Bayesian and Dominant Strategy Implementation Revisited

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Abstract

We consider a standard social choice environment with linear utility and one-dimensional types. We show by counterexample that, when there are at least three physical alternatives, Bayes-Nash Incentive Compatibility (BIC) and Dominant Strategy Incentive Compatibility (DIC) need no longer be equivalent. The example with three alternatives is minimal since we do obtain a general equivalence result for settings with only two social alternatives. Our negative result does not mathematically contradict the Manelli and Vincent [2010] equivalence obtained in a one-object auction setting, but it shows that BIC-DIC equivalence is only valid in restrictive environments. Our insights are based on mathematical results about the existence of monotone measures with given monotone marginals.

1 Introduction

In an important and surprising contribution Manelli and Vincent [2010] focus on the standard, not necessarily symmetric, one-object auction setting with private values and independent types, and show that for any Bayes-Nash

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Incentive Compatible (BIC) mechanism there exists a Dominant-Strategy Incentive Compatible (DIC) mechanism that yields, for each bidder and each type of this bidder, the same conditional expected probability of obtaining the object as in the original mechanism (and hence, by payoff equivalence, the same expected utility).

In this note we look at a standard social choice environment with linear utility and one-dimensional types which includes as a special case the one-object auction considered by Manelli and Vincent. We show by counterexample that, when there are at least three physical alternatives, BIC and DIC need no longer be equivalent: in other words, we construct a BIC mechanism that cannot be replicated by a DIC mechanism with the same conditional expected allocation functions. The example with three alternatives is minimal since we do obtain an equivalence result for settings with only two social alternatives.

The one-object auction model is restrictive because the mapping from type profiles to (ordinal) preference profiles over the physical alternatives takes a very special form: no matter what the type of an agent is, he always prefers the alternative where he gets the object (and is indifferent among all the rest); moreover, the preferred alternatives of any two agents are distinct. But the general independent private value model allows more general such maps, and this is how we obtain our counter-example. Thus, while our negative result does not mathematically contradict the Manelli-Vincent result, it shows that the BIC-DIC equivalence is only valid in restrictive settings, contrary to the impression one may otherwise get. The flexibility allowed by Bayes-Nash implementation versus dominant strategy implementation is generally pertinent in the independent private values model, and Bayes-Nash implementation should not be so quickly discarded.

Our analysis is based on mathematical results and ideas that, somewhat surprisingly, have been ignored so far in the Mechanism Design literature. The main role is played by a result due to Gutmann et al. [1991]: For any bounded, non-negative function of several variables that generates monotone, one dimensional marginals, it is possible to generate the same marginals with another non-negative function that is monotone in each coordinate, and that respects the same bound. It is important to note that the difficulty in this result stems from the constraint of keeping the same bound.

The connection to the BIC-DIC equivalence should be now obvious since in the independent private values model with quasi-linear utility and monetary transfers, DIC mechanisms are characterized by monotone allocations, which are described by probabilities of choosing various alternatives, while BIC mechanisms are characterized by monotone conditional expected allocations, which are obtained as marginals of the actual allocation.

The method of proof for our positive result about environments with two alternatives immediately yields BIC-DIC equivalence in the asymmetric two-bidder one-object auction case, and in the symmetric n-bidder case, as previously shown by Manelli and Vincent. We are not yet aware of a shorter proof method to supplant the Manelli-Vincent analysis for the n-bidder asymmetric one-object auction.

A further payoff from our enterprise is obtained by noting that the work of Gutmann et al. is based on several earlier, famous results which seek to characterize sets of measures that can be obtained as the set of marginals of some measure on the respective product space[1]: the classical references for this line of work are Lorenz [1949] for marginals of indicator functions, and Kellerer [1961] and Strassen [1965] for general measures[2]. These earlier results are related to Border's [1991], [2007] important characterizations of *reduced form* auctions (which was used as a step in the Manelli and Vincent construction).

2 Model and Preliminaries

There are K social alternatives and N agents. The utility of agent i in alternative k is given by $a_i^k x_i + c_i^k + t_i$ where $x_i \in [0,1]$ is agent i's private type, where $a_i^k, c_i^k \in \mathbb{R}$ with $a_i^k \geq 0$, and where $t_i \in \mathbb{R}$ is a monetary transfer. Types are drawn independently of each other, according to strictly increasing distributions F_i. Type x_i is private information of agent i.

Note that the one-object auction analyzed in Manelli and Vincent [2010] is the special case of the above model where $K = N$, where $a_i^i = 1$, $a_i^j = 0$ for any $j \neq i$, and where $c_i^k = 0$ for any i, k. Such a special specification reduces the kind of incentive problems that can be addressed, even within the private values paradigm.

A direct revelation mechanism $\mathbf{M}$ is given by K functions $q^k : [0,1]^N \to [0,1]$ and N functions $t_i : [0,1]^N \to \mathbb{R}$ where $q^k(x_1, ..., x_N)$ is the probability

[1]The beautiful connections to Optimal Transportation are reviewed in Villani [2003].

[2]It turns out that conditions are the same: if a set of measures constitute the marginals of some measure, then it is also the set of marginals of an indicator function. This observation can also be used to reduce probabilistic mechanisms to deterministic ones.

with which alternative k is chosen, and $t_i(x_1, ..., x_N)$ is the transfer to agent i if the agents report types $x_1, ..., x_N$. Note that $\sum_{k=1}^{K} q^k(x_1, ..., x_N) = 1$ for each vector of reports $\mathbf{x} = (x_1, ..., x_N) \in [0, 1]^N$.

A direct revelation mechanism $\mathbf{M}$ is *Dominant-Strategy Incentive Compatible* (DIC) if truth-telling constitutes a dominant strategy equilibrium in the game defined by $\mathbf{M}$ and the given utility functions. A direct revelation mechanism $\mathbf{M}$ is *Bayes-Nash Incentive Compatible* (BIC) if truth-telling constitutes a Bayes-Nash equilibrium in the game defined by $\mathbf{M}$ and the given utility functions. Obviously, a DIC mechanism is a fortiori BIC.

Given mechanism $\mathbf{M}$, define for each i, k,

$$Q_i^k(\widehat{x}_i) = \int_{[0,1]^{N-1}} q^k(x_1, ..., x_i, \widehat{x}_i, x_{i+1}, ..., x_N) dF_{-i}$$

where $dF_{-i} = dF_1 ... dF_{i-1} dF_{i+1} ... dF_N$. This is the expected probability that alternative k is chosen if all agents $j \neq i$ report truthfully while agent i reports type $\widehat{x}_i$.

A necessary condition for $\mathbf{M}$ to be BIC is that, for each agent i, the function $\sum_{k=1}^{K} a_i^k Q_i^k(x_i)$ is non-decreasing. Moreover, a standard argument that follows from the incentive compatibility constraint implies that any K functions q^k that satisfy this condition are part of a BIC mechanism.

Analogously, a necessary condition for $\mathbf{M}$ to be DIC is that, for each agent i , and for any signals of the other agents, the function $\sum_{k=1}^{K} a_i^k q^k(x_1, ..., x_N)$ is non-decreasing in x_i. Any K functions q^k that satisfy this condition are part of a DIC mechanism.

Definition 1 *Two mechanisms* $\mathbf{M}$ *and* $\widetilde{\mathbf{M}}$ *are equivalent if for each,* i, k *and* x_i *it holds that* $Q_i^k(x_i) = \widetilde{Q}_i^k(x_i)$*, where* Q_i^k *and* $\widetilde{Q}_i^k$ *are the conditional expected probabilities associated with* $\mathbf{M}$ *and* $\widetilde{\mathbf{M}}$*, respectively.*

In particular, the above condition implies that, for each agent i and for each type x_i of i, the expected utilities generated by two equivalent mechanisms can be equalized by choosing appropriate transfers. This is a straightforward consequence of the incentive constraints which allows us to define equivalence in terms of physical allocations only.

3 BIC-DIC Equivalence for Two Alternatives

In this Section we consider setting with two social alternatives only. In order to avoid trivial cases, we also assume that $a_i^1 \neq a_i^2$, for all agents i. Our main result is:

Proposition 2 *Assume that $K = 2$. Then for any BIC mechanism there exists an equivalent DIC mechanism.*

The main tool in the proof of the above Proposition is the following result, which is a simple consequence of an elegant result due to Gutmann et al. [1991][3]:

Theorem 3 *Consider an integrable function $0 \leq q \leq 1$ on $\mathbb{R}^N$ having non-decreasing one-dimensional marginals with respect to coordinates $i \in D \subseteq [1, 2, ..., N]$, and non-increasing one-dimensional marginals with respect to coordinates $j \in D^C = [1, 2, ..., N] \setminus D$. Then there exists a function $0 \leq \psi \leq 1$, with exactly the same marginals, such that ψ is non-decreasing in each coordinate $i \in D$, and is non-increasing in each coordinate $j \in D^C$.*

Proof of Proposition 2. Since $K = 2$, we must have $q^2(x_1, ..., x_n) = 1 - q^1(x_1, ..., x_n)$. Therefore, the allocation function in any mechanism can be represented by just one function $q^1(x_1, ..., x_n)$, the probability that alternative 1 is chosen. For BIC mechanisms we obtain for each i that the function $\sum_{k=1}^{K} a_i^k Q_i^k(x_i) = a_i^2 + (a_i^1 - a_i^2) Q_i^1(x_i)$ is non-decreasing. In particular, Q_i^1 is non-decreasing if $a_i^1 - a_i^2 > 0$ and Q_i^1 is non-increasing if $a_i^1 - a_i^2 < 0$. Thus, the function $q^1(x_1, ..., x_n)$ satisfies the conditions in the Theorem 3 with $D = \{i \mid a_i^1 - a_i^2 > 0\}$. We obtain another function $0 \leq \psi \leq 1$, with exactly the same marginals, such that ψ is non-decreasing in each coordinate $i \in D$, and non-increasing in each coordinate $j \in D^C$. As a consequence, $a_i^1 \psi(x_1, ..., x_n) + a_i^2 [1 - \psi(x_1, ..., x_n)] = a_i^2 + (a_i^1 - a_i^2) \psi(x_1, ..., x_n)$ is non-decreasing in x_i for any i. Together with appropriate transfers, $\psi(x_1, ..., x_n)$ defines a DIC mechanism which is equivalent to the given BIC mechanism. ■

Remark 4 *The same method of proof – an application of Theorem 3 – can be used for establishing the BIC-DIC equivalence in special settings with more*

[3]Gutman et al. formulate their result, Theorem 7, in terms of non-decreasing marginals only. Our extension is based on an immediate re-arrangement argument. Moreover, they only consider marginals with respect to the Lebesgue measure. A simple argument can be used to extend it to marginals with respect to product measures of the form $dF_{-i} = dF_1 ... dF_{i-1} dF_{i+1} ... dF_N$ as needed in our application. This is done by considering the change of variables $u_i = F(x_i)$.

than two alternatives. For such applications it is needed that all marginal conditions can be imposed on a single function. For example, consider the symmetric one-object auction setting with n bidders. Any symmetric mechanism can be summarized by a single function $q_1(x_1, ..., x_n)$*, the probability that bidder 1 obtains the object. The symmetry assumption on the mechanism, together with the assumptions that types are I.I.D. imply that there exists a single function* Q_1^1 *such that the marginals of* q_1 *are of the form* $Q_1^1(x_1), \frac{1-Q_1^1(x_2)}{n-1}, ..., \frac{1-Q_1^1(x_n)}{n-1}$[4]*. In a BIC mechanism, the function* Q_1^1 *, the conditional expected probability with which bidder 1 obtains the object, is non-decreasing. Applying Theorem 3, yields the existence of a DIC mechanism (not necessarily symmetric) with the same marginals. A symmetrization argument yields an equivalent and symmetric DIC mechanism.*

Remark 5 *Paralleling the application of Theorem 3 to the study of BIC-DIC equivalence in settings with two alternatives, consider the classical Lorenz-Kellerer-Strassen (LKS) necessary and sufficient condition for the existence of a measure with given marginals, where the measure is defined on the product of two given spaces. Border's characterization of reduced form auctions for the (possibly asymmetric) two-bidder case is a direct consequence*[5]*. Since we are not yet aware of a compact, explicit generalization of the LKS necessary and sufficient condition to the product of* $n > 2$ *spaces, it is not clear how to directly apply the LKS insights to other one-object auction settings.*

4 A Counterexample for Three Alternatives

In this section we construct a BIC mechanism that has no equivalent DIC mechanism. It will be clear that this is not a knife-edge phenomenon. We consider a setting with 2 agents, and 3 alternatives called A, B, V.

[4]We assume here that mechanisms are *maximal*, i.e., they allocate the object with probability 1. The proof of the claim about the marginals is in the Appendix.

[5]Border's condition for the symmetric n-bidder case has been recently translated into a second order stochastic dominance (SOSD) condition by Hart and Reny [2011]. It is worth noting that Gale [1957] and Ryser [1957] relate the existence of certain matrices with given row and column sums (i.e., with given "marginals") to a discrete majorization condition.

Agent 1 has types $x_1 > x_2$, agent 2 has types $y_1 > y_2$. Types are drawn independently from the uniform distribution, e.g. each type is drawn with probability $1/2$[6].

The utility of agent 1 with type x_i $(i = 1, 2)$, exclusive of transfers, is given by: $ax_i + c$ in alternative A; $x_i + d$ in alternative B; v in V. The utility for agent 2 is obtained by plugging y_i instead x_i in these expressions. We further assume that $0 < a < 1$[7].

An allocation is given by probabilities $\{q^A(x_i, y_j), q^B(x_i, y_j)\}_{1 \leq i,j \leq 2}$. Note that $q^A(x_i, y_j) + q^B(x_i, y_j) \leq 1$, where $q^V(x_i, y_j) = 1 - q^A(x_i, y_j) - q^B(x_i, y_j)$ represents the probability of choosing alternative V. In view of our previous result we need to ensure that q^V is not identically zero.

The construction is divided in several steps.

Step 1: **Equivalence and Symmetry.**

Consider an (agent) symmetric allocation function $\{\widehat{q}^A(x_i, y_j), \widehat{q}^B(x_i, y_j)\}_{1 \leq i,j \leq 2}$, i.e., an allocation where $\widehat{q}^A(x_1, y_2) = \widehat{q}^A(x_2, y_1)$ and $\widehat{q}^B(x_1, y_2) = \widehat{q}^B(x_2, y_1)$. Then, any equivalent allocation – that keeps all marginals fixed – has to be symmetric as well.

To see this, consider the conditional expected probabilities of choosing each alternative obtained from a given symmetric allocation. We have:

$$\begin{aligned}
Q_1^A(x_1) &= \frac{1}{2}[\widehat{q}^A(x_1, y_1) + \widehat{q}^A(x_1, y_2)] \\
&= \frac{1}{2}[\widehat{q}^A(x_1, y_1) + \widehat{q}^A(x_2, y_1)] = Q_2^A(y_1) \\
Q_1^B(x_1) &= \frac{1}{2}[\widehat{q}^B(x_1, y_1) + \widehat{q}^B(x_1, y_2)] \\
&= \frac{1}{2}[\widehat{q}^B(x_1, y_1) + \widehat{q}^B(x_2, y_1)] = Q_2^B(y_1)
\end{aligned}$$

Consider any other equivalent allocation rule $\{\widetilde{q}^A(x_i, y_j), \widetilde{q}^B(x_i, y_j)\}_{1 \leq i,j \leq 2}$.

[6] This discrete setting allows us to clearly illustrate the difficulty in the construction. The example can be extend to continuous distributions that, say, put almost all mass around two types. Note that a discrete setting allows even more flexibility when choosing transfers that complement a given monotonic allocation to form a DIC mechanism.

[7] A counter-example can be easily constructed also for asymmetric situations. But the main idea behind the construction is clearer in the present symmetric setting because less functions and parameters are involved.

By equivalence we must have

$$\begin{aligned} Q_1^A(x_1) &= \frac{1}{2}\widetilde{q}^A(x_1,y_1) + \frac{1}{2}\widetilde{q}^A(x_1,y_2) \\ Q_2^A(y_1) &= \frac{1}{2}\widetilde{q}^A(x_1,y_1) + \frac{1}{2}\widetilde{q}^A(x_2,y_1) \end{aligned}$$

Together with $Q_1^A(x_1) = Q_2^A(y_1)$, this yields $\widetilde{q}^A(x_1,y_2) = \widetilde{q}^A(x_2,y_1)$. An analogous argument yields $\widetilde{q}^B(x_2,y_1) = \widetilde{q}^B(x_1,y_2)$. Thus $\{\widetilde{q}^A(x_i,y_j), \widetilde{q}^B(x_i,y_j)\}_{1\le i,j\le 2}$ must be symmetric as well.

Step 2: Construction of a symmetric BIC mechanism.

Let s be a small positive number, say, $s = 1/15$, and consider the following symmetric allocation (i.e., with $\widehat{q}^A(x_1,y_2) = \widehat{q}^A(x_2,y_1)$, $\widehat{q}^B(x_1,y_2) = \widehat{q}^B(x_2,y_1)$) and corresponding conditional expected probabilities:

$$\begin{array}{cc} \widehat{q}^A(x_1,y_1) = 13s & \widehat{q}^B(x_1,y_1) = as \\ \widehat{q}^A(x_1,y_2) = s & \widehat{q}^B(x_1,y_2) = as \\ Q^A(x_1) = 7s & Q^B(x_1) = as \end{array} \tag{1}$$

and

$$\begin{array}{cc} \widehat{q}^A(x_2,y_1) = s & \widehat{q}^B(x_2,y_1) = as \\ \widehat{q}^A(x_2,y_2) = s & \widehat{q}^B(x_2,y_2) = 9as \\ Q^A(x_2) = s & Q^B(x_2) = 5as \end{array} \tag{2}$$

Note that $\widehat{q}^A(x_i,y_j), \widehat{q}^B(x_i,y_j) \in (0,1)$ and $\widehat{q}^A(x_i,y_j) + \widehat{q}^B(x_i,y_j) < 1$. This last inequality is precisely the degree of freedom gained by having more than 2 alternatives. Note also that the function $a\widehat{q}^A + \widehat{q}^B$ is not increasing in each coordinate separately, so that the constructed mechanism is not DIC. But, the constructed mechanism satisfies

$$aQ^A(x_1) + Q^B(x_1) = 8as > 6as = aQ^A(x_2) + Q^B(x_2)$$

and analogously for agent 2. This monotonicity on average property implies that we can construct appropriate transfers that, together with $\widehat{q}$, yield a symmetric BIC mechanism.

Step 3: An equivalent DIC mechanism does not exist.

Assume now the existence of a DIC mechanism that is equivalent to the BIC mechanism constructed in Step 2. This DIC mechanism needs to be symmetric, as explained in Step 1. Therefore, it consists of 6 non-negative numbers

$$\begin{aligned} q^A(x_1,y_1),\ q^A(x_1,y_2) &= q^A(x_2,y_1),\ q^A(x_2,y_2) \\ q^B(x_1,y_1),\ q^B(x_1,y_2) &= q^B(x_2,y_1),\ q^B(x_2,y_2) \end{aligned}$$

that satisfy the following system of equations:

$$\begin{array}{l} \frac{1}{2}q^A(x_i,y_1) + \frac{1}{2}q^A(x_i,y_2) = Q^A(x_i),\ i=1,2 \\ \frac{1}{2}q^B(x_i,y_1) + \frac{1}{2}q^B(x_i,y_2) = Q^B(x_i),\ i=1,2 \end{array} \tag{3}$$

where we omitted the redundant equations that need to hold for agent 2 with type y_i, $i = 1, 2$.

Let's define the function $\psi(x_i, y_j)$ as follows:

$$\psi(x_i,y_j) = aq^A(x_i,y_j) + q^B(x_i,y_j),\ i,j=1,2 \tag{4}$$

Because $q^A(x_i, y_j), q^B(x_i, y_j) \in [0, 1]$ and $q^A(x_i, y_j) + q^B(x_i, y_j) < 1$, we obtain that $\psi(x_i, y_j) \in [0, 1)$. By symmetry we have $\psi(x_1, y_2) = \psi(x_2, y_1)$. Since the underlying mechanism $\{q^A, q^B\}$ is assumed to be DIC, $\psi(x_i, y_j)$ must be non-decreasing in each coordinate:

$$\psi(x_1,y_1) \geq \psi(x_1,y_2) = \psi(x_2,y_1) \geq \psi(x_2,y_2). \tag{5}$$

It follows from (3) that

$$\begin{array}{l} \frac{1}{2}\psi(x_1,y_1) + \frac{1}{2}\psi(x_1,y_2) = aQ^A(x_1) + Q^B(x_1) = 8as \\ \frac{1}{2}\psi(x_1,y_2) + \frac{1}{2}\psi(x_2,y_2) = aQ^A(x_2) + Q^B(x_2) = 6as \end{array} \tag{6}$$

Together with inequalities (5), equations (6) yield the following necessary bounds:

$$\begin{array}{l} 8as \leq \psi(x_1,y_1) \leq 10as \\ 6as \leq \psi(x_1,y_2) = \psi(x_2,y_1) \leq 8as \\ 4as \leq \psi(x_2,y_2) \leq 6as \end{array} \tag{7}$$

Note that $\psi(x, y)$ is a solution to the (discrete) Gutmann et. al problem of finding a function, monotone in x and y separately, with given monotone marginals of the form $aQ^A(x_i) + Q^B(x_i)$, $aQ^A(y_i) + Q^B(y_i)$, $i = 1, 2$. In fact, any DIC mechanism (not necessarily equivalent to the BIC mechanism constructed at Step 2) with fixed marginals $aQ^A + Q^B$ yields such a solution. Since we know by the discrete version of Theorem 3 that a solution to this problem does exist[8], the construction of a counterexample must hinge on the additional constraints imposed by equivalence, i.e., by equations 3.

Fix then $q^A(x_1, y_2)$ and use equations (3) and (4) to write all other five q's in terms of $q^A(x_1, y_2)$:

$$\begin{aligned}
q^A(x_1,y_1) &= 2Q^A(x_1) - q^A(x_1,y_2) && (8)\\
q^B(x_1,y_2) &= \psi(x_1,y_2) - aq^A(x_1,y_2) && (9)\\
q^B(x_1,y_1) &= \psi(x_1,y_1) - aq^A(x_1,y_1) && \\
&= \psi(x_1,y_1) - 2aQ^A(x_1) + aq^A(x_1,y_2) && (10)\\
q^A(x_2,y_2) &= 2Q^A(x_2) - q^A(x_1,y_2) && (11)\\
q^B(x_2,y_2) &= \psi(x_2,y_2) - aq^A(x_2,y_2) && \\
&= \psi(x_2,y_2) - 2aQ^A(x_2) + aq^A(x_1,y_2) && (12)
\end{aligned}$$

We now check whether we can choose $q^A(x_1, y_2) \geq 0$, so that all other q's are also non-negative.

Equations (8), (9), and (11) yield the necessary condition:

$$q^A(x_1,y_2) \leq \min\left\{2Q^A(x_1), 2Q^A(x_2), \frac{1}{a}\psi(x_1,y_2)\right\}.$$

Equations (10) and (12) yield the necessary condition:

$$q^A(x_1,y_2) \geq \max\left\{2Q^A(x_1) - \frac{1}{a}\psi(x_1,y_1), 2Q^A(x_2) - \frac{1}{a}\psi(x_2,y_2)\right\}.$$

Therefore, putting these two conditions together yields the necessary condition:

$$\begin{aligned}
& \max\left\{2Q^A(x_1) - \frac{1}{a}\psi(x_1,y_1), 2Q^A(x_2) - \frac{1}{a}\psi(x_2,y_2)\right\} \\
\leq\ & \min\left\{2Q^A(x_1), 2Q^A(x_2), \frac{1}{a}\psi(x_1,y_2)\right\} \qquad (13)
\end{aligned}$$

By using now the the relations $Q^A(x_1) = 7s$, $Q^A(x_2) = s$, $\psi(x_1, y_1) = 16as - \psi(x_1, y_2)$, $\psi(x_2, y_2) = 12as - \psi(x_1, y_2)$,[9] we can rewrite the neces-

[8]The discrete version is also found in Gutmann et al.[1991]. In fact, the continuous result is obtained as a limit of the discrete one.

[9]See equations (1), (2) and (6).

sary condition (13) as

$$\begin{aligned}
\max\left\{-2s+\frac{1}{a}\psi(x_1,y_2),\ -10s+\frac{1}{a}\psi(x_1,y_2)\right\} &\leq \min\left\{14s,\ 2s,\ \frac{1}{a}\psi(x_1,y_2)\right\}\Leftrightarrow \\
-2s+\frac{1}{a}\psi(x_1,y_2) &\leq \min\left\{2s,\ \frac{1}{a}\psi(x_1,y_2)\right\}\Leftrightarrow \\
\frac{1}{a}\psi(x_1,y_2) &\leq 2s+\min\left\{2s,\ \frac{1}{a}\psi(x_1,y_2)\right\}
\end{aligned}$$

Since $2s+\min\left\{2s,\ \frac{1}{a}\psi(x_1,y_2)\right\}\leq 4s$, we obtain that a necessary condition for the above construction to be valid, i.e., for keeping all q's non-negative, is that $\psi(x_1,y_2)\leq 4as$. But this contradicts the requirement $\psi(x_1,y_2)\geq 6as$, which was obtained in inequalities (7) at Step 3. Thus, an equivalent DIC mechanism cannot be constructed here. It should be clear from the above that the inexistence is not a knife-edge phenomenon – there are several degrees of freedom here in the choice of parameters.

Q.E.D.

Remark 6 *What differentiates the general independent private values model, where the BIC-DIC equivalence does not generally hold, from the special one-object auction model where the equivalence is valid? In the more general model, the BIC monotonicity condition for each bidder i is imposed on the aggregated function $\sum_{k=1}^{K}a_i^kQ_i^k(x_i)$ (see the proof of Proposition 2), while the equivalence condition is imposed separately on each conditional expected probability of choosing alternative k, $Q_i^k(x_i)$. In contrast, in the the auction model, the BIC monotonicity condition is directly imposed on the same unique function as the equivalence condition: the agent's conditional probability of getting the object, $Q_i^k(x_i)$. Thus the BIC monotonicity conditions are more demanding in the auction case, and this reduces the flexibility of BIC versus DIC.*

It is also worth noting that Theorem 3 has no counterpart for higher-dimensional marginals (or projections). From this perspective it seems unlikely that BIC-DIC equivalence can hold in sufficiently interesting multidimensional models. Moreover, in the multi-dimensional case not every monotone allocation can be augmented by transfers in order to create an incentive compatible mechanism. Indeed, Jehiel, Moldovanu and Stacchetti [1999] analyzed a standard multi-dimensional, private values model, interpreted as an one-object auction with externalities and computed a Bayes-Nash equilibrium *in a two-bidder auction with a reserve price that cannot be replicated in dominant strategies.*

5 Appendix

Consider a symmetric n-bidder, one-object auction model with IID values (or types). A maximal, symmetric mechanism where the object is always allocated to one of the bidders is defined by $q_1(x_1, ..., x_n)$, the probability that bidder 1 obtains the object given reported types $x_1, ..., x_n$. We show here that for all $j \neq 1$, and for all z,

$$Q_1^j(z) = (1 - Q_1^1(z)/(n-1),$$

where Q_1^j is the marginal of q_1 with respect to the j'th coordinate. The claim is obvious for $n = 2$, and we assume below $n > 2$.

Symmetry means that

$$q_j(x_1, ..., x_{j-1,}x_j, x_{j+1}, ..., x_n) = q_1(x_j, ..., x_{j-1}, x_1, x_{j+1}, ..., x_n) \tag{14}$$

where q_j is the probability that bidder j gets the object. We have:

$$\sum_{i=1}^{n} q_i(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 1 = \sum_{i=1}^{n} q_i(x_1, x_3, x_2, ..., x_{n-1}, x_n). \tag{15}$$

By symmetry (14) we know that

$$\begin{aligned} q_2(x_1, x_2, x_3, ..., x_{n-1}, x_n) &= q_3(x_1, x_3, x_2, ..., x_{n-1}, x_n), \\ q_3(x_1, x_2, x_3, ..., x_{n-1}, x_n) &= q_2(x_1, x_3, x_2, ..., x_{n-1}, x_n). \end{aligned}$$

Thus, (15) becomes

$$\sum_{i \neq 2,3} q_i(x_1, x_2, x_3, ..., x_{n-1}, x_n) = \sum_{i \neq 2,3} q_i(x_1, x_3, x_2, ..., x_{n-1}, x_n). \tag{16}$$

Use again symmetry condition (14) to rewrite (16) as

$$\begin{aligned} &\sum_{i \neq 2,3} q_1(x_i, x_2, x_3, ..., x_{i-1}, x_1, x_{i+1} ..., x_{n-1}, x_n) \\ = &\sum_{i \neq 2,3} q_1(x_i, x_3, x_2, ..., x_{i-1}, x_1, x_{i+1} ..., x_{n-1}, x_n). \end{aligned} \tag{17}$$

Recall now that values are IID, and take the integral with respect to $x_1, x_3, x_4, ..., x_n$ in the above expression. This yields for any x_2:

$$(n-2)Q_1^2(x_2) = (n-2)Q_1^3(x_2) \Leftrightarrow Q_1^2(x_2) = Q_1^3(x_2), \tag{18}$$

where Q_1^j is the marginal of q_1 with respect to the j coordinate. Exactly the same proof works for any other pair of coordinates $i, j \neq 1$. Thus, for any z and for any $i, j \neq 1$, we have

$$Q_1^i(z) = Q_1^j(z) \tag{19}$$

Using symmetry condition (14) again, we obtain

$$1 = \sum_{i=1}^{n} q_i(x_1, x_2, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} q_1(x_i, x_2, ..., x_1, ..., x_{n-1}, x_n). \tag{20}$$

Taking the integrals with respect to $x_2, x_3, ..., x_n$ in the above expression yields for any x_1:

$$1 = \sum_{j=1}^{n} Q_1^j(x_1). \tag{21}$$

Together with (19), we now get

$$Q_1^j(z) = \frac{1 - Q_1^1(z)}{n-1}, \ j = 2, 3, ..., n \tag{22}$$

as promised. **Q.E.D.**

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