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A Virtual Stakes Approach to Measuring Competition In Product Markets[1]

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[* Abstract]**

1 Introduction

A primary goal of the study of industrial organization is to understand the nature of competition in product markets. The strength of competitive forces in a market constrains firms' ability to raise price above marginal cost, thereby enhancing distributional and allocative efficiency. Furthermore, competition *for* a market sharpens firms' incentives to innovate, driving long-run economic growth and living standards. However, *measuring* the degree of competition in product markets has proven to be difficult. Lacking direct observation, contemporary empirical research in industrial organization has focused on estimating econometric models of demand and "supply" (i.e. pricing) in specific industries under specific competition assumptions, yielding inferences about marginal cost, market power, and markups. While useful, in practice this approach is limiting, as it typically imposes one or a few models of static competitive interaction (e.g. Bertrand-Nash competition, perfect collusion, etc.). This is contrary to the diversity of competition behavior one might expect in real-world product markets.

The purpose of this paper is to introduce *Virtual Stakes equilibria*, a static Nash equilibrium concept designed to flexibly parameterize competition in product markets, and to demonstrate how Virtual Stakes can be used to assess the degree of competition or coordination among firms in such markets. The defining feature of a Virtual Stakes equilibrium is that it accommodates the possibility that firms may, to varying degrees, internalize the impact of their decisions on their rivals' profits. This behavior is captured by a matrix of Virtual Stakes parameters that measures for each pair of products (or firms) the degree of coordination between them.

We demonstrate that Virtual Stakes equilibria may represent a range of competitive outcomes, from Bertrand-Nash competition to perfect collusion (joint industry profit maximization). The Virtual Stakes approach to measuring and testing for the extent of competition in product markets has the advantage over existing approaches (e.g.,

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Bresnahan, 1989, Nevo, 2001) of admitting intermediate degrees of competition between the extremes of perfect collusion and full competition. Although this result perhaps suggests a differentiated-products analog to the method of conjectural variations (Iwata, 1974, Gollop and Roberts, 1979, Nevo, 1998), Virtual Stakes is instead a Nash equilibrium concept that can serve as a foundation upon which conjectural variations can be overlaid, analogous to the analysis of Daughety (1985), who studies consistent conjectural variations built upon Cournot-Nash equilibrium. As we show by applying Daughety's (1985) approach to Virtual Stakes, the unique consistent conjectural variations equilibrium associated with a given Virtual Stakes equilibrium is again the original Virtual Stakes equilibrium. Thus, whereas the only consistent conjectural variations equilibrium in the standard approach is the Cournot-Nash equilibrium as Daughety has shown, Virtual Stakes can support a wide spectrum of conjectural variations-consistent competitive behaviors.

Although Virtual Stakes equilibria are a static solution concept, we show that they can be used to represent the results of dynamic competition and provide information about dynamic interactions among firms. We consider a dynamic competition environment characterized by coordination enforced by credible threats to punish deviation (e.g., Stigler, 1964, Friedman, 1971, Green and Porter, 1984, Rotemberg and Saloner, 1986, Fershtman and Pakes, 2000). In this setting, firms coordinate with respect to their discounted sum of profits subject to the constraint that no firm has an incentive to deviate from the terms of coordination. Factors such as high discount rates can render some outcomes, including the joint profit-maximizing outcome for example, unattainable because deviations cannot be credibly deterred.

While many equilibria are still enforceable in this setting, as is common in the literature, we restrict attention to Pareto optimal equilibria in which the coordinated profit outcome is such that no firm can be made better off by a departure from this outcome without making some other firm worse off. We consider first the case in which all Pareto-efficient outcomes are enforceable, so that Pareto optimality is the binding constraint on admissible outcomes. We then consider by example the case in which some Pareto optimal outcomes may not be enforceable, and usefully construct the set of constrained Pareto optimal outcomes. Along the way, we provide two examples of how Pareto optimal equilibrium outcomes might be selected repeatedly over time in the face of shocks to demand and cost conditions.

We show that any Pareto optimal coordinated outcome that may result at any point in time from dynamic competition can be represented as a static Nash equilibrium where firms maximize their own profits subject to constraints related to other firms' coordinated profit levels and incentives to deviate. Our key result is that this static Nash equilibrium is also a Virtual Stakes equilibrium in which the virtual stakes parameters are equivalent to the Lagrange multipliers of the constrained maximization problems that generate the static Nash equilibrium. These virtual stakes parameters or Lagrange multipliers measure how severely a firm's pricing is constrained by the requirement to maintain the terms of the coordination, and thereby provide a measure of the extent of that coordination.

This equivalence between the outcomes of static Virtual Stakes equilibria and dynamic Pareto optimal coordination provides a basis for empirical analysis of competitive behavior in product markets. We discuss how one can exploit the adjustments made to sustain Pareto optimal equilibrium outcomes in the face of shocks to underlying demand and cost conditions to identify and estimate the values of the virtual stakes. The Pareto optimality property allows estimation of virtual stakes that vary with outcomes over time because their magnitudes map directly to the outcomes of firms' optimizing behavior. Estimated virtual stakes can indicate which firms are party to coordination and for which pairs of firms the constraints of coordination are most and least binding. These are valuable tools for the empirical analysis of issues in antitrust. Furthermore, with a time series of observations on a relevant product market, estimation of time-varying virtual stakes can distinguish between periods of stable coordination and periods of breakdowns or threat implementation, possibly involving price wars.

The balance of this paper proceeds as follows. Section 2 introduces Virtual Stakes equilibria, describes how it parameterizes competition, and relates it to the method of Conjectural Variations. Section 3 establishes the underlying environment of dynamic competition and demonstrates the equivalence between Virtual Stakes equilibria and Pareto optimal dynamic coordination outcomes, including those Pareto optimal outcomes constrained by enforceability requirements. Section 4 discusses econometric issues surrounding identification and estimation of virtual stakes. Section 5 concludes.

2 Virtual Stakes

2.1 Virtual Stakes Nash Equilibria

We model an n-firm differentiated products industry with multi-product firms. Let k_i be the number of goods produced by firm i, i = 1, ..., n, p_i the $k_i \times 1$ vector of prices for goods of firm i, and $p \equiv (p_1',..., p_n')'$ the $k \times 1$ price vector of all the products in the market, where $k \equiv \sum_{i=1}^{n} k_i$. Let $q_i(p,\delta_i)$ be the $k_i \times 1$ vector of demands for goods of firm i, with δ_i a vector of demand shifters for goods of firm i. Let $c_i(q_i,\gamma_i)$ be the cost function for firm i, with γ_i a vector of cost shifters for goods of firm i. Finally, let ν_i be the $n \times 1$ vector of *virtual stakes* for firm i, $\nu_i \equiv (\nu_{i1},\ldots,\nu_{in})'$ with $\nu_i \equiv (\nu_1,\ldots,\nu_n)$, an n x n matrix. We emphasize that virtual stakes could more generally be defined at the level of each product. Constraining such virtual stakes to be common to all products owned by a given firm would then be a testable restriction. We present the theory at the level of the firm for notational convenience.

Let the "own" profit function for firm i be given by

$$\pi_i(p_i, p_{(i)}, \gamma_i, \delta_i) = p_i' q_i(p,\delta_i) - c_i(q_i(p,\delta_i),\gamma_i) \tag{1}$$

and the "total" (virtual) profit function for firm i be given by

$$\bar{\pi}_i(p_i, p_{(i)}, \nu_i, \gamma, \delta) = \sum_{h=1}^{n} \nu_{ih} \pi_h(p_h, p_{(h)}, \gamma_h, \delta_h), \tag{2}$$

where $p_i = S_i p$, with S_i the k_i x k selection matrix that selects p_i from p, $p_{(i)} = S_{(i)} p$, with $S_{(i)}$ the $(k - k_i)$ x k selection matrix that selects all but p_i from p, and $k_{(i)} \equiv k - k_i$. Assume that the demand and cost shifters (i.e., $\delta_i, \gamma_i, i = 1, \ldots, n$) are known to all firms.

The virtual stakes parameter vector, ν_i, reflects firm *i*'s virtual ownership of other firms. In essence, virtual stakes permit us to bring other firms' profits into the objective function of firm *i*. We call this firm *i*'s virtual profit. We defer until the next section a detailed discussion of the interpretation of virtual stakes parameters. At this point, one can treat them as ownership interests of firm *i* in the other firms in the market. In what follows, we establish features of a Virtual Stakes Nash equilibrium.

The derivative of firm *i*'s virtual profit function is given by

$$\frac{\partial \overline{\pi}_i}{\partial p_i} = \nu_{ii} \frac{\partial \pi_i}{\partial p_i} + \sum_{h \neq i} \nu_{ih} \frac{\partial \pi_h}{\partial p_i}, \tag{3}$$

where the total profit derivative components are given by

$$\frac{\partial \pi_i}{\partial p_i} = q_i + \frac{\partial q_i}{\partial p_i}(p_i - \frac{\partial c_i}{\partial q_i}) \tag{4}$$

$$\frac{\partial \pi_h}{\partial p_i} = \frac{\partial q_h}{\partial p_i}(p_h - \frac{\partial c_h}{\partial q_h}),\ h \neq i \tag{5}$$

$\frac{\partial q_h}{\partial p_i}$ is $k_i \times k_h$, and $\frac{\partial c_i}{\partial q_i}$ is $k_i \times 1$. The necessary first-order condition for profit maximization is given by

$$\frac{\partial \overline{\pi}_i}{\partial p_i} = \nu_{ii} q_i + \sum_{h \neq i} \nu_{ih} \frac{\partial q_h}{\partial p_i}(p_h - \frac{\partial c_h}{\partial q_h}) = 0. \tag{6}$$

The sum of the first two terms in the middle expression represents total marginal revenue and the third term represents total marginal cost. From (6) one can solve for the optimal pricing relation

$$p_i = p_i^*(p_{(i)}, \nu_i, \gamma, \delta),\ \ i = 1, \ldots, n. \tag{7}$$

This constitutes a structural relationship. Under present assumptions, all quantities are known to firms. By making suitable assumptions about what is known to the econometrician, this relationship can be estimated by generalized method of moments or instrumental variables. In Section 4 we discuss how this estimation can be operationalized.

A Virtual Stakes Nash equilibrium occurs when the relations (7) hold jointly for all firms. That is, the equilibrium price vector p^e satisfies

$$p_1^e = p_1^*(p_{(1)}^e, \nu_1, \gamma, \delta),$$
$$\vdots \tag{8}$$
$$p_n^e = p_n^*(p_{(n)}^e, \nu_n, \gamma, \delta),$$

or, using obvious notation,

$$p^e = p^*\left(p^e, \nu, \gamma, \delta\right). \tag{9}$$

Provided the mapping p^* is continuous, the existence of the equilibrium p^e follows from Brouwer's fixed point theorem. Global or local uniqueness holds under additional conditions on p^*.

2.2 Measuring Competition with Virtual Stakes

Virtual Stakes equilibrium can describe a range of competitive outcomes within an industry. To see this, divide equation (6) by ν_{ii}:

$$\frac{\partial \bar{\pi}_i}{\partial p_i} = q_i + \sum_{h=1}^{n} \frac{\nu_{ih}}{\nu_{ii}} \frac{\partial q_h}{\partial p_i}\left(p_h - \frac{\partial c_h}{\partial q_h}\right) \tag{10}$$

$$= q_i + \frac{\partial q_i}{\partial p_i}\left(p_i - \frac{\partial c_i}{\partial q_i}\right) + \sum_{h \neq i} \theta_{ih} \frac{\partial q_h}{\partial p_i}\left(p_h - \frac{\partial c_h}{\partial q_h}\right) = 0,$$

where $\theta_{ih} \equiv \nu_{ih} / \nu_{ih}$, $\theta_{ii} = 1$ defines a matrix of *normalized virtual stakes parameters*.

Suppose the $[\theta_{ih}]$ are fixed, scalar parameters. It is easy to show that different levels of normalized virtual stakes yield different degrees of competitive interaction in equilibrium. If $\theta_{ih} = 0, \forall h \neq i,$ firms do not factor in the impact of their decisions on their rivals' profits, yielding just the own-profit first-order condition (cf Eq. (4) above) for each firm. In equilibrium, this corresponds to the standard Bertrand-Nash competitive outcome. By contrast, if $\theta_{ih} = 1, \forall h \neq i,$ firms count equally the impact of their decisions on their own and their rivals' profits, yielding the following first-order condition for each firm:

$$\frac{\partial \bar{\pi}_i}{\partial p_i} = q_i + \frac{\partial q_i}{\partial p_i}\left(p_i - \frac{\partial c_i}{\partial q_i}\right) + \sum_{h \neq i} \frac{\partial q_h}{\partial p_i}\left(p_h - \frac{\partial c_h}{\partial q_h}\right) = 0\,. \tag{11}$$

In equilibrium, this corresponds to the industry joint profit-maximizing outcome. Other values of θ_{ih} are also possible, yielding a continuum of equilibria with varying degrees of competitive interaction. We provide examples of coordinated equilibria yielding various values of θ_{ih} in the next section.

2.3 Conjectural Variations and Virtual Stakes

2.3.1 Conjectural Variations Equilibria

The Virtual Stakes framework shares certain features of the method of conjectural variations introduced by (Bowley, 1924) and (Frisch, 1933) for homogenous products industries. In the canonical conjectural variations model (e.g., Perry, 1982), there are *n* firms that simultaneously produce a homogenous product, q_i, to maximize their own profits in a single period. In choosing quantities, firm *i* forms a *conjectural variation* about the combined output response of the other firms to a unit change in his quantity: $\partial(\sum_{j\neq i} q_j)/\partial q_i$. The defining feature of a conjectural variations equilibrium is that different conjectures by firms about their rivals' responses yield different degrees of competitive interaction in equilibrium, from full competition to the Cournot competitive equilibrium, to the industry joint profit-maximizing equilibrium.
Conjectural variations equilibria have been attacked on grounds that the underlying behavioral assumptions are in general logically inconsistent. As noted above, the definition of a firm's conjectural variation is the response of his rivals' actual choices to firm *i*'s actual choice. However, since firms in a static model choose their outputs simultaneously, it is impossible to have actual choices dependent on one another. To make use of this traditional concept of conjectural variations, one has to consider a sequential game (e.g., Stackelberg equilibria) or a dynamic game where firms' choices today may depend on their choices yesterday (e.g., Riordan, 1985).

As made clear by Daughety (1985) properly modeling conjectural variations in a static game requires a definition of conjectural variations based on *perceptions* that firms have about other firms' decision processes rather than on actual choices. This implies an infinite-regress model of firms' decisions: in the duopoly case, firm *i*'s optimal choice depends on his perception of firm *j*'s choice (q_{ij}), which in turn will depend on firm *i*'s perception about what firm *j* believes firm *i* will produce (q_{iji}), and so on. A proper conjectural variation is then defined as the impact to *i*'s perception of *j*'s optimal choice, that is, of his perception of what his rival thinks he will produce (dq_{ij}/dq_{iji}). Using this definition, Daughety shows in the homogeneous goods case that only the Cournot equilibrium yields conjectures that are consistent with firms' optimal behavior.

Despite these problems with the conjectural variations approach, empirical researchers continue to estimate "conduct parameters" based (at least loosely) on conjectural variations notions (Iwata, 1974, Gollop and Roberts, 1979, Applebaum, 1982, Porter, 1983, Brander and Zhang, 1990,Rubinovitz, 1993, Graddy, 1995, Genesove and Mullin, 1998). The rationale is now descriptive; while unsupported by theory, conduct parameters are interpreted as evidence of how firms actually behave, regardless of their beliefs (Bresnahan 1989, p.1029). The lack of theoretical support, however, has made this approach increasingly less common. Recent empirical research in industrial organization, although emphasizing ever-greater flexibility in the estimation of demand, has taken a "menu approach" (e.g. pure competition or pure collusion) to the estimation of strategic interaction (e.g., Bresnahan, 1987, Nevo 2001).

2.3.2 A Comparison with Virtual Stakes

Although both conjectural variations and Virtual Stakes embody notions of firms' competitive responses to one another, there is a fundamental difference. This difference is best appreciated by viewing the Cournot-Nash equilibrium in the homogeneous goods case discussed above as a foundation upon which conjectural variations are overlaid. Similarly, the Virtual Stakes Nash equilibrium also provides a foundation upon which conjectural variations can be overlaid. As we now proceed to show, each Virtual Stakes Nash equilibrium supports a unique consistent conjectural variations equilibrium, namely, the original Virtual Stakes equilibrium. It follows that Virtual Stakes equilibria are well-supported equilibria for any values of the virtual stakes parameters, thereby admitting a broad range of logically consistent competitive behaviors.

To establish this result, consider a proper definition of conjectural variations in a differentiated product market. For simplicity, consider the case of duopoly with firm *i* and firm *j*. Firm *i's* optimal price p_i depends on his perception of firm *j*'s choice, which we label p_{ij}. Firm *i*'s perception of firm *j*'s price p_{ij} in turn will depend on his perception about firm *j's* belief of firm *i*'s price, which can be labeled as p_{iji}. This process continues ad infinitum. Firm *i*'s perceptions can be represented by the sequence (p_{ij}, p_{iji}, p_{ijij}, …), where the first term in the subscript indicates whose perception it is and the last term indicates whose price choice it assumes. Analogous to Daughety (1985), we define conjectural variations on firm *i*'s perceptions as $\partial p_{ij}/\partial p_{iji}$, $\partial p_{iji}/\partial p_{ijij}$, and so on. For example, $\partial p_{ij}/\partial p_{iji}$ is what firm *i* believes to be firm *j*'s price response to *i*'s perception of firm *j*'s perception of firm *i*'s price.

Consistent conjectural variations must reflect common beliefs about firms' payoff functions. In the standard Bertrand-Nash equilibrium, firms' payoffs are simply their own profits as given by equation (1). In Virtual Stakes Nash equilibria, firms' payoffs are given by their total virtual profits as given by equation (2). This implies that firm *i*'s belief about firm *j*'s price p_{ij} should be the price that maximizes firm *j*'s virtual profit while taking firm *i*'s price as p_{iji}.

Consistent conjectural variations equilibria also require that conjectural variations should be internally consistent, that is, $p_{ij} = p_{ijij} = \ldots = p_{iji\ldots j}$, so that firm *i*'s perception about firm *j*'s choice is always the same, regardless of the stage of the conjectural process. Further, in consistent conjectural variation equilibria, firms' perceptions are correct. That is, $p_{ij}^e = p_j^e$, so that firm *i*'s perception of firm *j*'s choice is what firm *j* actually chooses in equilibrium and *vice versa*.

Now suppose that (p_i^e, p_j^e) is a Virtual Stakes equilibrium. It is easy to see that this is a consistent conjectural variations equilibrium, in which firms all share common beliefs about firms' payoff functions as given by their virtual profits, and their perceptions are simply given by $p_{\ldots j}^e = p_j^e$ and $p_{\ldots i}^e = p_i^e$. On the other hand, any consistent conjectural equilibrium with such common beliefs about payoffs must be a Virtual Stakes equilibrium, because firms' perceptions must be correct in a consistent conjectural variations equilibrium.

3 Virtual Stakes and Dynamic Competition

In the previous section, virtual stakes equilibria were shown to be equivalent to consistent conjectural variations equilibria for any values the virtual stakes parameters might take on. In that respect, virtual stakes are superior to other formulations of conjectural variations because they are consistent across values representing a spectrum of competitive conduct. Although the basic virtual stakes notion that firms coordinate by internalizing some degree of interest in their rivals' welfare is intuitively appealing, we have not yet provided any sound explanation as to how the values of these stakes are determined. Is there in fact any justification for the assumption that firms arrive at outcomes "as if" their conduct were that which leads to virtual stakes equilibrium outcomes? Moreover, how can these static equilibria capture the dynamic process of coordination among firms?

We now formulate a theoretical foundation for Virtual Stakes equilibria that addresses these questions. From this foundation comes a rigorously founded interpretation of the values of virtual stakes parameters with intuitive appeal and with potentially broad applications. The basic intuition behind this result can be simply stated.

Firms engaged in dynamic competition with one another may arrive at coordinated outcomes sustained over time as the equilibria of certain dynamic games of interaction. These equilibria may periodically break down and become reestablished or the equilibria themselves may exhibit switching behavior between periods of high coordinated profits and periods of lower competitive profits. In any event, even during stable periods in which coordinated profit levels are maintained, short run shocks to underlying conditions will require adjustments in those profit outcomes in each period consistent with the long run equilibrium strategies of the firms.

For a broad class of coordinated outcomes that may arise from dynamic game equilibria, the adjusted outcomes in any period can be represented as the outcomes of a static Nash equilibrium arising from profit maximization by firms constrained by conditions pertaining to sustaining the underlying dynamic equilibrium. As we shortly demonstrate, the parameters of a Virtual Stakes equilibrium turn out to be equivalent to the Lagrange multipliers (or certain functions thereof) for the constrained maximization problems that lead to the static Nash equilibrium. As such, virtual stakes measure the intensity with which the constraints associated with sustaining the coordination are binding, and they therefore can be interpreted as meaningful measures of the extent of that coordination. This interpretation is justified because there is a theoretical basis for the assumption that firms behave "as if" their conduct led to static Nash or virtual stakes equilibria in the short run. Finally, by observing firms react to short run shocks in underlying conditions in order to sustain dynamic coordination over the long run, we can estimate values for virtual stakes parameters. These values provide information about the nature of the dynamic coordination and the role of each firm in it.

In this section, we first set the stage by describing the dynamic competition environment to which Virtual Stakes equilibria are relevant. We then reasonably limit the set of outcomes considered to equilibria supporting certain Pareto optimal allocations of profit among coordinating firms. We begin with the case in which all such Pareto optimal allocations are assumed to lie within the set of allocations that can be achieved as an equilibrium outcome of a dynamic game of repeated interaction among the firms. We refer to this case as unconstrained Pareto optimal coordination because only the Pareto optimality condition imposes a binding constraint on the set of possible outcomes. The usual dynamic game constraint that no firm has an incentive to deviate from the coordinated outcome is not binding because no firm is assumed to have an incentive to deviate from any of the Pareto optimal outcomes considered.

We then show that all such coordinated Pareto optimal outcomes at any given time correspond to the outcomes generated by static Nash or Virtual Stakes Nash equilibrium. The values of the virtual stakes parameters are the values associated with the Lagrange multipliers of the constrained profit maximization problems equivalently solved by the firms to reach the outcomes of the static Nash equilibrium. We next provide examples of how coordinating firms might select particular dynamic equilibria that generate Pareto optimal outcomes in every period of coordination, and how the associated static Nash equilibria might be used by those firms to compute outcomes in each period adjusted for short run shocks to underlying conditions.

We next demonstrate by example how the result can be extended to the case of constrained Pareto optimal coordination. Coordination is constrained in this case in the sense that not all Pareto optimal outcomes can be achieved as equilibria to particular dynamic games of interaction. Consequently, coordinated outcomes are limited to those that are Pareto optimal given the constraint that they also are enforceable as dynamic equilibria. This constraint in turn imposes additional constraints on firms in the constrained profit maximization problems that generate the associated static Nash equilibria. The result is a more complex set of virtual stakes parameters, which provide richer information about the coordination among firms. Finally, we conclude this section with a discussion of the interpretation of these virtual stakes parameters and the values they might be observed to take on.

3.1 Dynamic Competition Environment for Virtual Stakes

One environment relevant to Virtual Stakes equilibria is characterized by repeated interaction among a set of rival firms in a differentiated product market. We assume that price is the only strategic choice variable. Although we could allow the number of firms in the market to change over time in certain ways, for simplicity we consider a stable set of firms in all that follows. Our results could be extended to multi-product firms, but again for simplicity of notation, we assume single product firms. We assume that all firms have sufficient information about their own and rivals' costs and demands to make the computations required to solve the problems presented to them as described in what follows.

In the modern literature, firm interactions in this kind of dynamic competition environment have been modeled formally as infinitely repeated games by a number of researchers. Some important examples are found in Friedman (1971), Green and Porter (1984), Rotemberg and Saloner (1986), Abreu, Pearce and Stacchetti (1990) and Fershtman and Pakes (2000). In all of these models, each firm is assumed to maximize its discounted expected profits. All obtain the well-known dynamic game result that many equilibria are possible, with coordinated outcomes ranging from the most competitive (Bertrand or Cournot) to the most collusive (perfect joint profit maximizing).

These papers demonstrate that certain explicitly or tacitly coordinated equilibria yielding profits strictly above the most competitive outcome can be supported by "trigger strategies." Typically, a trigger strategy requires the firm to commit to produce at the competitive level for a fixed (possibly infinite) period in the event that a rival deviates from the conduct necessary to sustain the coordinated equilibrium allocation of profits superior to competitive profits. Such punishment threats reduce the incentive for firms to unilaterally depart from the coordinated equilibrium trigger strategy.

As noted by Green and Porter (1984), trigger strategies may appear to work all too well, implying no deviations from the superior profit coordinated outcome over time. Green and Porter introduce demand uncertainty and asymmetric information to show how these factors can, during periods of low demand, result in reversion to the threat outcome, a period commonly characterized as a "price war." Rotemberg and Saloner (1986) show a similar outcome due to exogenous demand shocks that change deviation incentives. Abreu, Pearce and Stacchetti (1990) extend the Green and Porter result to show how equilibrium strategies can incorporate switching between coordinated outcome and threat phases. Freshtman and Pakes (2000) show how periodic reversion to a Bertrand or threat outcome can provide information needed to reestablish the coordinated outcome. A developing theme in these results is that periodic switching between superior coordinated and competitive threat outcomes may not reflect departures from a dynamic equilibrium, but rather may be conduct necessary to support that equilibrium.

These models describe events consistent with those observed in the real world. Many product markets exhibit apparent patterns of alternating periods of "war" and "peace." Peaceful periods are characterized by stable adherence to superior coordinated outcomes and are punctuated by wartime periods of volatile pricing and lower profits. The well-known recent price-fixing conspiracy in the production of vitamins provides a salient example of this alternating war and peace phenomenon in a contemporary product market.[2] The conspiracy spanned 1990 to 1999, covered nearly all vitamin product markets, and included all the major producers of vitamins worldwide.

Figure 1 reports transaction prices for one product covered by the cartel – Vitamin C – from three major producers over a period of 14 years. **Cite? Greg, where did you get this? Can we use it?** These firms – BASF, Hoffman-LaRoche, and Takeda – pled guilty to fixing prices for the period January 1991 to Fall 1995.[3]

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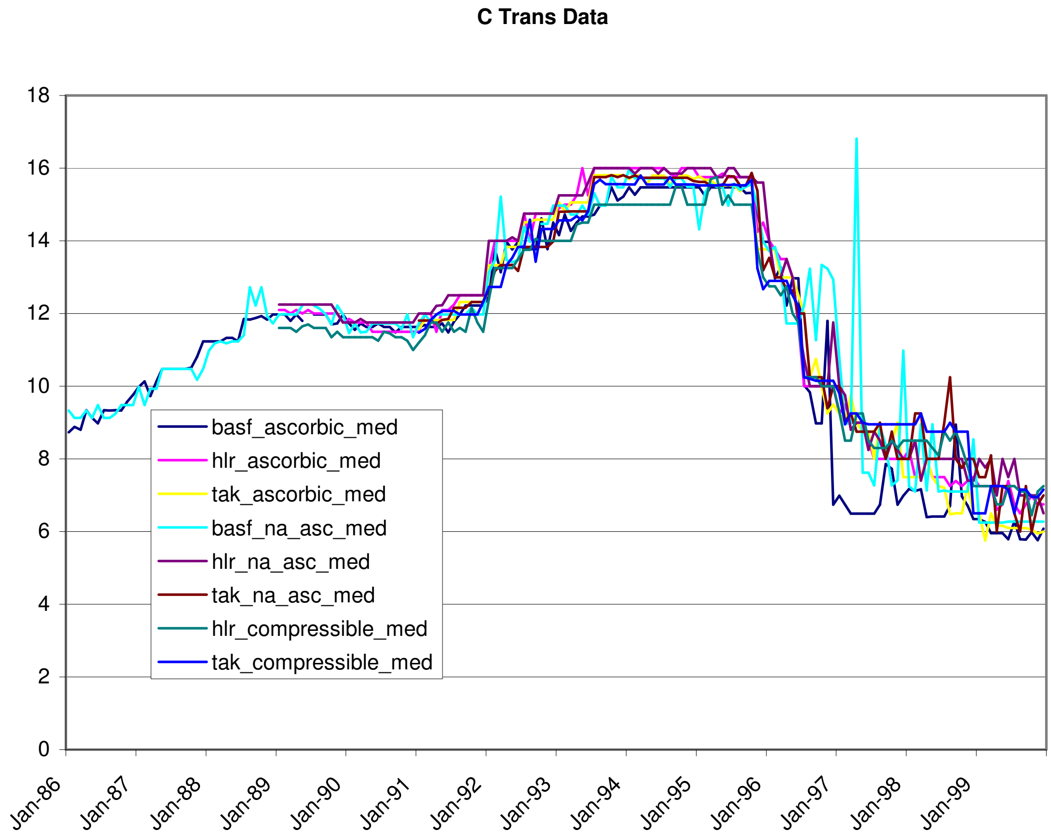
[2] The Vitamins cartel – dubbed "Vitamins, Inc." – was, in the words of Joel Klein, U.S. Assistant Attorney General for Antitrust, "the most pervasive and harmful criminal antitrust conspiracy ever uncovered." The U.S. Department of Justice fined the firms involved a record $1 billion (Barbosa (2001), *New York Times,* "Tearing Down the Facade of 'Vitamins Inc." October 10, 1999). Economic experts associated with and supported by Bates White provided testimony for plaintiffs in much of the related litigation.

[3] (DeRoos (2001) Note that this admission does not exclude the possibility of collusive behavior before or after the stated period.

Figure 1



As can be seen from the figure, prices began to rise after their agreement went into effect, peaking in the early summer of 1993 at a level 33% higher than previous (1990) levels. Prices remained stable until November 1995 at which point the agreement ceased. Prices quickly fell 50%, to levels lower even than those prevailing in 1990.[4] As modern theory has revealed, a price war need not reflect the breakdown of a coordinated equilibrium; it may in fact represent a phase necessary to sustaining the coordination over the longer run.

However, consider another feature of Figure 1. Even during the almost 5 year peaceful period during which the firms allegedly were able to sustain a superior profit coordinated outcome, we observe many smaller short run variations in prices. This highlights the fact that in order to sustain the stable superior coordinated phase over a significant period, firms also will likely find it necessary to adjust to short run shocks. Such shocks may not be sufficient to cause a breakdown in coordination or to trigger the switch to the alternative phase of the equilibrium, but nevertheless will require the firms to make finer

[4] Unlike many vitamins markets, there was significant entry in Vitamin C during the collusive period. DeRoos (2001) analyzes collusive behavior in the presence of a competitive fringe using data from this market.

adjustments to sustain the equilibrium and prevent mistaken triggering unwanted by all firms.

If firms believe they have enough information to adjust outcomes sufficiently to sustain the superior coordinated phase of the equilibrium in the face of small shocks, all will likely expect each firm to make the necessary calculations and adjustments. If it is perceived that some firm has not made the required adjustments, and has instead deviated from the equilibrium strategy, this could unnecessarily cause a breakdown or trigger a switch to the alternative and lower profit phase. The better the information, the greater will be the range of short run adjustments the firms can accommodate. With poor information, firms will likely find it necessary to adhere closely to fixed pricing to avoid the perception of deviation. Even with good information, when large shocks or large accumulated effects over time render the information available insufficient for adjustments that avoid the perception of deviation, it may then become necessary to reestablish the equilibrium or to "pull the trigger" that switches the equilibrium phase.

A major concern of this paper is these fine-tuning adjustments and what observation of them can reveal about the underlying long run equilibrium. Similar pricing patterns to those observed in vitamins are common in the study of many product markets. But when are these patterns evidence of coordinated conduct and how does one measure the extent of that coordination? The current literature (e.g. Bresnahan 1989, Nevo 2001) can only offer means of discriminating between the most and least competitive outcomes. By modeling these short run adjustments, virtual stakes offers a means of obtaining much richer information about the extent of long run coordination between these extremes.

3.2 Virtual Stakes and Pareto Optimal Coordination

We study a broad class of coordinated equilibria that can be supported by firms adopting trigger strategies. These equilibria are characterized by outcomes in which profits levels superior to competitive profits may be sustained indefinitely or may alternate with or switch to lower profit levels associated with the realization of the trigger strategy threat. We will refer to the equilibrium phase in which profits superior to competitive profits are sustained as the coordinated phase and the alternative state as the threat phase. The threat phase in our models, as in many such models in the literature, will correspond to the competitive or Bertrand equilibrium for these differentiated product markets.

As noted above, these models generally have many equilibria that are distinguished primarily by different allocations and levels of superior profits attained in the coordinated phase. Following much of the literature, we will limit the set of equilibria to those in which the profit allocation in the coordinated phase is Pareto optimal at any point of that phase (Green and Porter, 1984, Rotemberg and Saloner, 1986, Fershtman and Pakes, 2000). This yields the reasonable implication that any coordinated outcome reached among firms is such that the profit of any firm cannot be increased in the coordinated phase without reducing the coordinated profit of some other firm.

In the balance of this section, we mine the implications of Pareto optimality to establish the correspondence between the outcomes of static Nash or Virtual Stakes equilibria and the shock accommodating outcomes that sustain the coordinated phase of the equilibrium. We consider two forms of Pareto optimal coordination: constrained and unconstrained.

First, we consider unconstrained Pareto optimal coordination, which is unconstrained in the sense that all possible Pareto optimal allocations of profit are assumed to be enforceable by some trigger strategy set. While this strong assumption will generally not be met for many simple trigger strategies, augmenting such strategies with feasible side payments or additional penalties may render the assumption reasonable. After establishing the relationship to virtual stakes equilibria for unconstrained Pareto optimal coordination, we then offer an example of how the result can be extended to constrained Pareto optimal coordination in which the limitations on supportable allocations imposed by particular trigger strategies are explicitly taken into account. We also provide two examples of equilibrium selection that can sustain Pareto optimal outcomes in the face of shocks to underlying conditions, and show how the Pareto optimality condition can facilitate the computation of outcomes.

3.2.1 Unconstrained Pareto Optimal Coordination

We first show a key result concerning the character of any Pareto optimal allocation of profit among firms. We assume in all that follows that profit functions are strictly concave and monotonically increasing in the prices of other firms. Under these assumptions, for any allocation profit vector $\underline{\pi}^* = (\pi_1^*, \pi_2^*, \cdots, \pi_n^*)$, there is a unique price vector, $\underline{p}^*$, that supports these profits, that is, $\pi_i(\underline{p}^*) = \pi_i^*, \forall i = 1, \ldots, n$.

<u>Proposition 1</u>

A price vector $\underline{p}^*$ generates a Pareto optimal allocation of profits among *n* firms if and only if that price vector solves the following constrained maximization problem for each of the *n* firms. For the *i*-th firm, the problem is stated as:

$$\pi_i^* = \max_{p = (p_1, p_2, \ldots, p_n)} \pi_i\left(p_1, p_2, \ldots, p_n\right)$$

$$\text{s.}t.\ \pi_j\left(p_1, p_2, \ldots, p_n\right) \geq \pi_j^* \qquad \forall j \neq i \tag{12}$$

<u>Proof</u>

Suppose p^* solves the maximization problem for each firm but does not generate a Pareto optimal profit allocation. Then, there is a π generated by another price vector where some firms are better off. If so, p^* does not solve some of the maximization problems, a contradiction. Conversely, suppose p^* does produce a Pareto optimal profit allocation but it does not solve one of the maximization problems. Let p' be the solution to that problem. Then, p' makes one firm better off without reducing the profits of other firms below the levels set by p^*. So, p^* cannot be Pareto optimal. QED

In Equation (12), the *i*-th firm is selecting a vector of prices for *all* firms in order to maximize only its own profits subject to constraints that ensure all other firms earn profits no less than the amounts required by the coordinated allocation. But since the allocation is Pareto optimal, all firms must choose price vectors that force all other firms' profits to down to their constraint levels to maximize their own profits. With all constraints binding, there is a unique vector of *n* prices $\left(p_1^*, p_2^*, \ldots, p_n^*\right)$ that generates the constrained profit allocation $\left(\pi_1^*, \pi_2^*, \ldots, \pi_n^*\right)$ and solves the maximization problems of all firms. This price vector $\left(p_1^*, p_2^*, \ldots, p_n^*\right)$ satisfies the first-order conditions for the *i*th firm's constrained maximization problem as given by

$$\sum_j \theta_{ij}(\underline{p}^*)\pi^*_{jk}(\underline{p}^*) = 0 \qquad \forall k = 1,2,\ldots,n \tag{13}$$

where $\pi_{jk} = \frac{\partial \pi_j}{\partial p_k}$, $\theta_{ii} = 1$, and $\theta_{ij} > 0,\ j \neq i$ is the Lagrange multiplier for the constraint of the *j*th firm's profits on the *i*th firm's pricing. Clearly, the choice of the *i*th firm is arbitrary.

Thus, the prices that generate any Pareto optimal profit allocation can be represented as the solutions to the static profit maximization problem of any one firm constrained by the profit allocations to all other firms. This does not address how the firms select any particular coordinated allocation of profits, but that is not the primary problem for which we seek a solution. We presume that a selection mechanism or rule has been arrived at, either tacitly or explicitly by the firms, which will be used to repeatedly recompute the coordinated profit allocation as demand and cost shocks arrive that require adjustments to maintain the coordinated outcome. Again, we assume here that these short run adjustments generate no incentive for deviation from the coordinated outcome enforced by the trigger strategies. Given the adjusted allocation of profit, however determined, the prices that generate that allocation can be represented as the solutions to the individual firm constrained maximization problems provided the adjusted outcome remains Pareto optimal.

3.2.2 Coordinated Outcome Selection

We provide in this subsection a pair of examples of the kinds of rules that might be used to compute coordinated outcomes that retain the Pareto optimal property as adjustments are made in response to shocks to underlying conditions. In the first example, following Schmalensee (1987), we consider a coordinated outcome as a solution to a bargaining game. In particular, we adopt a solution concept from axiomatic bargaining theory, the asymmetric Nash bargaining solution. In the case of explicit collusion, such a solution can arise from any bargaining process that results in a reasonable bargaining solution as specified by the underlying axioms.[5] It is well known that asymmetric Nash bargaining solutions are (strongly) Pareto optimal (e.g. (Roth 1979, p.18)). Here we can also establish the following proposition.

[5] This solution concept satisfies three of the four Nash axioms, that is, Invariance to Equivalent Utility Utility Representation, Independence of Irrelevant Alternatives, and Pareto optimality. By not imposing the Symmetry axiom, we allow the possibility that firms may differ in their bargaining abilities.

Proposition 2

Under the market conditions specified above, any Pareto-optimal profit allocation in which all firms receive profits greater than those they would receive in a static Bertrand equilibrium can be obtained from an asymmetric Nash bargaining solution.

Proof
In our model, an asymmetric Nash bargaining solution yields a price vector, p, and is given by

$$\underline{p}(\underline{\alpha}) = \arg\max_{\underline{p}} \Pi_{j=1}^{n} \left(\pi_j(\underline{p}) - \pi_j^b \right)^{\alpha_j} \tag{14}$$

where π_j^b is the profit to firm *j* from the Bertrand (competitive) outcome and α_j represents firm *j*'s "bargaining ability" (Nash, 1950).[6] The objective function is the geometric mean of excess profits from coordination. The profits associated with the optimal prices, $\underline{p}(\underline{\alpha})$, are $\underline{\pi}(\underline{\alpha})$. We assume π_j^b is the disagreement outcome for firm *j* in the bargaining game. Without loss of generality, we can normalize $\underline{\alpha}$ such that $\sum_{j=1}^{n} \alpha_j = 1$.

As noted above, we know that $\underline{\pi}(\underline{\alpha})$ is a Pareto optimal allocation. By varying the weights $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, we can trace out the set of Pareto optimal allocations of profits that strictly dominate the Bertrand outcome, i.e., $\pi_i^* > \pi_i^b, \forall i = 1,2,\ldots,n.$ This is closely related to the standard result of equivalence between social welfare maximization and Pareto optimality in an exchange economy. For any Pareto optimal allocation of profits $\underline{\pi}^* = \left(\pi_1^*, \pi_2^*, \ldots, \pi_n^*\right)$ such that $\pi_i^* > \pi_i^b, \forall_i = 1,2,\ldots,n.$, we need to show that there is an $\underline{\alpha} = \left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) \in \left(0,1\right)^n$ and $\sum_{j=1}^{n} \alpha_j = 1$ such that the Nash solution is the same as $\underline{\pi}^*$. This can be easily seen by comparing the first-order conditions that characterize Pareto optimality with those that characterize the asymmetric Nash bargaining solution. The first-order conditions that characterize the asymmetric Nash bargaining solution to (14) are

[6] Formally, in a bargaining game of alternating offers, differences in α_i can be explained by differences in time preferences ((Osborne and Rubinstein 1990, Chapter 4)).

$$\sum_{j=1}^{n} \frac{\alpha_j}{\pi_j^* - \pi_j^b} \cdot \pi_{jk}^* = 0, \qquad k = 1,2,\ldots,n \tag{15}$$

Comparing (13), the first order conditions that characterize Pareto optimality, and (15) shows that $\underline{p}^*$ is clearly the asymmetric Nash bargaining solution if

$$\frac{\alpha_j}{\pi_j^* - \pi_j^b} \cdot s_i = \theta_{ij}, \qquad j = 1,2,\ldots,n \tag{16}$$

where s_i is an arbitrary scale factor. By properly choosing the scale factor such that $s_i = \sum_j \theta_{ij}(\pi_j^* - \pi_j^b)$, (16) gives the weights $\underline{\alpha} = \left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) \in \left(0,1\right)^n$ and $\sum_{j=1}^{n} \alpha_j = 1$ for which $\underline{p}^*$ gives the asymmetric Nash bargaining solution. QED

In the above, we derived the weights of the asymmetric Nash bargaining solution from the Lagrange multipliers of firm *i*'s constrained maximization problem. The choice of the *i*-th firm is arbitrary again because p^* solves the constrained maximization problems for *all* firms, which imposes restrictions on the $n * (n-1)$ Lagrange multipliers. In fact, Pareto optimality implies only *n* - 1 independent Lagrange multipliers. Given the *n* - 1 Lagrange multipliers in one firm's constrained maximization problem, the Lagrange multipliers in all other firms' constrained maximization problems are determined by Pareto optimality.

Another example of a decision rule for selecting the Pareto optimal coordinated outcomes is for firms to allocate profits among themselves in the same proportions they would receive in a Bertrand equilibrium. This is a much simpler rule and might be arrived at tacitly. Moreover, the constrained profit maximization problems could be used directly to solve for the new Pareto optimal allocations as adjustments to shocks were made. With the asymmetric Nash bargaining rule, the adjusted prices could be found by resolving the bargaining problem (given a fixed set of bargaining weights), so the constrained profit maximization problem remains an equivalent representation of the result.

With the Bertrand proportions rule, firms could adjust by first computing the Bertrand outcome for all firms under the new conditions and determining the correct proportions by which profits are to be allocated. Those proportion values would be incorporated into the constraint values of the constrained optimization problems of the individual firms that correspond to problems set out in (12) above. For the *i*th firm, the problem can be stated as

$$\pi_i^* = \max_{\underline{p}=(p_1,p_2,\ldots,p_n)} \pi_i\left(p_1, p_2, \ldots, p_n\right)$$

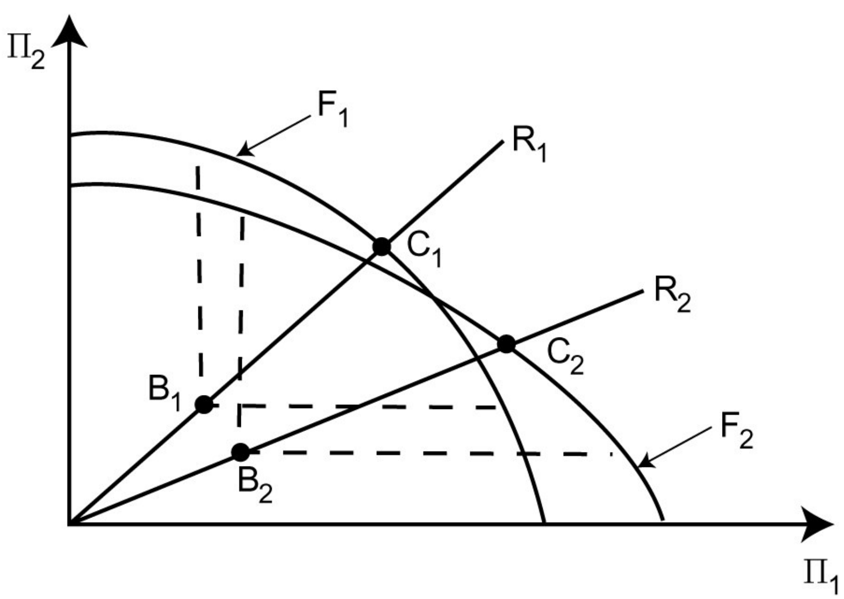
$$\text{s.}t.\,\pi_j\left(p_1, p_2, \ldots, p_n\right) \geq b_j \sum_k^n \pi_k^* \;\forall j \neq i \tag{17}$$

where b_j is the proportion of total profit allocated to firm *j*.

To solve the problem, in (17) each firm sets $\sum_k^n \pi_k^* = \bar{\pi}_T$, a fixed parameter, and solves for the optimal price vector $p_i^*(\bar{\pi}_T)$ for each firm as a function of $\bar{\pi}_T$. Then, given the fixed values for b_j, the firms then find π_T^*, the value of $\bar{\pi}_T$ for which the optimal price vectors are the same for all firms, so that $p_i^*(\pi_T^*) = p^*(\pi_T^*), \forall i$. From Proposition 1, if any single price vector p^* solves (17) for all firms, then the allocation of profits defined by π_T^* and the b_j must be a Pareto optimal allocation. Likewise, there must exist such a p^* if there exists a Pareto optimal profit allocation π_T^* for which the allocation to each firm is given by $b_j\pi_T^*$. The fact that all Pareto optimal allocations dominate the strictly positive Bertrand allocations for all firms and the Brouwer Fixed Point Theorem are sufficient to establish the existence of this particular Pareto allocation of profits.

Figure 2 below provides a graphical depiction of this adjustment solution in profit space for a duopoly. As the unconstrained Pareto frontier shifts from F_1 to F_2 as a result of shocks, the firms compute a new Bertrand equilibrium B_2 and a new proportional division of profits given by the slope of the ray R_2 from the origin through the B_2. The new coordinated allocation will be at C_2, where R_2 intersects F_2.

Figure 2



3.2.3 Equivalence of Virtual Stakes Equilbria

We now show the equivalence between static virtual stakes equilibria and the constrained optimization problems of Proposition 1.

<u>Proposition 3</u>

Pareto optimal profit allocations can also be expressed as a static Nash equilibrium in which each firm maximizes its own profits subject to the constraint that all other firms' profits be no less than their coordinated allocation given by $\underline{\pi}^* = \left(\pi_1^*, \pi_2^*, \ldots, \pi_n^*\right)$. This Nash equilibrium is equivalent to a static virtual stakes equilibrium.

<u>Proof</u>

Firm i's constrained maximization problem now becomes

$$\pi_i^N = \max_{p_i} \pi_i\left(p_1, p_2, \ldots, p_n\right) \tag{18}$$

$$s.t. \qquad \pi_j\left(p_1, p_2, \ldots, p_n\right) \geq \pi_j^* \qquad \forall j \neq i$$

The difference between (18) and (12) is that firm i now has control of only its own price rather than the prices of all firms. The equilibrium prices $\underline{p}^N$ are obtained from the first order conditions

$$\frac{\partial \pi_i(\underline{p})}{\partial p_i} + \sum_{j \neq i} \theta_{ij}(\underline{p}) \frac{\partial \pi_j(\underline{p})}{\partial p_i} = 0 \; \forall i = 1,2,\ldots,n \tag{19}$$

These are the same conditions as constructed by selecting the i-th first order condition from (13) for each of the firms $i = 1,2,\ldots,n$. This is because firms face the same constraints in (18) as in (12) so that the Lagrange multipliers $\theta_{ij}(\underline{p})$ are the same functions of prices. As shown above in Proposition 1, the price vector $\underline{p}^*$ for the Pareto optimal allocation $\underline{\pi}^*$ satisfies the first order conditions (13) for all $i = 1,2,\ldots,n$. It follows that $\underline{p}^*$ satisfies (19). Under our assumption on the properties of the profit functions, there is a unique price vector satisfying (19). Therefore, $\underline{p}^N = \underline{p}^*$ and the profits obtained in the Nash equilibrium are the same as the Pareto optimal allocation.

The first order conditions that characterize the Nash equilibrium are the same as the first-order conditions that characterize the Virtual Stakes equilibrium when each firm i has a virtual stake of θ_{ij} of each other firm *j*. Therefore, for any Pareto optimal coordinated outcome, there is an equivalent virtual stakes representation, where the virtual stakes parameters of the problem are simply the Lagrange multipliers of the constrained profit maximization problems that generate the coordinated outcome. QED

3.2.4 Constrained Pareto Optimal Coordination

We now turn to the case of coordinated outcomes constrained both by the condition of Pareto optimality and by the requirement that the outcome is enforceable, that is, sustainable by a trigger strategy equilibrium. In other words, there may be some unconstrained Pareto optimal allocations that cannot be supported by the trigger strategies for the firms. On the other hand, some outcomes may possess the Pareto property only under the constraint of enforceability. These latter outcomes, together with those unconstrained Pareto allocations that remain enforceable, compose the constrained Pareto optimal set of coordinated outcomes for any given trigger strategy set. The constrained Pareto optimal set, like the unconstrained Pareto optimal set, is a function of time because it depends on the shocks to the underlying demand and cost conditions at any time. Moreover, the constrained set at any time will depend on the particular trigger strategy set because those strategies will determine the enforceability constraints. In turn, as we shall see, the constrained Pareto set of outcomes at any time will depend on all future outcomes or expected future outcomes via the enforceability constraints.

We illustrate this set by means of a simple dynamic game with full information following Friedman (1971). Consider the supergame consisting of infinitely repeated static games among *n* single product firms in which the static game strategy at any time *t* for the *i*th firm is that firm's price p_{it}. Strategies in the supergame are sequences of strategies from the static game, that we denote $\rho_i = (p_{i0}, p_{i1}, ..., p_{it}, ...)$ for the *i*-th firm. We have represented the firms' strategies as prices for notational convenience. In this supergame context, it should be explicitly understood that strategies are generically *rules* for determining prices, not necessarily prices themselves. Two examples of such rules were provided in a previous section. This is of particular importance when the strategy is for some future date, and the information required to compute the price according to the strategy is not yet available. If, however, the information required to compute the prices resulting from applying the strategy is available, we will understand the strategy as being represented by the resulting price. This accommodates situations in which the strategy is a deterministic sequence of prices, as well as situations in which the strategy is "Bertrand in each period," or some other rule for setting prices.

A vector of supergame strategies for all players is given by $\rho = (\rho_1, ..., \rho_n)$. The repeated Bertrand strategy for firm *i* is given by the sequence $\rho_i^b = (p_{i0}^b, p_{i1}^b, ...)$, and the corresponding vector of Bertrand strategies for all firms is $\rho^b = (\rho_1^b, ..., \rho_n^b)$. It is easy to show that this is an equilibrium for the supergame.

Let $\rho_t = (\rho_{1t}, ..., \rho_{nt})$ be the strategies taken by all the firms in period *t*. Profits to any firm *i* in the supergame are given by the discounted sum of profits arising from a given strategy:

$$\sum_{t=0}^{\infty} \beta_i^t \pi_{it}(\rho_t, \gamma_t, \delta_t)$$

where β_i is firm *i*'s discount factor, and the vectors of demand and cost shifters affecting each of the *n* firms at time *t* are $\gamma_t = (\gamma_{1t}, \gamma_{2t}, \ldots, \gamma_{nt})$ and $\delta_t = (\delta_{1t}, \delta_{2t}, \ldots, \delta_{nt})$. In what follows, for notational convenience, we suppress the explicit dependence of firms' profits on the vectors of demand and cost shifters, γ_t and δ_t. For conciseness, we instead write $\pi_{it}(\rho_t) = \pi_{it}(\rho_t, \gamma_t, \delta_t)$.

We now consider a simple trigger strategy that will yield coordinated profits greater than those arising from playing repeated Bertrand strategies. Suppose the *n* firms have selected the trigger strategy set $\rho^x \equiv (\rho_1^x, \ldots, \rho_n^x)$, where $\rho_i^x \equiv (p_{i0}^x, p_{i1}^x, \ldots)$, such that for any firm *i*, the trigger strategy sets prices according to

$$\begin{aligned}
&p_{i0} = p_{i0}^x; \\
&p_{it} = p_{it}^x, \quad \textit{provided} \quad p_{j\tau} = p_{j\tau}^x, \textit{ for all } j, \textit{ all } \tau \le t-1 \\
&p_{it} = p_{it}^b \quad \textit{for all } t > \tau, \textit{ provided} \quad p_{j\tau} \ne p_{j\tau}^x, \textit{ for some } j, \textit{ for some } \tau > 0
\end{aligned}$$

The trigger strategy thus sets prices according to $\rho_t^x \equiv (p_{1t}^x, \ldots, p_{nt}^x)$, unless one or more firms deviated in the preceding period. If at some time (τ) a deviation occurs, all firms thereafter revert to the Bertrand-Nash price. The trigger strategy ρ^x will be an equilibrium strategy if the enforceability or No Deviation Conditions are met. These conditions ensure that no firm will wish to deviate from the equilibrium strategy at any time τ. The No Deviation condition $ND_{i\tau}$ for the i-th firm at any time τ is given by

$$\sum_{t=\tau}^{\infty} \beta_i^{t-\tau} \pi_{it|\tau}^x \ \ge\ \pi_{i\tau}^D(\rho_{(i)\tau}^x) + \sum_{t=\tau+1}^{\infty} \beta_i^{t-\tau} \pi_{it|\tau} \qquad i = 1, \ldots, n \qquad (20)$$

Here $\pi_{it|\tau}^x$ represents profits expected from the strategy ρ^x and $\pi_{it|\tau}^b$ represents profits expected from the Bertrand strategy, both conditional on information available at time τ; $\pi_{i\tau}^D(\rho_{(i)\tau}^x)$ represents the profit firm *i* will gain at time τ if it deviates from the equilibrium strategy to maximize profits at that time while all other firms play the strategy ρ^x. Note that $\pi_{i\tau}^D(\rho_{(i)\tau}^x)$ is known as of time τ and depends only on the prices of other firms at τ, since the price that maximizes profits for firm *i* is uniquely determined for any time, given the other firms strategies or prices.

The ND conditions for all firms collectively simply state that for each firm at every point in time, the expected discounted profits for the firm are greater if the firm maintains the equilibrium strategy ρ^x than if the firm departs from that strategy to gain the deviation profits for one period and then suffers Bertrand profits ever after.

It is well known that there are many possible equilibrium outcomes to such games. Again, it is natural to focus on equilibria that generate outcomes that are Pareto optimal, or, in this case, constrained Pareto optimal. By this we mean that the equilibrium strategy will generate constrained Pareto-optimal outcomes in every period in which the coordinated outcome is attained. This set of outcomes is the constrained Pareto-optimal frontier. At time τ this is formally given in price space by

$F_\tau = \{\ p_\tau \mid\ ND_{i\tau}$ holds for $i = 1,\ldots, n$; and there exists no p_τ' s.t. for some i

$$\pi_{i\tau}(p_\tau') > \pi_{i\tau}(p_\tau)$$

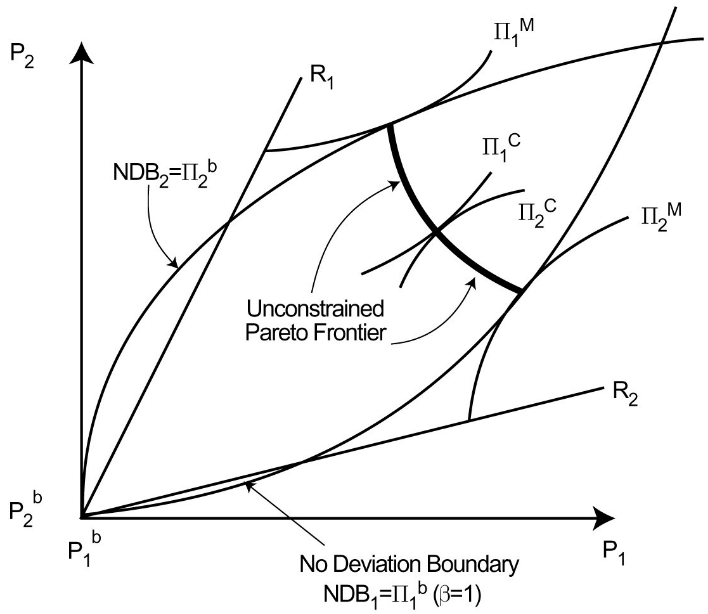
and for all $j \neq i$

$$\pi_{j\tau}(p_\tau') \geq \pi_{j\tau}(p_\tau) \quad \},$$

where p_τ and p_τ' are vectors of prices for all n firms.

We can now establish results for the equivalence of Virtual Stakes equilibria given constrained Pareto-optimal outcomes analogous to those derived for unconstrained Pareto-optimal outcomes. Before proceeding with that discussion, it is helpful to provide some diagrams that depict in a stylized way the constrained Pareto-optimal frontier. First, consider Figure 3, which depicts the unconstrained Pareto frontier in price space for a roughly symmetric duopoly. Here we show the unconstrained Pareto-optimal frontier as the limiting case when $\beta = 1$ in the dynamic game discussed above. R_1 and R_2 in Figure 3 are the static reaction curves of the firms intersecting at the origin, which is the Bertrand price pair, P_1^b and P_2^b. The No Deviation boundary curves, NDB_1 and NDB_2, bound the set of prices and associated profits that satisfy the ND Conditions. Note that the NDB curves for the firms are their respective isoprofit curves for profits equal to their Bertrand profits. This equivalence holds only when $\beta = 1$ because, if firms are perfectly patient, then any amount of profits above the Bertrand allocation is sustainable in an infinitely repeated game. Thus, the ND set consists of all profit allocations that strictly dominate the Bertrand allocation and is represented in price space by the interior of the lens shaped area bounded by the NDB curves in Figure 3.

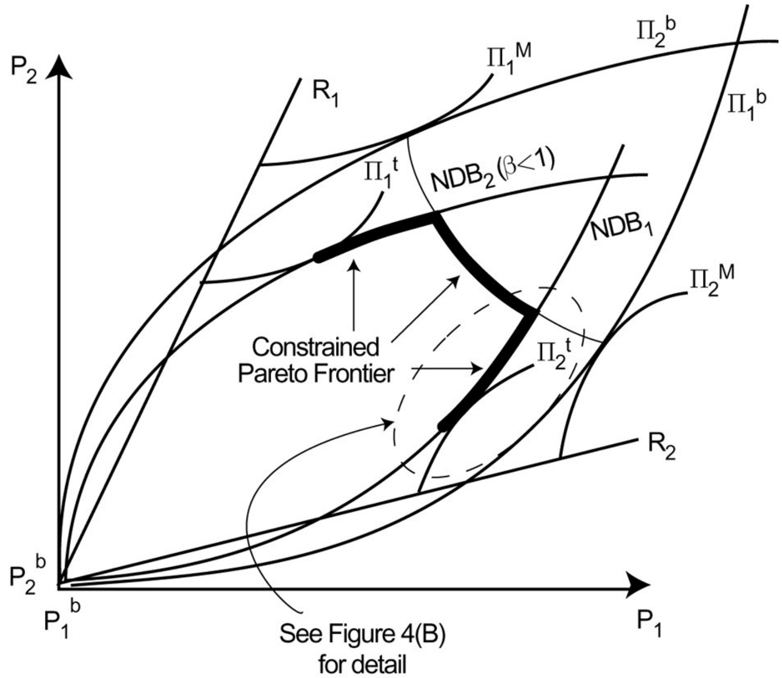
Figure 3



The unconstrained Pareto optimal set of allocations is the thick line within the ND set representing all points of tangency between the isoprofit curves of the firms. Only at such points of tangency is it not possible to move from that price point without reducing the profits of at least one of the firms. For example, the tangency of Π_1^C and Π_2^C depicts the joint profit maximization or perfect collusion price point. Π_1^M and Π_1^M identify the extremes of the unconstrained Pareto optimal frontier, in which one firm earns (almost) Bertrand profits and the other firm earns the maximum profits available to it given both the Pareto and ND conditions.

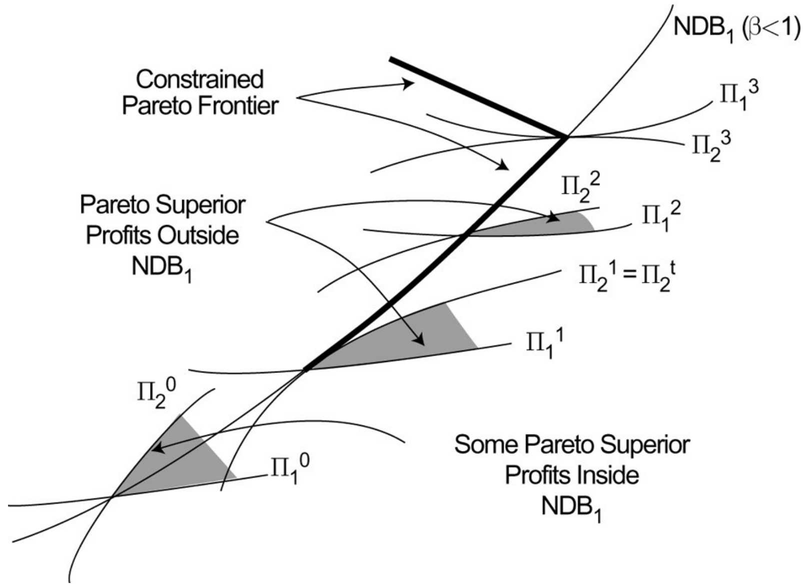
Contrast this with Figure 4 (A). This depicts the constrained Pareto optimal frontier for some value of $\beta < 1$. Note that now the NDB curves are not equal and are interior to the Bertrand isoprofit curves. This follows because allocations resulting from price points at the ends of the line of tangencies representing the unconstrained Pareto frontier will no longer be sustainable. With $\beta < 1$, firms become less patient, and the firm that earns little more than Bertrand profits will have an incentive to deviate. But note that the constrained Pareto frontier consists not only of a portion of the unconstrained frontier, but also part of the NDB curves.

Figure 4(A)



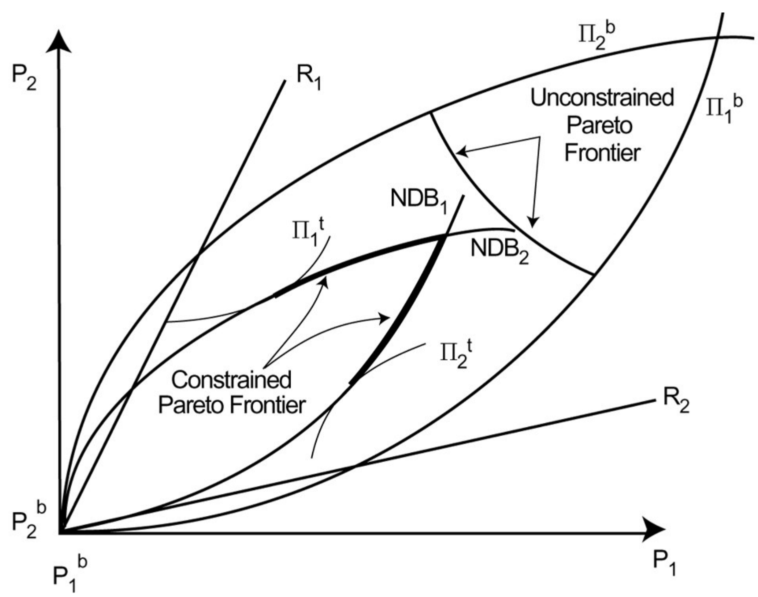
This is explained by Figure 4(B), which shows the detail for this part of NDB_1. Figure 4(B) illustrates that the portion of NDB_1 between the intersection with the unconstrained Pareto set line and the point of tangency with an isoprofit curve of firm 2, given by Π_2^t, is Pareto optimal given the ND constraint. This follows because all price points that would improve the profits of either firm without reducing the profits of the other lie outside the ND set, as indicated by the shaded areas. However, points of NDB_1 below the tangency with isoprofit curve Π_2^t do not have this property, again as shown by the shading in Figure 4(B).

Figure 4(B)



Now consider Figure 5.

Figure 5



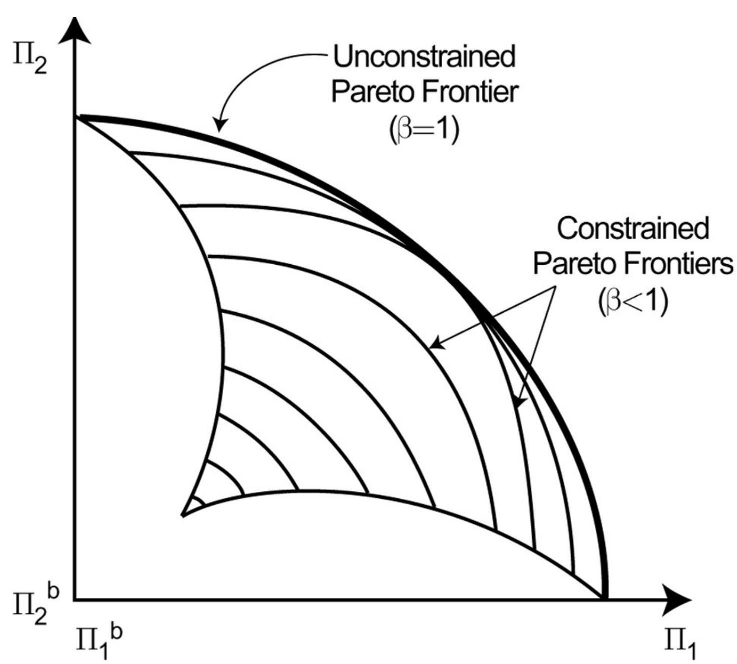
In Figure 5, we depict the situation where β becomes so small that no price points on the unconstrained Pareto frontier remain on the constrained Pareto frontier. However, the set is not empty, consisting of portions of the boundary of the ND set. As in Figure 4, these are points on the boundary of the ND set above the tangencies with isoprofit curves of the firms. Note that these price points are associated with profit allocations in which both firms earn less than they could earn at some point on the unconstrained frontier.

The constrained Pareto frontiers in profit space for different values of β are depicted in Figure 6. As β diminishes, the constrained frontier recedes into the fan shaped area below the unconstrained frontier. The intuition here can be seen by rewriting the ND Condition as

$$\sum_{t=\tau+1}^{\infty} \beta_i^{t-\tau} \left[\pi_{it|\tau}^{x} - \pi_{it|\tau}^{b} \right] \geq \pi_{i\tau}^{D}(p_{(i)\tau}) - \pi_{i\tau}(p_\tau) \qquad (21)$$

In this form, it is easy to see that as β goes to 1, the inequality is always met for any coordinated profits greater than Bertrand profits. However, as β gets small, meaning that firms become less patient, the firms become more concerned about the short term difference between the coordinated and deviation profits than the difference between the coordinated and Bertrand profits relevant in the longer run. To maintain the inequality, the difference between the coordinated and deviation profits must get small relative to the difference between the coordinated and Bertrand profits. This means lowering the deviation profits, which depend only on other firms' prices and are increasing in those prices. If all firms must reduce prices to ensure that no firm will deviate, then the coordinated profits of all firms have to become smaller, leading to points in the interior of the fan-shaped set. The set has this fan shape because disproportionate allocations between the symmetric firms in this example also could result in deviation. Note that as β gets smaller, the V-shaped constrained frontier in Figure 5 shrinks to just the point of the V at which isoprofit curves are tangent to the NDB curves only at that point, if at all. This occurs at a profit point strictly above the Bertrand point.

Figure 6



Thus, Figure 6 shows that virtual stakes parameters representing constrained Pareto-optimal outcomes have the ability to flexibly describe the maintenance of coordinated outcomes spanning a very large portion of the entire set outcomes bounded above by the unconstrained Pareto frontier. Figure 6 depicts approximately symmetric firms, but if the firms have different β values, then different areas inside the bounded area will be relevant.

With this illustrative background, we can now proceed by first establishing a result analogous to Proposition 1 for all equilibrium outcomes on the constrained Pareto optimal frontier.

Proposition 4

A price vector p_τ^x belongs to F_τ, the constrained Pareto optimal frontier, if and only if it solves the following constrained maximization problems for all firms $i = 1,2,...,n$:

$$\begin{aligned} \pi_{i\tau}^x \equiv \max\nolimits_{\mathrm{P}\tau} \ & \pi_{i\tau}(p_\tau) \\ s.t. \quad & \pi_{j\tau}(p_\tau) \geq \pi_{j\tau}^x \quad \text{for all } j \neq i \\ \text{and} \quad & \mathrm{ND}_{j\tau} \ \text{ holds for all } j = 1,...,n \end{aligned}$$

Proof

The proof is as in Proposition 1. If the profit allocation is in the constrained Pareto optimal frontier and one of the maximization problems is not solved by that associated price vector, then one of the firms could be better off at a different price vector in the feasible set where the ND conditions hold. This is a contradiction. If the price vector solves all the problems, and its outcome is not in the constrained Pareto frontier, then there exists some other price vector in the feasible set defined by the ND conditions at which some firm could be better off and no other firm worse off. This again is a contradiction. Again, the choice of firm is arbitrary. QED

It is important to understand the nature of the static optimization problem in Proposition 4 as it relates in particular to the ND conditions. In this static problem, the only variable controlled by the firm is all prices in the price vector p_τ. All the expected future prices and the associated expected outcomes that appear in the constraints expressing the ND conditions are fixed and not functions of current prices. These future outcomes are the results expected by the maximizing firm given that all firms play according to certain articulated strategies in the future. In contrast, in this static problem, all current prices are not viewed as the result of any of these or other strategies. They are decision variables free to be assigned any value selected by the firm to maximize its objective function in this static problem subject to the constraints imposed.

It is useful at this point to reconsider the ND constraint for the *i*-th firm in the form given by the inequality (21)

$$\sum_{t=\tau+1}^{\infty} \beta_i^{t-\tau} \left[\pi_{it|\tau}^x - \pi_{it|\tau}^b \right] \ \geq \ \pi_{i\tau}^D(p_{(i)\tau}) \ - \ \pi_{i\tau}(p_\tau) \qquad (21)$$

In this statement relevant to the constraints of the static optimization problem in Proposition 4, p_τ and $p_{(i)\tau}$ are the price vectors chosen by the *i*-th firm to maximize its own profit. Note that firm *i*'s own price $p_{i\tau}$ only appears in the last term in the vector p_τ. Consequently, the only variable affected in this constraint by firm *i*'s choice of its own price is its own profits. In maximizing its profits with respect to $p_{i\tau}$, if otherwise unconstrained, the firm will select that price determined by $p_{(i)\tau}$ (given prices for all other firms) and earn its deviation profits. Thus, this constraint alone is not binding on firm *i*'s efforts to maximize its profits with respect to its own price as can be seen from this statement of inequality (21). As profits go to deviation profits, the right hand side goes to zero and the constraint will always be met so long as coordination profits exceed Bertrand profits by any amount. This can also be seen graphically by inspection of Figure 4(A) for example. Deviation profits for the firm always lie on its reaction curve and its reaction curve is always interior to the feasible area defined by its own ND constraint. This turns out to be significant in our next result.

That result is that we can represent any constrained Pareto-optimal allocation as a static Nash equilibrium that is in turn equivalent to a Virtual Stakes equilibrium.

Proposition 5

Constrained Pareto-optimal profit allocations can also be expressed as a static Bertrand Nash equilibrium in which each firm maximizes its own profits with respect to its own price subject to the constraints that all other firms' profits be no less than their coordinated allocation given by $\pi_\tau^x \equiv (\pi_{1\tau}^x, ..., \pi_{n\tau}^x)$ and that the $ND_{i\tau}$ holds, $i = 1,\ldots,n$. This Bertrand Nash equilibrium is equivalent to a static Virtual Stakes equilibrium.

Proof

This result follows from Proposition 4 by the same argument that established Proposition 3 using Proposition 1. For each π_τ^x in the constrained Pareto optimal set, there is a unique p_τ^x belonging to F_τ that supports such profits. As before, the Bertrand Nash equilibrium is then characterized by the system obtained from the collection of the *i*th first-order condition from firm *i*'s constrained maximization problem in Proposition 4 for all $i = 1,\ldots,n$. These are the first order conditions in which firm *i* maximizes its profit function subject to the constraints with respect to its own price. As we have just shown, firm *i*'s own ND constraint $ND_{i\tau}$ is not binding under any circumstances on its choice of its own price $p_{i\tau}$, and therefore consideration of that constraint will be suppressed in what follows. QED

We may now write the Lagrangian expression for the i-th firm's constrained maximization problem with respect to its own price. This will generate the first order condition for the Bertrand Nash and Virtual Stakes equilibria for the i-th firm.

$$\underset{p_{i\tau}}{\text{MAX}}\ \boldsymbol{L}i = \pi_{i\tau}(p_\tau) + \quad (22)$$

$$\sum_{j\neq i}^{n-1}\theta_{ij}\left[\pi_{j\tau}(p_\tau) - \pi_{j\tau}(p_\tau^x)\right] + \sum_{j\neq i}^{n-1}\varsigma_{ij}\left\{\pi_j^E - \left[\pi_{j\tau}^D(p_{(j)\tau}) - \pi_{j\tau}(p_\tau)\right]\right\}$$

For conciseness of notation, the profit term $\pi_{j\tau}^E$ in (22) is equal to the entire left side of the inequality in (21), as it expresses the ND condition for the *j*-th firm. Since this term involves only future profits expected under given strategies, it is the constant for the constraint expressed by the last term in (22). Differentiating (22) with respect to $p_{i\tau}$ generates the following first order condition.

$$\frac{\partial\pi_{i\tau}(p_\tau)}{\partial p_{i\tau}} + \sum_{j\neq i}(\theta_{ij}+\zeta_{ij})[\frac{\partial\pi_{j\tau}(p_\tau)}{\partial p_{i\tau}}] - \sum_{j\neq i}\zeta_{ij}[\frac{\partial\pi_{j\tau}^D(p_{(j)\tau})}{\partial p_{i\tau}}] = 0 \quad (23)$$

The Virtual Stakes equilibrium is characterized by a system of *n* first order condition equations of this form, one for each firm.

As before, the θ's are Lagrange multipliers for the constraints that rival firms' profits are no less than the coordinated levels set by the strategy ρ^x. Now we have an additional set of Lagrange multipliers on the ND conditions, the ζ's. It is clear from the constraints that $\theta_{ij} \geq 0, \zeta_{ij} \geq 0$ for all *i,j*. When some of the ζ's are strictly positive, the related ND constraints are binding indicating that the coordinated outcome lies on the portion of the constrained Pareto frontier that is not also part of the unconstrained Pareto frontier. If all the ζ's are zero, then the outcome is on the unconstrained Pareto frontier, In this more complex Virtual Stakes equilibrium, each firm has a stake not only in the coordinated profits of its rivals but also a negative stake in the profits rivals would earn if they deviated.

The unconstrained Pareto equilibrium can be considered the limiting case for this simple game in which $\beta_i \to 1,\ i = 1,...,n$. If firms are patient enough, the ND conditions become trivially satisfied, so long as the coordinated profit outcome dominates the Bertrand outcome. The ND constraints are no longer binding, so that the ζ's vanish. We are then back to the simple Virtual Stakes equilibrium where firms are constrained only by rivals' coordinated profits. Note that the β_i do not appear in the first order conditions given in (23) because they are only part of the fixed future in the context of this solution and are unaffected by $p_{i\tau}$. However, they remain part of the solution, as do the fixed future profits, because they appear in the constraints and those constraints are the other set of first order conditions that result from differentiating the Lagrangian with respect to the multipliers. We further discuss the interpretation of these virtual stakes parameters and their values in more detail in the next section.

3.3 Interpretation of Virtual Stakes Parameters

In this section, we consider the interpretation of virtual stakes parameters and their associated Lagrange multipliers. First, we have so far assumed that all firms in the market are party to the modeled coordination. Clearly, we might well observe coalitions of coordinating firms within markets that include firms not party to that coordination. If so, we will find the value of all multipliers to be zero for firms that were not party to coordination with any firms and that could be reasonably modeled as engaging in Bertrand pricing conduct with respect to other firms. If some group of firms were coordinating, we will find the multipliers to be positive with respect to other firms in the group and zero with respect to firms outside the group. It is possible to have multiple coalitions coordinating separately, and the positive or null values of the multipliers observed would serve to identify those groups and their members.

Second, we might well find the parameter values changing over time for firms within a given coordinating group. Virtual stakes take on positive values during periods of sustained coordination to the superior profit outcome, but might well fall to zero during other periods in which the coordination broke down or entered the threat phase of the equilibrium. During such periods, the multiplier values might even become negative, reflecting punishment behaviors. One might also be able to detect when certain firms either joined or parted ways with persisting coordinating groups.

Third and quite significantly, Pareto optimality imposes strong restrictions on the virtual stakes parameters. To see this, write the first-order conditions for each firm's constrained maximization problems (19) in matrix form:

$$\begin{bmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1n} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2n} \\ & & \vdots & \\ & & \vdots & \\ \theta_{n1} & \theta_{n2} & \cdots & \theta_{nn} \end{bmatrix} \begin{bmatrix} \pi_{1j} \\ \pi_{2j} \\ \vdots \\ \vdots \\ \pi_{nj} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, j = 1,2,\ldots, n \quad (25)$$

where all Lagrange multipliers and first derivatives of profits ($\pi_{ij} \equiv \partial\pi_i / \partial p_j$) with respect to prices are all evaluated at equilibrium prices. Because the vector of first derivatives of profits with respect to prices may take on any value, in order for Equation (20) to hold, the rows of the Lagrange multipliers can differ only by a scale factor. This implies that given the Lagrange multipliers for firm 1, $(\theta_{11}, \theta_{12}, ..., \theta_{1n})$, the other Lagrange multipliers must satisfy

$$\theta_{i1} = \theta_{1i}^{-1} \quad \text{and} \quad \theta_{ij} = \theta_{1i}^{-1}\theta_{1j}, \; j = 2,\ldots,n; \; i = 1,\ldots.n. \quad (26)$$

A number of important properties come out of this analysis. First, only n-1 of the n(n-1) parameters are independent, which has obvious implications for estimation. Second, if the estimated parameter values do not exhibit these properties, the assumption of Pareto-optimal coordination is called into question. Third, consider the case of joint profit maximization assumed enforceable without side payments. In that case, each firm weighs all firms' profits equally with the result that all the Lagrange multipliers are unity. This would clearly satisfy the conditions stated in (26). If the firms chose to coordinate on some other Pareto allocation, that would necessarily reduce the profits of some firms at the expense of others. This suggests that the losers would be more constrained to maintain this outcome than the winners; that is, the losers would have more to gain by relaxing the constraints that require them to ensure high profits for the winners. Thus, noting how pairs of multipliers across firms take on reciprocal values, one might expect losers' multipliers to take values exceeding unity and winners' to take values less than one. Here winners and losers are measured pair-wise. Thus, the relative values of the parameters may serve as a metric as to how well firms do relative to each other, with the joint profit maximizing outcome as a benchmark.

Finally, there are at least two useful ways to interpret the values of the ζ's, the second type of multipliers that arises in the constrained Pareto- optimal coordination case. One way to think about these parameters is that each firm has a stake in the profits of its rivals in any state of the world that might come to pass. In the simple model presented here, only two alternative states of the world exist – one in which the rival deviates and one in which it doesn't. The firm has a positive stake in the former and a negative stake in the latter, to the extent that the firm wishes to sustain the coordinated outcome. If the alternative state in which deviation occurs does not exist for some firm because it will always have an incentive to coordinate, then all ζ multipliers of other firms with to respect to that firm will be zero. A positive stake is reflected in the sum of both multipliers, while the latter effect is captured by the ζ multiplier for the ND condition alone. This suggests the second interpretation.

If a firm wishes to maintain coordination, it must prevent coordination from breaking down in one of two ways. First, if the firm does not ensure that rivals receive the profits due them under the coordination, they will perceive the firm as deviating and "pull the trigger" to end coordination and switch the equilibrium to a new phase. A positive value for this multiplier reflects the extent of that constraint and involves only the level of coordinated profit due. Second, if the firm does not ensure that the outcome will not give the rivals an incentive to deviate, then the firm will be forced to "pull the trigger" on its rivals and end the high-profit coordination. A positive value for this multiplier reflects the extent of the constraint to avoid deviation by rivals, but involves both the coordinated profits and the profits from deviation, specifically, the difference between them. Thus, the two multipliers reflect the dual faces of the mutual forbearance involved in sustaining the coordinated outcome.

4 Identification and Econometric Estimation

In this section we discuss issues associated with identification and econometric estimation of virtual stakes parameters. For simplicity and brevity, we treat only the case in which firms are sufficiently patient that the non-deviation constraints are not binding, ensuring that firms only care about equilibrium and not deviation profits. The more complex situation in which the non-deviation constraints may be binding will be taken up elsewhere.

4.1 Virtual Stakes in a Linear Framework

To gain insight as readily as possible, we analyze in detail the case of a linear model. For this exercise, let the demand and cost functions be given by

$$q_i(p,\delta_i) = b_i' p + \delta_i = b_{ii}' p_i + \sum_{h\neq i} b_{hi}' p_h + \delta_i \qquad (27)$$

$$c_i(q_i,\gamma_i) = \gamma_i' q_i \qquad (28)$$

where b_i is a $k \times k_i$ matrix of price coefficients and $b_{hi} = S_h b_i$ is a $k_h \times k_i$ submatrix of b_i, with S_h the $k_h \times k$ selection matrix previously defined. We also have that γ_i and δ_i are $k_i \times 1$. The profit maximization condition requires the two derivative components

$$\frac{\partial q_h}{\partial p_i} = b_{ih} \qquad (29)$$

$$\frac{\partial c_h}{\partial q_h} = \gamma_h. \qquad (30)$$

Substituting into the first derivative expression (6) yields

$$\begin{aligned}\frac{\partial \overline{\pi}_i}{\partial p_i} &= \nu_{ii} b_{ii}' p_i + \nu_{ii} \sum_{h\neq i} b_{hi}' p_h + \nu_{ii}\delta_i + \sum_{h=1}^{n} \nu_{ih} b_{ih} p_h - \sum_{h=1}^{n} \nu_{ih} b_{ih} \gamma_h \\ &= \nu_{ii}(b_{ii}' + b_{ii}) p_i + \nu_{ii} \sum_{h\neq i} b_{hi}' p_h + \nu_{ii}\delta_i + \sum_{h\neq 1}^{n} \nu_{ih} b_{ih} p_h - \sum_{h=1}^{n} \nu_{ih} b_{ih} \gamma_h \\ &= \sum_{h=1}^{n} (\nu_{ii} b_{hi}' + \nu_{ih} b_{ih}) p_h + \nu_{ii}\delta_i - \sum_{h=1}^{n} \nu_{ih} b_{ih} \gamma_h. \end{aligned} \qquad (31)$$

The optimal p_i sets this first-order condition to zero. Imposing optimality and dividing by ν_{ii} to obtain an expression in terms of normalized virtual stakes $\theta_{ih} = \nu_{ih}/\nu_{ii}$ gives

$$\sum_{h=1}^{n} (b_{hi}' + \theta_{ih} b_{ih}) p_h + \delta_i - \sum_{h=1}^{n} \theta_{ih} b_{ih} \gamma_h = 0.$$

Stacking these equations delivers an expression that we can use to solve for the equilibrium price vector. To obtain a convenient expression, define the $k \times k$ matrices B

= [b_{ih}] and B_θ = [$\theta_{ih}\, b_{ih}$] and the $k \times 1$ vectors δ = [δ_i] and γ = [γ_i]. Stacking the equations above gives

$$(B'+B_\theta)p + \delta - B_\theta\gamma = 0,$$

which then yields the unique Virtual Stakes Nash equilibrium price vector

$$p = (B'+B_\theta)^{-1}B_\theta\gamma - (B'+B_\theta)^{-1}\delta. \qquad (32)$$

For notational simplicity, in this section we refer to equilibrium prices simply as p, and do not make further notational distinctions.

This expression compactly represents equilibrium prices in terms of the cost (γ) and demand (δ) shifters, the price response coefficients (B) and the normalized virtual stakes (θ). It is readily verified that this expression contains the standard expressions for the Bertrand-Nash equilibrium price vector and for the joint industry profit-maximizing price vector, each obtained by setting θ to the values appropriate for these cases. Equation (32) also serves as a convenient and powerful basis for comparative statics analysis.

4.2 Virtual Stakes as Time-Invariant Conduct Parameters

Previous approaches to the estimation of conduct parameters in the applied IO literature have often taken the conduct parameters to be time invariant (e.g., Nevo, 1998). Viewing the normalized virtual stakes as fixed conduct parameters to be estimated, parallel to this literature, we can use equation (32) as the basis for a reduced form parameterization supporting econometric estimation. But whereas the conduct parameters of the earlier literature are ad hoc and unsupported by theory, the normalized virtual stakes represent conduct parameters corresponding to conjectural variations-consistent Virtual Stakes Nash equilibria.

To obtain a useful reduced form from (32), one may proceed by specifying models for the cost and demand shifters, as these typically are not observed by the econometrician. For simplicity and concreteness, we suppose that the econometrician can observe cost and demand shifter component variables $Z_{00}, Z_{01}, Z_{11}, \ldots, Z_{1n}$ such that

$$E(\gamma \mid Z_{00}, Z_{01}, Z_{11}, \ldots, Z_{1n}) = A_{00}\, Z_{00}$$

$$E(\delta_i \mid Z_{00}, Z_{01}, Z_{11}, \ldots, Z_{1n}) = A_{0i}\, Z_{00} + A_{01i}\, Z_{01} + A_{1i}\, Z_{1i}, \qquad i = 1, \ldots, n. \qquad (33)$$

Here, Z_{00} is $l_{00} \times 1$, Z_{01} is $l_{01} \times 1$, and Z_{1i} is $l_{1i} \times 1$, $i = 1, \ldots, n$. The coefficient matrices A_{00} ($k \times l_{00}$), A_{0i} ($k_i \times l_{00}$), A_{01i} ($k_i \times l_{01}$), and A_{1i} ($k_i \times l_{1i}$), $i = 1, \ldots, n$, are unknown to the econometrician but may be estimated. The idea is that Z_{00} contains variables impacting cost (for all firms, although A_{00} may embody known zero restrictions), whereas the remaining variables contain variables impacting demand for firm i (through δ_i). The vector Z_{01} contains "common" demand shifters, i.e., shifters that impact demand not only for products of firm i, but also for products other than those produced by firm i. Examples are the price of energy or of other inputs common to the downstream customers of firm i and its competitors. (As for A_{00}, A_{0i} may embody known zero restrictions.) The demand shifters Z_{1i}, $i = 1, \ldots, n$, constitute demand shifter components that are unique to firm i. As we shall see, the presence of shifters unique to firm i's products is the key to identification of certain system parameters. Further, by including Z_{00} in (33), we permit δ_i to be affected by variables that are also cost shifters to firm i or to other firms, such as energy prices or input prices common to the firms in the industry and their downstream customers.

Defining $Z = (Z'_{00}, Z'_{01}, Z'_{11}, \ldots, Z'_{1n})'$, we further define

$$\eta_0 \equiv \gamma - E(\gamma \mid Z), \qquad \eta_1 \equiv \delta - E(\delta \mid Z),$$

so that

$$E(\eta_0 \mid Z) = 0, \qquad E(\eta_1 \mid Z) = 0,$$

and

$$\gamma = A_{00} Z_{00} + \eta_0 \tag{34}$$

$$\delta_i = A_{0i} Z_{00} + A_{01i} Z_{01} + A_{1i} Z_{1i} + \eta_{1i}, \quad i = 1, \ldots, n.$$

Stacking the equations for δ_i, we can write

$$\delta = A_0 Z_{00} + A_{01} Z_{01} + A_1 Z_1 + \eta_1, \tag{35}$$

where A_0 is the $k \times l_{00}$ matrix that stacks the $k_i \times l_{00}$ matrices A_{0i}, $i = 1, \ldots, n$; A_{01} is the $k \times l_{01}$ matrix that stacks the $k_i \times l_{01}$ matrices A_{01i}, $i = 1, \ldots, n$; A_1 is the $k \times l_1$ block diagonal matrix with $k_i \times l_{1i}$ diagonal blocks A_{1i}, with $l_1 \equiv \sum_{i=1}^{n} l_{1i}$; and Z_1 is the $l_1 \times 1$ vector $Z_1 = (Z'_{11}, \ldots, Z'_{1n})'$.

Next, we substitute (34) and (35) into (32) to obtain

$$\begin{aligned} p &= (B' + B_\theta)^{-1}(B_\theta A_{00} - A_0) Z_{00} - (B' + B_\theta)^{-1} A_{01} Z_{01} - (B' + B_\theta)^{-1} A_1 Z_1 + \varepsilon \\ &= \Pi_0 Z_0 + \Pi_1 Z_1 + \varepsilon \\ &= \Pi \ Z + \varepsilon, \end{aligned}$$

where

$$\varepsilon \equiv (B'+B_\theta)^{-1} B_\theta \eta_0 - (B'+B_\theta)^{-1} \eta_1$$
$$\Pi_0 \equiv [(B'+B_\theta)^{-1}(B_\theta A_{00} - A_0), \;\; -(B'+B_\theta)^{-1} A_{01}]$$
$$Z_0 \equiv [Z_{00}{}', Z_{01}{}']'$$
$$\Pi_1 \equiv -(B'+B_\theta)^{-1} A_1$$
$$\Pi \equiv [\Pi_1, \Pi_2].$$

Observe that by construction, we have $E(\varepsilon | Z) = 0$, ensuring that under standard conditions, ordinary least squares applied to this reduced form can deliver consistent estimates of Π.

We are now in a position to investigate identification of θ by examining the conditions under which the normalized virtual stakes can be recovered. It is most straightforward to attempt to recover θ from

$$\Pi_1 \equiv -(B'+B_\theta)^{-1} A_1.$$

Rearranging, we have

$$(B'+B_\theta)\Pi_1 = -A_1,$$

or

$$-(A_1 + B'\Pi_1) = B_\theta \Pi_1.$$

Next, it is convenient to isolate the virtual stakes for firm i, θ_{ih}, $h = 1, \ldots, n$. Recalling that S_i is the $k_i \times k$ selection matrix selecting p_i from p, we have the $k_i \cdot l_1$ equations

$$-S_i(A_1 + B'\Pi_1) = S_i B_\theta \Pi_1$$
$$= \sum_{h=1}^{n} b_{ih} \Pi_{h1} \theta_{ih}, \tag{36}$$

where $\Pi_1 \equiv [\Pi_{11}{}', \Pi_{21}{}', \ldots, \Pi_{n1}{}']'$, and each Π_{h1} is a $k_h \times l_1$ block of Π_1. These equations relate the normalized virtual stakes θ_{ih} to the reduced form coefficients Π_1 and to the demand coefficients B and A_1 (actually A_{1i}). We have seen that consistent estimation of Π_1 is generally possible, and, as we discuss further below, so is consistent estimation of B and A_{1i}. If we can recover θ_{ih} from equation (36), then it is identified and can be consistently estimated.

Examining equation (36), we see that we have $k_i \cdot l_1$ linear equations in n -1 unknowns (recalling that $\theta_{ii} = 1$). As $k_i \geq 1$, it follows that, provided that each firm's products have at least one demand shifter unique to that product, thereby ensuring $l_1 \geq n$, we generally have $k_i \cdot l_1 \geq n-1$. It follows that equation (36) is over-determined (apart from exceptional circumstances), so that one may identify and consistently estimate the normalized virtual stakes θ_{ih}, $h = 1, \ldots, n$. A useful practical estimator for $\theta_{i(i)}$, the vector containing $\theta_{ih},\ h \neq i$, can be based on the expression

$$\theta_{i(i)} = (\chi_i' \chi_i)^{-1} \chi_i \psi_i$$

where χ_i is the $k_i \cdot l_1 \times (n - 1)$ matrix with $(k_i \cdot l_1 \times 1)$ columns $vec(\, b_{ih} \Pi_{h1}\,),\ h \neq i$, and ψ_i is the $k_i \cdot l_1 \times 1$ vector

$$\psi_i \equiv -vec(S_i A_1 + B' \Pi_1 + b_{ii} \Pi_{i1})\,.$$

To see that B and A_{1i} can in principle be consistently estimated as claimed above, observe that with our linear demand specification and our specification for δ_i, we have that demand for firm i's products is given by

$$q_i = b_i' \, p + \mathrm{A}_{0i} \mathrm{Z}_{00} + \mathrm{A}_{01i} \mathrm{Z}_{01} + \mathrm{A}_{1i} \mathrm{Z}_{1i} + \eta_{1i}\,.$$

Examining this set of equations, we see that p is endogenous. Nevertheless, consistent estimation of the coefficients can be achieved by instrumental variables methods, using instrumental variables Z, and relying on the identification provided by the presence in Z of firm/product-specific demand shifters, Z_{1h}, $h = 1, \ldots, n$. Consistently estimating b_i, $i = 1, \ldots, n$, permits us to construct a consistent estimator of B, and combining this with consistent estimators of A_{1i} and Π_1 then permits construction of consistent estimators of the normalized virtual stakes θ_{ih}, $h = 1, \ldots, n$, as described above.

To summarize, our discussion shows that in the simple linear system analyzed here, it is possible to identify and consistently estimate time invariant conduct parameters that have a theoretically well-founded virtual stakes representation, in contrast to some earlier negative results for ad hoc conduct parameters (e.g., Nevo, 1998).

4.3 Pareto Optimality and Virtual Stakes Estimation

Our discussion in the previous section treated the virtual stakes as time invariant unknown coefficients to be estimated, parallel to standard approaches to the estimation of conduct parameters in the applied IO literature. Although this can provide a variety of useful insights and may serve as a useful basis for testing hypotheses about specific forms of competitive behavior (e.g., Bertrand-Nash competition or joint profit maximization), we saw earlier that as representations of Pareto-optimal outcomes over time arising from the maintenance of dynamic coordination, the resulting virtual stakes must depend on equilibrium prices (and thus ultimately on the cost and demand shifter outcomes). Consequently, it is of interest to estimate virtual stakes parameters that vary over time as outcomes and prices vary. We now consider the issue of identification and estimation of such virtual stakes. As we shall see, Pareto optimality plays the key role in making the identification and estimation of virtual stakes in this context quite straightforward.

Our earlier analysis of the virtual stakes associated with Pareto-optimal equilibrium outcomes led to restrictions on the virtual stakes stated in (25) and (26). These can be compactly represented as

$$\theta(p)\dot{\pi}(p) = 0, \tag{37}$$

where θ is the $n \times n$ matrix with rows $[\theta_{i1}, \theta_{i2}, \ldots, \theta_{in}]$ and $\dot{\pi}$ is the $n \times n$ matrix of profit derivatives with rows $[\pi_{i1},\ldots,\pi_{in}]$, $\pi_{ij} \equiv \partial\pi_i / \partial p_j$. The argument p appearing in θ and $\dot{\pi}$ makes explicit the dependence of these quantities on equilibrium prices, although we suppress this for notational convenience in what follows. Recall also that for simplicity in this context we assumed that each firm produced only one good, and we once again impose this simplifying assumption.

As we discussed above, equation (37) implies the following restrictions that must be satisfied by Pareto-optimal virtual stakes:

$$\theta_{i1} = \theta_{1i}^{-1}, \qquad \theta_{ij} = \theta_{1i}^{-1}\, \theta_{1j}, \qquad j = 2,\ldots, n;\ \ i = 1,\ldots, n. \tag{38}$$

It thus suffices to identify $\theta_{1(1)} \equiv (\theta_{12},\ldots,\theta_{1n})'$. For this, define $\dot{\pi}_1 \equiv (\pi_{11},\ldots,\pi_{1n})'$ and $\dot{\pi}_{(1)}$ such that $\dot{\pi}' \equiv [\dot{\pi}_1, \dot{\pi}_{(1)}]$. Extracting from (37) those elements corresponding to the first row of the matrix product (and transposing), we have

$$[\dot{\pi}_1, \dot{\pi}_{(1)}]\begin{bmatrix} 1 \\ \theta_{1(1)} \end{bmatrix} = 0,$$

so that

$$\dot{\pi}_{(1)}\theta_{1(1)} = -\dot{\pi}_1 .$$

This is a linear system of n equations in $n - 1$ unknowns, so it is in general over-determined. A convenient representation for $\theta_{1(1)}$ is then

$$\theta_{1(1)} = -[\dot{\pi}_{(1)}'\dot{\pi}_{(1)}]^{-1}\dot{\pi}_{(1)}'\dot{\pi}_1 . \tag{39}$$

It follows that the virtual stakes associated with Pareto-optimal outcomes are identified given $\dot{\pi}$, the matrix of profit derivatives with respect to price. We now show how these can be straightforwardly obtained.

In our linear example with each firm producing a single good, we have

$$\begin{aligned}\pi_{ij} \equiv \partial\pi_i / \partial p_j &= 1_{[i=j]}q_i + b_{ij}(p_i - \gamma_i) \\ &= 1_{[i=j]}q_i + b_{ij}\varphi_i ,\end{aligned} \tag{40}$$

where $1_{[i=j]}$ is the indicator equal to 1 if $i = j$ and equal to 0 otherwise, and φ_i represents firm i's equilibrium unit profit margin, $\varphi_i \equiv p_i - \gamma_i$. It thus suffices to observe equilibrium quantities q_i, equilibrium unit margins φ_i, and to estimate the price response coefficient matrix B, which, as above, can be consistently estimated from the demand equations, using instrumental variables methods. To the extent that each of these components for computing the virtual stakes is readily observable or estimable, obtaining the Pareto-optimal virtual stakes is straightforward, and requires no econometrics beyond the demand system estimation that is currently common.

Note also that the linear structure of our example is readily relaxed in the present context. It suffices simply to interpret b_{ij} as the demand function slope $\partial q_i / \partial p_j$ and to interpret γ_i as firm i's marginal cost $\partial c_i / \partial q_i$, so that φ_i is now firm i's equilibrium marginal profit margin. A convenient and insightful expression obtains by letting $\xi_{ij} \equiv (\partial q_i / \partial p_j)(p_j / q_i)$ denote the (cross-)price elasticity of demand and letting $\mu_i \equiv \varphi_i / p_i = (p_i - \partial c_i / \partial q_i) / p_i$ denote the equilibrium marginal profit margin as a proportion of price (firm i's "profit margin"). After some rearrangement we have

$$\pi_{ij} = q_i \left(1_{[i=j]} + \xi_{ij} \mu_i (p_i / p_j)\right). \tag{41}$$

Plugging (40) or (41) into (39) then delivers the desired equilibrium virtual stakes for firm 1, and these can then be translated into virtual stakes for the remaining firms using (38).

One can use virtual stakes obtained in this manner to obtain estimates of the other parameters of interest. The reduced form approach described above for estimation with virtual stakes assumed constant is no longer appropriate, because in the present Pareto-optimal setting the virtual stakes are endogenous. Moreover, the full reduced form that accounts for the endogeneity of the virtual stakes here is highly nonlinear. Nevertheless, in our linear example, we can make use of our expression

$$(B' + B_\theta) p + \delta - B_\theta \gamma = 0$$

to estimate parameters of interest in a straightforward manner. Specifically, we have from this expression that

$$B_\theta^{-1} (B' + B_\theta) p + B_\theta^{-1} \delta = \gamma .$$

Now $\delta = (q - B' p)$, so

$$\begin{aligned} & B_\theta^{-1} (B' + B_\theta) p + B_\theta^{-1} \delta \\ &= B_\theta^{-1} (B' + B_\theta) p + B_\theta^{-1} (q - B' p) \\ &= p + B_\theta^{-1} q , \end{aligned}$$

which yields

$$p + B_\theta^{-1} q = \gamma \\ = \mathrm{A}_{00}\mathrm{Z}_{00} + \eta_0, \tag{42}$$

where the last expression follows from our linear specification for γ. From this expression, we see that we can estimate A_{00} by ordinary least squares regression of $p + B_\theta^{-1} q$ (endogenous) on Z_{00} (exogenous). Although this is not literally feasible as B (required along with θ to construct B_θ) is unknown, replacing B with a consistent estimate (e.g., from demand equations estimated by instrumental variables) delivers asymptotically equivalent estimates of A_{00}. (This approach also provides a convenient way to recover A_{00} in the constant virtual stakes framework, once θ has been estimated.)

Finally, we note that the regression specified by (42) provides a convenient framework for testing restrictions of interest on θ, for example that certain subsets of virtual stakes values are zero, as occurs when only a specified subset of firms in an industry is behaving cooperatively.

5 Conclusion

[This section remains to be written **]**

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