambda_2^A} \frac{[\hat{z}^{A\ell}]^{\lambda^B - \lambda_2^A}}{[\hat{z}^{B\ell} + \hat{\alpha}_2^\ell]^{\lambda_1^A - \lambda_2^A}} \\ &\leq \left[\frac{z^{A\ell}}{z^{B\ell}}\frac{\pi^A(a)}{\pi^B(b)}\frac{z^{B\ell}\pi^B(b) + 1 - z^{B\ell}}{z^{A\ell}\pi^A(a) + (1 - z^{A\ell})\varphi^{A\ell}(a)}\right]^{\lambda_2^A} [\hat{z}^{A\ell}]^{\lambda^B - \lambda_2^A}, \end{aligned}$$

and

$$\lim_{\ell\to\infty} L^\ell \leq \left[R\,\frac{\pi^A(a)}{\pi^B(b)}\frac{1}{\varphi^{A\infty}(a)}\right]^{\lambda_2^A} \times \lim_{\ell\to\infty} [\hat{z}^{A\ell}]^{\lambda^B - \lambda_2^A} = 0$$

since $\lambda^B > \lambda_2^A$. Consequently, by Lemma 1, $\mu^{A\ell} \to 1$. □

Temporal Model.

Proof of Lemma 6
Let $\mu_1(t)$ and τ_1^S be defined by

$$v_1 e^{r_1^B t} = \mu_1(t)b + (1 - \mu_1(t))(1 - a) \quad \text{and} \quad \frac{\hat{z}^A}{1 - \mu_1(t)} e^{\lambda_1^A \tau_1^S} = 1.$$

That is, $\mu_1(t)$ is the immediate probability of concession by A that delivers the appropriate continuation value for $\mathcal{B}_1$ and τ_1^S is the length of the corresponding WOA between $\mathcal{A}$ and $\mathcal{B}_1$ (alone). Note that when $\varphi_1(b,t|a) > 0$, whether $\varphi_2(b,t|a) > 0$ or $\varphi_2(b,t|a) = 0$, $\mu_1(t)$ is always uniquely defined by the first equation above, a fact we will use later. Let

$$E = e^{-(\lambda_1^A + r_2^B)\tau_1^S} = \left[\frac{\hat{z}^A}{1 - \mu_1(t)}\right]^{(\lambda_1^A + r_2^B)/\lambda_1^A}.$$

For $t < t^*$, $(1 - \mu_1(t)) > \hat{z}^A$ so $\tau_1^S > 0$ and $E < 1$. Then

$$\begin{aligned} \underline{v}_2(t) &= \mu_1(t)b + (1 - \mu_1(t))C \quad \text{where} \\ C &= b\int_0^{\tau_1^S} e^{-r_2^B s}\lambda_1^A e^{-\lambda_1^A s}ds + (1 - a)\,e^{-(\lambda_1^A + r_2^B)\tau_1^S} \\ &= b\frac{\lambda_1^A}{\lambda_1^A + r_2^B}(1 - E) + (1 - a)E. \end{aligned}$$

As in Step 3 of Lemma 3

$$C = (1-a)+b(1-E)\left[\frac{\lambda_1^A}{\lambda_1^A + r_2^B} - \frac{\lambda_1^A}{\lambda_1^A + r_1^B}\right] = (1-a)+(1-a)(1-E)\frac{r_1^B - r_2^B}{\lambda_1^A + r_2^B} > (1-a).$$

It follows that $\underline{v}_2(t) > v_1 e^{r_1^B t}$. However, at t^*, $1-\mu_1(t^*) = \hat{z}^A$ so $\tau_1^S = 0$, and in this case $C = 1-a$ and $\underline{v}_2(t^*) = v_1 e^{r_1^B t^*}$, as required. This establishes (i).

We now establish (ii). Let

$$\Omega = -r_2^B \underline{v}_2(t) + \underline{v}_2'(t) = -r_2^B[\mu_1(t)(b-C)+C] + \mu_1'(t)(b-C) + (1-\mu_1(t))\frac{dC}{dt}.$$

We first show that $\Omega > 0$. Since

$$\mu_1(t) = \frac{v_1 e^{r_1^B t} - (1-a)}{a+b-1} \quad \Longrightarrow \quad \mu_1'(t) = r_1^B \mu_1(t) + \lambda_1^A,$$
$$\frac{dE}{dt} = \frac{E}{1-\mu_1(t)}\left[\frac{\lambda_1^A + r_2^B}{\lambda_1^A}\right]\mu_1'(t) \quad \text{and} \quad \frac{dC}{dt} = -(1-a)\left[\frac{r_1^B - r_2^B}{\lambda_1^A + r_2^B}\right]\frac{dE}{dt},$$

we obtain that

$$\Omega = (r_1^B - r_2^B)\left[\mu_1(t)(b-C) - \frac{(1-a)E}{\lambda_1^A}(r_1^B \mu_1(t) + \lambda_1^A)\right] + \lambda_1^A(b-C) - r_2^B C.$$

Using the expression for C we deduced above, one can check that $\lambda_1^A(b-C) - r_2^B C = (1-a)(r_1^B - r_2^B)E$. Thus

$$\begin{aligned}\Omega &= (r_1^B - r_2^B)\left[\mu_1(t)(b-C) - \frac{(1-a)E}{\lambda_1^A} r_1^B \mu_1(t)\right] \\ &= (r_1^B - r_2^B)\frac{\mu_1(t)(1-a)}{\lambda_1^A}\left[r_1^B - \left(\frac{\lambda_1^A}{\lambda_1^A + r_2^B}(r_1^B - r_2^B)(1-E) + r_1^B E\right)\right] > 0\end{aligned}$$

since $[\lambda_1^A/(\lambda_1^A + r_2^B)](r_1^B - r_2^B)(1-E) + r_1^B E < (r_1^B - r_2^B)(1-E) + r_1^B E < r_1^B$. Assume that $v_2 e^{r_2^B t} \leq \underline{v}_2(t)$. Then

$$\underline{v}_2'(t) > r_2^B \underline{v}_2(t) \geq r_2^B\left[v_2 e^{r_2^B t}\right] = \frac{d}{dt}\left[v_2 e^{r_2^B t}\right]. \qquad \square$$

Densities.

Assume that A has demanded $a \in \mathbb{A}$, and that in equilibrium $\varphi^A(a) > 0$, so the posterior $\hat{z}^A$ that A is behavioral is given by (1). For any $b \in \mathbb{B}$ with $a + b > 1$, we now derive the densities with which $\mathcal{B}_k$'s choose b at various times.

By Lemma 1 of the atemporal types model, the WOA $\Gamma(r, \hat{\alpha}, \hat{z}^A, \hat{z}^B, a, b)$ has a unique equilibrium and hence a unique equilibrium value v_k^B for each $\mathcal{B}_k$ with $\hat{\alpha}_k > 0$, $k = 1, 2$. Hereafter, fix $(\hat{z}^A, z^B, a, b)$ once and for all, where $a + b > 1$ (possibly, $b = \hat{b}_2 = \lfloor b_2^*(a) \rfloor$). Consider the WOA that arises after the counteroffer b at time t, when $\mathcal{A}$ believes that $\mathcal{B}_k$ counteroffers (b, t) with probability $\varphi_k^B(t)$, $k = 1, 2$.[9] By (2)–(3), $(\hat{z}^B(t), \hat{\alpha}_1(t), \hat{\alpha}_2(t))$ are functions of $(\varphi_1^B(t), \varphi_2^B(t))$,[10] and thus $(\varphi_1^B(t), \varphi_2^B(t))$ leads to a unique equilibrium value $v_k^B(t)$ for each $\mathcal{B}_k$ with $\varphi_k^B(t) > 0$, $k = 1, 2$, in the corresponding WOA. In any equilibrium, it must be the case that $v_k^B(t) = v_k e^{r_k t}$ for some fixed v_k, $k = 1, 2$. Given (v_1, v_2), the equilibrium value functions can be inverted to construct the functions $(\varphi_1^B(t), \varphi_2^B(t))$ so that for each t, the corresponding WOA delivers the equilibrium value $v_k e^{r_k t}$ for each $k = 1, 2$ with $\varphi_k^B(t) > 0$.

Fix (v_1, v_2) so that $1 - a \le v_1 \le v_2 < b_z$.[11] We now solve for $(\varphi_1^B(t), \varphi_2^B(t))$ in each one of the separating and pooling intervals. Consider the WOA after the counteroffer (b, t) with $t > 0$. Let $\mu_k(t)$ be such that

$$v_k e^{r_k t} = \mu_k(t) b + (1 - \mu_k(t))(1 - a), \quad k = 1, 2.$$

When $\varphi_1^B(t) > 0$, $\mu_1(t)$ is the required probability of immediate concession by A to deliver $\mathcal{B}_1$ his corresponding expected payoff. Similarly, when $\varphi_1^B(t) = 0$ and $\varphi_2^B(t) > 0$, $\mu_2(t)$ is the required probability of immediate concession by A to deliver $\mathcal{B}_2$ his corresponding expected payoff. When $\varphi_k^B(t) > 0$ it must be that $1 - a < v_k e^{r_k t} < b$, hence

$$\mu_k(t) = \frac{v_k e^{r_k t} - (1 - a)}{a + b - 1} \in (0, 1), \ \ k = 1, 2.$$

It is also useful to define the function $\bar{\mu}_k(t) = 1 - \mu_k(t)$, $k = 1, 2$. Since $v_k e^{r_k^B t} \in (1 - a, b)$, $\bar{\mu}_k(t)$ is the distance from $v_k e^{r_k^B t}$ to b relative to the total distance from $1 - a$ to b.

Recall the various pooling and separating intervals from Lemma 9.

Separating Interval for $\mathcal{B}_1$: Let t be such that $\varphi_1^B(t) > 0$ and $\varphi_2^B(t) = 0$. Then, By Lemma 1, the length of the WOA τ_1 must be such that

$$1 = \frac{\hat{z}^A}{\bar{\mu}_1(t)} e^{\lambda_1^A \tau_1} = \frac{z^B \pi^B(b, t)}{z^B \pi^B(b, t) + (1 - z^B) \alpha_1 \varphi_1^B(t)} e^{\lambda^B \tau_1}.$$

[9] To simplify notation here, since we fix b and focus only on the time dimension, we we write $\varphi_k^B(t)$ instead of $\varphi_k^B(t | \hat{z}^A, z^B, a, b, v_1, v_2)$.

[10] They are also functions of (a, b), but since these variables have been fixed, we omit them here.

[11] In equilibrium, $v_1 \ge 1 - a$ always. If $v_1 \ge b_z$, then $\varphi_k^B(t) \equiv 0$ for $k = 1, 2$. If $1 - a \le v_1 < b_z \le v_2$, then $\varphi_2^B(t) \equiv 0$ and only equation (4) below is relevant.

Let $Z(z^B, b, t) = z^B \pi^B(b,t)/(1-z^B)$. It follows that

$$\varphi_1^B(t) = \frac{Z(z^B, b, t)}{\alpha_1} \left[\left[\frac{\bar{\mu}_1(t)}{\hat{z}^A} \right]^{\lambda^B/\lambda_1^A} - 1 \right]. \tag{4}$$

Separating Interval for $\mathcal{B}_2$: Let t be such that $\varphi_1^B(t) = 0$ and $\varphi_2^B(t) > 0$. By a similar argument, we now obtain that

$$\varphi_2^B(t) = \frac{Z(z^B, b, t)}{\alpha_2} \left[\left[\frac{\bar{\mu}_2(t)}{\hat{z}^A} \right]^{\lambda^B/\lambda_2^A} - 1 \right]. \tag{5}$$

Pooling Interval: Let t be such that $\varphi_1^B(t) > 0$ and $\varphi_2^B(t) > 0$. Here again, $\mu_1(t)$ must be the probability that A concedes immediately. By Lemma 1, there exist $0 < \tau_1 < \tau_2$ such that

$$[\hat{z}^B(t) + \hat{\alpha}_2(t)] e^{\lambda^B \tau_1} = 1, \quad \hat{z}^B(t) e^{\lambda^B \tau_2} = 1, \quad \text{and} \quad \frac{\hat{z}^A}{\bar{\mu}_1(t)} e^{\lambda_1^A \tau_1 + \lambda_2^A(\tau_2 - \tau_1)} = 1.$$

Therefore

$$\bar{\mu}_1(t) = \hat{z}^A \left[\frac{1}{\hat{z}^B(t) + \hat{\alpha}_2(t)} \right]^{\lambda_1^A/\lambda^B} \left[\frac{\hat{z}^B(t) + \hat{\alpha}_2(t)}{\hat{z}^B(t)} \right]^{\lambda_2^A/\lambda^B}. \tag{6}$$

The WOA should also deliver $\mathcal{B}_2$ his expected value $v_2 e^{r_2^B t}$. An optimal strategy for $\mathcal{B}_2$ is to concede at $t + \tau_1$ if A has not conceded yet. Therefore, $\mathcal{B}_2$'s expected value in the WOA is

$$\begin{aligned} v_2 e^{r_2^B t} &= \mu_2(t) b + (1 - \mu_2(t))(1-a), \\ &= \left[\mu_1(t) + (1 - \mu_1(t)) \frac{\lambda_1^A}{\lambda_1^A + r_2^B} (1 - E) \right] b + (1 - \mu_1(t)) E (1 - a) \end{aligned}$$

where

$$E = e^{-(\lambda_1^A + r_2^B)\tau_1} \qquad \text{so} \qquad \int_0^{\tau_1} e^{-r_1^B \tau} \lambda_1^A e^{-\lambda_1^A \tau} d\tau = \frac{\lambda_1^A}{\lambda_1^A + r_2^B} (1 - E).$$

Subtracting b from sides in the previous equation, we obtain

$$\bar{\mu}_2(t)(a + b - 1) = \bar{\mu}_1(t) \left[\left[1 - \frac{\lambda_1^A}{\lambda_1^A + r_2^B} (1 - E) \right] b - E(1 - a) \right]. \tag{7}$$

Let $D = z^B \pi^B(b,t) + (1 - z^B)[\alpha_1^B \varphi_1(t) + \alpha_2 \varphi_2^B(t)]$ and $U = z^B \pi^B(b,t) + (1 - z^B)\alpha_2 \varphi_2^B(t)$ so that $\hat{z}^B(t) + \hat{\alpha}_2(t) = U/D$. Substituting the expressions for the corresponding posteriors in (6) we obtain

$$\bar{\mu}_1(t) = \frac{\hat{z}^A D^{\lambda_1^A/\lambda^B}}{U^{(\lambda_1^A - \lambda_2^A)/\lambda^B} [z^B \pi^B(b,t)]^{\lambda_2^A/\lambda^B}}. \tag{8}$$

Solving for E from (7), we conclude that $\mathcal{B}_2$'s expected value is attained when

$$E = \frac{(\lambda_1^A + r_2^B)(a+b-1)\bar{\mu}_2(t) - r_2^B b\bar{\mu}_1(t)}{[\lambda_1^A b - (\lambda_1^A + r_2^B)(1-a)]\bar{\mu}_1(t)}$$
$$= e^{-(\lambda_1^A + r_2^B)\tau_1} = [\hat{z}^B(t) + \hat{\alpha}_2(t)]^{[\lambda_1^A + r_2^B]/\lambda^B} = \left[\frac{U}{D}\right]^{\frac{\lambda_1^A + r_2^B}{\lambda^B}} \tag{9}$$

Let $c = (\lambda_1^A + r_2^B)(a+b-1)$ and $d = r_2^B b$. Note that $c > d$ since $\lambda_1^A > \lambda_2^A$. Also let $\gamma_k = \lambda_k^A/\lambda^B$, $k = 1, 2$, and $\rho = r_2^B/\lambda^B$. Then (8) and (9) imply

$$\left[\frac{D^{\gamma_1}}{U^{\gamma_1-\gamma_2}}\right]^{\frac{\gamma_1+\rho}{\gamma_1-\gamma_2}} \left[\frac{U}{D}\right]^{\gamma_1+\rho} = D^{\frac{\gamma_2(\gamma_1+\rho)}{\gamma_1-\gamma_2}} = \left[\frac{\bar{\mu}_1[z^B\pi^B(b,t)]^{\gamma_2}}{\hat{z}^A}\right]^{\frac{\gamma_1+\rho}{\gamma_1-\gamma_2}} \left[\frac{c\bar{\mu}_2 - d\bar{\mu}_1}{(c-d)\bar{\mu}_1}\right]$$
$$\left[\frac{D^{\gamma_1}}{U^{\gamma_1-\gamma_2}}\right]^{\frac{\gamma_1+\rho}{\gamma_1}} \left[\frac{U}{D}\right]^{\gamma_1+\rho} = U^{(\gamma_1+\rho)(\gamma_1+\gamma_2)/\gamma_1} = \left[\frac{\bar{\mu}_1[z^B\pi^B(b,t)]^{\gamma_2}}{\hat{z}^A}\right]^{\frac{\gamma_1+\rho}{\gamma_1}} \left[\frac{c\bar{\mu}_2 - d\bar{\mu}_1}{(c-d)\bar{\mu}_1}\right]$$

which can be solved for D and U to get

$$D = \left[\frac{\bar{\mu}_1[z^B\pi^B(b,t)]^{\gamma_2}}{\hat{z}^A}\right]^{1/\gamma_2} \left[\frac{c\bar{\mu}_2 - d\bar{\mu}_1}{(c-d)\bar{\mu}_1}\right]^{\frac{\gamma_1-\gamma_2}{\gamma_2(\gamma_1+\rho)}}$$
$$U = \left[\frac{\bar{\mu}_1[z^B\pi^B(b,t)]^{\gamma_2}}{\hat{z}^A}\right]^{1/\gamma_2} \left[\frac{c\bar{\mu}_2 - d\bar{\mu}_1}{(c-d)\bar{\mu}_1}\right]^{\frac{\gamma_1}{\gamma_2(\gamma_1+\rho)}}$$

Finally $(1-z^B)\alpha_1\varphi_1^B(b,t|a;v) = D - U$ and $(1-z^B)\alpha_2\varphi_2^B(b,t|a;v) = U - z^B\pi^B(b,t)$. It follows that

$$\varphi_1^B(t) = \frac{Z(z^B,b,t)}{\alpha_1}[g(t) - h(t)] \quad \text{and} \tag{10}$$
$$\varphi_2^B(t) = \frac{Z(z^B,b,t)}{\alpha_2}[h(t) - 1], \quad \text{where} \tag{11}$$
$$N = \frac{\lambda^B}{\lambda_2^A}, \quad m_1 = N\,\frac{\lambda_1^A - \lambda_2^A}{\lambda_1^A + r_2^B}, \quad m_2 = N\,\frac{\lambda_1^A}{\lambda_1^A + r_2^B},$$
$$g(t) = \left[\frac{\bar{\mu}_1(t)}{\hat{z}^A}\right]^N \left[\frac{c\bar{\mu}_2(t) - d\bar{\mu}_1(t)}{(c-d)\bar{\mu}_1(t)}\right]^{m_1} \quad \text{and} \tag{12}$$
$$h(t) = \left[\frac{\bar{\mu}_1(t)}{\hat{z}^A}\right]^N \left[\frac{c\bar{\mu}_2(t) - d\bar{\mu}_1(t)}{(c-d)\bar{\mu}_1(t)}\right]^{m_2} \tag{13}$$

It is clear that $\varphi_k^B(t)$, $k = 1, 2$, (formulas (4)–(5) and (10)–(11)) depend on the equilibrium values (v_1, v_2). They also depend on the offers (a, b) (assumed fixed early on) and the reputations $(\hat{z}^A, z^B)$. We may find it convenient to make this dependence explicit sometimes and write instead $\varphi_k(t|\hat{z}^A, z^B, a, b, v_1, v_2)$ (or $\varphi_k(t|v_1, v_2)$ if we only want to highlight the dependence on (v_1, v_2)). The function $\varphi_k(t|\hat{z}^A, z^B, a, b, v_1, v_2)$ is continuous in $(t, \hat{z}^A, z^B, a, b, v_1, v_2)$.

By definition, $\bar{\mu}_2(t_3) = \hat{z}^A$, so $\varphi_2^B(t_3) = 0$. Since $\bar{\mu}_2(t)$ is a decreasing function of t, (5) implies that $\varphi_2^B(t)/\pi^B(b,t)$ is decreasing in $t \in (t_2, t_3)$. Tedious but simple computations also show that $h'(t) > 0$, so by (11), $\varphi_2^B(t)/\pi^B(b,t)$ is strictly increasing in $t \in (t_1, t_2)$. Hence, $\varphi_2^B(t)/\pi^B(b,t)$ is single-peaked at $t = t_2$. Also, $h(t_1) = 1$, so $\varphi_2^B(t_1) = 0$, as required.

For completeness, let us also include here a simple expression for the sneaking-in value function that we developed in Lemma 6 above:

$$\underline{v}_2(t) = (1 - \hat{b}_2) - \frac{\bar{\mu}_1(t)}{\lambda_1^A + r_2^B}\left[d + (c-d)\left[\frac{\hat{z}^A}{\bar{\mu}_1(t)}\right]^{1 + r_2^B/\lambda_1^A}\right]. \tag{14}$$

Linearization.

In this section we develop approximations for $\varphi_k^B(t)$, $k = 1, 2$, and their integrals. As in the previous subsection, we maintain fixed $a \in \mathbb{A}$ and $b \in \mathbb{B}$, where $a + b > 1$ and $\varphi^A(a) > 0$. As always, let $b_z = (1 - \hat{z}^A)b + \hat{z}^A(1-a)$. The values of $\varphi_1^B(t)$ and $\varphi_2^B(t)$ depend on (v_1, v_2) nonlinearly, through $\bar{\mu}_k(t)$, $k = 1, 2$. Fix $v_1 \in [1-a, b_z)$ and let t^* be such that $v_1 e^{r_1^B t^*} = b_z$. We now develop linear approximations for $\bar{\mu}_1(t)$ and $\bar{\mu}_2(t)$ near t^*. When the times t_j, $j = 1, 2, 3$, are close to t^*, the linear approximations produce good approximations for $\varphi_1^B(t)$ and $\varphi_2^B(t)$.

Here we restrict to the case where $v_2 > v_1$ and $v_2 e^{r_2^B t^*} = b_z - \Delta$ for some $\Delta > 0$. In this case, $t_1, t_2)$ and (t_2, t_3) are nonempty intervals. Note that adjusting v_2 is equivalent to adjusting Δ. Hereafter we assume that Δ is small,. It follows that $|t_1 - t^*|$ and $|t_3 - t^*|$ are order $O(\Delta)$. Therefore, by Taylor series expansion around t^*, $e^{r_k^B t} = e^{r_k^B t^*}[1 + r_k^B(t - t^*)] + O(|t - t^*|^2)$ for all $t \in (t_1, t_3)$. Let $s = t - t^*$. Hence

$$v_1 e^{r_1^B t} = b_z(1 + r_1^B s) + O(s^2) \quad \Longrightarrow \quad \bar{\mu}_1(t) = -\frac{r_1^B b_z s}{a+b-1} + O(s^2)$$

$$v_2 e^{r_2^B t} = (b_z - \Delta)(1 + r_2^B s) + O(s^2) \quad \Longrightarrow \quad \bar{\mu}_2(t) = \frac{\Delta - r_2^B b_z s}{a+b-1} + O(s^2).$$

Let $s_i = t_i - t^*, i = 1, 2, 3$. The equation $\bar{\mu}_1(t_2) = \bar{\mu}_2(t_2)$ implies that $-r_1^B b_z s_2 = \Delta - r_2^B b_z s_2 + O(s_2^2)$, or

$$s_2 = \tilde{s}_2 + O(\Delta^2) \quad \text{where} \quad \tilde{s}_2 = \frac{-\Delta}{b_z(r_1^B - r_2^B)}. \tag{15}$$

Similarly, $v_2 e^{r_2^B t_3} = b_z$ leads to $(b - \Delta)(1 + r_2^B s_3) + O(s_3^2) = b_z$, or

$$s_3 = \tilde{s}_3 + O(\Delta^2) \quad \text{where} \quad \tilde{s}_3 = \frac{\Delta}{r_2^B b_z}. \tag{16}$$

If $v_2 \geq \underline{v}_2(0)$ the equation $\underline{v}_2(t_1) = v_2 e^{r_2^B t_1}$ leads to the approximation

$$\frac{r_1^B b_z s_1}{(\lambda_1^A + r_2^B)(a+b-1)} [d + (c-d)W] = -\Delta + b_z r_2^B s_1 + O(s_1^2)$$

$$\text{where} \qquad W = \left[\frac{\hat{z}^A(a+b-1)}{-r_1^B b_z s_1}\right]^{1+r_2^B/\lambda_1^A}.$$

This is a nonlinear equation in s_1. If we view W as an exogenous parameter, the solution s_1 of this equation increases with W. When we make $W = 0$, we obtain the approximation

$$\tilde{s}_1 = \frac{(\lambda_1^A + r_2^B)}{r_2^B}\tilde{s}_2, \tag{17}$$

and hence $\tilde{s}_1 + O(\Delta^2) \leq s_1 \leq s_2$. Though this does not establish a tight estimate for s_1, it does establish that $|s_1| = |t_1 - t^*| = O(\Delta)$, as claimed earlier. If $v_2 > \underline{v}_2(0)$, then $t_1 = 0$ (by definition) and consequently we define $\tilde{s}_1 = \max\{(\lambda_1^A + r_2^B)\tilde{s}_2/r_2^B, -t^*\}$ in general. Obviously, when $-t^* > (\lambda_1^A + r_2^B)\tilde{s}_2/r_2^B$, we still have that $\tilde{s}_1 = O(\Delta)$. We discuss $\tilde{s}_1$ later in more detail, after we obtain an approximation for $\varphi_2^B(t)$.

Let

$$X = \frac{\lambda_1^A + r_2^B}{(r_1^B - r_2^B)\lambda_1^A b_z}, \qquad Y = \frac{r_2^B}{\lambda_1^A}, \quad \text{and} \quad n_j = N - m_j,\ j = 1, 2.$$

If $\xi(x, y) = x^n y^m$, then by Taylor approximation,

$$\xi(x + \epsilon_1, y + \epsilon_2) = \xi(x, y)\left[1 + \frac{n\epsilon_1}{x} + \frac{m\epsilon_2}{y}\right].$$

We use this approximation when $x = -s$, $y = X\Delta + Ys$, $\epsilon_1 = O(s^2) = \epsilon_2$, and $s \in [s_1, s_2]$. Since $s_1 < s_2 < 0$ and $s_2 = O(\Delta)$, $\xi(-s + O(s^2), X\Delta + Ys + O(s^2)) = \xi(-s, X\Delta + Ys)(1 + O(\Delta)$. Therefore, for each function $\gamma \in \{g, h, \varphi_1^B, \varphi_2^B\}$

$$\gamma(t^* + s) = \tilde{\gamma}(s)(1 + O(\Delta)) \quad \text{for all } s \in [\tilde{s}_1, \tilde{s}_3],$$

where the corresponding functions $\tilde{\gamma}$ are defined as follows

$$\tilde{g}(s) = \left[\frac{r_1^B b_z}{\hat{z}^A(a+b-1)}\right]^N [-s]^{n_1}[X\Delta + Ys]^{m_1} \tag{18}$$

$$\tilde{h}(s) = \left[\frac{r_1^B h_z}{\hat{z}^A(a+b-1)}\right]^N [-s]^{n_2}[X\Delta + Ys]^{m_2} \tag{19}$$

$$\tilde{\varphi}_1^B(s) = \frac{Z(z^B, b, t^* + s)}{\alpha_1}[\tilde{g}(s) - \tilde{h}(s)] \qquad s \in [\tilde{s}_1, \tilde{s}_2] \tag{20}$$

$$\tilde{\varphi}_2^B(s) = \begin{cases} \dfrac{Z(z^B, b, t^* + s)}{\alpha_2}\tilde{h}(s) & s \in [\tilde{s}_1, \tilde{s}_2] \\[2ex] \dfrac{Z(z^B, b, t^* + s)}{\alpha_2}\left[\dfrac{\Delta - r_2^B b_z s}{\hat{z}^A(a+b-1)}\right]^N & s \in [\tilde{s}_2, \tilde{s}_3], \end{cases} \tag{21}$$

and $\tilde{\varphi}_1^B(s) = 0$ for $s \in [s_1, s_3] \backslash [\tilde{s}_1, \tilde{s}_2]$ and $\tilde{\varphi}_2^B(s) = 0$ for $s \in [s_1, s_3] \backslash [\tilde{s}_1, \tilde{s}_3]$.

We now discuss the precision of the approximation $\tilde{s}_1$ when $v_2 \geq \underline{v}_2(0)$. Then, the point t_1 is also the left limit of the pooling region. Hence, $\varphi_2^B(t_1) = 0$. Since $\tilde{\varphi}_2^B(s)$ is a good approximation for $\varphi_2^B(t^* + s)$, an approximation for s_1 is obtained when we solve the equation $\tilde{\varphi}_2^B(s) = 0$. The approximation $\tilde{s}_1$ we obtained earlier is precisely the solution of this last equation. When $\hat{z}^A/\Delta = O(\Delta^2)$, $[\hat{z}^A]^{1+r_2^B/\lambda_1^A}/|s|^{r_2^B/\lambda_1^A} = o(\Delta^2)$ for all $s < s_2$ (note that $s_1 < s_2 < 0$). In this case $W = o(\Delta^2)$ and $s_1 = \tilde{s}_1 + O(\Delta^2)$. The case in which $b = \hat{b}_2 = \lfloor b_2^*(a) \rfloor$ is particularly important for our analysis. Here, a minimum requirement for equilibrium is that $\Phi_2^B(\hat{z}^A, z^B, a, b, v_1, v_2) \leq 1$. If $\varphi^A(a) > 0$ and $(z^A, z^B) \in K(R, \bar{z})$ for $\bar{z}$ small, this inequality is satisfied only if $\Delta^{N+1}/[\hat{z}^A]^{N-1} = O(1)$ (see equation (23) below). Since $N > 1$ when $b = \hat{b}_2$, Δ is small when $\hat{z}^A$ is small. Moreover, if the grid $\mathbb{B}$ is sufficiently fine so that $N = r^A(1 - \hat{b}_2)/[r_2^B(1 - a)] \leq 2$, this implies that $\hat{z}^A/\Delta$ is of order less than $O(\Delta^2)$.

We would also like to obtain good approximations for the integrals of φ_k^B, $k = 1, 2$. We do so by integrating $\tilde{\varphi}_k^B$, $k = 1, 2$. For the rest of this section, we assume that $v_2 \geq \underline{v}_2(0)$ and hence assume that $\tilde{s}_1 = (\lambda_1^A + r_2^B)\tilde{s}_2/r_2^B$. We consider the case $v_2 < \underline{v}_2(0)$ in Lemma 11, and there we only obtain a bound for the relevant integrals.

Both $\tilde{g}$ and $\tilde{h}$ are functions of the form $f(s) = (-s)^n(X\Delta + Ys)^m$ for some constants $n > 0$ and $m > 0$. We have that

$$\int_{\tilde{s}_1}^{\tilde{s}_2} f(s)ds = [X\Delta]^m \int_{\tilde{s}_1}^{\tilde{s}_2} (-s)^n \left[1 + \frac{Y}{X\Delta}s\right]^m ds = \frac{[X\Delta]^{n+m+1}}{Y^{n+1}} \int_{1+\frac{Y}{X\Delta}\tilde{s}_1}^{1+\frac{Y}{X\Delta}\tilde{s}_2} (1-t)^n\, t^m dt,$$

with the change of variables $t = 1 + Ys/[X\Delta]$. Now

$$\int (1-t)^n\, t^m dt = \frac{t^{m+1}}{m+1} H(t, m, n), \quad 1 + \frac{Y}{X\Delta}\tilde{s}_1 = 0 \quad \text{and} \quad 1 + \frac{Y}{X\Delta}\tilde{s}_2 = \frac{\lambda_1^A}{\lambda_1^A + r_2^B},$$

where $H(t, m, n)$ is the hypergeometric function (usually denoted by ${}_2F_1(m+1, -n, m+2, t)$) defined by the series expansion:

$$H(t, m, n) = 1 + \sum_{k=1}^{\infty} h_k t^k \quad \text{where} \quad h_k = \frac{m+1}{m+k+1} \frac{(-n)(1-n)\cdots(k-1-n)}{k!}.$$

Since $H(0) = 1$,

$$\int_{\tilde{s}_1}^{\tilde{s}_2} f(s)ds = \frac{[X\Delta]^{n+m+1}}{(m+1)Y^{n+1}} \Big[t^{m+1} H(t, m, n)\Big]_{t=\lambda_1^A/(\lambda_1^A + r_2^B)}.$$

Since $\Delta - r_2^B b_z \tilde{s}_3 = 0$, it follows that

$$\begin{aligned}\int_{\tilde{s}_2}^{\tilde{s}_3} [\Delta - r_2^B b_z s]^N ds &= \frac{[\Delta - r_2^B b_z \tilde{s}_2]^{N+1}}{[N+1] r_2^B b_z} \\ &= \frac{1}{r_2^B b_z} \frac{\lambda_2^A}{\lambda_2^A + \lambda^B} \left[\frac{r_1^B \Delta}{r_1^B - r_2^B}\right]^{N+1}.\end{aligned}$$

We assumed that $\pi^B(b,t) \geq \underline{\pi}$ is continuous in $t \in [0,T]$ for each b. Thus, $Z(z^B, b, t)$ is absolutely continuous in t. Note that $\tilde{s}_1 = -\omega_1\Delta$ and $\tilde{s}_3 = \omega_2\Delta$ for some positive constants ω_1 and ω_2. Thus, for each $\delta > 0$ there exists $\Delta \in (0,\delta)$ such that $Z(z^B, b, t+s) = Z(z^B, b, t)(1 + O(\delta))$ for all $s \in [\tilde{s}_1, \tilde{s}_3]$ and all $t \in [-\tilde{s}_1, T - \tilde{s}_3]$. Therefore

$$\begin{aligned}\int_{s_1}^{s_2} \varphi_1^B(t^*+s)ds &= \int_{\tilde{s}_1}^{\tilde{s}_2} \frac{Z(z^B, b, t^*+s)}{\alpha_1}[\tilde{g}(s) - \tilde{h}(s)](1 + O(\Delta))ds \\ &= \frac{Z(z^B, b, t^*)}{\alpha_1}(1 + O(\epsilon)) \int_{\tilde{s}_1}^{\tilde{s}_2} [\tilde{g}(s) - \tilde{h}(s)]ds.\end{aligned}$$

Thus, for any $\delta > 0$, there exists $\Delta \in (0,\delta)$ such that

$$\int_{s_1}^{s_2} \varphi_1^B(t^*+s)ds = \frac{Z(z^B, b, t^*)}{\alpha_1[\hat{z}^A]^N}\Delta^{N+1}\theta_1(1 + O(\delta)) \tag{22}$$

$$\int_{s_1}^{s_3} \varphi_2^B(t^*+s)ds = \frac{Z(z^B, b, t^*)}{\alpha_2[\hat{z}^A]^N}\Delta^{N+1}\theta_2(1 + O(\delta)) \quad \text{where} \tag{23}$$

$$x = \frac{\lambda_1^A}{\lambda_1^A + r_2^B}, \quad \rho(m,n) = \frac{x^{m+1}H(x,m,n)}{(m+1)Y^{n+1}}, \quad J = \frac{\lambda_1^A}{(r_1^B - r_2^B)(1-a)}$$

$$\theta_1 = X\left[\frac{\lambda_1^A + r_2^B}{(r_1^B - r_2^B)(1-a)}\right]^N [\rho(m_1,n_1) - \rho(m_2,n_2)]$$

$$\theta_2 = X\left[\frac{\lambda_1^A + r_2^B}{(r_1^B - r_2^B)(1-a)}\right]^N \rho(m_2,n_2) + \frac{[J]^N\lambda_1^A}{(\lambda_2^A + \lambda^B)(r_1^B - r_2^B)b_z}.$$

Define

$$\theta(a,b) = \frac{\theta_1}{\theta_2} \qquad (a,b) \in \mathbb{A} \times \mathbb{B} \quad \textit{with} \quad a + b > 1.$$

The functions θ_1 and θ_2 depend on a and b, and are independent of (v_1, v_2). Furthermore, they also depend on $\hat{z}^A$ but only through b_z (note that X depends on b_z). Since b_z cancels out when we take the ratio, $\theta(a,b)$ is indeed a function of (a,b) alone.

Temporal Model.

Proof of Lemma 10

Let $b = \hat{b}_2$ and $\Delta = b_z - v_2 e^{r_2^B t^*}$. By assumption, $0 < \Delta < \delta$. In the previous linearization section we defined $s_i = t_i - t^*$ and show that $s_i = O(\Delta)$ for $i = 1,2,3$ (see equations (15)–(17)). Therefore, we can choose $\delta > 0$ sufficiently small so that $|s_i| < \epsilon$ for $i = 1,2,3$. Now, if $v_2 \leq \underline{v}_2(0)$, then $t_1 = 0$ (by definition) and $t^* = t^* - t_1$. Hence, $t^* = O(\Delta)$ and $v_1 = e^{-r_1^B t^*} b_z \geq b_z(1 - O(\Delta))$. Again, we can choose $\delta > 0$ sufficiently small so that $v_1 \geq b_z - \epsilon$. $\square$

Proof of Lemma 11

Choose $\delta > 0$ as required by Lemma 10 so that $|t_i - t^*| < \epsilon$, $i = 1, 2, 3$. Since $b_2 = \hat{b}_2 = \lfloor b_2^*(a) \rfloor$, $theta(a) \equiv \theta(a, \hat{b}_2)$ is only a function of a. By assumption, $v_2 \geq \underline{v}_2(0)$, so $\tilde{s}_1 < -t^*$ and the equations (22)–(23) are valid. Thus,

$$\frac{\int_{t_1}^{t_2} \varphi_1^B(t|\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)dt}{\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, v_1, v_2)} = \frac{\alpha_2}{\alpha_1}\,\theta(a) + O(\delta),$$

and we can choose $\delta > 0$ small enough so that $|O(\delta)| < \epsilon$. □

Proof of Lemma 12

Assume that $v_2 < \underline{v}_2(0)$, so that $v_2 e^{r_2^B t}$ and $\underline{v}_2(t)$ do not intersect in $[0, t^*)$. In this case $t_1 = 0$ and $s_1 = -t^*$ is not the left limit of integration assumed in equations (22)–(23), and $(0, t_1) = \emptyset$. When $s_1 = -t^*$, the pooling region is "truncated" in the left at $t = 0$. For $s > 0$, let $(\tilde{v}_1, \tilde{v}_2)$ be such that $\tilde{v}_k e^{r_k^B s} = v_k$ for $k = 1, 2$. Then $\tilde{v}_k e^{r_k^B (s+t)} = v_k e^{r_k^B t}$ for $t \geq 0$, $k = 1, 2$. Thus, adjusting $(\tilde{v}_1, \tilde{v}_2)$ this way corresponds to "moving the vertical axes" and extending the original diagram to the left. Corresponding to $(\tilde{v}_1, \tilde{v}_2)$ there is a new sneaking-in function that also satisfies $\underline{v}_2(s + t|\tilde{v}) = \underline{v}_2(t|v)$ for all $t \geq 0$. Since the indifference curve $\tilde{v}_2 e^{r_2^B t}$ is steeper than the sneaking-in function $\underline{v}_2(t|\tilde{v})$ at each t, there exists $s^* > 0$ large enough so that when $s = s^*$, $\tilde{v}_2 = \underline{v}_2(0|\tilde{v})$. Let v_k^* be such that $v_k^* e^{r_k^B s^*} = v_k$, $k = 1, 2$. Then $\varphi_k^B(s^* + t|v^*) = \varphi_k^B(t|v)$ for all $t \geq 0$, $k = 1, 2$. By Lemma 11, for any $\epsilon > 0$ small we can choose $\delta > 0$ such that

$$\frac{\Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, v^*)}{\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, v^*)} < \frac{\alpha_2}{\alpha_1}\,\theta(a) + \epsilon < 1.$$

Assume by contradiction that $\Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, v)/\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, v) \geq 1$. The function $\varphi_1^B(\cdot|v^*) : [0, s^* + t_2]$ is quasiconcave and attains a maximum at some point in $(0, s^* + t_2)$. The function $\varphi_2^B(\cdot|v^*) : [0, s^* + t_3]$ is quasiconcave and attains its maximum at $s^* + t_2$. Moreover, $\varphi_1^B(0|v^*) > 0 = \varphi_2^B(0|v^*)$ and $\varphi_2^B(s^* + t_2|v^*) > 0 = \varphi_1^B(s^* + t_2|v^*)$. Hence, there exists a unique $t_0 \in (0, s^* + t_2)$ such that $\varphi_1^B(t_0|v^*) = \varphi_2^B(t_0|v^*)$, and $\varphi_1^B(t|v^*) > \varphi_2^B(t|v^*)$ for $t \in (0, t_0)$ and $\varphi_1^B(t|v^*) < \varphi_2^B(t|v^*)$ for $t \in (t_0, s^* + t_3)$. Therefore, $\Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, v)/\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, v) \geq 1$ implies that $t_0 - s^* > 0$.

As we move the vertical axes to the left, s increases from 0 to s^* and $(\tilde{v}_1, \tilde{v}_2)$ decreases from (v_1, v_2) to (v_1^*, v_2^*). We now study how the ratio

$$\gamma(s) = \frac{\Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, \tilde{v})}{\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, \tilde{v})}$$

varies with s. By assumption, when $s = 0$, $\tilde{v} = v$ and $\gamma(0) \geq 1$. We now argue that $\gamma(s) > 1$ for all $s \in (0, s^*]$. Suppose that at some $s \in (0, s^*]$, $\gamma(s) = 1$. Let $\underline{s}$ be the smallest such point and $\underline{v}_k$ be such that $\underline{v}_k e^{r_k^B \underline{s}} = v_k$, $k = 1, 2$. Then, $\gamma(s) > 1$ for $s \in (0, \underline{s})$, and

$$\begin{aligned}\gamma'(\underline{s}) &= \frac{\varphi_1^B(s^* - \underline{s}|v^*)\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, \underline{v}) - \Phi_1^B(\hat{z}^A, z^B, a, \hat{b}_2, \underline{v})\varphi_2^B(s^* - \underline{s}|v^*)}{\Phi_2^B(\hat{z}^A, z^B, \underline{v})^2} \\ &= \frac{1}{\Phi_2^B(\hat{z}^A, z^B, a, \hat{b}_2, \underline{v})}[\varphi_1^B(s^* - \underline{s}|v^*) - \gamma(\underline{s})\varphi_2^B(s^* - \underline{s}|v^*)] > 0.\end{aligned}$$

Thus, $\gamma(s) < 1$ for s just smaller than $\underline{s}$, which is a contradiction. Therefore, $\gamma(s) > 1$ for all $s \in (0, s^*]$. But this implies that $\gamma(s^*) > 1$ which is also a contradiction because the assumption $\alpha_2\theta(a)/\alpha_1 < 1$ implies that $\gamma(s^*) < 1$, as we argued above. □

Proof of Lemma 13

Assume that $\epsilon < \underline{\pi}$. Then, $\hat{z}^A \geq z^A\pi^A(a)/\epsilon$. Since $\lambda_1^A > \lambda^B$ and and $z^A \geq z^B/R$, (5) implies that there exists $\bar{z}$ such that for all $z \in K(R, \bar{z})$ and each $t \in (0, t_1)$,

$$\varphi_1^B(t) \leq \left[\frac{R\epsilon}{\pi^A(a)}\right]^{\lambda^B/\lambda_1^A} \frac{\pi^B(\hat{b}_2, t)}{\alpha_1}[\bar{\mu}_1(t)]^{\lambda^B/\lambda_1^A}\bar{z}^{1-\lambda^B/\lambda_1^A}.$$

Therefore, for some constant $M > 0$,

$$\int_0^{t_1^\ell} \varphi_1^B(t)dt \leq M\,\bar{z}^{1-\lambda^B/\lambda_1^A},$$

and clearly we can choose $\bar{z}$ small enough so that $M\,\bar{z}^{1-\lambda^B/\lambda_1^A} \leq \epsilon$. □

References

1. D. Abreu and F. Gul, *Bargaining and Reputation*, Econometrica **68** (2000), 85–117.
2. D. Abreu and D. Pearce, *Bargaining, Reputation, and Equilibrium Selection in Repeated Games with Contracts*, Econometrica **75** (2007), 653–710.
3. L. Ausubel and R. Deneckere, *Reputation in Bargaining and Durable Goods Monopoly*, Econometrica **57** (1989), 511–531.
4. J. Bulow, *Durable Goods Monopolists*, Journal of Political Economy **90** (1982), 314-322.
5. K. Chatterjee and L. Samuelson, *Bargaining with Two-Sided Incomplete Information: An Infinite Horizon Model with Alternating Offers*, Review of Economic Studies **54** (1987), 175–192.
6. K. Chatterjee and L. Samuelson, *Bargaining under Two-Sided Incomplete Information: The Unrestricted Offers Case*, Operations Research **36** (1988), 605–618.
7. R. Coase, *Durability and Monopoly*, Journal of Law and Economics **15** (1971), 143–149.
8. P. Cramton, *Bargaining with Incomplete Information: An Infinite-Horizon Model with Continuous Uncertainty*, Review of Economic Studies **51** (1984), 579–593.
9. D. Fudenberg and D. Levine, *Reputation and Equilibrium Selection in Games with a Patient Player*, Econometrica **57** (1989), 759–778.
10. D. Fudenberg, D. Levine and J. Tirole, *Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information*, Game-Theoretic Models of Bargaining (A. Roth, eds.), Cambridge University Press, 1985, pp. 73–98.
11. D. Fudenberg and J. Tirole, *Sequential Bargaining with Incomplete Information*, Review of Economic Studies **50** (1983), 221-247.
12. F. Gul and H. Sonnenschein, *On Delay in Bargaining with One-Sided Uncertainty*, Econometrica **56** (1988), 601–611.
13. F. Gul, H. Sonnenschein and R. Wilson, *Foundations of Dynamic Monopoly and the Coase Conjecture*, Journal of Economic Theory **39** (1986), 155–190.
14. R. Inderst, *Bargaining with a Possibly Committed Seller*, Review of Economic Dynamics **8** (2005), 927–944.
15. K. Kim, *The Coase Conjecture with Incomplete Information on the Monopolist's commitment*, Theoretical Economics **4** (2009), 17–44.
16. D. Kreps and R. Wilson, *Reputation and Imperfect Information*, Journal of Economic Theory **27** (1982), 253–279.
17. D. Kreps, P. Milgrom, J. Roberts and R. Wilson, *Rational Cooperation in the Finitely Repeated Prisoner's Dilemma*, Journal of Economic Theory **27** (1982), 245–252.
18. G. Mailath and L. Samuelson, *Repeated Games and Reputations: Long-Run Relationships*, Oxford University Press, 2006.
19. P. Milgrom and J. Roberts, *Predation, Reputation and Entry Deterrence*, Journal of Economic Theory **27** (1982), 280–312.
20. R. Myerson, *Game Theory: Analysis of Conflict*, Harvard University Press, 1991.
21. A. Rubinstein, *Perfect Equilibrium in a Bargaining Model*, Econometrica **50** (1982), 97–109.
22. A. Rubinstein, *A Bargaining Model with Incomplete Information about Time Preferences*, Econometrica **53** (1985), 1151–1172.
23. J. Sobel and I. Takahashi, *A Multi-Stage Model of Bargaining*, Review of Economic Studies **50** (1983), 411–426.
24. N. Stokey, *Rational Expectations and Durable Goods Pricing*, Bell Journal of Economics **12** (1981), 112–128.