International market links and volatility transmission

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Abstract: This paper gauges volatility transmission between stock markets by testing conditional independence of their volatility measures. In particular, we check whether the conditional density of the volatility changes if we further condition on the volatility of another market. We employ nonparametric methods to estimate the conditional densities and model-free realized measures of volatility, allowing for both microstructure noise and jumps. We establish the asymptotic normality of the test statistic as well as the first-order validity of the bootstrap analog. Finally, we uncover significant volatility spillovers between the stock markets in China, Japan, UK and US.

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1 Introduction

Even though testing for the presence of international market links has a long history in asset pricing (see survey by Roll, 1989), the literature has been gaining momentum since the October 1987 crash. The main interest lies on the analysis of volatility transmission across markets. King and Wadhwani (1990) argue that the strength of international market links depends mainly on volatility. As the latter declines, the correlation between price changes in the different markets also decreases and so market links become weaker. In contrast, international market links become stronger in periods of high volatility. The idea is that, with Bayesian update of beliefs about variances, a common shock to all markets would result in an increase in the perceived variance of any common factor and hence of the correlation.

This paper develops formal statistical tools for testing conditional independence between volatility processes. We propose a nonparametric approach in stark contrast with most papers in the literature. In particular, under the assumption that asset prices follow a multivariate jump-diffusion process, we show how to test whether the conditional distribution of asset A's integrated variance (say, over a day) also depends on asset B's integrated variance. The procedure is nonparametric in that, apart from some mild regularity conditions, we impose no parametric assumption on the functional form of the drift and diffusive term as well as on the presence of leverage or jumps. Broadly speaking, our testing procedure checks whether conditioning also on asset B's integrated variance, rather than exclusively on asset A's past integrated variance, entails a different conditional distribution for asset A's integrated variance. As we do not observe daily variance, we rely on model-free realized measures based on intraday returns. We focus on the conditional distribution for two reasons. It allows for nonlinear channels of volatility transmission in contrast to the standard practice of carrying out pointwise analyses based on the conditional mean of the volatility. In addition, the distribution of the daily integrated variance is of particular interest for pricing variance swap contracts (Carr, Geman, Madan and Yor, 2005).

The asymptotic theory we develop proceeds in three steps. First, we establish the asymptotic normality of the unfeasible statistics based on unobservable integrated variances using either kernel or local linear smoothing. Second, we provide conditions on the rate of growth of intraday observations relative to the number of days under which the feasible statistic, based on realized measures, is asymptotically equivalent to its unfeasible counterpart. Our setting is general enough to allow for microstructure noise as well as for jumps. Third, we establish the first-order validity of bootstrap-based critical values based on the m out of n (henceforth, moon) bootstrap (Bretagnolle, 1983; Bickel, Götze and van Zwet, 1997). Bootstrap inference is typically more robust to variations in the bandwidth as the latter plays roughly the same role on both the original and bootstrap statistics. Monte Carlo simulations indeed reveal that the moon bootstrap works reasonably well even in relatively small samples.

We carry out the nonparametric analysis using different realized measures as a means to discern the main channels of quadratic variation transmission between stock markets. The idea is similar to Aït-Sahalia and Jacod's (2010) analogy of turning the knobs of a measurement device running spectrographic analysis. Note that we consider only two tuning parameters. The first is the type of the realized measure. It essentially combines the truncation and power knobs of Aït-Sahalia and Jacod (2010), enabling us to eliminate the contribution of either jumps or market microstructure noise to the quadratic variation. The second is the sampling frequency, which we vary as another means to alleviate market microstructure effects. For instance, the tripower variation is robust to jumps and hence it would capture only the contribution of the diffusive volatility to the quadratic variation if using returns at a sufficiently low frequency, say 30 minutes. In turn, the realized kernel approach kills the contribution of the market microstructure noise to the quadratic variation, but not that of jumps. We thus view the differences in the test results using the above realized measures as very informative. Suppose we find no significant transmission using the tripower variation, but significant evidence of spillovers using the realized kernel measure. This would imply that the main channel of transmission is likely through jumps given that rejection of the null occurs only if we do not exclude the jump contribution to the quadratic variation.

We investigate the links between international stock markets using intraday data from China, Japan, UK, and US from January 2000 to December 2005. By varying the realized measure, we show that the primary channel of transmission within the quadratic variation of asset prices is through spillovers in the integrated variance. The empirical evidence of volatility spillovers becomes indeed stronger once we control for jumps and/or market microstructure effects. The only exception is the transmission from China to the US, which runs mainly through price jumps. There is to some extent significant interdependence between all of the stock market indices we consider. Further, the volatility transmission mechanisms are mostly symmetrical in that we normally reject the null of conditional independence in both directions using almost the same realized measures. Finally, we show that the financial markets seemingly price these interconnections in view that there are also volatility spillovers to the options-implied market volatility in the US. In other words, investors form their expectations about the future volatility of the S&P 500 index accounting for the stock market information coming from China, Japan and, especially, the UK.

To ensure that the volatility spillovers we uncover are indeed genuine, we carry out a series of robustness checks. We first examine whether the evidence is spurious due to time-series persistence. We redo the spillover analysis for the US stock market adding an extra control, namely, the VIX index. The latter is a market volatility index from the Chicago Board Options Exchange (CBOE) that gauges the options-implied volatility of the S&P 500 index. Conditioning on the VIX index should effectively control for further serial dependence in the integrated variance given the strong link between the actual volatility and its risk-neutral expectation (Bandi and Perron, 2006). There are no qualitative changes in our main results; in fact, the evidence of significant transmission from the UK and Japan become stronger. We then investigate whether the volatility spillovers running from China and Japan to the US are truly about Asia-specific shocks by further conditioning on the FTSE 100 realized measure over the same day. The idea is that the FTSE 100 index should reflect any global shock, though not necessarily Asia-specific shocks that may affect the US. We observe no qualitative change in the results for Japan. In stark contrast, we now find much more significative volatility transmission from China to the US. Further conditioning on the realized measure of the FTSE 100 index seems to help disentangle the China-specific spillovers from the global volatility shocks. Finally, we also carry out a similar analysis considering hourly realized measures to check for a swifter reaction time possibly through jumps in prices or in volatility (King and Wadhwani, 1990). We find strong evidence of volatility spillovers across all markets, especially if controlling for market microstructure noise by means of the realized kernel approach. The only exception is that we find no significant volatility transmission from Japan to the UK.

There are several papers in the literature that carry out similar, though mostly parametric, analyses of volatility transmission across international stock markets. Engle and Ng (1988), Hamao, Masulis and Ng (1990), Engle, Ito and Lin (1990), King, Sentana and Wadhwani (1994), Lin, Engle and Ito (1994), Karolyi (1995), and Wongswan (2006) employ multivariate GARCH models to show that volatility spillovers indeed occur across foreign exchange markets as well as international stock markets, notably, between Japan, UK and US. In contrast, Cheung and Ng (1996), Hong (2001), Pantelidis and Pittis (2004), Sensier and van Dijk (2004), and van Dijk, Osborne and Sensier (2005) propose simple tests of noncausality in variance based on the cross-correlation between leads and lags of squared GARCH-standardized residuals. More recently, Gourieroux and Jasiak (2007) address causality in variance (or even in higher-order moments) by means of approximate conditional log-Laplace transforms of compound autoregressive processes. The testing strategy of the above papers mainly differs from ours in three respects. First, they assume a discrete-time data generating mechanism in which the conditional variance is a measurable function of past asset returns. In contrast, we assume that asset prices follow a multivariate jump-diffusion process and then test for spillovers within the quadratic variation. Second, their tests are sensitive to misspecifications in the conditional mean and variance equations, whereas the nonparametric nature of our tests minimizes any misspecification risk. Third, they do not contemplate any sort of nonlinear dependence between variances as opposed to our testing procedures, whose nontrivial power against nonlinear channels of volatility transmission results from looking at the whole volatility distribution.

The testing strategy put forth by Diebold and Yilmaz (2009, 2011) is the closest to ours. They examine volatility spillovers by means of a VAR approach using range-based measures of volatility. The latter belongs to the class of realized measures robust to market microstructure noise and hence provides a consistent nonparametric estimator of the quadratic variation of the underlying diffusion process. However, Diebold and Yilmaz do not study how the realized estimation of the quadratic variation affects the subsequent VAR analysis. In addition, their approach also differs from ours in that they focus on linear spillovers in the conditional mean of the quadratic variation of asset prices.

The remainder of this paper ensues as follows. Section 2 describes the data generating process we assume for asset prices and discusses the hypotheses of interest. Section 3 establishes the asymptotic normality of the unfeasible statistic based on integrated variances. Section 4 derives the asymptotic equivalence of feasible test statistics that substitute realized measures for integrated variances. Section 5 first establishes the first-order validity of bootstrap-based critical values and then examines their finite-sample properties of the resulting test through Monte Carlo simulations. Section 6 investigates whether there are significant volatility spillovers across international stock markets using data from China, Japan, UK and US. Section 7 offers some closing remarks, whereas the Appendix collects all technical proofs.

2 Volatility transmission: Setup and issues

In this section, we discuss how to analyze volatility transmission through nonparametric tests of conditional independence. For notational simplicity, we restrict attention to testing whether the daily variance of asset B affects the dynamic of asset A's daily variance. It is straightforward to consider more than two assets, though the usual concern with the curse of dimensionality applies. See also Section 6.1.2 for empirical applications with three stock markets.

Let $p_{A,t}$ and $p_{B,t}$ denote the log-prices of assets A and B with instantaneous (stochastic) volatility given by $\sigma^2_{A,t}$ and $\sigma^2_{B,t}$, respectively. We assume that asset prices follow a continuous-time semimartingale process, as implied by a very weak form of the no-arbitrage property (Delbaen and Schachermayer, 1994). Note that this is not a very stringent requirement given that semimartingales nest almost every continuous-time model in finance (e.g., Harrison and Pliska, 1981; Andersen, Bollerslev and Dobrev, 2007). To fix ideas, we consider a simple example in which asset prices follow a multivariate stochastic-volatility process with jumps:

$$\begin{pmatrix} \mathrm{d}p_{A,t} \\ \mathrm{d}p_{B,t} \\ \mathrm{d}\sigma^2_{A,t} \\ \mathrm{d}\sigma^2_{B,t} \end{pmatrix} = \begin{pmatrix} \mu_{A,t} \\ \mu_{B,t} \\ b_{1,A}(\sigma^2_{A,t},\sigma^2_{B,t}) \\ b_{1,B}(\sigma^2_{A,t},\sigma^2_{B,t}) \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \mathrm{d}J_{1,t} \\ \mathrm{d}J_{2,t} \\ \mathrm{d}J_{3,t} \\ \mathrm{d}J_{4,t} \end{pmatrix} + \begin{pmatrix} \sqrt{1-\rho_A^2}\,\sigma_{A,t} \\ \sigma_{BA,t} \\ 0 \\ 0 \end{pmatrix} \mathrm{d}W_{1,t} + \begin{pmatrix} \sigma_{AB,t} \\ \sqrt{1-\rho_B^2}\,\sigma_{B,t} \\ 0 \\ 0 \end{pmatrix} \mathrm{d}W_{2,t}$$
$$+ \begin{pmatrix} \rho_A\,\sigma_{A,t} \\ 0 \\ b_{2,A}(\sigma^2_{A,t},\sigma^2_{B,t}) \\ b_{2,BA}(\sigma^2_{A,t},\sigma^2_{B,t}) \end{pmatrix} \mathrm{d}W_{3,t} + \begin{pmatrix} 0 \\ \rho_B\,\sigma_{B,t} \\ b_{2,AB}(\sigma^2_{A,t},\sigma^2_{B,t}) \\ b_{2,B}(\sigma^2_{A,t},\sigma^2_{B,t}) \end{pmatrix} \mathrm{d}W_{4,t}, \qquad (1)$$

where $(W_{1,t},\dots,W_{4,t})$ are independent standard Brownian motions and $(J_{1,t},\dots,J_{4,t})$ are jump processes. The latter is such that $\mathrm{d}J_{i,t} = \int_{\mathcal{A}} c_i(u)\,N_i(\mathrm{d}t,\mathrm{d}u)$, for $i = 1,\dots,4$, where $N_i([t_1,t_2],\mathcal{A})$ is a Poisson measure that counts the number of jumps between t_1 and t_2, whose size $c_i(u)$ is an iid random variable in $\mathcal{A}$. The jump process is such that, over a finite time span, there is only a finite number of jumps. Although we allow for leverage effects between asset prices and their own stochastic volatility through the correlation coefficients ρ_A and ρ_B, for the sake of simplicity, we assume away cross-leverage effects by imposing zero correlation between one asset price and the stochastic volatility of the other asset.[1] We

[1] Cross-leverage effects would only add other possible channels of volatility transmission between assets A and B. Although this would bring about additional misspecification risk in any parametric approach to test for volatility transmission, it does not affect in any way the nonparametric procedure we propose.

also assume that the drift components $\mu_{A,t}$ and $\mu_{B,t}$ follow predictable processes. This is not stringent for their role is asymptotically negligible in the context of volatility transmission. Finally, as standard in multivariate stochastic volatility models, we suppose for simplicity that asset prices do not directly affect volatility.

It is possible to decompose the quadratic variation process $\langle\cdot\rangle_t$ of a given asset price, say p_A, over the time interval $[0,t]$ into the part due to the discontinuous jump component p_A^D and the part due to the continuous diffusive component p_A^C. In particular, $\langle p_A\rangle_t = \langle p_A^C\rangle_t + \langle p_A^D\rangle_t$, where $\langle p_A^D\rangle_t \equiv \int_0^t \int_{\mathcal{A}} c_1^2(u)\, N_1(\mathrm{d}s, \mathrm{d}u)$ and $\langle p_A^C\rangle_t$ corresponds to the integrated variance over the time interval $[0,t]$, namely, $IV_{A,t} = \int_0^t \sigma_{A,s}^2\,\mathrm{d}s + \int_0^t \sigma_{AB,s}^2\,\mathrm{d}s$. Now, recall that

$$\begin{aligned}\int_0^t \sigma_{A,s}^2\,\mathrm{d}s &= \int_0^t \left(\int_0^s b_{1,A}(\sigma_{A,u}^2, \sigma_{B,u}^2)\,\mathrm{d}u\right)\mathrm{d}s + \int_0^t \left(\int_0^s \mathrm{d}J_{3,u}\right)\mathrm{d}s \\ &\quad + \int_0^t \left(\int_0^s b_{2,A}(\sigma_{A,u}^2, \sigma_{B,u}^2)\,\mathrm{d}W_{3,u}\right)\mathrm{d}s + \int_0^t \left(\int_0^s b_{2,AB}(\sigma_{A,u}^2, \sigma_{B,u}^2)\,\mathrm{d}W_{4,u}\right)\mathrm{d}s\end{aligned}$$

and that $\langle p_A^C, p_B^C\rangle_t = \int_0^t (\sigma_{A,s}\sigma_{BA,s} + \sigma_{B,s}\sigma_{AB,s})\,\mathrm{d}s$ is the integrated covariation between p_A and p_B over the $[0,t]$ interval. It is easy to see that IV_B does not affect $IV_{A,t}$ if and only if

(i) $\sigma_{AB,s} = 0$ a.s.
(ii) $J_{3,s}$ is independent of $J_{4,t}$
(iii) $b_{1,A}(\sigma_{A,u}^2, \sigma_{B,u}^2) = b_{1,A}(\sigma_{A,u}^2)$ a.s.
(iv) $b_{2,A}(\sigma_{A,u}^2, \sigma_{B,u}^2) = b_{2,A}(\sigma_{A,u}^2)$ a.s.
(v) $b_{2,AB}(\sigma_{A,u}^2, \sigma_{B,u}^2) = 0$ a.s.

If any of the above conditions fails to hold, then $IV_{A,t}$ remains dependent upon $IV_{B,t}$ even after conditioning on its own past values. Regardless of whether condition *(i)* holds, volatility interdependence may arise even in the case that volatility is a measurable function of past asset prices due to a violation of condition *(iii)*, which reduces to $b_{1,A}(p_{A,u}, p_{B,u}) = b_{1,A}(p_{A,u})$ almost surely. It thus turns out that it does not suffice, nor it is necessary, to impose that the cross-variation process $\langle p_A^C, p_B^C\rangle_t$ is zero almost surely. In principle, it is possible to test directly whether conditions *(i)* to *(v)* hold if one is ready to specify the parametric functional forms of the drift, diffusive, and jump terms. The outcome would however depend heavily on the correct specification of the data generating process in (1). To minimize the risk of misspecification, we take a nonparametric route.

Our goal is to check whether the daily variance of asset B helps predict the daily variance of asset A. We thus formulate a testing procedure that focuses on the density restrictions implied by conditional independence:

$$\mathbb{H}_0: \quad f_{IV_{A,t+1}|\boldsymbol{IV}_{A,t}^{(q_A)}, \boldsymbol{IV}_{B,t+k}^{(q_B)}}\left(y \,\middle|\, \boldsymbol{IV}_{A,t}^{(q_A)}, \boldsymbol{IV}_{B,t+k}^{(q_B)}\right) = f_{IV_{A,t+1}|\boldsymbol{IV}_{A,t}^{(q_A)}}\left(y \,\middle|\, \boldsymbol{IV}_{A,t}^{(q_A)}\right) \quad \text{a.s. for all } y, \tag{2}$$

where $f_{IV_{A,t+1}|\boldsymbol{IV}_{A,t}^{(q_A)}}$ and $f_{IV_{A,t+1}|\boldsymbol{IV}_{A,t}^{(q_A)}, \boldsymbol{IV}_{B,t+k}^{(q_B)}}$ denote the conditional density of $IV_{A,t+1}$ given $\boldsymbol{IV}_{A,t}^{(q_A)}$ and $\left(\boldsymbol{IV}_{A,t}^{(q_A)}, \boldsymbol{IV}_{B,t+k}^{(q_B)}\right)$, with $\boldsymbol{IV}_{A,t}^{(q_A)} \equiv (IV_{A,t}, \ldots, IV_{A,t-q_A+1})'$ and $\boldsymbol{IV}_{B,t+k}^{(q_B)} \equiv (IV_{B,t}, \ldots, IV_{B,t+k-q_B+1})'$

standing for vectors of dimension q_A and q_B concerning the information about the integrated variances of assets A and B, respectively. We allow for $k \in \{0, 1\}$ so as to control for time differences between the markets under consideration.[2] As usual, we define the alternative hypothesis as the negation of the null hypothesis.

In general, the integrated variance does not follow a finite-order Markov process.[3] This means that, to test for noncausality in variance, one would have to let the number of conditioning variables (q_A and q_B) to increase with the sample size. This is unfeasible due to the curse of dimensionality and hence we consider the less ambitious null of conditional independence by fixing the number of conditioning variables in (2) to a finite (and, possibly, small) figure.[4]

To implement a nonparametric test for $\mathbb{H}_0$, we propose a statistic that gauges the discrepancy between the nonparametric estimates of the density functions that appear in (2). In particular, our test statistic hinges on the sample counterpart of the following integrated square relative distance

$$\int \left[\frac{f_{IV_{A,t+1}|\boldsymbol{IV}_{A,t}^{(q_A)},\boldsymbol{IV}_{B,t+k}^{(q_B)}}(y|\boldsymbol{x}^{(q_A)},\boldsymbol{x}^{(q_B)}) - f_{IV_{A,t+1}|\boldsymbol{IV}_{A,t}^{(q_A)}}(y|\boldsymbol{x}^{(q_A)})}{f_{IV_{A,t+1}|\boldsymbol{IV}_{A,t}^{(q_A)}}(y|\boldsymbol{x}^{(q_A)})} \right]^2 \pi(y, \boldsymbol{x}^{(q_A)}, \boldsymbol{x}^{(q_B)}) \, \mathrm{d}y \, \mathrm{d}\boldsymbol{x}^{(q_A)} \, \mathrm{d}\boldsymbol{x}^{(q_B)}, \tag{3}$$

where $\boldsymbol{IV}_{i,t}^{(q_i)} \equiv (IV_{i,t}, \ldots, IV_{i,t-q_i+1})$ with $IV_{i,t}$ denoting asset i's integrated variance over day t ($i = A, B$). We employ a weighting scheme $\pi(\cdot, \cdot, \cdot)$ so as to avoid the lack of precision that afflicts conditional density estimation in areas of low density of the conditioning variables. The integrated square distance that we adopt in (3) is convenient because it facilitates the derivation of the asymptotic theory. Bickel and Rosenblatt (1973), Aït-Sahalia (1996), Aït-Sahalia, Bickel and Stoker (2001), Amaro de Matos and Fernandes (2007), and Aït-Sahalia, Fan and Peng (2009) use similar squared distance measures, though one could likewise employ entropic pseudo-distance measures as in, e.g., Robinson (1991) and Hong and White (2004). Alternatively, one could also implement testing procedures based on the cumulative distribution functions, such as the Kolmogorov-Smirnov and Cramér-von Mises tests. However, Fan (1996) provides ample evidence that these goodness-of-fit tests have little power against local deviations such as bumps and global deviation at higher frequencies of the Fourier expansion of the cumulative distribution function.

We derive the limiting distribution of the test statistic in (3) in Sections 3 and 4.

3 Asymptotic theory for the unfeasible case

To simplify notation, we denote by Y_t the integrated variance of interest $IV_{A,t+1}$, whereas we denote the conditioning vectors $\boldsymbol{IV}_{A,t}^{(q_A)}$ and $\left(\boldsymbol{IV}_{A,t}^{(q_A)}, \boldsymbol{IV}_{B,t+k}^{(q_B)}\right)$ respectively by $\boldsymbol{X}_t^{(q_A)}$ and $\boldsymbol{X}_t^{(q)}$, with $q = q_A + q_B$

[2] For instance, as the Tokyo Stock Exchange closes before the opening of the New York Stock Exchange, one may condition on the same day information ($k = 1$) rather than on information from the previous trading day ($k = 0$).

[3] Meddahi (2003) shows, for instance, that the CIR specification entails an ARMA(1,1) process for the integrated variance.

[4] In the empirical application in Section 6, we add the VIX index as an extra control so as to accommodate for the non-Markovian nature of the data. Alternatively, one could adapt our asymptotic theory to deal with dimension reduction techniques as in, e.g., Hall and Yao (2005) and Fan, Peng, Yao and Zhang (2009).

standing for the higher dimension. The null hypothesis of conditional independence in (2) now reads

$$\mathbb{H}_0: \quad f_{Y|\boldsymbol{X}^{(q)}}(y \,|\, \boldsymbol{x}^{(q)}) = f_{Y|\boldsymbol{X}^{(q_A)}}(y \,|\, \boldsymbol{x}^{(q_A)}) \qquad \text{for all } (y, \boldsymbol{x}^{(q)}). \tag{4}$$

We employ local linear smoothing to estimate both the right- and left-hand sides of (4). The sample analog of (3) then is

$$\sum_{t=1}^{T} \left[\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t \,|\, \boldsymbol{X}_t^{(q)}) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)})} \right]^2 \pi(Y_t, \boldsymbol{X}_t^{(q)}), \tag{5}$$

where the conditional density estimates $\widehat{f}_{Y|\boldsymbol{X}^{(q)}}$ and $\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}$ derive from local linear smoothing using different sets of bandwidths.

Denote by $\widehat{\boldsymbol{\beta}}_T(y, \boldsymbol{x}^{(q)}) = \left(\widehat{\beta}_{0T}(y, \boldsymbol{x}^{(q)}), \widehat{\beta}_{1T}(y, \boldsymbol{x}^{(q)}), \ldots, \widehat{\beta}_{qT}(y, \boldsymbol{x}^{(q)})\right)'$ with $\boldsymbol{x}^{(q)} = (x_1, \ldots, x_q)$ the argument that minimizes

$$\frac{1}{T} \sum_{t=1}^{T} \Big[K_b(Y_t - y) - \beta_0 - \beta_1(X_{1t} - x_1) - \ldots - \beta_q(X_{qt} - x_q) \Big]^2 \prod_{j=1}^{q} W_{h_q}(X_{jt} - x_j), \tag{6}$$

where $K_b(u) = b^{-1}K(u/b)$ and $W_{h_q}(u) = h_q^{-1}W(u/h)$ are symmetric kernels. The local linear estimator of the conditional density function $f_{Y|\boldsymbol{X}^{(q)}}$ is given by $\widehat{f}_{Y|\boldsymbol{X}^{(q)}}(y \,|\, \boldsymbol{x}^{(q)}) = \widehat{\beta}_{0T}(y, \boldsymbol{x}^{(q)})$. The local linear estimator $\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}$ of the lower dimensional conditional density is analogous for $\boldsymbol{x}^{(q_A)} = (x_1, \ldots, x_{q_A})$.

To establish the limiting behavior of the test statistic in (5), we shall rely on the following assumptions.

Assumption A1: The product kernels $\boldsymbol{W}(\boldsymbol{u}) = \prod_{j=1}^{q} W(u_j)$ and $\widetilde{\boldsymbol{W}}(\boldsymbol{u}) = \prod_{j=1}^{q_A} W(u_j)$ rest on a symmetric, nonnegative, continuous univariate kernel W of second order with bounded support $[-\Delta, \Delta]$ for $1 \leq j \leq q$. The kernel W is also at least twice differentiable on the interior of its support. The symmetric kernel K is of order $s \geq 2$ (even integer) and at least twice differentiable on the interior of its bounded support $[-\Delta, \Delta]$.

Assumption A2: The density functions $f_{Y|\boldsymbol{X}^{(q)}}(y \,|\, \boldsymbol{x}^{(q)})$ and $f_{Y\boldsymbol{X}^{(q)}}(y, \boldsymbol{x}^{(q)})$ are r-times continuously differentiable in $(y, x^{(q)})$ with bounded derivatives and with $r \geq s$. The same condition also holds for the lower-dimensional density functions $f_{Y|\boldsymbol{X}^{(q_A)}}(y \,|\, \boldsymbol{x}^{(q_A)})$ and $f_{Y\boldsymbol{X}^{(q_A)}}(y, \boldsymbol{x}^{(q_A)})$.

Assumption A3: The weighting function $\pi\left(y, \boldsymbol{x}^{(q)}\right)$ is continuous and integrable, with second derivatives in a compact support.

Assumption A4: The stochastic process $\left(Y_t, \boldsymbol{X}_t^{(q)}\right)$ is strictly stationary and β-mixing with $\beta_\tau = O\left(\rho^\tau\right)$, where $0 < \rho < 1$.

Assumptions A1 to A4 are quite standard in the literature on local linear smoothing (see, e.g., Fan, Yao and Tong, 1996) and hence we only briefly discuss them in what follows. Assumption A1 rules out higher-order kernels for W essentially to ensure the positivity of the criterion function in (6). Assumptions A2 and A3 require that the weighting scheme and the density functions are both well defined and smooth enough to admit functional expansions. Assumption A4 restricts the amount of data dependence, requiring that the stochastic process is absolutely regular with geometric decay rate. Alternatively, one could assume $\alpha-$mixing conditions as in Aït-Sahalia et al. (2009) and Gao and Hong (2008), though the conditions under which the quadratic variation of a jump-diffusion process satisfies Assumption A4 are quite weak (see discussion in Corradi, Distaso and Swanson, 2011). See also Chen, Linton and Robinson (2001) for some advantages of the $\beta-$mixing assumption relative to the $\alpha-$mixing condition in the context of nonparametric density estimation.

The scaled and centered version of (5) reads as

$$\widehat{\Lambda}_T = \widehat{\Omega}_T^{-1} \left\{ \begin{array}{l} h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left[\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} \right]^2 \pi(Y_t, \boldsymbol{X}_t^{(q)}) \\ \qquad - h_q^{-q/2}\, b^{-1/2}\, \widehat{\mu}_{1,T} - h_q^{q/2}\, h_{q_A}^{-q_A}\, b^{-1/2}\, \widehat{\mu}_{2,T} + 2\, h_q^{q/2-q_A}\, b^{-1/2}\, \widehat{\mu}_{3,T} \end{array} \right\}, \tag{7}$$

where $\widehat{\mu}_{1,T}$, $\widehat{\mu}_{2,T}$, $\widehat{\mu}_{3,T}$, and $\widehat{\Omega}_T$ are consistent estimators of the asymptotic bias terms and variance, respectively (see Appendix A for definitions).

To ensure the asymptotic standard normality of $\widehat{\Lambda}_T$ in (7), we must impose some conditions on the rates at which the bandwidths shrink to zero in Lemmata 1 to 2 (see Appendix B). It turns out that there are no bandwidth rates that jointly meet the conditions (v) and (vi) in Lemma 1 for $q > 2$ if K is of second order ($s = 2$) and for $q > 3$ if K is of higher order ($s \geq 4$). The main snag is that, as aforementioned, one cannot increase the order of the kernel function W in the local-linear smoothing. In turn, restricting attention to at most two conditioning variables and to second-order kernels, the bandwidth conditions in Lemmata 1 and 2 hold if $b = h_1 = O\left(T^{-1/5}\right)$ and $h_2 = o\left(T^{-1/m}\right)$ with $30/7 < m \leq \frac{50}{9}$. Note that b and h_1 have optimal rates, whereas the bandwidth for the full conditioning set entails some degree of undersmoothing. As expected , the bandwidth conditions in Lemmata 1 and 2 in general require more undersmoothing as the dimensionality of the conditioning sets increases.

We are now ready to state our main result concerning the asymptotic behavior of the normalized test statistic in (7) under both the null and alternative hypotheses.

Theorem 1: Let Assumptions A1 to A4 hold as well as the bandwidth conditions (i) to (vi) in Lemmata 1 and 2. It follows for $q \leq 3$ and $q_A \leq 2$ that

(i) Under the null hypothesis $\mathbb{H}_0$, $\widehat{\Lambda}_T \xrightarrow{d} N(0,1)$.

(ii) Under the alternative hypothesis $\mathbb{H}_A$, $\Pr\left(T^{-1}\, h_q^{-q/2}\, b^{-1/2} \left|\widehat{\Lambda}_T\right| > \varepsilon\right) \longrightarrow 1$ for any $\varepsilon > 0$.

Part (i) of Theorem 1 provides the means to compute the asymptotic critical values of the test, whereas (ii) establishes consistency. If one restricts attention to the case in which $q = 1$ and $q_A = 0$, the above result follows almost immediately from Aït-Sahalia et al.'s (2009) Corollary to Theorem 1. Yet, even in this simple case, it is necessary to account for the bias component that arises due to the nonparametric estimation of the lower-dimensional model.

We now deal with higher dimensions (i.e., $q > 3$) by employing a Nadaraya-Watson estimator based on higher-order kernels. More specifically, consider

$$\bar{f}_{Y|\boldsymbol{X}^{(q)}}(y|\boldsymbol{x}^{(q)}) = \frac{\bar{f}_{Y,\boldsymbol{X}^{(q)}}(y,\boldsymbol{x}^{(q)})}{\bar{f}_{\boldsymbol{X}^{(q)}}(\boldsymbol{x}^{(q)})} = \frac{\frac{1}{T}\sum_{t=1}^{T} \bar{\boldsymbol{W}}_{h_q}(\boldsymbol{X}_t^{(q)} - \boldsymbol{x}^{(q)})\, K_b(Y_t - y)}{\frac{1}{T}\sum_{t=1}^{T} \bar{\boldsymbol{W}}_{h_q}(\boldsymbol{X}_t^{(q)} - \boldsymbol{x}^{(q)})},$$

where $\bar{\boldsymbol{W}}_{h_q}(\boldsymbol{u}) = h_q^{-p} \prod_{j=1}^{q} \bar{W}(u_j/h_q)$. Define $\bar{f}_{Y|\boldsymbol{X}^{(q_A)}}$ analogously using the product kernel given by $\widetilde{\bar{\boldsymbol{W}}}_{h_{q_A}}(\boldsymbol{u}) = h_{q_A}^{-q_A} \prod_{j=1}^{q_A} \bar{W}(u_j/h_{q_A})$. We next modify Assumption A1 to accommodate higher-order kernels.

Assumption A5: The kernel functions K and $\bar{W}$ are of even integer orders $(s, \bar{s} \geq 2)$, symmetric, continuous, and at least twice differentiable on the interior of their bounded support $[-\Delta, \Delta]$.

The kernel-based test statistic is essentially analogous to the one based on local linear smoothing:

$$\bar{\Lambda}_T = \bar{\Omega}_T^{-1} \left\{ \begin{array}{l} h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left[\frac{\bar{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) - \bar{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})}{\bar{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} \right]^2 \pi(Y_t, \boldsymbol{X}_t^{(q)}) \\ \qquad - h_q^{-q/2}\, b^{-1/2}\, \bar{\mu}_{1,T} - h_q^{q/2}\, h_{q_A}^{-q_A}\, b^{-1/2}\, \bar{\mu}_{2,T} + 2\, h_q^{q/2 - q_A}\, b^{-1/2}\, \bar{\mu}_{3,T} \end{array} \right\}. \quad (8)$$

We provide expressions for the bias and scaling terms in Appendix A. As before, to ensure the asymptotic standard normality of $\bar{\Lambda}_T$ in (8), we must impose the bandwidth conditions in Lemmata 4 and 5 (see Appendix B). The fact that we can now increase the order of the kernel that we employ to smooth the conditioning set is very advantageous. We are now able to find bandwidths that satisfy the conditions for $q = 4$ and $q_A = 3$ as long as $\bar{W}$ is of fourth order. This is a huge improvement relative to the local linear case. Surprisingly, it turns out that it pays off to use a second-order kernel for K if we fix both b and h_{q_A} to their optimal rates. This is the configuration that entails the least undersmoothing. For instance, in the event that $b = O\left(T^{-1/5}\right)$ and $h_1 = O\left(T^{-1/9}\right)$, we require only a bit of undersmoothing for the larger conditioning set, viz., $h_2 = o\left(T^{-1/10}\right)$. As before, we have to increase the degree of undersmoothing as the dimensionality of the conditioning set grows: e.g., for $q = 3$ and $q_A = 2$, the best configuration for the bandwidths is $b = O\left(T^{-1/5}\right)$, $h_2 = O\left(T^{-1/10}\right)$, and $h_3 = o\left(T^{-9/95}\right)$.

The next result documents the asymptotic standard normality of the kernel-based test statistic in (8) under the null as well as the consistency of the resulting test.

Theorem 2: Let the bandwidth conditions (i) to (vi) in Lemmata 4 and 5 hold as well as Assumptions A2 to A5. It then follows that

(i) Under the null hypothesis $\mathbb{H}_0$, $\bar{\Lambda}_T \xrightarrow{d} N(0,1)$.

(ii) Under the alternative hypothesis $\mathbb{H}_A$, $\Pr\left(T^{-1}\, h_q^{-q/2}\, b^{-1/2}\, |\bar{\Lambda}_T| > \varepsilon\right) \longrightarrow 1$ for any $\varepsilon > 0$.

A nontrivial alternative is to develop the asymptotic theory for higher-order local polynomials with K of order s (and a second-order kernel W). For instance, a local polynomial of second or third order would entail a bias of the same order of magnitude as the kernel-based approach with both K and W of order s, thereby meeting the conditions for Theorem 2. However, the order of the local polynomial would affect the kernel constant in the asymptotic variance of the density estimator even if it does not affect its order of magnitude (Fan and Gijbels, 1996). For instance, the variance of the second- and third-order local polynomial estimators is 1.68 times the variance of the local linear polynomial for a Gaussian kernel. Note also that local polynomial smoothing is only free of boundary bias for polynomials of odd order.

Theorems 1 and 2 form the basis for asymptotically locally strictly unbiased tests for the conditional independence null $\mathbb{H}_0$ in (4) based on local linear and kernel smoothing, respectively. To alleviate the boundary bias that haunts kernel smoothing, one could always take the log of the daily variances before testing for conditional independence as an alternative to weighting down by means of $\pi(Y_t, \boldsymbol{X}_t^{(q)})$ any realized measure close to zero.

4 Accounting for the realized measure estimation

The asymptotic theory so far considers the unfeasible test statistic in (7). In this section, we show the asymptotic equivalence of the corresponding feasible test statistic that replaces integrated variances by realized measures. To discuss the impact of estimating the integrated variance, we must first establish some notation that makes explicit the dependence on the number of intraday observations that we employ to compute the realized measure. We thus denote the time series of realized measures by $Y_{t,M}$ and $\boldsymbol{X}_{t,M}^{(d)}$, where M is the number of intraday observations and $d \in \{q, q_A\}$.

Let $\widehat{\boldsymbol{\beta}}_T^{(M)}(y, \boldsymbol{x}^{(d)}) = \left(\widehat{\beta}_{0T}^{(M)}(y, \boldsymbol{x}^{(d)}), \ldots, \widehat{\beta}_{dT}^{(M)}(y, \boldsymbol{x}^{(d)})\right)'$ denote the argument that minimizes

$$\frac{1}{T}\sum_{t=1}^{T}\Big[K_b(Y_{t,M} - y) - \beta_0 - \beta_1(X_{1t,M} - x_1) - \ldots - \beta_d(X_{dt,M} - x_d)\Big]^2 \prod_{j=1}^{d} W_{h_d}(X_{jt,M} - x_j).$$

The local linear estimator of the conditional density is $\widehat{f}_{Y|\boldsymbol{X}^{(d)}}^{(M)}(y \,|\, \boldsymbol{x}^{(d)}) = \widehat{\beta}_{0T}^{(M)}(y, \boldsymbol{x}^{(d)})$, yielding the following feasible test statistic

$$\widehat{\Lambda}_{M,T} = \widehat{\Omega}_{M,T}^{-1}\left\{\begin{array}{c} h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T}\left[\dfrac{\widehat{f}_{Y|\boldsymbol{X}^{(q)}}^{(M)}(Y_{t,M} \,|\, \boldsymbol{X}_{t,M}^{(q)}) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}^{(M)}(Y_{t,M} \,|\, \boldsymbol{X}_{t,M}^{(q_A)})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}^{(M)}(Y_{t,M} \,|\, \boldsymbol{X}_{t,M}^{(q_A)})}\right]^2 \pi(Y_{t,M}, \boldsymbol{X}_{t,M}^{(q)}) \\ - h_q^{-q/2}\, b^{-1/2}\, \widehat{\mu}_{1,T}^{(M)} - h_q^{q/2}\, h_{q_A}^{-q_A}\, b^{-1/2}\, \widehat{\mu}_{2,T}^{(M)} + 2\, h_q^{q/2 - q_A}\, b^{-1/2}\, \widehat{\mu}_{3,T}^{(M)} \end{array}\right\}, \tag{9}$$

where $\widehat{\mu}_{\cdot,T}^{(M)}$ differs from $\widehat{\mu}_{\cdot,T}$ because it employs realized measures rather than the true integrated variance. Let $N_{0,t,M} = Y_t - Y_{t,M}$ and $N_{j,t,M} = X_{j,t} - X_{j,t,M}$ for $1 \le j \le d \in \{q, q_A\}$ denote the errors stemming from the estimation of the integrated variance. To ensure the asymptotic equivalence between the unfeasible and feasible test statistics, we must restrict the rate at which the moments of the estimation errors converge to zero. We do that by constraining the moments of the drift and diffusive functions as well as of the noise due to market-microstructure effects.

Assumption A6: The drift terms of (1) are continuous locally bounded processes with $\mathbb{E}\,|\mu_{i,t}|^{2k} < \infty$, whereas the diffusive functions are *càdlàg* with $\mathbb{E}\,|\sigma_{ij,t}|^{2k} < \infty$ and the jump components $c_i(t)$ are iid with $\mathbb{E}\,|c_i(t)|^{2k} < \infty$ for some $k \geq 2$ and $i, j \in \{A, B\}$. In addition, the microstructure noise is iid with symmetric distribution around zero and with finite $2k$th moment for some $k \geq 2$.

Assumption A6 ensures that the conditions in Corradi et al.'s (2011) Lemma 1 hold and hence that there exists a sequence a_M, with $a_M \to \infty$ as $M \to \infty$, such that $\mathbb{E}\,|N_{j,t,M}|^k = O\left(a_M^{-k/2}\right)$ for some $k \geq 2$ and $1 \leq j \leq d \in \{q, q_A\}$. Note that this establishes a bound to the kth moment of the absolute estimation error that depends on the realized measure we employ to estimate the integrated variance. In particular, $a_M = M$ for the realized variance (Andersen, Bollerslev, Diebold and Labys, 2001; Barndorff-Nielsen and Shephard, 2002) and tripower variation (Barndorff-Nielsen and Shephard, 2004), whereas $a_M = M^{1/3}$ for the two-scale realized variance (Zhang, Mikland and Aït-Sahalia, 2005) and $a_M = \sqrt{M}$ for the multi-scale realized variance (Zhang, 2006; Aït-Sahalia, Mykland and Zhang, 2011) and the realized kernel estimator (Barndorff-Nielsen, Hansen, Lunde and Shephard, 2008).

Theorem 3: Let Assumptions A1 to A4 and A6 hold as well as the bandwidth conditions (i) to (vi) in Lemmata 1 and 2. Let also $T^{\frac{k+1}{2(k-1)}}\,(\ln T)^{1/2}\,a_M^{-1/2}(h_q^{-1} + h_{q_A}^{-1} + b^{-1}) \to 0$ as $T, M \to \infty$ for k as defined in Assumption A6. It follows for $q \leq 3$ that

(i) Under the null hypothesis $\mathbb{H}_0$, $\widehat{\Lambda}_{M,T} \xrightarrow{d} N(0,1)$.

(ii) Under the alternative hypothesis $\mathbb{H}_A$, $\Pr\left(T^{-1}\,h_q^{-q/2}\,b^{-1/2}\,|\,\widehat{\Lambda}_{M,T}| > \varepsilon\right) \longrightarrow 1$ for any $\varepsilon > 0$.

As for the feasible kernel-based statistic, define $\bar{\Lambda}_{M,T}$ analogously to $\bar{\Lambda}_T$ in (8) but replacing $(Y_t, \boldsymbol{X}_t^{(q)})$ with $(Y_{t,M}, \boldsymbol{X}_{t,M}^{(q)})$. The next results documents asymptotic equivalence in the context of kernel density estimation.

Theorem 4: Let Assumptions A2 to A6 hold as well as the bandwidth conditions (i) to (vi) in Lemmata 4 and 5. Also, let $T^{\frac{k+1}{2(k-1)}}\,(\ln T)^{1/2}\,a_M^{-1/2}(h_q^{-1} + h_{q_A}^{-1} + b^{-1}) \to 0$ as $T, M \to \infty$ for k as defined in Assumption A6. It follows that

(i) Under the null hypothesis $\mathbb{H}_0$, $\bar{\Lambda}_{M,T} \xrightarrow{d} N(0,1)$.

(ii) Under the alternative hypothesis $\mathbb{H}_A$, $\Pr\left(T^{-1}\,h_q^{-q/2}\,b^{-1/2}\,|\,\bar{\Lambda}_{M,T}| > \varepsilon\right) \longrightarrow 1$ for any $\varepsilon > 0$.

Theorems 3 and 4 establish that the asymptotic equivalence between unfeasible and feasible test statistics necessitates that the number of intraday observations M grows fast enough relative to the number of days T. As usual, there is a tradeoff between using a non-robust realized measure with $a_M = M$ at a frequency for which microstructure noise is negligible and employing a microstructure-robust realized measure with $a_M = \sqrt{M}$ at the highest available frequency.

5 Bootstrap critical values

It is well known that the asymptotic behavior of nonparametric tests does not always entail a reasonable approximation in finite samples and that their results may heavily depend on the bandwidth choice (see, e.g., Fan, 1995; Fan and Linton, 2003). In what follows, we aim to alleviate such concerns by employing resampling techniques. There are a number of issues that one must bear in mind, though. First, given the nonparametric nature of the null hypothesis, we cannot rely on standard resampling algorithms based on either parametric or wild bootstrap methods (Härdle and Mammen, 1993; Andrews, 1997; Aït-Sahalia et al., 2009). Second, integrated variance does not follow a Markov process, ruling out as well bootstrap algorithms for Markov sequences (Rajarshi, 1990; Paparoditis and Politis, 2002; Horowitz, 2003).[5] Third, a standard nonparametric bootstrap would also fail to mimic the limiting distribution of our test statistics for they involve degenerate U-statistics (Bretagnolle, 1983; Arcones and Giné, 1992).

To circumvent the above issues, we resort to a variation of the standard moon bootstrap by Bretagnolle (1983) and Bickel et al. (1997). We sample $\mathcal{T}$ out of T daily realized measures by blocks (rather than individually) so as to cope with time-series dependence. In addition, for each bootstrap sample, we compute the test statistics using a bandwidth vector (h_{*q}, h_{*q_A}, b_*) that shrinks to zero at the same as before rate, but depending on $\mathcal{T}$ (rather than T). This implies distinct orders of magnitude for the bias terms in the original and bootstrap statistics and thence they do not cancel out. This is in stark contrast with the bias cancelation that happens within the context of parametric and wild bootstrap. It nonetheless remains unnecessary to compute the scaling term corresponding to the asymptotic variance of the test statistic.

The (unscaled) bootstrap counterparts of (7) and (8) then are respectively

$$\widehat{\Lambda}^*_{\mathcal{T}} = \left\{ \begin{array}{l} h_{*q}^{q/2}\, b_*^{1/2} \sum_{t=1}^{\mathcal{T}} \left[\frac{\widehat{f}^*_{Y|\boldsymbol{X}^{(q)}}(Y_t^*|\boldsymbol{X}_t^{*(q)}) - \widehat{f}^*_{Y|\boldsymbol{X}^{(q_A)}}(Y_t^*|\boldsymbol{X}_t^{*(q_A)})}{\widehat{f}^*_{Y|\boldsymbol{X}^{(q_A)}}(Y_t^*|\boldsymbol{X}_t^{*(q_A)})} \right]^2 \pi(Y_t^*, \boldsymbol{X}_t^{*(q)}) \\ \qquad - h_{*q}^{-q/2}\, b_*^{-1/2}\, \widehat{\mu}^*_{1,\mathcal{T}} - h_{*q}^{q/2}\, h_{*q_A}^{-q_A}\, b_*^{-1/2}\, \widehat{\mu}^*_{2,\mathcal{T}} + 2\, h_{*q}^{q/2-q_A}\, b_*^{-1/2}\, \widehat{\mu}^*_{3,\mathcal{T}} \end{array} \right\} \tag{10}$$

and

$$\bar{\Lambda}^*_{\mathcal{T}} = \left\{ \begin{array}{l} h_{*q}^{q/2}\, b_*^{1/2} \sum_{t=1}^{\mathcal{T}} \left[\frac{\bar{f}^*_{Y|\boldsymbol{X}^{(q)}}(Y_t^*|\boldsymbol{X}_t^{*(q)}) - \bar{f}^*_{Y|\boldsymbol{X}^{(q_A)}}(Y_t^*|\boldsymbol{X}_t^{*(q_A)})}{\bar{f}^*_{Y|\boldsymbol{X}^{(q_A)}}(Y_t^*|\boldsymbol{X}_t^{*(q_A)})} \right]^2 \pi(Y_t^*, \boldsymbol{X}_t^{*(q)}) \\ \qquad - h_{*q}^{-q/2}\, b_*^{-1/2}\, \bar{\mu}^*_{1,\mathcal{T}} - h_{*q}^{q/2}\, h_{*q_A}^{-q_A}\, b_*^{-1/2}\, \bar{\mu}^*_{2,\mathcal{T}} + 2\, h_{*q}^{q/2-q_A}\, b_*^{-1/2}\, \bar{\mu}^*_{3,\mathcal{T}} \end{array} \right\}. \tag{11}$$

As before, we provide in Appendix A the expressions for the bias terms in (10) and (11).

We next establish the first-order validity of the moon bootstrap only for the unfeasible test statistic given that the asymptotic equivalence between the feasible and unfeasible bootstrap statistics ensues along the same lines as in Theorem 3 provided that Assumption A6 holds.

[5] Restricting attention to the class of eigenfunction stochastic volatility models would actually yield integrated variances with a finite ARMA representation (Meddahi, 2003) and hence approximately Markov (even if of higher order).

Theorem 5: Let Assumptions A1 to A4 hold and let the bandwidth conditions (i) to (vi) in Lemmata 1 and 2 hold for (h_{*q}, h_{*q_A}, b_*) and $\mathcal{T}$ in lieu of (h_q, h_{q_A}, b) and T, respectively. Letting $T, \mathcal{T}, T/\mathcal{T} \to \infty$ yields, for $q \leq 3$ and for any $\varepsilon > 0$, $\Pr\left(\sup_{v\in\mathbb{R}}\left|\Pr_*(\widehat{\Lambda}^*_{\mathcal{T}} \leq v) - \Pr(\widehat{\Omega}_T\,\widehat{\Lambda}_T \leq v)\right| > \varepsilon\right) \longrightarrow 0$ under the null, whereas $\Pr\left(\Pr_*\left(\left|\mathcal{T}^{-1}\, h_{*q}^{-q/2}\, b_*^{-1/2}\,\widehat{\Lambda}^*_{\mathcal{T}}\right| > \varepsilon\right)\right) \longrightarrow 1$ under the alternative. In addition, replacing Assumption A1 with A5 and letting bandwidth conditions (i) to (vi) in Lemmata 4 and 5 hold ensure the first-order validity of the moon bootstrap for the kernel-based test statistic in (11) as well.

It is immediate to see that the sample and bootstrap statistics have the same limiting distribution under the null, whereas they diverge at different rates under the alternative. In particular, (10) and (11) diverge at a slower rate $\mathcal{T}\, h_{*q}^{q/2}\, b_*^{1/2}$ relative to their sample counterparts. In practice, one must deal with the feasible bootstrap test statistics $\widehat{\Lambda}^*_{M,\mathcal{T}}$ and $\bar{\Lambda}^*_{M,\mathcal{T}}$ that replace $(Y^*_t, \boldsymbol{X}^{*(q)}_t)$ with the corresponding realized measures $(Y^*_{M,t}, \boldsymbol{X}^{*(q)}_{M,t})$. Assumption A6 ensures that the statement in Theorem 5 also applies if one substitutes $\widehat{\Lambda}^*_{M,\mathcal{T}}$ and $\bar{\Lambda}^*_{M,\mathcal{T}}$ for $\widehat{\Lambda}^*_{\mathcal{T}}$ and $\bar{\Lambda}^*_{\mathcal{T}}$, respectively. The bootstrap critical values for $\widehat{\Omega}_{M,T}\,\widehat{\Lambda}_{M,T}$ are readily available from the empirical distribution of $\widehat{\Lambda}^*_{M,\mathcal{T}}$ using a large number, say B, of bootstrap statistics.

To check whether the moon block-bootstrap indeed entails accurate critical values, we run a limited Monte Carlo study. In particular, we simulate intraday returns from two independent mean-reverting CIR processes (Cox, Ingersoll and Ross, 1985) and then examine how the empirical size of our two-step testing procedure varies according to the bandwidth choice. We employ the CIR process not only because it is a standard model in finance, but also because it implies a simple ARMA(1,1) process for the integrated variance (Meddahi, 2003). For each of the 1,000 Monte Carlo replications, we simulate intraday data from

$$\mathrm{d}P_{At} = \kappa_A\,(\mu_A - P_{At})\,\mathrm{d}t + \varsigma_A\sqrt{P_{At}}\,\mathrm{d}W_{At}$$
$$\mathrm{d}P_{Bt} = \kappa_B\,(\mu_B - P_{Bt})\,\mathrm{d}t + \varsigma_B\sqrt{P_{Bt}}\,\mathrm{d}W_{Bt},$$

where W_{At} and W_{Bt} are two independent Brownian motions, using an Euler discretization scheme with a reflection device to ensure positivity. To entail realistic asset price processes, we fix the parameter vectors to $(\kappa_A, \mu_A, \varsigma_A) = (0.080, 0.150, 0.011)$ and $(\kappa_B, \mu_B, \varsigma_B) = (0.120, 0.200, 0.013)$. After burning the first 200 observations of the sample, we employ the last $M\,T$ intraday observations, where M and T correspond respectively to the number of intraday observations within a day and to the number of days. We focus on the relatively small sample sizes of $M = 144$ and $T \in \{400, 600\}$ so as to assess how important is the condition in Theorem 3 that calls for M to grow at a faster rate than T.

From the intraday log-returns, we retrieve the daily realized variances RV_{Ad} and RV_{Bd} for each day $d = 1, \ldots, T$ and then test for conditional independence of asset A's daily variance with respect to asset B's daily variance by looking at the conditional density of $X = \ln RV_{Ad}$ given $Y = \ln RV_{Ad-1}$ and $Z = \ln RV_{Bd}$. We first standardize the data by their mean and standard deviation and then estimate the conditional densities using local linear smoothing with Gaussian kernels. To comply with the bandwidth conditions, we first adjust the rule-of-thumb bandwidths with a Gaussian reference to the appropriate rate, resulting in $b = h_1 = (4/3)^{1/5}\, T^{-1/5}$ and $h_2 = T^{-9/50}/\ln\ln T$. We then multiply these bandwidths by scaling factors $\kappa_b \in \{1/2, 3/4, 1\}$ and $\kappa_h \in \{3/4, 1, 3/2, 5/2\}$, with $\kappa_b < \kappa_h$, so as to assess sensitivity. For simplicity, we employ a weighting scheme relying on a standard multivariate normal density: $\pi_{XYZ}(x, y, z) = \phi(x, y, z) = \phi(x)\phi(y)\phi(z)$. Given that the distribution of the realized variance logarithm is typically close to normal (Andersen et al., 2001, 2003), such a weighting function keeps the focus on the bulk of the data rather than on extreme levels of volatility.

To ensure a reasonable number of daily observations in the bootstrap artificial samples, we consider $\mathcal{T} = 100$, though further simulations show that the results are quite robust to variations in the bootstrap sample size. Table 1 reports the results for $B = 300$ bootstrap samples using a block length of 4 daily observations (i.e., approximately $\mathcal{T}^{1/4}$). Despite the fact that $M < T$, we find that empirical size is close to nominal as long as κ_h is not too high relative to κ_b. All in all, fixing $\kappa_b = 3/4$ and $\kappa_h = 1$ yields very encouraging results, thereby providing some guidance for the bandwidth choice in practice.

6 Spillovers across international stock markets

We examine whether there are volatility spillovers between China, Japan, UK and US using data from their main stock market indices. In particular, we collect ultra-high-frequency data for the SSE B share index, the Topix 100 index, the FTSE 100 index, and the S&P 500 index from Reuters, available at the Securities Industry Research Centre of Asia-Pacific (`www.sirca.org.au`).

Before describing the data, it is important to justify our index selection by establishing some background. We adopt the S&P 500 index to measure the movements in the US stock market because it is one of the main bellwethers for the US economy. In addition, the CBOE also publishes a volatility index (VIX) that measures market expectations of the near-term volatility implied by the S&P 500 index options. This is convenient because it provides an extra control variable to cope with the time-series persistence in the daily volatility of the S&P 500 index. We also consider the UK stock market, as represented by the FTSE 100 index, because it is the main financial hub in Europe.

As for the Topix 100 index, it is a weighted index gauging the performance of the 100 most liquid stocks with the largest market capitalization on the Tokyo Stock Exchange (TSE). There are two continuous trading sessions on the TSE, with a call auction-procedure determining their opening prices. The morning session runs from 9:00 to 11:00, whereas the afternoon session is from 12:30 to 15:00. In view of the time difference, there is no overlapping trading hours between Tokyo and the US stock markets. The same applies to the Shanghai Stock Exchange (SSE), whose morning and afternoon consecutive bidding sessions run from 9:30 to 11:30 and from 13:00 to 15:00. One of the particular features of the Chinese stock market is the relative importance of individual investors despite the fact they face substantial trading restrictions, e.g., a very stringent short-sale constraint (Hertz, 1998; Feng and Seasholes, 2008). In addition, local investors could not own B shares before March 2001 and, even though they may now purchase them using foreign currency, capital controls still restrict their ability to do so. See Allen, Qian and Qian (2007) and

Mei, Scheinkman and Xiong (2009) for more details on the institutional background. Our motivation to include the SSE B share index in the analysis is twofold. First, because pricing of trading for B shares is in US dollars, there is no room for exchange rate movements to blur (or to cause spurious) stock market links. Second, albeit its stock market is relatively young, dating back only to November 1990, China is becoming a major player in the world economy and hence it is interesting to study the role it plays within the context of volatility transmission. The fact that B shares are not as liquid as A shares in the Shanghai Stock Exchange means that controlling for market-microstructure noise is essential for China.

The sample runs from January 3, 2000 to December 30, 2005 with 1,301 common trading days. To compute the realized measures of daily integrated variance, we first compute continuously compounded returns over regular time intervals of 1, 5, 15, and 30 minutes. The sample does not include overnight returns in that the first intraday return refers to the opening price that ensues from the pre-sessional auction, if any. Similarly, we also exclude returns over the lunch break for China and Japan, though they do not affect in any way the qualitative results (see Section 6.1).

Table 2 reports the descriptive statistics for the 1-minute and 30-minute returns. The average intraday return is slightly negative for every stock market, though relative lower for Japan and China. This is to some extent surprising in view that the Topix 100 and, especially, the Shanghai B share indices exhibit larger standard deviations. As usual, index returns exhibit substantial excess kurtosis, which rapidly increases with the sampling frequency. This is especially the case for the FTSE 100 index, which climbs from 21 at the 30-minute frequency to 213 at the 1-minute frequency. As for skewness, it is strongly negative for the S&P 500 at the 1-minute frequency, though slightly positive at the 30-minute frequency. The opposite applies to the FTSE 100 and Topix 100 indices, namely, skewness is positive at the 1-minute frequency, whereas negative at the 30-minute frequency. The skewness coefficient increases with the sampling frequency from nearly zero to 0.91 for the SSE B shares index. The differences between the skewness and kurtosis of the 1-minute and 30-minute returns are possibly due to liquidity issues. The proportion of zero returns is indeed much higher at the 1-minute frequency, especially for the SSE B share index. Not surprisingly, we also observe strong first-order autocorrelation for every stock market index at the 1-minute frequency as well as for the 30-minute returns on the SSE B share indices. Further analysis shows that the liquidity of the SSE B share index, as measured by the proportion of nonzero returns, increases over time, especially after March 2001.

In what follows, we carry out an empirical analysis of volatility transmission using daily realized measures. We consider the realized variance, the tripower variation, the two-scale realized variance, and the realized kernel based on 1-minute and 5-minute returns. In addition, we also compute the realized variance and tripower variation using 15-minute and 30-minute returns. As in Aït-Sahalia and Jacod (2010), we vary the realized measure we employ so as to emphasize different aspects of the quadratic variation of the index returns. The realized variance based on 1-minute and 5-minute returns essentially gauges the overall quadratic variation, including not only information about the daily variance but also about price jumps and microstructure noise. As the sampling frequency decreases, reducing the market microstructure effects, the realized variance starts reflecting only the diffusive and jump contributions to the quadratic variation. The tripower variation excludes the contribution of price jumps to the quadratic variation and hence provide a reasonable estimator for the daily variance if based on 15-minute and 30-minute returns. Finally, the two-scale and realized kernel approaches eliminate the contribution of the microstructure noise to the quadratic variation, capturing only its jump and diffusive components.

Figure 1 plots the time series of the realized variance and tripower variation based on the 1-minute and 30-minute returns as well as the realized kernel estimates at the 1-minute frequency. It is interesting to observe that controlling for market microstructure noise affects in a substantial manner the estimates of the daily variance, especially for the SSE B share index.

Tables 3 to 6 report the test results for the null hypothesis of conditional independence using bootstrap critical values. As the null of conditional independence is invariant to monotonic transformations, we first standardize the logarithms of the realized measures by their mean and standard deviation and then estimate the conditional densities using kernel-based smoothing with Gaussian-type kernels.[6] In particular, we consider the standard Gaussian kernel for K and the fourth-order kernel derived from the Gaussian density for $\bar{W}$ regardless of the dimension of the conditioning set. The bandwidths are as in the Monte Carlo study, with scaling factors set to $\kappa_b = 3/4$ and $\kappa_h = 1$, though the results remain qualitatively the same for virtually every combination between $\kappa_b \in \{0.5, 0.75, 1\}$ and $\kappa_h \in \{0.75, 1, 1.25\}$ as long as $\kappa_b < \kappa_h$. As before, we employ a weighting scheme based on the standard multivariate normal density. To obtain bootstrap critical values, we construct $B = 500$ bootstrap artificial samples of size $\mathcal{T} = 250$ by resampling blocks of 4 daily observations.

Table 3 documents that there is some strong evidence of volatility spillovers running from the UK to the US. At the 5% significance level, we reject the null for the realized variance and tripower variation estimates based on 1-minute returns, and for the two-scale realized variance at the 5-minute frequency. In addition, we also reject the null at the 10% level of significance for the realized kernel using 1-minute returns as well as for the tripower variation at the 30-minute frequency. This seems to indicate that the transmission channel from the UK to the US is mainly through the integrated variance given that accounting for jumps and/or microstructure noise yields stronger results. The evidence is weaker for Japan. Volatility transmission from the Topix 100 index to the S&P 500 index is significant at the 5% level only for the realized variance and tripower variation at the 1-minute frequency. We also observe some borderline results at the 10% level of significance for the tripower variation and the two-scale realized variance using 5-minute returns. Finally, we also uncover some weak evidence of volatility transmission from China to the US. In particular, we reject the null at the 5% level of significance for the realized variance at the 1- and 5-minute frequencies and for the tripower variation and realized kernel measures based on 5-minute returns.

[6] We do not employ local-linear smoothing as in the previous section because we estimate in section 6.1 conditional densities given three state variables. At any rate, the p-values of the tests based on local linear smoothing are only marginally different from the p-values of the kernel-based tests.

We then ask whether the diffusive and jump components of the quadratic variation in the UK, Japan, or China affect the options-implied market volatility as measured by the VIX index. A positive answer would mean that investors price these spillovers. This is exactly what happens for the FTSE 100 index. Table 3 reveals significant volatility transmission to the VIX index for almost every realized measure we employ. In fact, the UK effect seems stronger on the risk-neutral expected volatility than on the realized measures of the S&P 500 index, especially if one controls for both jumps and market microstructure noise. In contrast, we observe no change in the qualitative results as what concerns spillovers from the Topix 100 and SSE B share indices. As per the former, we reject conditional independence for every realized measure based on 1-minute returns apart from the realized kernel. As for the SSE B share index, we find evidence of significant spillovers to the VIX index for the realized variance at the 1- and 30-minute frequencies and for the tripower variation using 5-minute returns.

Table 4 displays the results for the daily volatility transmission to the FTSE 100 index. For the S&P 500 index, we reject the null for every realized measure at the 1-minute frequency as well as for the realized variances based on 5- and 30-minute returns and for the realized kernel measure at the 5-minute frequency. We interpret these results as strong evidence of spillovers in the quadratic variation. To identify whether the transmission is through the diffusive or discontinuous part of the quadratic variation, we turn our attention to the tests using tripower variation. We take their rejections at the 1-, 5-, and 15-minute frequencies as a clue that the FTSE 100 integrated variance depends on the past S&P 500 integrated variance. We obtain similar results for Japan and China in that accounting for jumps and/or microstructure noise seems to strengthen the evidence of spillovers in the quadratic variation.

Table 5 reveals that the dependence structure between the S&P 500 index and the Topix 100 index is approximately symmetric in that significant transmission seems to eventuate in both directions for almost the same realized measures. We indeed reject conditional independence using the tripower variation only at the 1-minute frequency, whereas we reject the null if we employ either the realized variance using 1-minute returns or the noise-robust realized measures at the 5-minute frequency. A similar symmetric pattern arises for the FTSE 100 index. We find significant spillovers from the UK to Japan for essentially the same realized measures for which we observe significant spillovers in the opposite direction. As before, this seems to indicate the presence of spillovers in the integrated variance. Finally, volatility shocks in the SSE B share index also affect the Topix 100 index according to the tests based on the realized variance at the 1-minute frequency as well as on the tripower variation and realized kernel estimates at the 1- and 5-minute frequencies. Notice that the evidence is stronger once we account for jumps. This suggests that spillovers are mainly through the integrated variance, with jumps to some extent concealing the volatility transmission from China to Japan.

Table 6 documents a somewhat different pattern for the volatility transmission to China. We find significant spillovers from the S&P 500 index only for the realized variance and two-scale realized variance at the 1-minute frequency and for the tripower variation at the 5-minute frequency. The evidence is much stronger for the FTSE 100 index, especially if we control for jumps and/or microstructure effects. As for the Topix 100 index, we unveil significant spillovers for the realized variance and tripower variation based on 1-minute returns and for the tripower variation and realized kernel estimates at the 5-minute frequency. These findings suggest that the main channel of transmission from Japan is also through the diffusive component of the quadratic variation, given that the evidence becomes stronger for the realized measures that are robust either to jumps or to microstructure noise.

Altogether, it seems that the links are generally symmetric in that we normally reject the null of conditional independence in both directions for almost exactly the same realized measures. To exemplify the volatility transmission we uncover, Figure 2 illustrates how the conditional density of the daily tripower variation of the FTSE 100 index at the 15-minute frequency changes if we also condition on the corresponding realized measure of the SSE B share index. The plot evaluates the conditioning tripower variations at their first quartile, median, and third quartile. It is apparent that the imprint of the China effect is in every quartile. Further conditioning on the first quartile of the Shanghai tripower variation shifts the density to the left, whereas it shifts to the right if evaluated at the third quartile. Finally, it seems to reduce the spread of the conditional density of the FTSE 100 tripower variation once we evaluate the tripower variation of the SSE B share index at its median value. This is interesting because we would most likely fail to capture this sort of nonlinear distributional impacts using the usual parametric approaches in the literature.

6.1 Robustness analysis

In what follows, we carry out some robustness analysis by conducting three inspections. First, we assess how pivotal is the assumption that, under the null hypothesis, the past integrated variance suffices to control for the persistence in the data by also conditioning on the past implied volatility (and vice-versa). Second, we investigate whether the evidence we find supporting spillovers effects from Japan and China to the US are robust to further conditioning on realized measures of the FTSE 100 index. Third, we examine how fast the volatility transmission occurs by looking at the quadratic variation over shorter periods of time. In particular, we focus on the conditional distribution of the quadratic variation over the first hour of the trading day given the last hour of previous trading day and the last hour of trading on the other stock market.

6.1.1 Data persistence

The conditional independence restriction we test does not exactly correspond to a null of noncausality in variance given the non-Markovian character of the daily integrated variance. In particular, the empirical analysis in Section 6 controls only for the own integrated variance in the previous day rather than on the whole history of daily variances. This raises a concern on whether our findings are in fact genuine or just an artifact due to higher-order dependence in the data.

To assess robustness against such concerns, we redo our empirical analysis of volatility transmission to the US including the past VIX index as an additional control. The latter measures the options-implied volatility of the S&P500 index and hence should provide information about the future integrated variance.[7] In addition, Bandi and Perron (2006) find strong evidence of fractional cointegration between implied and realized variances and so conditioning on the VIX index should effectively control for any high-order dependence implied by the non-Markovian nature of the integrated variance, regardless of whether the null of conditional independence is true or not.

Table 7 shows that the spillover effects we uncover are not an artifact due to persistence. Adding the VIX index to the conditioning set does not alter much the qualitative results. It actually strengthens the evidence of quadratic variation transmission from the UK and Japan. In contrast, accounting for the VIX index somewhat affects the results for the SSE B share index. We now reject the null of conditional independence for the realized variance at every frequency as well as for the two-scale realized variance at the 1-minute frequency. Interestingly, controlling for jumps by means of the tripower variation measure actually weakens the evidence of transmission. This seems to indicate that spillovers eventuate from China to the US mainly through jumps.

Similarly, we revisit the evidence of volatility transmission to the VIX index by further conditioning on the realized measure of the S&P 500 index in the previous day. Table 7 reveals that, all in all, there is not much change in the qualitative results. The evidence that the realized measure of the FTSE 100 index significantly affects on the VIX index is virtually the same regardless of whether we control or not for the realized measure of the S&P 500 index in the previous day. As for spillovers from the Topix 100 and SSE B share indices, better accounting for data persistence slightly strengthens the statistical evidence of volatility transmission. As before, the China effect is weaker for the tripower variation measures, corroborating our indirect evidence of jump spillovers.

6.1.2 Asia effect given the UK

Most recent episodes of high uncertainty in the global financial markets are mainly due to shocks in Europe and the US, with little action taking place in China and Japan.[8] We thus check whether the evidence of volatility transmission running from China and Japan to the US is genuinely about shocks in Asia or about common shocks to the global financial markets. To do this, we further condition the distribution of the daily realized measures of the S&P 500 index on the corresponding realized measure of the FTSE 100 index over the same trading day (up to 14:30 London time).

The motivation lies in the fact that the UK stock market should reflect any global shock, but not necessarily Asia-specific shocks that may affect the US. Examining the volatility transmission from Asia

[7] Note that it is not necessary to assume that the VIX index is the best forecast for future realizations of the integrated variance nor that it is unbiased or efficient (see, among others, Christensen and Prabhala, 1998).

[8] We thank an anonymous referee for calling our attention to this issue. Note that, although our sample does not include the subprime and transatlantic sovereign debt crises, it covers the burst of the dotcom bubble in the US and its aftermath. This is a period of high volatility in the US stock market due mostly to domestic factors.

(either China or Japan) to the US given the UK realized measure allows us to better understand the geographic nature of the volatility spillovers. Finding significant effects regardless of whether we further condition on the realized measure of the FTSE 100 index indicates that there are Asia-specific shocks that affect the uncertainty in the US stock markets. Rejecting the null only if disregarding the stock market volatility in the UK suggests there is no Asia-specific transmission channel. The S&P 500 index realized measure is then reacting to global (volatility) shocks that are common to the UK and to Asia. Finally, rejecting the null only if accounting for the FTSE 100 realized measure attests that filtering the global effect by means of the variation in the FTSE 100 index helps identify as well spillovers due to Asia-specific shocks.

Table 7 shows that further conditioning on the FTSE 100 realized measure does not change much the qualitative results concerning spillovers from Japan. We still reject the null of conditional independence for both realized variance and tripower variation estimates at the 1-minute frequency as well as for the two-scale realized variance using 5-minute returns. The differences are that we now reject the null also for the realized variance at the 15-minute frequency and for the tripower variation based on 30-minute returns. This seems to suggest that the variation in the FTSE 100 index does not suffice to fully capture the Japan effect on the US stock market volatility. Finally, we observe a very interesting result for the spillovers running from China. Once we control for the FTSE 100 realized measure, the evidence of a significant China effect in the US stock market volatility becomes much stronger, especially if we control for jumps and/or market microstructure noise. All in all, these findings appears to illustrate that accounting for the realized measure of the FTSE 100 index helps individuate Asia-specific spillovers from global volatility shocks.

Figure 3 illustrates how accounting for the realized measure of the SSE B share index alters the conditional density of the S&P 500 realized measure given its past realization and the FTSE 100 realized measure over the same day. It is striking how the information that the SSE B share realized measure conveys is almost exclusively about the dispersion in the distribution of the quadratic variation of the S&P 500 index. It does no shift to the right or left depending on the quartile we condition upon as in Figure 2; we observe across the board only a reduction in the spread of the distribution.

6.1.3 Reaction time

King and Wadhwani (1990) derive an imperfectly revealing equilibrium model to explain contemporaneous transmission of volatility between stock markets. Their framework posits that price jumps will take place as soon as a market reopens so as to reflect changes in both idiosyncratic and common factors since last trade. Given that other stock market indices also depend on the common factors, contagion will result in immediate spillovers from one market to another as the latter reopens for trading. This is in stark contrast with our empirical study focusing on daily quadratic variation (rather than over a shorter interval of time) in that we could well miss the almost instantaneous reaction that King and Wadhwani (1990) predict.

We thus investigate in this section whether reaction time is indeed an issue. To examine spillovers from the UK to the US, we condition the realized measure of the S&P 500 index over its first hour of trading on the realized measures of the FTSE 100 index over the one-hour interval immediately before (i.e., 13:30 and 14:30 London time) and of the S&P 500 index over the last hour of trading in the previous day. For the transmission from the US to the UK, we look at the conditional distribution of the realized measure over the first hour of trading on the London Stock Exchange given the realized measures over the previous day's last hour of trading on the New York Stock Exchange and on the London Stock Exchange. To test for spillovers from Asia to the UK and to the US, we check whether the realized measures over the first hour of trading in the UK/US depends on the corresponding realized measures over the last hour of trading in China/Japan in that same day given the realized measures over the last hour of trading in the UK/US in the previous day. To examine spillovers running in the opposite direction, we investigate whether the realized measures of the FTSE 100 index and of the S&P 500 index over the last hour of trading in the previous day affects the realized measures over the first hour of trading in Asia even after controlling for the latter's realized measures over the last hour of trading in the previous day. The same applies if testing for transmission from China to Japan given that the former shuts before the opening of the latter. Finally, given the one-hour difference between Shanghai and Tokyo, we test for spillovers from Japan to China by looking at whether the realized measure of the SSE B share index over the first hour of trading depends on the realized measures of the Topix 100 index over the one-hour period immediately before the opening of the Shanghai Stock Exchange (i.e., from 9:30 to 10:30 Tokyo time) given the realized measures of the SSE B share index over the last hour of trading in the previous day.

As before, asymptotic equivalence between the feasible and unfeasible statistics allows us to interpret realized variance results as concerning the total quadratic variation, including the contributions of the integrated variance, jumps, and market microstructure noise. Tripower variation purges the influence of price jumps, whereas the realized kernel estimator measures the contributions of the jump and diffusive components to the quadratic variation. Note that we do not employ the two-scale realized variance in the hourly transmission study because of the limited number of intra-hour observations. The measurement error of the two-scale estimator converges at the slower rate $a_M = M^{1/3}$ rather than at $a_M = \sqrt{M}$ as in the realized kernel approach. Given that we compute the hourly realized measures using 1-minute returns (and hence $M = 60$), the magnitude of the measurement error of the two-scale realized variance could well compromise inference.

Panel A in Table 8 reveals significant evidence of volatility transmission from the FTSE 100 index to the S&P 500 index only after controlling for microstructure noise. This suggests that the market microstructure noise blurs the evidence of spillovers in the tests using the realized variance and tripower variation. In turn, we find significant spillovers from Japan to the US only if we employ the tripower variation. The transmission channel seems mainly through the integrated variance, with the jump component only obscuring the volatility spillovers. In addition, we find spillovers running from the SSE B share index to the S&P

500 index at the 10% level of significance for the realized variance and realized kernel measures. Given that the transmission at the daily level is via integrated variance, we interpret the failure to reject using the hourly tripower variation as evidence of jump-in-volatility spillovers (King and Wadhwani, 1990). Panel B uncovers notable spillovers from China and the US to the UK. In contrast, we cannot reject the absence of spillovers from Japan to the UK regardless of the realized measure we employ. Panel C makes plain once more the importance of accounting for the market microstructure imprint when testing for spillovers to the Topix 100 index. As before, this is somewhat consistent with volatility spillovers through jumps in the volatility. Finally, Panel D establishes that the quadratic variation transmission to the SSE B share index is significant at the usual levels for the realized variance and tripower variation estimates for every stock market.

7 Conclusion

This paper develops formal statistical tools for nonparametric tests of conditional independence between integrated variances. Under the assumption that asset prices follow a multivariate jump-diffusion processes with stochastic volatility, we show how to test whether the conditional distribution of asset A's integrated variance also depends on information concerning asset B's integrated variance. Our testing procedure involves two steps. In the first stage, we estimate the integrated variances using intraday returns data by means of realized measures so as to avoid misspecification risks. In the second step, we then test for conditional independence between the resulting realized measures. Although asymptotic critical values are not very reliable in finite samples, we show how to construct more accurate critical values by means of a simple bootstrap algorithm.

Our contribution to the literature on nonparametric density-based tests is twofold. First, our asymptotic theory specifically accounts for the impact of the estimation error in the first step of the testing procedure. Second, we also consider a more general setup in which the conditional distribution may depend on a state vector of any dimension. It turns out that such a generalization is not so straightforward as it seems at first glance. In particular, one must employ kernel-based methods rather than local-linear smoothing if the dimension of the conditioning set is large enough.

We also contribute to the literature on international market links by investigating volatility transmission between China, Japan, UK and US. Our empirical findings reveal that these stock markets display significant interconnection. The evidence is particularly stronger for the realized measures that are robust to jumps and/or market microstructure noise and hence it seems that the principal channel of transmission is through the integrated variance. The only exception is for spillovers from China to the US, which take place predominantly through price jumps. Finally, China and Japan effects on the US stock market volatility are more pronounced if one further controls for the quadratic variation in the UK. The FTSE 100 realized measure thus helps demarcate Asia-specific effects in the US in the presence of global shocks.

Appendix

A Bias and scaling terms

Let $C_1(K) \equiv \int K(u)^2 \, du$ and $C_2(K) \equiv \int \left(\int K(u)K(u+v)\, du\right)^2 \, dv$. Define $C_1(\boldsymbol{W})$, $C_2(\boldsymbol{W})$, $C_1(\widetilde{\boldsymbol{W}})$, and $C_2(\widetilde{\boldsymbol{W}})$ analogously. The bias and scaling terms that appear in (7) are given by

$$
\begin{aligned}
\widehat{\mu}_{1,T} &= C_1(K)\, C_1(\boldsymbol{W})\, \frac{1}{T} \sum_{t=1}^{T} \frac{\pi(Y_t, \boldsymbol{X}_t^{(q)})}{\bar{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t, \boldsymbol{X}_t^{(q)})} - b\, C_1(\boldsymbol{W})\, \frac{1}{T} \sum_{t=1}^{T} \frac{\frac{1}{T}\sum_{s=1}^{T} \boldsymbol{W}_{h_q}(\boldsymbol{X}_s^{(q)} - \boldsymbol{X}_t^{(q)})\, \pi(Y_s, \boldsymbol{X}_s^{(q)})}{\bar{f}_{\boldsymbol{X}^{(q)}}(\boldsymbol{X}_t^{(q)}) \frac{1}{T}\sum_{s=1}^{T} \boldsymbol{W}_{h_q}(\boldsymbol{X}_s^{(q)} - \boldsymbol{X}_t^{(q)})} \\
\widehat{\mu}_{2,T} &= C_1(K)\, C_1(\widetilde{\boldsymbol{W}})\, \frac{1}{T} \sum_{t=1}^{T} \frac{\frac{1}{T}\sum_{s=1}^{T} K_b(Y_s - Y_t) \widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_s^{(q_A)} - \boldsymbol{X}_t^{(q_A)})\, \pi(Y_s, \boldsymbol{X}_s^{(q)})}{\bar{f}_{Y,\boldsymbol{X}^{(q_A)}}(Y, \boldsymbol{X}_t^{(q_A)}) \frac{1}{T}\sum_{s=1}^{T} K_b(Y_s - Y_t) \widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_s^{(q_A)} - \boldsymbol{X}_t^{(q_A)})} \\
&\quad - b\, C_1(\widetilde{\boldsymbol{W}})\, \frac{1}{T} \sum_{t=1}^{T} \frac{\frac{1}{T}\sum_{s=1}^{T} \widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_s^{(q_A)} - \boldsymbol{X}_t^{(q_A)})\, \pi(Y_s, \boldsymbol{X}_s^{(q)})}{\bar{f}_{\boldsymbol{X}^{(q_A)}}(\boldsymbol{X}_t^{(q_A)}) \frac{1}{T}\sum_{s=1}^{T} \widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_s^{(q_A)} - \boldsymbol{X}_t^{(q_A)})} \\
\widehat{\mu}_{3,T} &= C_1(K)\, \widetilde{\boldsymbol{W}}(0)\, \frac{1}{T} \sum_{t=1}^{T} \frac{\frac{1}{T}\sum_{s=1}^{T} K_b(Y_s - Y_t) \widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_s^{(q_A)} - \boldsymbol{X}_t^{(q_A)})\, \pi(Y_s, \boldsymbol{X}_s^{(q)})}{\bar{f}_{Y,\boldsymbol{X}^{(q_A)}}(Y_t, \boldsymbol{X}_t^{(q_A)}) \frac{1}{T}\sum_{s=1}^{T} K_b(Y_s - Y_t) \widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_s^{(q_A)} - \boldsymbol{X}_t^{(q_A)})} \\
&\quad - b\, \widetilde{\boldsymbol{W}}(0)\, \frac{1}{T} \sum_{t=1}^{T} \frac{\frac{1}{T}\sum_{s=1}^{T} \widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_s^{(q_A)} - \boldsymbol{X}_t^{(q_A)})\, \pi(Y_s, \boldsymbol{X}_s^{(q)})}{\bar{f}_{\boldsymbol{X}^{(q_A)}}(\boldsymbol{X}_t^{(q_A)}) \frac{1}{T}\sum_{s=1}^{T} \widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_s^{(q_A)} - \boldsymbol{X}_t^{(q_A)})} \\
\widehat{\Omega}_T^2 &= 2\, C_2(K)\, C_2(\boldsymbol{W})\, \frac{1}{T} \sum_{t=1}^{T} \frac{\pi^2(Y_t, \boldsymbol{X}_t^{(q)})}{\bar{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t, \boldsymbol{X}_t^{(q)})}
\end{aligned}
$$

with $\boldsymbol{W}_{h_q}(\boldsymbol{u}) = h_q^{-q} \prod_{i=1}^{q} W(u_i/h_q)$ and $\widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{u}) = h_{q_A}^{-q_A} \prod_{i=1}^{q_A} W(u_i/h_{q_A})$.

To estimate the asymptotic bias and variance of the integrated squared relative difference statistic based on kernel smoothing, we employ similar bias and scaling terms in (8). The only difference is that we replace the second-order univariate kernel W with the s-order kernel function $\bar{W}$. For instance,

$$
\bar{\mu}_{1,T} = C_1(K)\, C_1(\bar{\boldsymbol{W}})\, \frac{1}{T} \sum_{t=1}^{T} \frac{\pi(Y_t, \boldsymbol{X}_t^{(q)})}{\bar{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t, \boldsymbol{X}_t^{(q)})} - b\, C_1(\bar{\boldsymbol{W}})\, \frac{1}{T} \sum_{t=1}^{T} \frac{\frac{1}{T}\sum_{s=1}^{T} \bar{\boldsymbol{W}}_{h_q}(\boldsymbol{X}_s^{(q)} - \boldsymbol{X}_t^{(q)})\, \pi(Y_s, \boldsymbol{X}_s^{(q)})}{\bar{f}_{\boldsymbol{X}^{(q)}}(\boldsymbol{X}_t^{(q)}) \frac{1}{T}\sum_{s=1}^{T} \bar{\boldsymbol{W}}_{h_q}(\boldsymbol{X}_s^{(q)} - \boldsymbol{X}_t^{(q)})}.
$$

Finally, for the bootstrap test statistics in (10) and (11), we obtain $(\widehat{\mu}_{1,\mathcal{T}}^*, \widehat{\mu}_{2,\mathcal{T}}^*, \widehat{\mu}_{3,\mathcal{T}}^*)$ and $(\bar{\mu}_{1,\mathcal{T}}^*, \bar{\mu}_{2,\mathcal{T}}^*, \bar{\mu}_{3,\mathcal{T}}^*)$ by replacing the sample quantities in $(\widehat{\mu}_{1,T}, \widehat{\mu}_{2,T}, \widehat{\mu}_{3,T})$ and in $(\bar{\mu}_{1,T}, \bar{\mu}_{2,T}, \bar{\mu}_{3,T})$ with their bootstrap counterparts, that is, we substitute $(Y_t^*, X_t^{*(q)}, \mathcal{T}, b_*, h_{*q}, h_{*q_A})$ for $(Y_t, X_t^{(q)}, T, b, h_q, h_{q_A})$. For instance,

$$
\widehat{\mu}_{1,\mathcal{T}}^* = C_1(K)\, C_1(\boldsymbol{W})\, \frac{1}{\mathcal{T}} \sum_{t=1}^{\mathcal{T}} \frac{\pi(Y_t^*, \boldsymbol{X}_t^{*(q)})}{\bar{f}^*_{Y,\boldsymbol{X}^{(q)}}(Y_t^*, \boldsymbol{X}_t^{*(q)})} - b_*\, C_1(\boldsymbol{W})\, \frac{1}{\mathcal{T}} \sum_{t=1}^{\mathcal{T}} \frac{\frac{1}{\mathcal{T}}\sum_{s=1}^{\mathcal{T}} \boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_s^{*(q)} - \boldsymbol{X}_t^{*(q)}) \pi(Y_s^*, \boldsymbol{X}_s^{*(q)})}{\bar{f}^*_{\boldsymbol{X}^{(q)}}(\boldsymbol{X}_t^{*(q)}) \frac{1}{\mathcal{T}}\sum_{s=1}^{\mathcal{T}} \boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_s^{*(q)} - \boldsymbol{X}_t^{*(q)})}.
$$

B Proofs

B.1 Lemmata

The proof of Theorem 1 relies heavily on Lemmata 1 to 3, whereas we employ the results in Lemmata 4 to 6 in the proof of Theorem 2.

Lemma 1: Assume that there are at most three conditioning variables in the higher dimensional density ($q \leq 3$) and that the bandwidths satisfy the following conditions: (i) $T(\ln T)^{-1/2} h_{q_A}^{q_A/2}\, h_q^{q/2}\, b \to \infty$, (ii) $T h_q^{q/2}\, b^{2s+1/2} \to 0$, (iii) $T h_q^{4+q/2}\, b^{1/2} \to 0$, (iv) $h_q^{-q}\, b^{2s-1} \to 0$, (v) $h_q^{4-q}\, b^{-1} \to 0$, (vi) $T h_q^{3q/2}\, b^{3/2} \to \infty$. It then follows from Assumptions A1 to A4 that, under the null $\mathbb{H}_0$,

$$
\Omega^{-1} \left\{ h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}_t^{(q)}) \left[\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) - f_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} \right]^2 - h_q^{-q/2}\, b^{-1/2}\, \mu_1 \right\} \xrightarrow{d} N(0,1)
$$

where $\Omega^2 \equiv 2\, C_2(K)\, C_2(\boldsymbol{W}) \int \pi^2(y, \boldsymbol{x})\, dy\, d\boldsymbol{x}$ and

$$
\mu_1 = C_1(K)\, C_1(\boldsymbol{W}) \int \pi(y, \boldsymbol{x}^{(q)})\; dy\; d\boldsymbol{x}^{(q)} - b\, C_1(\boldsymbol{W}) \int \mathbb{E}\Big[\pi(Y, \boldsymbol{X}^{(q)}) \big| \boldsymbol{X}^{(q)} = \boldsymbol{x}^{(q)}\Big]\, d\boldsymbol{x}^{(q)}.
$$

Lemma 2: Assume that there are at most three conditioning variables in the higher dimensional density ($q \leq 3$) and that the bandwidths are such that: (i) $T(\ln T)^{-1/2} h_{q_A}^{2q_A}\, h_q^{-q}\, b \to \infty$, (ii) $T h_q^{q/2}\, b^{2s+1/2} \to 0$, (iii) $T h_{q_A}^{4}\, h_q^{q/2}\, b^{1/2} \to 0$, (iv) $h_{q_A}^{-2q_A}\, h_q^{q}\, b^{2s-1} \to 0$, (v) $h_{q_A}^{4-2q_A}\, h_q^{q}\, b^{-1} \to 0$, and (vi) $T h_{q_A}^{5q_A/2}\, h_q^{-q}\, b^{3/2} \to \infty$. Assumptions A1 to A4 then ensures that

$$\Omega^{-1}\left\{ h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}_t^{(q)}) \left[\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)}) - f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} \right]^2 - h_q^{q/2}\, h_{q_A}^{-q_A}\, b^{-1/2}\, \mu_2 \right\} = o_p(1)$$

where

$$\mu_2 = C_1(K)\, C_1(\widetilde{\boldsymbol{W}}) \int \mathbb{E}\Big[\pi(Y, \boldsymbol{X}^{(q)}) \big| Y = y, \boldsymbol{X}^{(q_A)} = \boldsymbol{x}^{(q_A)}\Big]\, \mathrm{d}y\, \mathrm{d}\boldsymbol{x}^{(q_A)} \\ - b\, C_1(\widetilde{\boldsymbol{W}}) \int \mathbb{E}\Big[\pi(Y, \boldsymbol{X}^{(q)}) \big| \boldsymbol{X}^{(q_A)} = \boldsymbol{x}^{(q_A)}\Big]\, \mathrm{d}\boldsymbol{x}^{(q_A)}.$$

Lemma 3: Let the bandwidth conditions (i) to (vi) in Lemmata 1 and 2 hold. Assumptions A1 to A4 ensure that, under the null $\mathbb{H}_0$,

$$\Omega^{-1}\left[h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}_t^{(q)})\, \frac{\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})\, \widehat{\epsilon}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} - h_q^{q/2-q_A}\, b^{-1/2}\, \mu_3 \right] = o_p(1),$$

where $\widehat{\epsilon}_{Y|\boldsymbol{X}^{(\cdot)}}(Y_t|\boldsymbol{X}_t^{(\cdot)}) \equiv \widehat{f}_{Y|\boldsymbol{X}^{(\cdot)}}(Y_t|\boldsymbol{X}_t^{(\cdot)}) - f_{Y|\boldsymbol{X}^{(\cdot)}}(Y_t|\boldsymbol{X}_t^{(\cdot)})$ and

$$\mu_3 = C_1(K)\, \widetilde{\boldsymbol{W}}(0) \int \mathbb{E}\Big[\pi(Y, \boldsymbol{X}^{(q)}) \big| Y = y, \boldsymbol{X}^{(q_A)} = \boldsymbol{x}^{(q_A)}\Big]\, \mathrm{d}y\, \mathrm{d}\boldsymbol{x}^{(q_A)} \\ - b\, \widetilde{\boldsymbol{W}}(0) \int \mathbb{E}\Big[\pi(Y, \boldsymbol{X}^{(q)}) \big| \boldsymbol{X}^{(q_A)} = \boldsymbol{x}^{(q_A)}\Big]\, \mathrm{d}\boldsymbol{x}^{(q_A)}.$$

Lemma 4: Let Assumptions A2 to A5 hold as well as the following bandwidths conditions: (i) $T(\ln T)^{-1/2} h_{q_A}^{q_A/2}\, h_q^{q/2}\, b \to \infty$, (ii) $T h_q^{q/2}\, b^{2s+1/2} \to 0$, (iii) $T h_q^{2s+q/2}\, b^{1/2} \to 0$, (iv) $h_q^{-q}\, b^{2s-1} \to 0$, (v) $h_q^{2s-q}\, b^{-1} \to 0$, (vi) $T h_q^{3q/2}\, b^{3/2} \to \infty$. It then follows that, under the null $\mathbb{H}_0$,

$$\bar{\Omega}^{-1}\left\{ h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}_t^{(q)}) \left[\frac{\bar{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) - f_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})}{\bar{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} \right]^2 - h_q^{-q/2}\, b^{-1/2}\, \mu_1 \right\} \xrightarrow{d} N(0, 1),$$

where $\bar{\Omega}^2 \equiv 2\, C_2(K)\, C_2(\bar{\boldsymbol{W}}) \int \pi^2(y, \boldsymbol{x})\, \mathrm{d}y\, \mathrm{d}\boldsymbol{x}$.

Lemma 5: Let Assumptions A2 to A5 hold as well as the following bandwidths conditions: (i) $T(\ln T)^{-1/2} h_{q_A}^{2q_A}\, h_q^{-q}\, b \to \infty$, (ii) $T h_q^{q/2}\, b^{2s+1/2} \to 0$, (iii) $T h_{q_A}^{2s}\, h_q^{q/2}\, b^{1/2} \to 0$, (iv) $h_{q_A}^{-2q_A}\, h_q^{q}\, b^{2s-1} \to 0$, (v) $h_{q_A}^{2s-2q_A}\, h_q^{q}\, b^{-1} \to 0$, and (vi) $T h_{q_A}^{5q_A/2}\, h_q^{-q}\, b^{3/2} \to \infty$. It then follows that

$$\bar{\Omega}^{-1}\left\{ h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}_t^{(q)}) \left[\frac{\bar{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)}) - f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)}}{\bar{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})}) \right]^2 - h_q^{q/2}\, h_{q_A}^{-q_A} b^{-1/2} \mu_2 \right\} = o_p(1).$$

Lemma 6: Let the bandwidth conditions (i) to (vi) in Lemmata 4 and 5 hold. It then follows from Assumptions A2 to A5 that, under the null $\mathbb{H}_0$,

$$\bar{\Omega}^{-1}\left[h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}_t^{(q)})\, \frac{\bar{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})\, \bar{\epsilon}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})}{\bar{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} - h_q^{q/2-q_A}\, b^{-1/2} \mu_3 \right] = o_p(1),$$

where $\bar{\epsilon}_{Y|\boldsymbol{X}^{(\cdot)}}(Y_t|\boldsymbol{X}_t^{(\cdot)}) \equiv \bar{f}_{Y|\boldsymbol{X}^{(\cdot)}}(Y_t|\boldsymbol{X}_t^{(\cdot)}) - f_{Y|\boldsymbol{X}^{(\cdot)}}(Y_t|\boldsymbol{X}_t^{(\cdot)})$.

B.2 Proof of Theorem 1

(i) We first observe that

$$\widehat{\Omega}_T^2 - \Omega^2 = 2\,C_2(K)\,C_2(\boldsymbol{W})\,\left[\frac{1}{T}\sum_{t=1}^{T}\frac{\pi^2(Y_t,\boldsymbol{X}_t^{(q)})}{f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})} - \int \pi^2(y,\boldsymbol{x}^{(q)})\,\mathrm{d}y\,\mathrm{d}\boldsymbol{x}^{(q)} \right.$$
$$\left. + \frac{1}{T}\sum_{t=1}^{T}\frac{\pi^2(Y_t,\boldsymbol{X}_t^{(q)})}{f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})\widehat{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})}\left(\widehat{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)}) - f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})\right)\right].$$

is of order $o_p(1)$ and hence we treat them as asymptotically equivalent in what follows. Under the null that $f_{Y|X^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) = f_{Y|X^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})$ almost surely, it follows that, up to a term of order $o_p(1)$,

$$\begin{aligned}
\Lambda_T =\ & \Omega^{-1}\left[h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})}{f^2_{Y|\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})}\,\widehat{\epsilon}^{\,2}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) - h_q^{-q/2}\,b^{-1/2}\,\mu_1\right]\\
& + \Omega^{-1}\left[h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})}{f^2_{Y|\boldsymbol{X}^{(q_A)}}(Y_t,\boldsymbol{X}_t^{(q_A)})}\,\widehat{\epsilon}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)}) - h_q^{q/2}\,h_{q_A}^{-q_A}\,b^{-1/2}\,\mu_2\right]\\
& - 2\,\Omega^{-1}\left[h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})}{f^2_{Y|\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})}\,\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})\,\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)}) - h_q^{q/2-q_A}\,b^{-1/2}\,\mu_3\right]\\
& + \Omega^{-1}\,h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\pi(Y_t,\boldsymbol{X}_t^{(q)})\left[\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) - \widehat{\epsilon}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})\right]^2\\
& \qquad\qquad \times\left[\frac{1}{\widehat{f}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} - \frac{1}{f^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})}\right]\\
& - \Omega^{-1}\left[h_q^{-q/2}\,b^{-1/2}\left(\widehat{\mu}_{1,T}-\mu_1\right) - h_q^{q/2}\,h_{q_A}^{-q_A}\,b^{-1/2}\left(\widehat{\mu}_{2,T}-\mu_2\right) + 2\,h_q^{q/2-q_A}\,b^{-1/2}\left(\widehat{\mu}_{3,T}-\mu_3\right)\right]\\
=\ & \Lambda_{1,T}^{(0)} + \Lambda_{2,T}^{(0)} + \Lambda_{3,T}^{(0)},
\end{aligned} \tag{12}$$

where $\Lambda_{1,T}^{(0)}$ is the sum of the first three terms on the right-hand side of (12). Lemmata 1 to 3 yield the asymptotic normality of $\Lambda_{1,T}^{(0)}$ under the null and ensure that $\Lambda_{2,T}^{(0)} = o_p(1)$.

It thus remains to show that $\Lambda_{3,T}^{(0)}$ is also of order $o_p(1)$. We start with

$$\begin{aligned}
h_q^{-q/2}\,b^{-1/2}\left(\widehat{\mu}_{1,T}-\mu_1\right) =\ & C_1(K)\,C_1(\boldsymbol{W})\,h_q^{-q/2}\,b^{-1/2}\,\frac{1}{T}\sum_{t=1}^{T}\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})\left(\widehat{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)}) - f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})\right)}{\widehat{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})}\\
& + h_q^{-q/2}\,b^{1/2}\,C_1(\boldsymbol{W})\,\frac{1}{T}\sum_{t=1}^{T}\frac{1}{f_{\boldsymbol{X}^{(q)}}(\boldsymbol{X}_t^{(q)})}\left\{\mathbb{E}\Big[\pi(Y_s,\boldsymbol{X}_s^{(q)})|\boldsymbol{X}_s^{(q)} = \boldsymbol{X}_t^{(q)}\Big]\right.\\
& \qquad\qquad \left. - \frac{\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{W}_{h_q}(\boldsymbol{X}_s^{(q)}-\boldsymbol{X}_t^{(q)})\,\pi(Y_s,\boldsymbol{X}_s^{(q)})}{\frac{1}{T}\sum_{s=1}^{T}\boldsymbol{W}_{h_q}(\boldsymbol{X}_s^{(q)}-\boldsymbol{X}_t^{(q)})}\right\}\\
& + h_q^{-q/2}\,b^{1/2}\,C_1(\boldsymbol{W})\,\frac{1}{T}\sum_{t=1}^{T}\frac{\mathbb{E}\Big[\pi(Y_s,\boldsymbol{X}_s^{(q)})|\boldsymbol{X}_s^{(q)} = \boldsymbol{X}_t^{(q)}\Big]\left(\widehat{f}_{\boldsymbol{X}^{(q)}}(\boldsymbol{X}_t^{(q)}) - f_{\boldsymbol{X}^{(q)}}(\boldsymbol{X}_t^{(q)})\right)}{f_{\boldsymbol{X}^{(q)}}(\boldsymbol{X}_t^{(q)})\widehat{f}_{\boldsymbol{X}^{(q)}}(\boldsymbol{X}_t^{(q)})}\\
=\ & o_p(1).
\end{aligned}$$

The last equality follows from the fact that the second and third terms are of smaller order than the first term, whereas the quantity $\inf_{\mathcal{C}(Y,\boldsymbol{X}^{(q)})} f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})$ is bounded away from zero in a compact set $\mathcal{C}(Y,\boldsymbol{X}^{(q)}) \subset \mathbb{R}^{q+1}$ and the degenerate U-statistic

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})\left(\widehat{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)}) - f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})\right)}{\widehat{f}_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})} = o_p(h_q^{q/2}\,b^{1/2}).$$

In addition, it follows along the same lines that $h_q^{q/2}\,h_{q_A}^{-q_A}\,b^{-1/2}\left(\widehat{\mu}_{2,T}-\mu_2\right)$ and $h_q^{q/2-q_A}\,b^{-1/2}\left(\widehat{\mu}_{3,T}-\mu_3\right)$ are also of order $o_p(1)$.

(ii) Consider the following expansion under the alternative hypothesis $\mathbb{H}_A$

$$\begin{aligned}
\Lambda_T &= \Omega^{-1}\left[h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \frac{\pi(Y_t, \boldsymbol{X}_t^{(q)})}{\widehat{f}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t, \boldsymbol{X}_t^{(q_A)})}\, \widehat{\epsilon}^{\,2}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) - h_q^{-q/2}\, b^{-1/2}\, \widehat{\mu}_{1,T}\right] \\
&\quad + \Omega^{-1}\left[h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \frac{\pi(Y_t, \boldsymbol{X}_t^{(q)})}{\widehat{f}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t, \boldsymbol{X}_t^{(q_A)})}\, \widehat{\epsilon}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)}) - h_q^{q/2}\, h_{q_A}^{-q_A}\, b^{-1/2}\, \widehat{\mu}_{2,T}\right] \\
&\quad - 2\,\Omega^{-1}\left[h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \frac{\pi(Y_t, \boldsymbol{X}_t^{(q)})}{\widehat{f}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t, \boldsymbol{X}_t^{(q_A)})}\, \widehat{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})\, \widehat{\epsilon}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)}) - h_q^{q/2-q_A}\, b^{-1/2}\, \widehat{\mu}_{3,T}\right] \\
&\quad + \Omega^{-1}\, h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \frac{\pi(Y_t, \boldsymbol{X}_t^{(q)})}{\widehat{f}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t, \boldsymbol{X}_t^{(q_A)})} \left[f_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)}) - f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})\right]^2 \\
&= \Lambda_{1,T}^{(1)} + \Lambda_{2,T}^{(1)} + \Lambda_{3,T}^{(1)} + \Lambda_{4,T}^{(1)}.
\end{aligned}$$

The asymptotic behavior of $\Lambda_{i,T}^{(1)}$ for $i \in \{1,2,3\}$ is the same under both hypotheses. However, under the alternative, $f_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})$ differs from $f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})$ and thus $\Lambda_{4,T}^{(1)}$ is of order $O_p\left(T\, h_q^{q/2}\, b^{1/2}\right)$, which ensures unit asymptotic power. ■

B.3 Proof of Theorem 2

The result ensues from Lemmata 4 to 6 along the same lines as in the proof of Theorem 1. ■

B.4 Proof of Theorem 3

(i) The local linear estimator based on realized measures (rather than on integrated variances) reads

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}_T^{(M)}(y, \boldsymbol{x}^{(q)}) &= \widehat{\boldsymbol{\beta}}_T(y, \boldsymbol{x}^{(q)}) + \left(\mathcal{H}'_{\boldsymbol{x}^{(q)}} \mathcal{W}_{\boldsymbol{x}^{(q)}} \mathcal{H}_{\boldsymbol{x}^{(q)}}\right)^{-1} \left(\mathcal{H}'_{\boldsymbol{x}_M^{(q)}} \mathcal{W}_{\boldsymbol{x}_M^{(q)}} \mathcal{Y}_{y_M} - \mathcal{H}'_{\boldsymbol{x}^{(q)}} \mathcal{W}_{\boldsymbol{x}^{(q)}} \mathcal{Y}_y\right) \\
&\quad + \left[\left(\frac{1}{T}\, \mathcal{H}'_{\boldsymbol{x}_M^{(q)}} \mathcal{W}_{\boldsymbol{x}_M^{(q)}} \mathcal{H}_{\boldsymbol{x}_M^{(q)}}\right)^{-1} - \left(\frac{1}{T} \mathcal{H}'_{\boldsymbol{x}_M^{(q)}} \mathcal{W}_{\boldsymbol{x}^{(q)}} \mathcal{H}_{\boldsymbol{x}^{(q)}}\right)^{-1}\right] \frac{1}{T}\, \mathcal{H}'_{\boldsymbol{x}^{(q)}} \mathcal{W}_{\boldsymbol{x}^{(q)}} \mathcal{Y}_y \\
&\quad + \left[\left(\frac{1}{T} \mathcal{H}'_{\boldsymbol{x}_M^{(q)}} \mathcal{W}_{\boldsymbol{x}_M^{(q)}} \mathcal{H}_{\boldsymbol{x}_M^{(q)}}\right)^{-1} - \left(\frac{1}{T}\, \mathcal{H}'_{\boldsymbol{x}^{(q)}} \mathcal{W}_{\boldsymbol{x}^{(q)}} \mathcal{H}_{\boldsymbol{x}^{(q)}}\right)^{-1}\right] \frac{1}{T} \left(\mathcal{H}_{\boldsymbol{x}_M^{(q)}} \mathcal{W}_{\boldsymbol{x}_M^{(q)}} \mathcal{Y}_{y_M} - \mathcal{H}'_{\boldsymbol{x}^{(q)}} \mathcal{W}_{\boldsymbol{x}^{(q)}} \mathcal{Y}_y\right),
\end{aligned}$$

where the index M denotes reliance on realized measures and $\frac{1}{T}\left(\mathcal{H}'_{\boldsymbol{x}_M^{(q)}} \mathcal{W}_{\boldsymbol{x}_M^{(q)}} \mathcal{Y}_{y_M} - \mathcal{H}'_{\boldsymbol{x}^{(q)}} \mathcal{W}_{\boldsymbol{x}^{(q)}} \mathcal{Y}_y\right)$ is a column vector given by

$$\begin{pmatrix}
\frac{1}{T}\sum_{t=1}^{T}\left[\prod_{j=1}^{q} W_{h_q}(X_{jt,M} - x_j) K_b(Y_{t,M} - y) - \prod_{j=1}^{q} W_{h_q}(X_{jt} - x_j) K_b(Y_t - y)\right] \\
\frac{1}{T}\sum_{t=1}^{T}\left[\prod_{j=1}^{q} W_{h_q}(X_{jt,M} - x_j) K_b(Y_{t,M} - y)(X_{1t,M} - x_1) - \prod_{j=1}^{q} W_{h_q}(X_{jt} - x_j) K_b(Y_t - y)(X_{1t} - x_1)\right] \\
\vdots \\
\frac{1}{T}\sum_{t=1}^{T}\left[\prod_{j=1}^{q} W_{h_q}(X_{jt,M} - x_j) K_b(Y_{t,M} - y)(X_{qt,M} - x_q) - \prod_{j=1}^{q} W_{h_q}(X_{jt} - x_j) K_b(Y_t - y)(X_{qt} - x_q)\right]
\end{pmatrix} \tag{13}$$

We start by bounding the first term on the right-hand side of (13), namely,

$$\begin{aligned}
&\sup_{\mathcal{C}(Y, \boldsymbol{X}^{(q)})} \left|\frac{1}{T}\sum_{t=1}^{T}\left(\prod_{j=1}^{q} W_{h_q}(X_{jt,M} - x_j) K_b(Y_{t,M} - y) - \prod_{j=1}^{q} W_{h_q}(X_{jt} - x_j) K_b(Y_t - y)\right)\right| \\
&\quad \leq \sup_{\mathcal{C}(Y, \boldsymbol{X}^{(q)})} \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{q}\left[\frac{\partial}{\partial \widetilde{X}_{jt,M}} \prod_{j=1}^{q} W_{h_q}(\widetilde{X}_{jt,M} - x_j)\right] K_b(\widetilde{Y}_{t,M} - y)\, N_{i,t,M}\right| \\
&\qquad + \sup_{\mathcal{C}(Y, \boldsymbol{X}^{(q)})} \frac{1}{T}\sum_{t=1}^{T}\left|\prod_{j=1}^{q} W_{h_q}(\widetilde{X}_{jt,M} - x_j)\left[\frac{\partial}{\partial \widetilde{Y}_{t,M}}\, K_b(\widetilde{Y}_{t,M} - y)\right] N_{0,t,M}\right| \\
&\qquad + \sup_{\mathcal{C}(Y, \boldsymbol{X}^{(q)})} \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{q}\prod_{j=1}^{q}\left[\frac{\partial}{\partial \widetilde{X}_{jt,M}} \prod_{j=1}^{q} W_{h_q}(\widetilde{X}_{jt,M} - x_j)\right]\left[\frac{\partial}{\partial \widetilde{Y}_{t,M}}\, K_b(\widetilde{Y}_{t,M} - y)\right] N_{0,t,M}\, N_{i,t,M}\right|,
\end{aligned} \tag{14}$$

where $\widetilde{X}_{jt,M} \in (X_{jt,M}, X_{jt})$ and $\widetilde{Y}_{t,M} \in (Y_{t,M}, Y_t)$. As for the first term on the right-hand side of (14), it turns out that

$$
\begin{aligned}
\sup_{\mathcal{C}(Y,\boldsymbol{X}^{(q)})} & \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{q}\left[\frac{\partial}{\partial \widetilde{X}_{jt,M}}\prod_{j=1}^{q} W_{h_q}(\widetilde{X}_{jt,M}-x_j)\right] K_b(\widetilde{Y}_{t,M}-y)\, N_{i,t,M}\right| \\
&\leq q \sup\left|N_{i,t,M}\right| \sup_{\mathcal{C}(Y,\boldsymbol{X}^{(q)})} \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{q}\left[\frac{\partial}{\partial \widetilde{X}_{jt,M}}\prod_{j=1}^{q} W_{h_q}(\widetilde{X}_{jt,M}-x_j)\right] K_b(\widetilde{Y}_{t,M}-y)\, N_{i,t,M}\right| \\
&= O_p(h_q^{-1}) \sup\left|N_{i,t,M}\right|
\end{aligned}
$$

and analogously the second term on the right-hand side of (13) is of order $O_p(b^{-1}) \sup\left|N_{i,t,M}\right|$. In view of Assumption A6 and Corradi et al.'s (2011) Lemma 1,

$$
\begin{aligned}
\Pr\left(\sup_{1\leq t\leq T} T^{-\frac{1}{k-1}}\, a_M^{1/2}\, \left|N_{i,t,M}\right| > \varepsilon\right) &\leq \sum_{t=1}^{T}\Pr\left(T^{-\frac{1}{k-1}}\, a_M^{1/2}\, \left|N_{i,t,M}\right| > \varepsilon\right) \;\leq\; \varepsilon^{-k}\, T^{1-\frac{k}{k-1}}\, a_M^{k/2}\, \mathbb{E}\left|N_{i,t,M}\right|^k \\
&\leq \varepsilon^{-k}\, T^{-\frac{1}{k-1}}\, a_M^{k/2}\, O(a_M^{-k/2}) \to 0, \qquad \text{as } M, T\to\infty
\end{aligned}
$$

meaning that $\sup_{1\leq t\leq T}\left|N_{i,t,M}\right| = O_p\left(T^{\frac{1}{k-1}}\, a_M^{-1/2}\right)$. It is straightforward to show that the third term on the right-hand side of (14) is of smaller order that the first and third terms. It then follows that

$$
\sup_{\mathcal{C}(Y,\boldsymbol{X}^{(q)})}\left|\widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q)}}(y|\boldsymbol{x}^{(q)}) - \widehat{f}_{Y|\boldsymbol{X}^{(q)}}(y|\boldsymbol{x}^{(q)})\right| = O_p\left(T^{\frac{1}{k-1}}\, a_M^{-1/2}(h_q^{-1}+b^{-1})\right) \tag{15}
$$

and, analogously,

$$
\sup_{\mathcal{C}(Y,\boldsymbol{X}^{(q_A)})}\left|\widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q_A)}}(y|\boldsymbol{x}^{(q_A)}) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(y|\boldsymbol{x}^{(q_A)})\right| = O_p\left(T^{\frac{1}{k-1}}\, a_M^{-1/2}(h_{q_A}^{-1}+b^{-1})\right). \tag{16}
$$

It is now immediate to see that, given bandwidth condition (vi),

$$
\begin{aligned}
\Omega\left(\widehat{\Lambda}_T^{(M)} - \widehat{\Lambda}_T\right) &= h_q^{q/2}\, b^{1/2}\sum_{t=1}^{T}\left[\left(\frac{\widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q)}}(Y_{t,M}|\boldsymbol{X}^{(q)}_{t,M}) - \widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q_A)}}(Y_{t,M}|\boldsymbol{X}^{(q_A)}_{t,M})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)}\right)^2 \pi(Y_{t,M}, \boldsymbol{X}^{(q)}_{t,M})\right. \\
&\qquad \left. - \left(\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}^{(q)}_t) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)}\right)^2 \pi(Y_t, \boldsymbol{X}^{(q)}_t)\right] \\
&\quad + h_q^{q/2}\, b^{1/2}\sum_{t=1}^{T}\left(\widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q)}}(Y_{t,M}|\boldsymbol{X}^{(q)}_{t,M}) - \widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q_A)}}(Y_{t,M}|\boldsymbol{X}^{(q_A)}_{t,M})\right)^2 \pi(Y_{t,M}, \boldsymbol{X}^{(q)}_{t,M}) \\
&\qquad \times\left[\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t) - \widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q_A)}}(Y_{t,M}|\boldsymbol{X}^{(q_A)}_{t,M})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)\widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q_A)}}(Y_{t,M}|\boldsymbol{X}^{(q_A)}_{t,M})}\right] \\
&\quad + O_p\left(h_q^{-q/2}\, b^{-1/2}\, T^{\frac{1}{k-1}}\, a_M^{-1/2}(h_q^{-1}+h_{q_A}^{-1}+b^{-1})\right) \\
&= A_{T,M} + B_{T,M} + O_p\left(h_q^{-q/2}\, b^{-1/2}\, T^{\frac{1}{k-1}}\, a_M^{-1/2}(h_q^{-1}+h_{q_A}^{-1}+b^{-1})\right) \\
&= A_{T,M} + B_{T,M} + o_p\left(T^{\frac{k+1}{2(k-1)}}\, a_M^{-1/2}(h_q^{-1}+h_{q_A}^{-1}+b^{-1})\right),
\end{aligned} \tag{17}
$$

where the last term captures the contribution of the bias terms, namely, $(\widehat{\mu}_{i,T,M} - \widehat{\mu}_{i,T}) = O_p\left(T^{\frac{1}{k-1}}\, a_M^{-1/2}(h_q^{-1}+h_{q_A}^{-1}+b^{-1})\right)$ for $i\in\{1,2,3\}$. Now,

$$
\left|\pi(Y_{t,M}, \boldsymbol{X}^{(q)}_{t,M}) - \pi(Y_t, \boldsymbol{X}^{(q)}_t)\right| \leq \left(\sup_{\mathcal{C}(Y,\boldsymbol{X}^{(q)})}\sum_{i=0}^{q}\partial_i \pi_{t,M}\right)\left(\sup_{i,t}\left|N_{i,t,M}\right|\right) = O_p\left(T^{\frac{1}{k-1}}\, a_M^{-1/2}\right),
$$

and so letting $\pi_{t,M} \equiv \pi(Y_{t,M}, \boldsymbol{X}^{(q)}_{t,M})$ and $\pi(Y_t, \boldsymbol{X}_t) \equiv \pi(Y_t, \boldsymbol{X}^{(q)}_t)$ yields

$$
h_q^{q/2}\, b^{1/2}\sum_{t=1}^{T}\left(\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}^{(q)}_t) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)}\right)^2 (\pi_{t,M} - \pi(Y_t, \boldsymbol{X}_t)) = O_p\left(h_q^{-q/2}\, b^{-1/2}\, T^{\frac{1}{k-1}}\, a_M^{-1/2}\right), \tag{18}
$$

which is of order $o_p(1)$. It also follows from (15) and (16) that

$$h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left(\frac{\widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q)}}(Y_{t,M}|\boldsymbol{X}^{(q)}_{t,M}) - \widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_{t,M}|\boldsymbol{X}^{(q)}_{t,M})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)} \right)^2 \pi_{t,M} = O_p\left(h_q^{q/2}\, b^{1/2}\, T^{\frac{k+1}{k-1}}\, a_M^{-1} (h_q^{-2} + b^{-2}) \right) \tag{19}$$

$$h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left(\frac{\widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(q_A)}}(Y_{t,M}|\boldsymbol{X}^{(q_A)}_{t,M}) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_{t,M}|\boldsymbol{X}^{(q_A)}_{t,M})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)} \right)^2 \pi_{t,M} = O_p\left(h_q^{q/2}\, b^{1/2}\, T^{\frac{k+1}{k-1}}\, a_M^{-1} (h_{q_A}^{-2} + b^{-2}) \right), \tag{20}$$

$$\tag{21}$$

whereas

$$\begin{aligned} &h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left(\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_{t,M}|\boldsymbol{X}^{(q)}_{t,M}) - \widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}^{(q)}_t)}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)} \right)^2 \pi_{t,M} \\ &\quad \le \left(\sup_{i,t} N^2_{i,t,M} \right) h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left(\frac{\sum_{i=0}^{q} \partial_i \widehat{\boldsymbol{\beta}}_{0T}(\widetilde{Y}_{t,M}|\widetilde{\boldsymbol{X}}^{(q)}_{t,M})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)} \right)^2 \pi_{t,M} = O_p\left(h_q^{q/2}\, b^{1/2}\, T^{\frac{k+1}{k-1}}\, a_M^{-1} (h_q^{-2} + b^{-2}) \right) \end{aligned} \tag{22}$$

$$\begin{aligned} &h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left(\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_{t,M}|\boldsymbol{X}^{(q_A)}_{t,M}) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)} \right)^2 \pi_{t,M} \\ &\quad \le \left(\sup_{i,t} N^2_{i,t,M} \right) h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left(\frac{\sum_{i=0}^{q} \partial_i \widehat{\boldsymbol{\beta}}_{0T}(\widetilde{Y}_{t,M}|\widetilde{\boldsymbol{X}}^{(q_A)}_{t,M})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)} \right)^2 \pi_{t,M} = O_p\left(h_q^{q/2}\, b^{1/2}\, T^{\frac{k+1}{k-1}}\, a_M^{-1} (h_{q_A}^{-2} + b^{-2}) \right), \end{aligned} \tag{23}$$

and hence that

$$\begin{aligned} &h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \left(\frac{\widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}^{(q)}_t) - \widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)} \right) \left(\frac{\widehat{f}^{(M)}_{Y|\boldsymbol{X}^{(d)}}(Y_{t,M}|\boldsymbol{X}^{(d)}_{t,M}) - \widehat{f}_{Y|\boldsymbol{X}^{(q)}}(Y_{t,M}|\boldsymbol{X}^{(d)}_{t,M})}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)} \right) \\ &\quad = O_p\left(T^{\frac{k+1}{2(k-1)}}\, \sqrt{\ln T}\, a_M^{-1/2} (h_q^{-1} + h_{q_A}^{-1} + b^{-1}) \right). \end{aligned} \tag{24}$$

Altogether, (18) to (24) imply that $A_{T,M} = O_p\left(T^{\frac{k+1}{2(k-1)}}\, \sqrt{\ln T}\, a_M^{-1/2} (h_q^{-1} + h_{q_A}^{-1} + b^{-1}) \right)$. Finally, given that $B_{T,M}$ is of smaller probability order than $A_{T,M}$, it suffices to follow the same steps as in the proof of Theorem 1(i) to complete the proof of statement (i).

(ii) Under the alternative $\mathbb{H}_A$, $f_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}^{(q)}_t)$ and $f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}^{(q_A)}_t)$ differ in a subset of nonzero Lebesgue measure. This implies that the terms in (18) and (24) become of order $O_p\left(h_q^{q/2}\, b^{1/2}\, T^{\frac{k}{k-1}}\, a_M^{-1/2} (h_q^{-1} + h_{q_A}^{-1} + b^{-1}) \right)$ under the alternative, though there is no change in the probability orders of (19) to (23). Altogether, this shows that $\widehat{\Lambda}^{(M)}_T - \widehat{\Lambda}_T = O_p\left(h_q^{q/2}\, b^{1/2}\, T^{\frac{k}{k-1}}\, a_M^{-1/2} (h_q^{-1} + h_{q_A}^{-1} + b^{-1}) \right)$ under $\mathbb{H}_A$. This completes the proof due to the fact that $a_M^{-1/2}(h_q^{-1} + h_{q_A}^{-1} + b^{-1}) = o_p(1)$. ■

B.5 Proof of Theorem 4

The result ensues along the same lines as in the proof of Theorem 3.

B.6 Proof of Lemma 1

Under the null $\mathbb{H}_0$, $f_{Y|\boldsymbol{X}^{(q)}}(y|\boldsymbol{x}^{(q)})$ coincides almost surely with $f_{Y|\boldsymbol{X}^{(q_A)}}(y|\boldsymbol{x}^{(q_A)})$ and hence

$$\begin{aligned} \widehat{\Lambda}_{1,T} &\equiv \Omega^{-1} \left\{ h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}^{(q)}_t) \left[\frac{\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t \mid \boldsymbol{X}^{(q)}_t)}{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \mid \boldsymbol{X}^{(q_A)}_t)} \right]^2 - h_q^{-q/2}\, b^{-1/2}\, \mu_1 \right\} \\ &= \Omega^{-1} \left\{ h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}^{(q)}_t) \left[\frac{\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t \mid \boldsymbol{X}^{(q)}_t)}{f_{Y|\boldsymbol{X}^{(q)}}(Y_t \mid \boldsymbol{X}^{(q_A)}_t)} \right]^2 - h_q^{-q/2}\, b^{-1/2}\, \mu_1 \right\} \\ &\quad + \frac{h_q^{q/2}\, b^{1/2}}{\Omega} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}^{(q)}_t)\, \widehat{\epsilon}^{\,2}_{Y|\boldsymbol{X}^{(q)}}(Y_t \mid \boldsymbol{X}^{(q)}_t) \left[\frac{1}{\widehat{f}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \mid \boldsymbol{X}^{(q_A)}_t)} - \frac{1}{f^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \mid \boldsymbol{X}^{(q_A)}_t)} \right] \\ &= \widehat{\Lambda}_{11,T} + \widehat{\Lambda}_{12,T}. \end{aligned}$$

As per $\widehat{\Lambda}_{12,T}$,

$$\begin{aligned}
\widehat{\Lambda}_{12,T} &= \frac{h_q^{q/2}\, b^{1/2}}{\Omega} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}_t^{(q)})\, \widehat{\epsilon}^{\,2}_{Y|\boldsymbol{X}^{(q)}}(Y_t \,|\, \boldsymbol{X}_t^{(q)}) \\
&\quad \times \frac{\left(\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) - f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)})\right)\left(\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) + f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)})\right)}{\widehat{f}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) f^{2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)})} \\
&\leq \frac{T h_q^{q/2}\, b^{1/2}}{\Omega} \sup_{C(Y,\boldsymbol{X}^{(q)})} \left| \frac{\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) + f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)})}{\widehat{f}^{\,2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) f^{2}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)})} \right| \left[\frac{1}{T} \sum_{t=1}^{T} \pi^2(Y_t, \boldsymbol{X}_t^{(q)})\, \widehat{\epsilon}^{\,4}_{Y|\boldsymbol{X}^{(q)}}(Y_t \,|\, \boldsymbol{X}_t^{(q)}) \right]^{1/2} \\
&\quad \times \left\{ \frac{1}{T} \sum_{t=1}^{T} \left[\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) - f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) \right]^2 \right\}^{1/2}.
\end{aligned}$$

Now, in view that

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^{T} \pi^2(Y_t, \boldsymbol{X}_t^{(q)})\, \widehat{\epsilon}^{\,4}_{Y|\boldsymbol{X}^{(q)}}(Y_t \,|\, \boldsymbol{X}_t^{(q)}) &\leq \sup_{C(Y,\boldsymbol{X}^{(q)})} \widehat{\epsilon}^{\,2}_{Y|\boldsymbol{X}^{(q)}}(Y_t \,|\, \boldsymbol{X}_t^{(q)})\; \frac{1}{T} \sum_{t=1}^{T} \pi^2(Y_t, \boldsymbol{X}_t^{(q)})\, \widehat{\epsilon}^{\,2}_{Y|\boldsymbol{X}^{(q)}}(Y_t \,|\, \boldsymbol{X}_t^{(q)}) \\
&= O_p\left(T^{-1} \ln T h_q^{-q}\, b^{-1}\right) \;\times\; O_p\left(T^{-1} h_q^{-q/2}\, b^{-1/2}\right)
\end{aligned}$$

and that

$$\frac{1}{T} \sum_{t=1}^{T} \left(\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) - f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t \,|\, \boldsymbol{X}_t^{(q_A)}) \right)^2 = O_p\left(T^{-1} h_{q_A}^{-q_A/2}\, b^{-1/2}\right),$$

$\widehat{\Lambda}_{12,T} = o_p(1)$ due to bandwidth condition (i).

As $\widehat{\Lambda}_{11,T}$ concerns only $\boldsymbol{X}^{(q)}$, we hereafter suppress the superscript index from the conditioning state vector and let $m(\boldsymbol{x}, y) = \mathbb{E}\big[K_b(Y_t - y) \,\big|\, \boldsymbol{X}_t = \boldsymbol{x}\big]$. By the same reasoning as in the proof of Theorem 1 in Fan et al. (1996), the bandwidth conditions (i) to (v) ensure that

$$\begin{aligned}
I_T &= h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \pi(Y_t, \boldsymbol{X}_t^{(q)}) \left[\frac{\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t \,|\, \boldsymbol{X}_t^{(q)})}{f_{Y|\boldsymbol{X}^{(q)}}(Y_t \,|\, \boldsymbol{X}_t^{(q)})} \right]^2 \\
&= h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \frac{\pi(Y_t, \boldsymbol{X}_t)}{f^2_{Y,\boldsymbol{X}}(Y_t, \boldsymbol{X}_t)} \left(\frac{1}{T} \sum_{\tau=1}^{T} \boldsymbol{W}_{h_q}(\boldsymbol{X}_\tau - \boldsymbol{X}_t) \Big[K_b(Y_\tau - Y_t) - m(\boldsymbol{X}_t, Y_t) \Big] \right)^2 + O_p(T^{-1/2} \sqrt{\ln T}\, h_q^{-q/2}\, b^{-1/2}) \\
&= \widetilde{I}_T + o_p(1).
\end{aligned}$$

Letting now

$$\phi(t, \tau, k) = \frac{1}{T^2} \frac{\pi(Y_t, \boldsymbol{X}_t)}{f^2_{Y,\boldsymbol{X}}(Y_t, \boldsymbol{X}_t)} \boldsymbol{W}_{h_q}(\boldsymbol{X}_\tau - \boldsymbol{X}_t) \Big[K_b(Y_\tau - Y_t) - m(\boldsymbol{X}_\tau, Y_t) \Big] \boldsymbol{W}_{h_q}(\boldsymbol{X}_k - \boldsymbol{X}_t) \Big[K_b(Y_k - Y_t) - m(\boldsymbol{X}_k, Y_t) \Big]$$

and $\bar{\phi}(t, \tau, k) = \phi(t, \tau, k) + \phi(t, k, \tau) + \phi(\tau, t, k) + \phi(\tau, k, t) + \phi(k, t, \tau) + \phi(k, \tau, t)$ yields

$$\begin{aligned}
\widetilde{I}_T &= h_q^{q/2}\, b^{1/2} \sum_{t<\tau<k}^{T} \bar{\phi}(t, \tau, k) + h_q^{q/2}\, b^{1/2} \sum_{t \neq \tau}^{T} \Big[\phi(t, \tau, \tau) + \phi(\tau, t, \tau) + \phi(\tau, \tau, t) \Big] + h_q^{q/2}\, b^{1/2} \sum_{t=1}^{T} \phi(t, t, t) \\
&= \widetilde{I}_{1,T} + \widetilde{I}_{2,T} + \widetilde{I}_{3,T}.
\end{aligned}$$

As in Aït-Sahalia et al. (2009), we must now demonstrate the following statements to conclude the proof; the only difference is that we must also account for the higher dimensionality of the conditioning set ($q > 1$).

(a) $\widetilde{I}_{1,T} = (T - 2)\, h_q^{q/2}\, b^{1/2} \sum_{t<\tau} \bar{\phi}(t, \tau) + o_p(1)$, where $\bar{\phi}(t, \tau) = \int \bar{\phi}(t, \tau, k)\, \mathrm{d}F(y_k, \boldsymbol{x}_k)$.

(b) $\widetilde{I}_{2,T} = \frac{1}{2}\, T(T - 1)\, h_q^{q/2}\, b^{1/2}\, \widetilde{\phi}(0) + o_p(1)$, where $\widetilde{\phi}(0) = \mathbb{E}[\widetilde{\phi}(t)]$, $\widetilde{\phi}(t) = \int \widetilde{\phi}(t, \tau)\, \mathrm{d}F(y_\tau, \boldsymbol{x}_\tau)$, and $\widetilde{\phi}(t, \tau) = \phi(t, t, \tau) + \phi(t, \tau, t) + \phi(\tau, t, t) + \phi(\tau, t, \tau) + \phi(\tau, \tau, t) + \phi(t, \tau, \tau)$.

(c) $\widetilde{I}_{3,T} = o_p(1)$.

(d) It also holds that

$$\tfrac{1}{2}\,T(T-1)\,h_q^{q/2}\,b^{1/2}\,\widetilde{\phi}(0) = h_q^{-q/2}\,b^{-1/2}\,C_1(K)\,C_1(\boldsymbol{W})\int \pi(y,\boldsymbol{x})\,\mathrm{d}y\,\mathrm{d}\boldsymbol{x} \\ -\,h_q^{-q/2}\,b^{1/2}\,C_1(\boldsymbol{W})\int \mathbb{E}\big[\pi(Y,\boldsymbol{X})\,\big|\,\boldsymbol{X}=\boldsymbol{x}\big]\,\mathrm{d}\boldsymbol{x} + o(1) \tag{25}$$

and that

$$\Omega^2 = \lim_{T\to\infty} \mathrm{Var}\left[(T-2)h_q^{q/2}\,b^{1/2}\sum_{t<\tau}\bar{\phi}(t,\tau)\right] \;=\; 2\,C_2(K)\,C_2(\boldsymbol{W})\int \pi^2(y,\boldsymbol{x})\,\mathrm{d}y\,\mathrm{d}\boldsymbol{x}. \tag{26}$$

(e) $(T-2)\,h_q^{q/2}\,b^{1/2}\sum_{t<\tau}\bar{\phi}(t,\tau) \xrightarrow{d} N(0,\Omega^2)$.

B.6.1 Proof of statement (a)

It follows from the Hoeffding decomposition that

$$\widetilde{I}_{1,T} = h_q^{q/2}\,b^{1/2}\sum_{t<\tau<k}\Phi(t,\tau,k) + (T-2)\,h_q^{q/2}\,b^{1/2}\sum_{t<\tau}\bar{\phi}(t,\tau), \tag{27}$$

where $\Phi(t,\tau,k) = \bar{\phi}(t,\tau,k) - \bar{\phi}(t,\tau) - \bar{\phi}(t,k) - \bar{\phi}(\tau,k)$. To show that the first term on the right-hand side of (27) is of order $o_p(1)$, it suffices to apply Lemma 5(i) in Aït-Sahalia et al. (2009) with $\delta = 1/3$. This results in $\mathbb{E}\Big(\widetilde{I}_{1,T}^{\,2}\Big) = O\left(T^{-1}\,h_q^{3q/2}\,b^{-3/2}\right)$, which is of order $o(1)$ by condition (vi). ■

B.6.2 Proof of statement (b)

As before, applying the Hoeffding decomposition yields

$$h_q^{q/2}\,b^{1/2}\,\widetilde{I}_{2,T} = h_q^{q/2}\,b^{1/2}\sum_{t<\tau}\widetilde{\phi}(t,\tau) \\ = h_q^{q/2}\,b^{1/2}\sum_{t<\tau}\Big[\widetilde{\phi}(t,\tau) - \widetilde{\phi}(t) - \widetilde{\phi}(\tau) - \widetilde{\phi}(0)\Big] + (T-1)\,h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\Big[\widetilde{\phi}(t) - \widetilde{\phi}(0)\Big] \\ +\,\tfrac{1}{2}\,T(T-1)\,h_q^{q/2}\,b^{1/2}\,\widetilde{\phi}(0).$$

Lemma 5(ii) in Aït-Sahalia et al. (2009) with $\delta = 1$ then dictates that

$$h_q^{q/2}\,b^{1/2}\sum_{t<\tau}\Big[\widetilde{\phi}(t,\tau) - \widetilde{\phi}(t) - \widetilde{\phi}(\tau) - \widetilde{\phi}(0)\Big] = O_p\left(T^{-1}\,h_q^{-5q/4}\,b^{-5/4}\right),$$

which is of order $o_p(1)$ due to the bandwidth condition (vi). Under Assumption A4, the central limit for β-mixing processes ensures that

$$(T-1)\,h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\Big[\widetilde{\phi}(t) - \widetilde{\phi}(0)\Big] = O_p\left(T^{-1}\,h_q^{-q}\,b^{-1}\right) = o_p(1).$$ ■

B.6.3 Proof of statement (c)

It is immediate to see that

$$\widetilde{I}_{3,T} = h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\phi(t,t,t) = O_p\left(T\,h_q^{-3q/2}\,b^{-3/2}\right),$$

which is of order $o_p(1)$ by condition (vi). ■

B.6.4 Proof of statement (d)

As for (25) and (26), the result follows along similar lines of the proof of claim (d) in Aït-Sahalia et al. (2009). ■

B.6.5 Proof of statement (e)

It suffices to apply Fan and Li's (1999) central limit theorem for degenerate U-statistics of absolutely regular processes to obtain the desired result (see Amaro de Matos and Fernandes, 2007). See also Aït-Sahalia et al. (2009) and Gao and Hong (2008) for alternative central limit theorems that deal with degenerate U-statistics of α-mixing processes. ■

B.7 Proof of Lemma 2

Let $\bar{\psi}(t,\tau)$ and $\widetilde{\psi}(0)$ respectively denote the counterparts of $\bar{\phi}(t,\tau)$ and $\widetilde{\phi}(0)$ once we substitute

$$\psi(t,\tau,k) = \frac{1}{T^2}\,\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})}{f^2_{Y,\boldsymbol{X}^{(q_A)}}(Y_t,\boldsymbol{X}_t^{(q_A)})}\,\Big\{\widetilde{\boldsymbol{W}}_{h_q}(\boldsymbol{X}_\tau^{(q_A)}-\boldsymbol{X}_t^{(q_A)})\Big[K_b(Y_\tau-Y_t)-m(\boldsymbol{X}_\tau^{(q_A)},Y_t)\Big]$$
$$\times\ \widetilde{\boldsymbol{W}}_{h_q}(\boldsymbol{X}_k^{(q_A)}-\boldsymbol{X}_t^{(q_A)})\Big[K_b(Y_k-Y_t)-m(\boldsymbol{X}_k^{(q_A)},Y_t)\Big]\Big\}$$

for $\phi(t,\tau,k)$. Applying the same argument we put forth in the proof of Lemma 1 then yields

$$\begin{aligned} J_T &= h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})}{f^2_{Y,\boldsymbol{X}^{(q_A)}}(Y_t,\boldsymbol{X}_t^{(q_A)})}\Big[\widehat{f}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})-f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})\Big]^2 \\ &= (T-2)\,h_q^{q/2}\,b^{1/2}\sum_{t<\tau}\bar{\psi}(t,\tau)+\tfrac{1}{2}\,T(T-1)\,h^{q/2}\,b^{1/2}\,\widetilde{\psi}(0)+o_p(1), \end{aligned}$$

whose first term on the right-hand side satisfies the central limit theorem for U-statistics. In addition,

$$\begin{aligned} \tfrac{1}{2}\,\widetilde{\psi}(0) &= \frac{1}{T^2}\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f^2(y_i,\boldsymbol{x}_i^{(q_A)})}\,\widetilde{\boldsymbol{W}}^{\,2}_{h_{q_A}}(\boldsymbol{x}_j^{(q_A)}-\boldsymbol{x}_i^{(q_A)})\Big(K_b(y_j-y_i)-m(\boldsymbol{x}_j^{(q_A)},y_i)\Big)^2\,\mathrm{d}F(y_i,\boldsymbol{x}_i^{(q)}\,\mathrm{d}F(y_j,\boldsymbol{x}_j^{(q_A)}) \\ &= \frac{1}{T^2}\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f^2(y_i,\boldsymbol{x}_i^{(q_A)})}\,\widetilde{\boldsymbol{W}}^{\,2}_{h_{q_A}}(\boldsymbol{x}_j^{(q_A)}-\boldsymbol{x}_i^{(q_A)})K_b^2(y_j-y_i)\,\mathrm{d}F(y_i,\boldsymbol{x}_i^{(q)})\,\mathrm{d}F(y_j,\boldsymbol{x}_j^{(q_A)}) \\ &\quad-\frac{2}{T^2}\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f^2(y_i,\boldsymbol{x}_i^{(q_A)})}\,\widetilde{\boldsymbol{W}}^2_{h_{q_A}}(\boldsymbol{x}_j^{(q_A)}-\boldsymbol{x}_i^{(q_A)})K_b(y_j-y_i)m(\boldsymbol{x}_j^{(q_A)},y_i)\,\mathrm{d}F(y_i,\boldsymbol{x}_i^{(q)})\,\mathrm{d}F(y_j,\boldsymbol{x}_j^{(q_A)}) \\ &\quad+\frac{1}{T^2}\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f^2(y_i,\boldsymbol{x}_i^{(q_A)})}\,\widetilde{\boldsymbol{W}}^2_{h_{q_A}}(\boldsymbol{x}_j^{(q_A)}-\boldsymbol{x}_i^{(q_A)})m^2(\boldsymbol{x}_j^{(q_A)},y_i)\,\mathrm{d}F(y_i,\boldsymbol{x}_i^{(q)})\,\mathrm{d}F(\boldsymbol{x}_j^{(q_A)}) \\ &= \frac{1}{T^2}\,\frac{1}{h_{q_A}^{q_A}\,b}\int\widetilde{\boldsymbol{W}}^2(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\int\boldsymbol{K}^2(v)\,\mathrm{d}v\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f(y_i,\boldsymbol{x}_i^{(q_A)})}\,f(y_i,\boldsymbol{x}_i^{(q)})\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q)}\,\Big\{1+O\Big(h_{q_A}^2+b^s\Big)\Big\} \\ &\quad-\frac{1}{T^2}\,\frac{1}{h_{q_A}^{q_A}}\int\widetilde{\boldsymbol{W}}^2(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f^2(y_i,\boldsymbol{x}_i^{(q_A)})}\,f(y_i,\boldsymbol{x}_i^{(q)})\,m^2(\boldsymbol{x}_i^{(q_A)},y_i)\,f(\boldsymbol{x}_i^{(q_A)})\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q)}\,\Big\{1+O(h_{q_A}^2+b^s)\Big\}, \end{aligned}$$

where the last equality follows from a Taylor expansion with $\boldsymbol{u}=(\boldsymbol{x}_j^{(q_A)}-\boldsymbol{x}_i^{(q_A)})/h_{q_A}$ and $v=(\boldsymbol{y}_j-\boldsymbol{y}_i)/b$ given that

$$\mathbb{E}\Big[K_b(y_j-y_i)|Y=y_i,\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]=m(\boldsymbol{x}_i^{(q_A)},y_i)=f(y_i|\boldsymbol{x}_i^{(q_A)})\Big\{1+O(h_{q_A}^2+b^s)\Big\}.$$

In addition, it ensues from

$$\begin{aligned} \int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f(y_i,\boldsymbol{x}_i^{(q_A)})}\,f(y_i,\boldsymbol{x}_i^{(q)})\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q)} &= \int\pi(y_i,\boldsymbol{x}_i^{(q)})\,f(\boldsymbol{x}_i^{(q_B)}|y_i,\boldsymbol{x}_i^{(q_A)})\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q)} \\ &= \int\mathbb{E}\Big[\pi(y_i,\boldsymbol{x}_i^{(q)})|Y=y_i,\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q_A)} \end{aligned}$$

and

$$\begin{aligned} \int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f^2(y_i,\boldsymbol{x}_i^{(q_A)})}\,f^2(y_i|\boldsymbol{x}_i^{(q_A)})\,f(y_i,\boldsymbol{x}_i^{(q)})\,f(\boldsymbol{x}_i^{(q_A)})\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q)} &= \int\pi(y_i,\boldsymbol{x}_i^{(q)})\,f(y_i,\boldsymbol{x}_i^{(q_B)}|\boldsymbol{x}_i^{(q_A)})\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q)} \\ &= \int\mathbb{E}\Big[\pi(y_i,\boldsymbol{x}_i^{(q)})|\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]\,\mathrm{d}\boldsymbol{x}_i^{(q_A)}, \end{aligned}$$

that

$$\begin{aligned} \tfrac{1}{2}\,T(T-1)\,h_q^{q/2}\,b^{1/2}\,\widetilde{\psi}(0) &= h_q^{q/2}\,h_{q_A}^{-q_A}\,b^{-1/2}\,C_1(K)\,C_1(\widetilde{\boldsymbol{W}})\int\mathbb{E}\Big[\pi(y_i,\boldsymbol{x}_i^{(q)}|Y=y_i,\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]\,\mathrm{d}\boldsymbol{x}_i^{(q_A)}\,\mathrm{d}y_i \\ &\quad+h_q^{q/2}\,h_{q_A}^{-q_A}\,b^{1/2}\,C_1(\widetilde{\boldsymbol{W}})\int\mathbb{E}\Big[\pi(y_i,\boldsymbol{x}_i^{(q)})|\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]\,\mathrm{d}\boldsymbol{x}_i^{(q_A)}+o(1), \end{aligned}$$

completing the proof. ■

B.8 Proof of Lemma 3

Let

$$\varphi(t,\tau,k) = \frac{1}{T^2}\,\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})}{f_{Y,\boldsymbol{X}^{(q)}}(Y_t,\boldsymbol{X}_t^{(q)})\,f_{Y,\boldsymbol{X}^{(q_A)}}(Y_t,\boldsymbol{X}_t^{(q_A)})}\,\boldsymbol{W}_{h_q}(\boldsymbol{X}_\tau^{(q)}-\boldsymbol{X}_t^{(q)})\Big[K_b(Y_\tau-Y_t)-m(\boldsymbol{X}_\tau^{(q)},Y_t)\Big]$$
$$\times\,\widetilde{\boldsymbol{W}}_{h_{q_A}}(\boldsymbol{X}_k^{(q_A)}-\boldsymbol{X}_t^{(q_A)})\Big[K_b(Y_k-Y_t)-m(\boldsymbol{X}_k^{(q_A)},Y_t)\Big].$$

Proceeding along the same line as in the proof of Lemma 1 then yields

$$h_q^{q/2}\,b^{1/2}\sum_{t=1}^{T}\frac{\pi(Y_t,\boldsymbol{X}_t^{(q)})\,\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})\,\widehat{\epsilon}_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})}{f_{Y|\boldsymbol{X}^{(q)}}(Y_t|\boldsymbol{X}_t^{(q)})\,f_{Y|\boldsymbol{X}^{(q_A)}}(Y_t|\boldsymbol{X}_t^{(q_A)})} = (T-2)\,h_q^{q/2}\,b^{1/2}\sum_{t<\tau}\bar{\varphi}(t,\tau)$$
$$+\tfrac{1}{2}\,T(T-1)\,h_q^{q/2}\,b^{1/2}\,\widetilde{\varphi}(0)+o_p(1).$$

Let now $\boldsymbol{u}=(\boldsymbol{x}_j^{(q_A)}-\boldsymbol{x}_i^{(q_A)})/h_{q_A}$, $v=(y_j-y_i)/b$, and $\boldsymbol{z}=(\boldsymbol{x}_j^{(q)}-\boldsymbol{x}_i^{(q)})/h_q$. Given that under the null $\mathbb{H}_0$ $\mathbb{E}\Big[K_b(y_j-y_i)|Y=y_i,\boldsymbol{X}^{(q)}=\boldsymbol{x}_i^{(q)}\Big]=m(\boldsymbol{x}_i^{(q)},y_i)=m(\boldsymbol{x}_i^{(q_A)},y_i)$, it follows that

$$\begin{aligned}
\tfrac{1}{2}\,\widetilde{\varphi}(0) &= T^{-2}h_q^{-q_A}\,b^{-1}\,C_1(K)\,\widetilde{\boldsymbol{W}}(0)\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f(y_i,\boldsymbol{x}_i^{(q_A)})}\,f(y_i,\boldsymbol{x}_i^{(q)})\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q)}\left\{1+O\left(h_{q_A}^2+b^s\right)\right\}\\
&\quad - T^{-2}h_q^{-q_A}\,\widetilde{\boldsymbol{W}}(0)\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f^2(y_i,\boldsymbol{x}_i^{(q_A)})}\,f(y_i,\boldsymbol{x}_i^{(q)})\,m^2(\boldsymbol{x}_i^{(q_A)},y_i)\,f(\boldsymbol{x}_i^{(q_A)})\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q)}\left\{1+O\left(h_{q_A}^2+b^s\right)\right\}\\
&= T^{-2}h_q^{-q_A}\,b^{-1}\,C_1(K)\,\widetilde{\boldsymbol{W}}(0)\int\mathbb{E}\Big[\pi(y_i,\boldsymbol{x}_i^{(q)})|Y=y_i,\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q_A)}\left\{1+O\left(h_{q_A}^2+b^s\right)\right\}\\
&\quad - T^{-2}h_q^{-q_A}\,\widetilde{\boldsymbol{W}}(0)\int\mathbb{E}\Big[\pi(y_i,\boldsymbol{x}_i^{(q)})|\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]\,\mathrm{d}\boldsymbol{x}_i^{(q_A)}\left\{1+O\left(h_{q_A}^2+b^s\right)\right\}.
\end{aligned}$$

This means that

$$\tfrac{1}{2}\,T(T-1)\,h_q^{q/2}\,b^{1/2}\,\widetilde{\varphi}(0) = h_q^{q/2-q_A}\,b^{-1/2}\,C_1(K)\,\widetilde{\boldsymbol{W}}(0)\int\mathbb{E}\Big[\pi(y_i,\boldsymbol{x}_i^{(q)})|Y=y_i,\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]\,\mathrm{d}y_i\,\mathrm{d}\boldsymbol{x}_i^{(q_A)}$$
$$-\,h_q^{q/2-q_A}\,\widetilde{\boldsymbol{W}}(0)\int\Big[\pi(y_i,\boldsymbol{x}_i^{(q)})|\boldsymbol{X}^{(q_A)}=\boldsymbol{x}_i^{(q_A)}\Big]\,\mathrm{d}\boldsymbol{x}_i^{(q_A)},$$

which completes the proof. ■

B.9 Proofs of Lemmata 4 to 6

We omit the proofs because they are almost exactly the same as the proofs of Lemmata 1 to 3. It indeed suffices to apply the same line of reasoning to derive the results in a straightforward manner. ■

B.10 Proof of Theorem 5

We denote by $\Pr_*$ the probability distribution induced by the bootstrap sampling, with expectation and variance operators given by $\mathbb{E}_*$ and Var_*, respectively. In addition, we also let $O_p^*(1)$ and $o_p^*(1)$ denote the orders of magnitude according to the bootstrap-induced probability law.

Both local-linear and kernel smoothing results follow straightforwardly once we prove the bootstrap versions of Lemmata 1 to 3 and of Lemmata 4 to 6, respectively. As the proofs are very similar, in what follows, we restrict attention to the bootstrap test based on local linear smoothing. We start with the bootstrap counterpart of Lemma 1. As in the latter's proof, it turns out that

$$\begin{aligned}
\widehat{\Lambda}_{1,\mathcal{T}}^* &= h_{*q}^{q/2}\,b_*^{1/2}\sum_{t=1}^{\mathcal{T}}\pi(Y_t^*,\boldsymbol{X}_t^{*(q)})\left[\frac{\widehat{f}^*_{Y|\boldsymbol{X}^{(q)}}(Y_t^*\mid\boldsymbol{X}_t^{*(q)})-f_{Y|\boldsymbol{X}^{(q)}}(Y_t^*\mid\boldsymbol{X}_t^{*(q)})}{\widehat{f}^*_{Y|\boldsymbol{X}^{(q_A)}}(Y_t^*\mid\boldsymbol{X}_t^{*(q_A)})}\right]^2 - h_{*q}^{-q/2}\,b_*^{-1/2}\,\mu_1\\
&= h_{*q}^{q/2}\,b_*^{1/2}\sum_{t=1}^{\mathcal{T}}\frac{\pi(Y_t^*,\boldsymbol{X}_t^{*(q)})}{f^2_{Y,\boldsymbol{X}^{(q)}}(Y_t^*,\boldsymbol{X}_t^{*(q)})}\left(\frac{1}{\mathcal{T}}\sum_{\tau=1}^{\mathcal{T}}\boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_\tau^{*(q)}-\boldsymbol{X}_t^{*(q)})\Big[K_{b_*}(Y_\tau^*-Y_t^*)-m(\boldsymbol{X}_\tau^{*(q)},Y_t^*)\Big]\right)^2\\
&\quad -h_{*q}^{-q/2}\,b_*^{-1/2}\,\mu_1+o_p^*(1).
\end{aligned}$$

Let now

$$\phi_*(k,j,i) = \mathcal{T}^{-2}\,\frac{\pi(Y_k^*,\boldsymbol{X}_k^{*(q)})}{f^2_{Y,\boldsymbol{X}^{(q)}}(Y_k^*,\boldsymbol{X}_k^{*(q)})}\,\boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_j^{*(q)}-\boldsymbol{X}_k^{*(q)})\Big[K_{b_*}(Y_j^*-Y_k^*)-m(\boldsymbol{X}_j^{*(q)},Y_k^*)\Big]$$
$$\times\,\boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_i^{*(q)}-\boldsymbol{X}_k^{*(q)})\Big[K_{b_*}(Y_i^*-Y_k^*)-m(\boldsymbol{X}_i^{*(q)},Y_k^*)\Big].$$

Taking conditional expectation over bootstrap samples given $(Y_k, \boldsymbol{X}_k^{(q)})$ then yields

$$\begin{aligned}
\phi_*(j,i) &= \mathbb{E}_*\left[\phi_*(k,j,i)\,\Big|\,Y_k,\boldsymbol{X}_k^{(q)}\right] \\
&= \mathcal{T}^{-2}\,\frac{1}{T}\sum_{k=1}^{T}\frac{\pi(Y_k,\boldsymbol{X}_k^{(q)})}{f^2_{Y,\boldsymbol{X}^{(q)}}(Y_k,\boldsymbol{X}_k^{(q)})}\,\boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_j^{*(q)}-\boldsymbol{X}_k^{(q)})\Big[K_{b_*}(Y_j^*-Y_k)-m(\boldsymbol{X}_j^{*(q)},Y_k)\Big] \\
&\qquad\qquad\times\boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_i^{*(q)}-\boldsymbol{X}_k^{(q)})\Big[K_{b_*}(Y_i^*-Y_k)-m(\boldsymbol{X}_i^{*(q)},Y_k)\Big] \\
&= \mathcal{T}^{-2}\int\frac{\pi(y_k,\boldsymbol{x}_k^{(q)})}{f^2_{Y,\boldsymbol{X}^{(q)}}(y_k,\boldsymbol{x}_k^{(q)})}\,\boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_j^{*(q)}-\boldsymbol{x}_k^{(q)})\Big[K_{b_*}(Y_j^*-y_k)-m(\boldsymbol{X}_j^{*(q)},y_k)\Big] \\
&\qquad\times\boldsymbol{W}_{h_{*q}}(\boldsymbol{X}_i^{*(q)}-\boldsymbol{x}_k^{(q)})\Big[K_{b_*}(Y_i^*-y_k)-m(\boldsymbol{X}_i^{*(q)},y_k)\Big]\,\mathrm{d}\boldsymbol{x}_k^{(q)}\,\mathrm{d}y_k+o_p^*(\mathcal{T}^{-2}h_{*q}^{-q/2}b_*^{-1/2}),
\end{aligned}$$

and so

$$\begin{aligned}
\phi_*(0) &= \mathbb{E}_*[\phi_*(i,j,j)+\phi_*(j,i,i)] \\
&= 2\,\mathcal{T}^{-2}\,\frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T}\frac{\pi(Y_i,\boldsymbol{X}_i^{(q)})}{f^2_{Y,\boldsymbol{X}^{(q)}}(Y_i,\boldsymbol{X}_i^{q})}\,\boldsymbol{W}^2_{h_{*q}}(\boldsymbol{X}_j^{(q)}-\boldsymbol{X}_i^{(q)})\left(K_{b_*}(Y_j-Y_i)-m(\boldsymbol{X}_j^{(q)},Y_i)\right)^2 \\
&= 2\,\mathcal{T}^{-2}\int\frac{\pi(y_i,\boldsymbol{x}_i^{(q)})}{f^2_{Y,\boldsymbol{X}^{(q)}}(y_i,\boldsymbol{x}_i^{(q)})}\,\boldsymbol{W}^2_{h_{*q}}(\boldsymbol{x}_j^{(q)}-\boldsymbol{x}_i^{(q)})\left(K_{b_*}(y_j-y_i)-m(\boldsymbol{x}_j^{(q)},y_i)\right)^2\mathrm{d}F_{Y,\boldsymbol{X}^{(q)}}(y_i,\boldsymbol{x}_i^{(q)})\,\mathrm{d}F_{Y,\boldsymbol{X}^{(q)}}(y_j,\boldsymbol{x}_j^{(q)}) \\
&\quad+o_p\left(\mathcal{T}^{-2}h_{*q}^{-q/2}b_*^{-1/2}\right).
\end{aligned}$$

As in statement **(d)** in the proof of Lemma 1, it then follows that

$$\tfrac{1}{2}\,\mathcal{T}(\mathcal{T}-1)\,h_{*q}^{q/2}\,b_*^{1/2}\,\phi_*(0)=h_{*q}^{-q/2}b_*^{-1/2}\mu_1+o_p^*(1)$$

and, as $\mathcal{T}/T\to 0$,

$$(\mathcal{T}-2)\,h_{*q}^{q/2}\,b_*^{1/2}\sum_{j<i}^{\mathcal{T}}\phi_*(j,i)=(\mathcal{T}-2)\,h_{*q}^{q/2}\,b_*^{1/2}\sum_{j<i}^{\mathcal{T}}\Big(\phi_*(j,i)-\mathbb{E}_*\left[\phi_*(j,i)\right]\Big)+o_p^*(1). \tag{28}$$

In view that $\mathrm{Var}_*\left\{(\mathcal{T}-2)\,h_{*q}^{q/2}\,b_*^{1/2}\sum_{t<\tau}\Big(\phi_*(j,i)-\mathbb{E}_*[\phi_*(j,i)]\Big)\right\}=\Omega+o_p(1)$, the first term on the right-hand side of (28) weakly converges to $N(0,\Omega)$ as both T and $\mathcal{T}$ go to infinity, thus mimicking the limiting distribution of $(T-2)\,h_{*q}^{q/2}\,b_*^{1/2}\sum_{j<i}\phi(j,i)$.

Define next $\psi_*(k,j,i)$, $\psi_*(j,i)$ and $\psi_*(0)$ analogously to $\phi_*(k,j,i)$, $\phi_*(j,i)$ and $\phi_*(0)$ for $\boldsymbol{X}^{(q_A)}$, with $\widetilde{\boldsymbol{W}}_{h_{*q_A}}$ replacing $\boldsymbol{W}_{h_{*q}}$. As $\mathcal{T}/T\to 0$, it is possible to show that $(\mathcal{T}-2)h_{*q}^{q/2}b_*^{1/2}\sum_{t<\tau}^{\mathcal{T}}\psi_*(j,i)=o_p^*(1)$ and $\frac{1}{2}\,\mathcal{T}(\mathcal{T}-1)\,h_{*q}^{q/2}\,b_*^{1/2}\,\psi_*(0)=h_{*q}^{q/2}\,h_{*q_A}^{-q_A}\,b_*^{-1/2}\mu_2+o_p^*(1)$. It is also straightforward to derive the bootstrap counterpart of Lemma 3 as well. The statement under the null thereby follows by noting that $\widehat{\mu}^*_{1,\mathcal{T}}=\widehat{\mu}_{1,\mathcal{T}}+o_p(h_{*q}^{-p/2}\,b_*^{-1/2})$, whereas it is immediate to see that $\widehat{\Lambda}^*_{\mathcal{T}}$ diverges at most at rate $O_p\left(\mathcal{T}\,h_{*q}^{q/2}\,b_*^{1/2}\right)$ under the alternative. ■

C Realized measures

According to the data generating process, one may employ different realized measures to estimate the daily integrated variance from a sample of M intraday regularly-spaced-in-time observations under very mild conditions. Andersen et al. (2001) and Barndorff-Nielsen and Shephard (2002) propose the realized variance

$$RV_{i,t,M}\equiv\sum_{j=1}^{M-1}\left(p_{i,t+(j+1)/M}-p_{i,t+j/M}\right)^2, \tag{29}$$

whereas Barndorff-Nielsen and Shephard (2004) suggests the tripower variation

$$TV_{i,t,M}=\mu_{2/3}^{-3}\sum_{j=1}^{M-3}\left|\Delta p_{i,(j+3)/M}\right|^{2/3}\left|\Delta p_{i,(j+2)/M}\right|^{2/3}\left|\Delta p_{i,(j+1)/M}\right|^{2/3}, \tag{30}$$

where μ_k is the k-th moment of a standard normal distribution. The former is a consistent estimator for the quadratic variation of the process, and hence it estimates consistently the integrated variance only in the absence of jumps. In contrast, the latter entails a consistent estimate of the integrated variance even in the presence of jumps.

The above realized measures implicitly assumes the absence of market frictions. Decomposing the observed asset price $p_{i,t}$ into the true price $p^*_{i,t}$ and a noise $\epsilon_{i,t}$ arising from generic market frictions yields $p_{i,t} = p^*_{i,t} + \epsilon_{i,t}$, $i = A, B$. The resulting sample of M intraday regularly-spaced-in-time observations over T days then is

$$p_{i,t+j/M} = p^*_{i,t+j/M} + \epsilon_{i,t+j/M}, \qquad i = A, B \tag{31}$$

where $\epsilon_{i,t+j/M}$ is by assumption a zero-mean geometric $\alpha-$mixing process. It is possible to estimate the integrated variance at day t from the noisy price data $\{p_{i,t+j/M};\ j = 1, \ldots, M;\ t = 1, \ldots, T\}$ using appropriate realized measures. Zhang et al. (2005) introduce the two-scale realized variance

$$TS_{i,t,L,M} = \overline{RV}_{i,t,L,M} - \frac{L}{M} RV_{i,t,M}, \tag{32}$$

where

$$\overline{RV}_{i,t,L,M} = \frac{L}{M} \sum_{\ell=1}^{M/L} \sum_{j=1}^{L-1} \left(p_{i,t+\frac{(j+1)M/L+\ell}{M}} - p_{i,t+\frac{jM/L+\ell}{M}} \right)^2$$

gauges the average realized variance across M/L subsamples of size $L = O\left(M^{1/3}\right)$. Similarly, the multi-scale realized variance put forth by Zhang (2006) and Aït-Sahalia et al. (2011) considers a weighted average of realized variances over different sampling frequencies. In particular,

$$MS_{i,t,L,M} = \sum_{\ell=1}^{L} a_\ell \left[\frac{1}{L_\ell} \sum_{j=1}^{M-L_\ell} \left(p_{i,t+\frac{j+L_\ell}{M}} - p_{i,t+\frac{j}{M}} \right)^2 \right] + \frac{RV_{i,t,M}}{M}, \tag{33}$$

where a_ℓ is such that $\sum_{\ell=1}^{L} a_\ell = 1$ and $\sum_{\ell=1}^{L} a_\ell/\ell = 0$. For instance, if one considers $L_\ell = \ell$, then

$$a_\ell = 12 \frac{\ell}{L^2} \frac{\ell/L - 1/2 - 1/(2L)}{1 - 1/L^2}.$$

Barndorff-Nielsen et al. (2008) show that both the two- and multi-scale realized volatility estimators are asymptotic equivalent to realized measures belonging to the class of kernel-based estimators given by

$$RK_{i,t,H,M} = \sum_{\ell=1}^{L} \kappa\left(\frac{\ell-1}{L}\right) \left(\gamma_{i,t,\ell} + \gamma_{i,t,-\ell}\right), \tag{34}$$

where $\gamma_{i,t,\ell} = \sum_{j=\ell}^{M-L-1} (p_{i,t+(j+1)/M} - p_{i,t+j/M})(p_{i,t+(j+1-\ell)/M} - p_{i,t+(j-\ell)/M})$, and the kernel is such that $\kappa(0) = 1$ and $\kappa(1) = \kappa'(0) = \kappa'(1) = 0$. More specifically, the two-scale realized volatility corresponds to a realized kernel estimator with $\kappa(x) = 1 - x$ and $L = M^{2/3}$, whereas the multi-scale realized volatility to a realized kernel measure with $\kappa(x) = 1 - 3x^2 + 2x^3$ and $L = M^{1/2}$.

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References

Aït-Sahalia, Y., 1996, Testing continuous-time models of the spot interest rate, Review of Financial Studies 9, 385–426.

Aït-Sahalia, Y., Bickel, P. J., Stoker, T. M., 2001, Goodness-of-fit tests for kernel regression with an application to option implied volatilities, Journal of Econometrics 105, 363412.

Aït-Sahalia, Y., Fan, J.-i., Peng, H., 2009, Nonparametric transition-based tests for diffusions, Journal of the American Statistical Association 104, 1102–1116.

Aït-Sahalia, Y., Jacod, J., 2010, Analyzing the spectrum of asset returns: Jump and volatility components in high frequency data, forthcoming in the Journal of Economic Literature.

Aït-Sahalia, Y., Mykland, P., Zhang, L., 2011, Ultra high frequency volatility estimation with dependent microstructure noise, Journal of Econometrics 160, 160–175.

Allen, F., Qian, J., Qian, M., 2007, China's financial system: Past, present, and future, in: L. Brandt T. Rawski (eds), China's Great Economic Transformation, Cambridge University Press, Cambridge.

Amaro de Matos, J., Fernandes, M., 2007, Testing the Markov property with high frequency data, Journal of Econometrics 141, 44–64.

Andersen, T. G., Bollerslev, T., Diebold, F. X., Labys, P., 2001, The distribution of realized exchange rate volatility, Journal of the American Statistical Association 96, 42–55.

Andersen, T. G., Bollerslev, T., Diebold, F. X., Labys, P., 2003, Modeling and forecasting realized volatility, Econometrica 71, 529–626.

Andersen, T. G., Bollerslev, T., Dobrev, D., 2007, No-arbitrage semi-martingale restrictions for continuous-time volatility models subject to leverage effects, jumps and i.i.d. noise: Theory and testable distributional assumptions, Journal of Econometrics 138, 125–180.

Andrews, D. W., 1997, A conditional Kolmogorov test, Econometrica 65, 1097–1128.

Arcones, M. A., Giné, E., 1992, On the bootstrap of U and V statistics, Annals of Statistics 20, 655–674.

Bandi, F. M., Perron, B., 2006, Long memory and the relation between implied and realized volatility, Journal of Financial Econometrics 4, 636–670.

Barndorff-Nielsen, O. E., Hansen, P. H., Lunde, A., Shephard, N., 2008, Designing realized kernels to measure the ex-post variation of equity prices in the presence of noise, Econometrica 76, 1481–1536.

Barndorff-Nielsen, O. E., Shephard, N., 2002, Econometric analysis of realised volatility and its use in estimating stochastic volatility models, Journal of the Royal Statistical Society, series B 64, 243–280.

Barndorff-Nielsen, O. E., Shephard, N., 2004, Power and bipower variation with stochastic volatility and jumps (with discussion), Journal of Financial Econometrics 2, 1–48.

Bickel, P. J., Götze, F., van Zwet, W. R., 1997, Resampling fewer than n observations: Gains, losses, and remedies for losses, Statistica Sinica 7, 1–31.

Bickel, P. J., Rosenblatt, M., 1973, On some global measures of the deviations of density function estimates, Annals of Statistics 1, 1071–1095.

Bretagnolle, J., 1983, Lois limites du bootstrap de certaines fonctionnelles, Annales de l'Institut Henri Poincaré, Section B, Probabilités et Statistiques 19, 281–296.

Carr, P., Geman, H., Madan, D. B., Yor, M., 2005, A causality in variance test and its applications to financial market prices, Finance and Stochastics 9, 453–475.

Chen, X., Linton, O., Robinson, P. M., 2001, The estimation of conditional densities, in: M. L. Puri (ed.), Asymptotics in Statistics and Probability, VSP International Science Publishers, Leiden, pp. 71–84.

Cheung, Y. W., Ng, L. K., 1996, A causality in variance test and its applications to financial market prices, Journal of Econometrics 72, 33–48.

Christensen, B. J., Prabhala, N. R., 1998, The relation between implied and realized volatility, Journal of Financial Economics 50, 125–150.

Corradi, V., Distaso, W., Swanson, N. R., 2011, Predictive inference for integrated volatility, Journal of the American Statistical Association 106, 1496.

Cox, J. C., Ingersoll, J. E., Ross, S. A., 1985, A theory of the term structure of interest rates, Econometrica 53, 385–407.

Delbaen, F., Schachermayer, W., 1994, A general version of the fundamental theorem of asset pricing, Mathematische Annalen 300, 463–520.

Diebold, F. X., Yilmaz, K., 2009, Measuring financial asset return and volatility spillovers, with application to global equity markets, Economic Journal 119, 158–171.

Diebold, F. X., Yilmaz, K., 2011, Equity market spillovers in the Americas, in: R. Alfaro (ed.), Financial Stability, Monetary Policy, and Central Banking, Vol. 15, Bank of Chile, Santiago, pp. 199–214.

Engle, R. F., Ito, T., Lin, W.-L., 1990, Meteor showers or heat waves: Heteroskedastic intradaily volatility in the foreign exchange market, Econometrica 58, 525–542.

Engle, R. F., Ng, V., 1988, Measuring and testing the impact of news on volatility, Journal of Finance 48, 1749–1778.

Fan, J., 1996, Test of significance based on wavelet thresholding and Neymans truncation, Journal of the American Statistical Association 91, 674–688.

Fan, J., Gijbels, I., 1996, Local Polynomial Modelling and its Applications, Chapman and Hall, London.

Fan, J., Peng, L., Yao, Q.-W., Zhang, W.-Y., 2009, Approximating conditional density functions using dimension reduction, Acta Mathematicae Applicatae Sinica 25, 445–456.

Fan, J., Yao, Q., Tong, H., 1996, Estimation of conditional densities and sensitivity measures in nonlinear dynamical systems, Biometrika 83, 189–206.

Fan, Y., 1995, Bootstrapping a consistent nonparametric goodness-of-fit test, Econometric Reviews 14, 367–382.

Fan, Y., Li, Q., 1999, Central limit theorem for degenerate U-statistics of absolutely regular processes with applications to model specification tests, Journal of Nonparametric Statistics 10, 245–271.

Fan, Y., Linton, O., 2003, Some higher-order theory for a consistent non-parametric model specification test, Journal of Statistical Planning and Inference 109, 125–154.

Feng, L., Seasholes, M. S., 2008, Individual investors and gender similarities in an emerging stock market, Pacific Basin Finance Journal 16, 44–60.

Gao, J., Hong, Y., 2008, Central limit theorems for generalized U-statistics with applications in nonparametric specification, Journal of Nonparametric Statistics 20, 61–73.

Gourieroux, C., Jasiak, J., 2007, Nonlinear causality, with applications to liquidity and stochastic volatility, working paper, CREST, University of Toronto, and York University.

Hall, P., Yao, Q.-W., 2005, Conditional distribution function approximation, and prediction, using dimension reduction, Annals of Statistics 33, 1404–1421.

Hamao, Y., Masulis, R. W., Ng, V., 1990, Correlations in price changes and volatility across international stock markets, Review of Financial Studies 3, 281–308.

Härdle, W., Mammen, E., 1993, Comparing nonparametric vs. parametric regression fits, Annals of Statistics 21, 1926–1947.

Harrison, M., Pliska, S., 1981, Martingales and stochastic integrals in the theory of continuous trading, Stochastic Processes and Their Applications 11, 215–260.

Hertz, E., 1998, The Trading Crowd: An Ethnography of the Shanghai Stock Market, Cambridge University Press, Cambridge.

Hong, Y., 2001, A test for volatility spillover with application to exchange rates, Journal of Econometrics 103, 183–224.

Hong, Y., White, H., 2004, Asymptotic distribution theory for nonparametric entropy measures of serial dependence, Econometrica 73, 837–901.

Horowitz, J. L., 2003, Bootstrap methods for Markov processes, Econometrica 71, 1049–1082.

Karolyi, G. A., 1995, A multivariate GARCH model of international transmissions of stock returns and volatility: The case of the United States and Canada, Journal of Business and Economic Statistics 13, 11–25.

King, M., Sentana, E., Wadhwani, S., 1994, Volatility and links between national stock markets, Econometrica 62, 901–933.

King, M., Wadhwani, S., 1990, Transmission of volatility between stock markets, Review of Financial Studies 3, 5–33.

Lin, W.-L., Engle, R. F., Ito, T., 1994, Do bulls and bears move across borders? International transmission of stock returns and volatility, Review of Financial Studies 7, 507–538.

Meddahi, N., 2003, ARMA representation of integrated and realized variances, Econometrics Journal 6, 334–355.

Mei, J., Scheinkman, J. A., Xiong, W., 2009, Speculative trading and stock prices: Evidence from Chinese A-B share premia, Annals of Economics and Finance 10, 225–255.

Pantelidis, T., Pittis, N., 2004, Testing Granger causality in variance in the presence of causality in mean, Economic Letters 85, 201–207.

Paparoditis, E., Politis, D. N., 2002, The local bootstrap for Markov processes, Journal of Statistical Planning and Inference 108, 301–328.

Rajarshi, M. B., 1990, Bootstrap in Markov-sequences based on estimates of transition density, Annals of the Institute of Statistical Mathematics 42, 253–268.

Robinson, P. M., 1991, Consistent nonparametric entropy-based testing, Review of Economic Studies 58, 437–453.

Roll, R., 1989, Price volatility, international markets link and implications for regulatory policies, Journal of Financial Services Research 3, 211–246.

Sensier, M., van Dijk, D., 2004, Testing for volatility changes in U.S. macroeconomic time series, Review of Economics and Statistics 86, 833–839.

van Dijk, D., Osborne, D. R., Sensier, M., 2005, Testing for causality in variance in the presence of breaks, Economic Letters 89, 193–199.

Wongswan, J., 2006, Transmission of information across international equity markets, Review of Financial Studies 19, 1157–1189.

Zhang, L., 2006, Efficient estimation of stochastic volatility using noisy observations: A multi-scale approach, Bernoulli 12, 1019–1043.

Zhang, L., Mikland, P., Aït-Sahalia, Y., 2005, A tale of two time scales: Determining integrated volatility with noisy high frequency data, Journal of the American Statistical Association 100, 1394–1411.

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Table 1
Empirical size using bootstrap critical values

To examine empirical size, we simulate intraday returns from two independent CIR processes and then test for conditional independence in variance using bootstrap critical values. We consider tests at the 5% and 10% levels of significance for sample sizes of 400 and 600 daily realized variances based on $M = 144$ intraday observations. We set the bandwidth scaling factors to $\kappa_b \in \{1/2, 3/4, 1\}$ and $\kappa_h \in \{3/4, 1, 3/2, 5/2\}$, with $\kappa_b < \kappa_h$. All results rest on 1,000 Monte Carlo replications and 300 bootstrap artificial samples of 100 daily observations.

κ_b		5%				10%		
κ_h	3/4	1	3/2	5/2	3/4	1	3/2	5/2
Panel A: $T = 400$								
1/2	0.065	0.055	0.050	0.047	0.121	0.107	0.134	0.162
3/4		0.052	0.048	0.049		0.128	0.117	0.121
1			0.067	0.063			0.133	0.156
Panel B: $T = 600$								
1/2	0.071	0.065	0.061	0.058	0.124	0.116	0.129	0.142
3/4		0.054	0.057	0.056		0.109	0.114	0.132
1			0.058	0.059			0.120	0.137

Table 2
Descriptive statistics for index returns

We collect transactions data for the S&P 500, FTSE 100, SSE B share, and Topix 100 indices. The sample spans the period ranging from January 3, 2000 to December 30, 2005. We document the main descriptive statistics for the index percentage returns with continuously compounding at regular sampling intervals of 1 and 30 minutes. The sample does not include overnight returns, so that the first intraday return refers to the opening price that ensues from the pre-sessional auction.

	S&P 500	FTSE 100	Topix 100	SSE B share
sampling frequency: 1 minute				
mean	−0.0001	−0.0001	−0.0003	−0.0004
standard deviation	0.0448	0.0403	0.0531	0.0525
minimum	−1.9020	−2.4636	−1.4095	−1.5255
maximum	1.5562	2.4611	1.0710	2.3179
skewness	−0.1149	0.2698	0.0429	0.9108
kurtosis	34.4874	212.918	21.4390	58.9127
zero returns	3.58%	5.24%	5.57%	19.39%
sampling frequency: 30 minutes				
mean	−0.0014	−0.0021	−0.0084	−0.0098
standard deviation	0.2910	0.2691	0.3170	0.5516
minimum	−3.3035	−4.1526	−4.8155	−6.4782
maximum	3.9838	2.9476	3.5572	5.0368
skewness	0.0459	−0.2768	−0.2314	0.0450
kurtosis	13.1125	20.8049	14.6988	15.3793
zero returns	1.72%	1.66%	1.04%	2.51%

Table 3

Daily volatility transmission to the US

We report the outcome of the bootstrap test for conditional independence of the S&P 500 index daily realized measures and of the VIX index with respect to the daily realized measures of the FTSE 100, Topix 100, and SSE B share indices. We employ the following realized measures based on 1-minute and 5-minute returns: realized variance (RV), tripower variation (TV), two-scale realized variance (TS), and realized kernel (RK). In addition, we also compute the realized variance and tripower variation using 15-minute and 30-minute returns. We first standardize the logarithm of the data by their mean and standard deviation and then estimate the conditional densities by means of kernel smoothing. As per the weighting function, we employ a standard multivariate normal density. To obtain critical values, we construct $B = 500$ bootstrap artificial samples of size $\mathcal{T} = 250$ by resampling blocks of 4 daily observations.

	S&P 500 index				VIX index			
	RV	TV	TS	RK	RV	TV	TS	RK
FTSE 100 index								
1 minute	0.024	0.016	0.306	0.084	0.064	0.090	0.002	0.000
5 minutes	0.294	0.274	0.032	0.284	0.044	0.160	0.004	0.002
15 minutes	0.306	0.274			0.086	0.002		
30 minutes	0.448	0.068			0.094	0.002		
Topix 100 index								
1 minute	0.014	0.014	0.190	0.350	0.000	0.004	0.064	0.160
5 minutes	0.354	0.104	0.108	0.168	0.302	0.320	0.136	0.162
15 minutes	0.350	0.956			0.192	0.228		
30 minutes	0.124	0.202			0.422	0.410		
SSE B share index								
1 minute	0.014	0.280	0.210	0.136	0.038	0.178	0.140	0.188
5 minutes	0.070	0.010	0.406	0.052	0.160	0.044	0.384	0.142
15 minutes	0.412	0.240			0.234	0.194		
30 minutes	0.194	0.422			0.082	0.184		

Table 4

Daily volatility transmission to the UK

We report the outcome of the bootstrap test for conditional independence of the FTSE 100 index daily realized measures with respect to the daily realized measures of the S&P 500, Topix 100, and SSE B share indices. The test details are exactly as in Table 3.

	RV	TV	TS	RK
S&P 500 index				
1 minute	0.032	0.016	0.082	0.038
5 minutes	0.092	0.084	0.822	0.100
15 minutes	0.190	0.000		
30 minutes	0.090	0.200		
Topix 100 index				
1 minute	0.044	0.078	0.058	0.210
5 minutes	0.132	0.318	0.022	0.014
15 minutes	0.044	0.012		
30 minutes	0.256	0.304		
SSE B share index				
1 minute	0.036	0.094	0.046	0.000
5 minutes	0.342	0.304	0.582	0.122
15 minutes	0.216	0.002		
30 minutes	0.120	0.034		

Table 5

Daily volatility transmission to Japan

We report the outcome of the bootstrap test for conditional independence of the daily realized measures of the Topix 100 index with respect to the daily realized measures of the S&P 500, FTSE 100 and SSE share B indices. The test details are exactly as in Table 3.

	RV	TV	TS	RK
S&P 500 index				
1 minute	0.002	0.018	0.474	0.284
5 minutes	0.176	0.178	0.012	0.070
15 minutes	0.842	0.592		
30 minutes	0.244	0.342		
FTSE 100 index				
1 minute	0.002	0.052	0.156	0.182
5 minutes	0.074	0.402	0.050	0.084
15 minutes	0.032	0.010		
30 minutes	0.496	0.530		
SSE B share index				
1 minute	0.018	0.012	0.770	0.290
5 minutes	0.242	0.054	0.366	0.182
15 minutes	0.492	0.442		
30 minutes	0.120	0.816		

Table 6

Daily volatility transmission to China

We report the outcome of the bootstrap test for conditional independence of the daily realized measures of the SSE B share index with respect to the daily realized measures of the S&P 500, FTSE 100 and Topix 100 indices. The test details are exactly as in Table 3.

	RV	TV	TS	RK
S&P 500 index				
1 minute	0.018	0.778	0.084	0.232
5 minutes	0.214	0.052	0.232	0.376
15 minutes	0.122	0.140		
30 minutes	0.536	0.476		
FTSE 100 index				
1 minute	0.020	0.002	0.684	0.004
5 minutes	0.140	0.086	0.218	0.282
15 minutes	0.082	0.006		
30 minutes	0.112	0.432		
Topix 100 index				
1 minute	0.012	0.010	0.172	0.130
5 minutes	0.438	0.096	0.266	0.084
15 minutes	0.428	0.674		
30 minutes	0.458	0.138		

Table 7

Daily volatility transmission to the US using extra controls

We report the outcome of the bootstrap test for conditional independence of the S&P 500 index daily realized measures and of the VIX index with respect to the daily realized measures of the FTSE 100, Topix 100, and SSE B share indices. To better account for data persistence, we further condition the distribution of the S&P 500 realized measure on the VIX index and vice-versa. To filter global volatility shocks, we also consider tests for which we further condition the distribution of the S&P 500 realized measure on the FTSE 100 realized measure. The test details are as in Table 3.

	S&P 500 index				VIX index			
	RV	TV	TS	RK	RV	TV	TS	RK
FTSE 100 index	(+ VIX index)				(+ S&P 500 index)			
1 minute	0.024	0.024	0.044	0.116	0.024	0.014	0.026	0.120
5 minutes	0.072	0.082	0.034	0.002	0.066	0.066	0.028	0.004
15 minutes	0.020	0.016			0.016	0.004		
30 minutes	0.090	0.000			0.102	0.000		
Topix 100 index	(+ VIX index)				(+ S&P 500 index)			
1 minute	0.002	0.004	0.086	0.084	0.000	0.000	0.072	0.056
5 minutes	0.072	0.056	0.054	0.040	0.152	0.026	0.062	0.028
15 minutes	0.196	0.068			0.162	0.034		
30 minutes	0.038	0.042			0.018	0.142		
Topix 100 index	(+ FTSE 100 index)							
1 minute	0.060	0.004	0.130	0.180				
5 minutes	0.166	0.298	0.018	0.124				
15 minutes	0.040	0.310						
30 minutes	0.136	0.024						
SSE B share index	(+ VIX index)				(+ S&P 500 index)			
1 minute	0.024	0.284	0.090	0.242	0.040	0.346	0.008	0.274
5 minutes	0.082	0.210	0.176	0.210	0.294	0.382	0.394	0.462
15 minutes	0.014	0.260			0.002	0.352		
30 minutes	0.066	0.184			0.040	0.196		
SSE B share index	(+ FTSE 100 index)							
1 minute	0.032	0.000	0.294	0.000				
5 minutes	0.258	0.010	0.136	0.102				
15 minutes	0.052	0.030						
30 minutes	0.772	0.082						

Table 8
Hourly volatility transmission

We report the outcome of the bootstrap test for conditional independence using three hourly realized measures based on 1-minute returns: realized variance (RV), tripower variation (TV), and realized kernel (RK). The test details are exactly as in Table 3.

	RV	TV	RK
Panel A: Transmission to the S&P 500 index			
FTSE 100 index	0.282	0.766	0.000
Topix 100 index	0.160	0.068	0.268
SSE B share index	0.064	0.148	0.054
Panel B: Transmission to the FTSE 100 index			
S&P 500 index	0.040	0.066	0.118
Topix 100 index	0.560	0.722	0.462
SSE B share index	0.086	0.086	0.078
Panel C: Transmission to the Topix 100 index			
S&P 500 index	0.138	0.220	0.008
FTSE 100 index	0.058	0.188	0.040
SSE B share index	0.164	0.226	0.006
Panel D: Transmission to the SSE B share index			
S&P 500 index	0.014	0.072	0.534
FTSE 100 index	0.022	0.014	0.206
Topix 100 index	0.010	0.042	0.150

Figure 1

Realized measures of the daily variance of the index returns

The plots display the time series of the realized variance and tripower variation based on the 1-minute and 30-minute index returns as well as the realized kernel estimate of the daily variance based on 1-minute index returns. Intraday index returns refer to continuously compounded returns on the SSE B share, Topix 100, S&P 500, and FTSE 100 indices from January 3, 2000 to December 30, 2005.

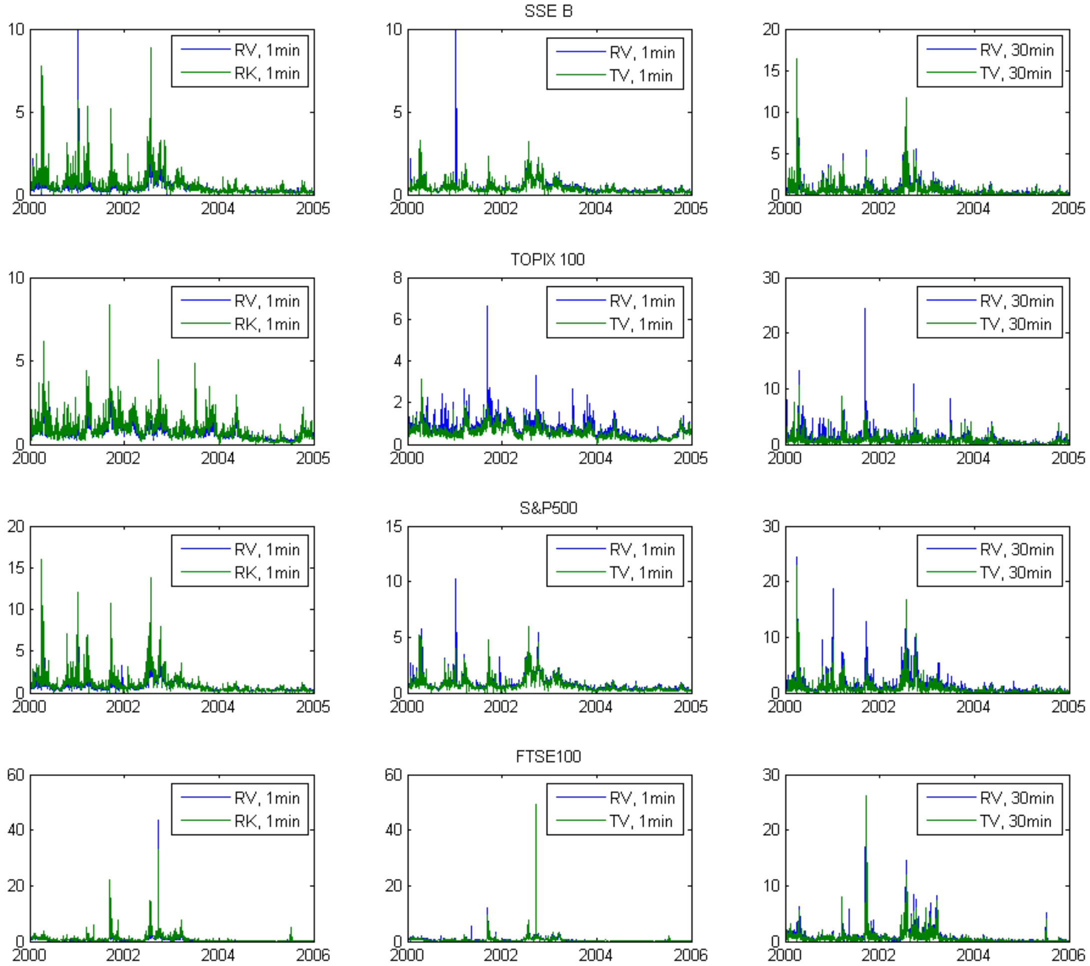
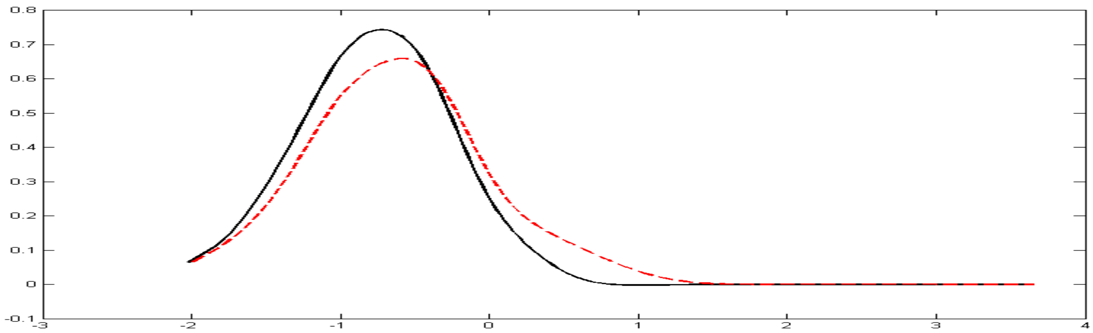
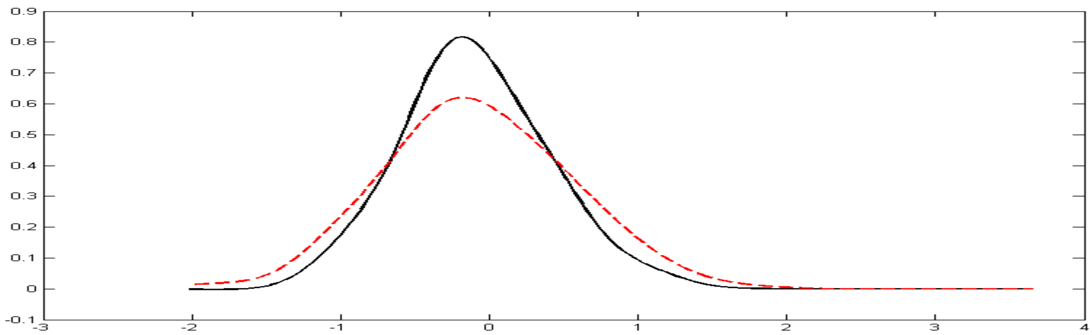


Figure 2
China spillovers to the UK

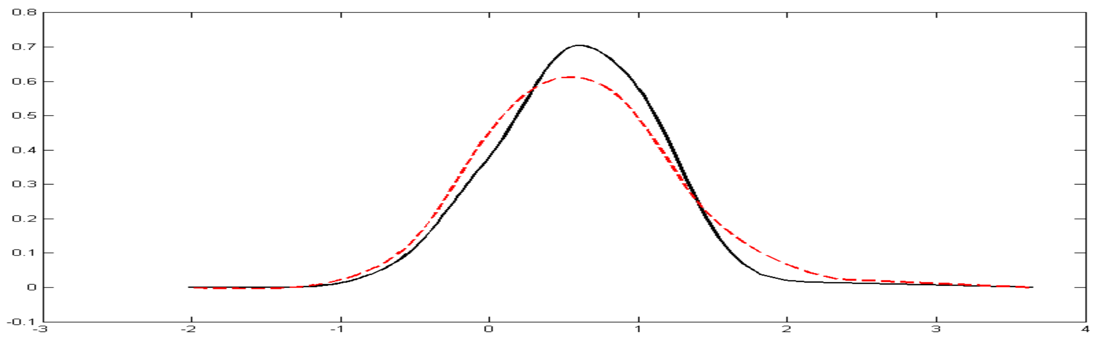
The dashed red line plots the conditional density of the log of the quadratic variation of the FTSE 100 index given its past realization, whereas the solid black line depicts a similar conditional density, but further conditioning on the log of the quadratic variation of the SSE B share index. We evaluate the conditioning variables at their (a) first quartile, (b) median, and (c) third quartile values. We measure quadratic variation using the tripower variation estimator at the 15-minute frequency. The kernel density estimation employs the same bandwidths we use in the tests; see Table 3 for details.



(a) first quartile



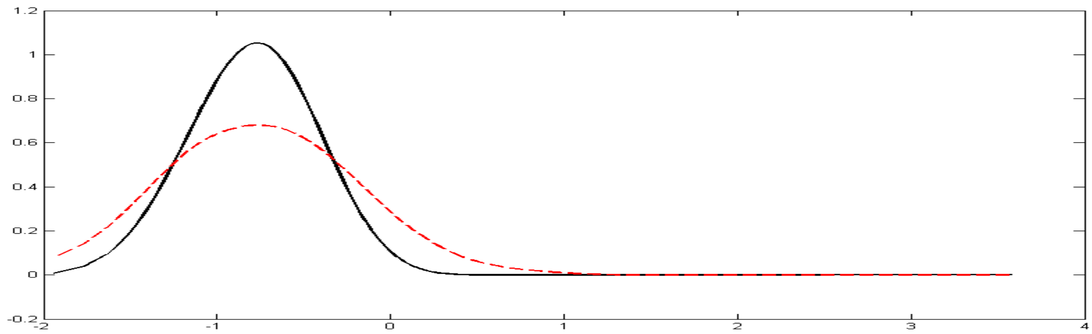
(b) median



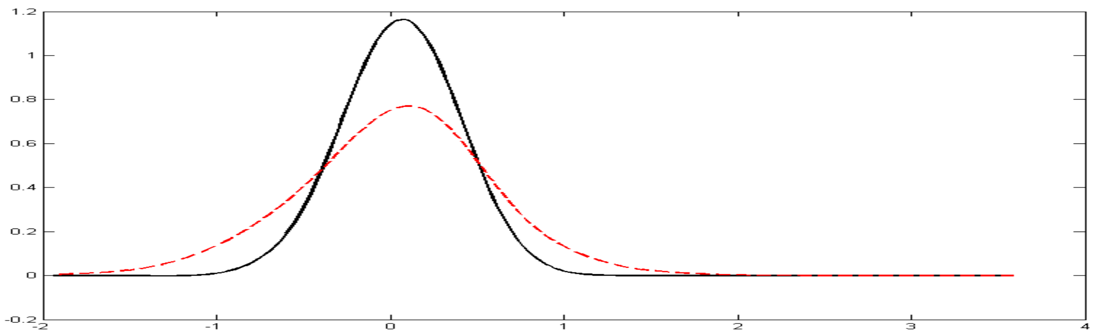
(c) third quartile

Figure 3
China spillovers to the US given the UK

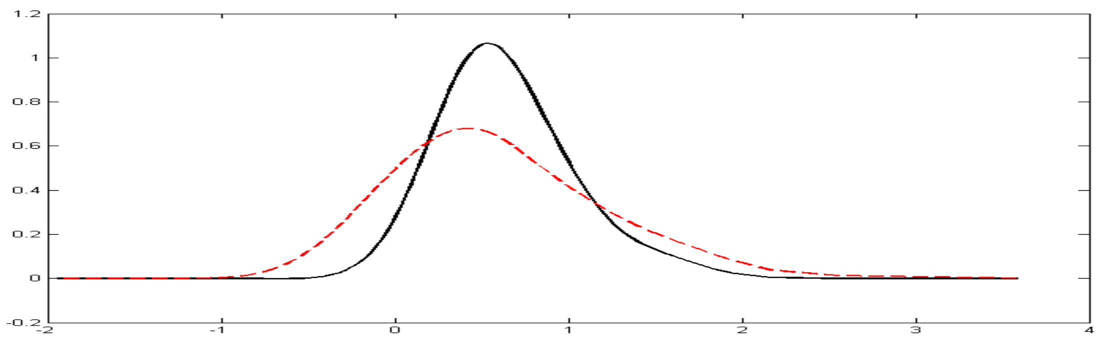
The dashed red line portrays the conditional density of the log of the quadratic variation of the S&P 500 index given its past realization and the log of the quadratic variation of the FTSE 100 index. The solid black line displays a similar conditional density, but further conditioning on the log of the quadratic variation of the SSE B share index. We evaluate the conditioning variables at their (a) first quartile, (b) median, and (c) third quartile values. We measure the daily quadratic variation using the realized kernel estimator at the 1-minute frequency. The kernel density estimation employs the same bandwidths we use in the tests; see Table 8 for details.



(a) first quartile



(b) median



(c) third quartile