The Demand for Lotto: The Role of Conscious Selection

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This article presents estimates of the elasticity of demand for lottery tickets using time series data in which there is variation in the expected value of a lottery ticket induced by rollovers. An important feature of our data is that there are far more rollovers than expected given the lottery design. We find strong evidence that individuals do not choose their lottery numbers uniformly from a uniform distribution—that is, conscious selection. We use our estimates to derive the inverse supply function for the industry, and this enables us to identify the demand elasticity. We find the price elasticity to be close to unity, which implies that the operator is revenue maximizing—which is the regulator's objective.

KEY WORDS: Lotto; Lottery; Simulation.

Many countries have lotteries that raise considerable amounts of revenue. In many cases, these are state-owned monopolies in which the revenue in excess of costs is used as tax (sometimes hypothecated) revenue, and in other cases these are regulated with tax (and other deductions) being a contractually specified proportion of revenue. The largest examples of lotteries are to be found in Spain, Australia, Ireland, Canada, and the United Kingdom, and in the United States there are large games in some states and some large consortia that sell across several states. Our analysis is based on the U.K. National Lottery that was introduced in November 1994 although, because game designs are similar the world over, our methodology and findings are more widely relevant. We refer to the online draw game as the National Lottery (NL), which is run by a private company franchised, from 1994 to 2001, by the U.K. government and regulated by the Office of the National Lottery (OFLOT). The NL online "lotto" draw rapidly became the fourth largest in the world in terms of weekly sales revenue per capita. In its first year, NL sales were estimated to be \$4.39 billion (this is approximately 1% of annual retail sales), it raised \$527 million in direct government tax revenue (i.e., not allowing for reductions in revenue from other sources induced by changes in expenditure patterns), and, in addition, \$1.296 billion in revenue was hypothecated for charitable purposes (known as "good causes"). The NL operator also markets Instants, a "scratchcard" game that was introduced just six months after the launch of the online draw. Instants revenue also benefits good causes. Fitzherbert, Giussani, and Hunt (1996) provided useful background to the U.K. National Lottery; see LaFleurs (1998) for an international perspective.

The "price" elasticity of demand for lottery tickets shows how demand varies with the expected value of the return from a ticket, and it is this elasticity that is relevant in assessing the merits of the design of the lottery and the attractiveness of potential reforms to the design. That is, it tells us how demand would vary in response to changes in the design of the lottery—in particular, the tax rate on the lottery or the nature of the prizes. Moreover, an estimate of the elasticity would enable us to see the extent to which the regulator, OFLOT, is succeeding in getting the operator to revenue maximize as opposed to profit maximize.

Previous work has attempted to estimate this elasticity by looking at how demand varies in response to actual changes in lottery design across time or differences across states (see Vrooman 1976; DeBoer 1985; Beenstock, Goldin, and Haitovsky 1997). These have been few and far between, however, and limited attempts have been made to control for other changes and differences that may have occurred. Moreover, it seems likely that these design changes have themselves been endogenous—for example, motivated by flagging sales. An important exception is found in the work of Clotfelter and Cook (1993), who estimated the elasticity of sales with respect to variations in the size of the major draw prize (the jackpot). We also exploit the changes in the return to a ticket induced by "rollovers" that occur when the jackpot in one week is not won and gets added to the jackpot prize pool for the subsequent week. This changes the expected return to a ticket in a very specific way. In particular, the expected return rises in a way that cannot be arbitraged away by the behavior of agents. Moreover, it is only by comparing rollover with nonrollover weeks that we can reveal the appropriate demand elasticity: Changes in the expected return that occur because of demand shocks result in movements along the (inverse) supply curve, not movements along the demand curve; only rollovers cause a shift in the (inverse) supply schedule so that sales rise along the demand curve.

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The fact that we rely on rollovers for identification causes us a difficulty: Rollovers should occur with relative infrequency, so we may not have sufficient variance in our data to obtain a reliable estimate. One of the startling features of our data, however, is that it exhibits many more rollovers than could have been generated by statistical chance. The frequency of rollovers is obviously a property of the lottery design. In general, if players choose randomly, the probability of a rollover is given by $(1-\pi)^n$, where n is the number of tickets sold, so that π is approximately 1 in 14 million for a lottery design that features 6 balls drawn from 49 without replacement. For the U.K. lottery, this implies a rollover probability of approximately 1% when ticket sales are at their mean levels of 65 million. In fact, in our data there have been 19 rollovers in 116 draws, and a high rollover probability is a phenomenon common to all lotto games. For example, the U.S. Powerball lotto game recently rolled over in 17 *consecutive* draws, eventually generating a jackpot in excess of $200 million.

Rollover frequency depends on the game design and the level of sales, but in the U.K. case the exceptionally high frequency of rollovers can only arise from individuals choosing the numbers on the lottery tickets that they buy in a nonuniform way. That is, many more individuals choose the same combinations of numbers than would occur by chance if individuals selected their numbers uniformly. Clotfelter and Cook (1993) referred to this nonuniformity as "conscious selection." The result of it is that the probability distribution of numbers chosen by players does not follow a uniform distribution, and the probability of each number being chosen in the draw is the inverse of the number of balls drawn—for example, 1/49 in the NL case. Thus, the tickets sold cover a smaller set of possible combinations than would have been the case had individuals chosen their numbers in a uniform way—thus, there will be more occasions when there are no jackpot prizes. There will also be more occasions when there are many jackpot winners. That is, the variance in the number of jackpot winners will be higher under nonuniform choice. The implications of this nonuniformity and (unintentional) coordination between players are important. If players realize that such nonuniformity is occurring, then they will expect the return for holding a ticket to be smaller (for any given size of rollover) than it would be if individuals were choosing their numbers uniformly. Essentially, the nonuniformity increases the probability that there will be a rollover, and we would expect this to change the behavior of potential ticket purchasers.

Lotteries are typically operated to maximize the resulting tax (or "good-causes") revenue, which is usually a fixed proportion of sales. Thus, knowledge of the price elasticity is central to choosing an appropriate "take-out" rate. The aim of this article is to recover the elasticity of demand for lottery tickets from data on sales and on the number of prize winners of each type. The latter is one important innovation of the article. The observed number of prize winners enables us to deduce the extent to which actual behavior departs from random choice from a uniform distribution of numbers and this allows us to compute the expected value of holding a ticket, given the way in which numbers are chosen. Indeed, a knowledge of the number of winners of each prize pool and the winning numbers enables us to estimate the probability distribution of the numbers chosen and test for nonuniformity.

The second innovation in the article is that it addresses a deficiency in the existing estimation of the demand elasticity. In particular, earlier work has, at best, regressed sales against the expected value. As we shall show, however, the expected value is a complicated nonlinear function of sales and the value of the rollover. Thus, nonlinear maximum likelihood methods need to be applied.

The plan of the article is as follows. In Section 1, we outline the theory of the determination of the expected value of holding a lottery ticket that allows for the nonuniform nature of individual choices of numbers. This is essentially the inverse supply function for the market. In Section 2, we outline the statistical methodology required to estimate the extent of nonuniformity to enable us to compute this inverse supply function. In Section 3, we describe the data and produce results from estimating precisely how individuals are nonuniform. That is, we produce estimates of the probability distribution for the numbers chosen and generate the inverse supply function from this. In Section 4, we use these estimates of the extent of nonuniformity to infer the expected value of holding a ticket and apply nonlinear maximum likelihood to estimate from this the elasticity of demand from the observed relationship between sales and the expected value.

1. THE EXPECTED VALUE OF A LOTTERY TICKET

The expected value (V) of holding a lottery ticket was first derived by Sprowls (1970) and has subsequently been used by Scoggins (1995) and Clotfelter and Cook (1993). These articles only considered the case in which individuals choose their numbers uniformly and in which there are no fixed prizes (i.e., winners share a prize pool so that the size of all the prizes won depends on the number who win). There is considerable evidence, however, that many players choose their numbers themselves, and there is a strong possibility that they do so using rules that are shared by other players. This opens the possibility that the proportion of certain six-number combinations covered by the tickets sold is less than would be expected under uniform choice. Clotfelter and Cook (1993) asserted that the qualitative properties of V under the assumption of uniform selection hold in this more general case of "conscious selection." We know of no proofs of these properties in this more general framework, however, and we shall turn to this later.

In addition to observing the frequency of rollovers the lottery operator records, for each draw, we have information on the "coverage rate" that shows the proportion of the 14 million possible combinations of numbers that have been chosen by at least one player. We would expect the coverage rate to be close to unity when sales are on the order of $60 million. Although the coverage data themselves are regarded as commercially sensitive information by the operator, we can reveal that in the early draws of the game the coverage rate was less than .75 and has drifted upward to approximately .9 toward the end of the period. In addition to this systematic trend, there is a marked increase in coverage when there is a rollover. This presumably occurs because players wish to purchase more than their normal number of tickets and may therefore choose additional combinations less systematically. In addition, there was a small increase in coverage associated with the introduction of the "Lucky Dip" facility on the sales terminals that permitted players to instruct the terminal to use its pseudorandom number generator to pick the combination. The use of the Lucky Dip facility has been modest, with less than 10% using the facility shortly after its introduction rising slowly to around 15% at the end of our data period.

1.1 Expected Value Under Nonuniform Choice

The expected monetary value of a lottery ticket depends on the size of the prizes, which, for a pari-mutuel lottery, are functions of the number of winners. Consequently, deriving a formula for the expected value of a ticket requires that we specify a model of how participants select the numbers on their tickets. We will make the simple assumption that individual selections are independent realizations of the same (typically, nonuniform) probability distribution, π. More particularly, if S is the set of permitted selections of numbers (6 out of 49 for the U.K. lottery), the probability that any participant selects $\sigma \in S$ is $\pi(\sigma)$, independently of all other choices.

Participants wishing to make a uniform choice are often offered the facility of a random-number generator. This was introduced in week 71 of the U.K. game. Such a possibility can be accommodated by using a mixture distribution $\pi(\sigma) = \delta/|S| + (1-\delta)\pi_0(\sigma)$, where δ is the proportion using Lucky Dip (which is publicly available information for each draw) so that π_0 is the distribution used by those eschewing Lucky Dip.

With n tickets sold, the total revenue is n (taking the cost of the ticket as numeraire) of which a proportion τ is taken for "tax" and operating costs. Many lotteries, including the U.K. National Lottery, offer at least one class of prize that is fixed in value irrespective of the number of winners. We shall assume that there is only one such prize worth a (although the results are readily extended to cover more general cases) and write $\pi_F(\sigma)$ for the probability that a participant who selects according to the distribution π will win such a prize when $\sigma \in S$ is drawn. In the U.K. lottery, $a = \$10$ (except that this may be reduced in the unlikely event that the prize pool is not large enough to pay this to all winners). The probability of such an event is so small (less than 1 in 1,500) that it has no significant effect on the expected value calculations, however, and we ignore it. If there are n_F winners of the fixed prizes, we are left with a residual pool of $(1-\tau)n - an_F$, a proportion ρ of which (augmented by any sum, $R \geq 0$, rolled over from the previous draw) forms the jackpot prize pool that is distributed equally among all of the jackpot prizewinners. We write $\pi_J(\sigma)$, which is equal to $\pi(\sigma)$ in the U.K. NL, for the probability that a participant selecting according to π will win the jackpot pool when $\sigma \in S$ is drawn. If there are no winners, the jackpot pool is rolled over to the next draw. We assume that all the remaining prize money is distributed as prizes in the current draw: Rollovers of prize pools other than the jackpot pool are excluded. The actual rules are usually a good deal more complicated than this. For example, in the NL, if no participant matches four numbers drawn, this part of the prize pool is used to augment the fixed prizes. In many lotteries the number of consecutive draws in which the jackpot pool can be rolled over is limited (to 3 in the NL). For the participation levels observed in practice, however, all deviations from our simplified story are very low-probability events, which we ignore. For example, in the NL the most likely candidate for a rollover, after the jackpot, is matching five numbers out of six drawn plus the bonus ball that is also drawn. With $n > 10$ million, the probability of a rollover of this prize pool is less than .01, and with $n = 60$ million this falls to less than 10^{-11}. Nevertheless the formulas derived later may need modification and should at least be treated with circumspection for "small" n, where smallness will depend on the lottery design. Because our choice model assumes that players are homogeneous, the expected money value of the nonfixed prizes other than the jackpot will be $(1-\rho)\{(1-\tau)n - an_F\}/(n - n_J - n_F)$, where n_J is the number of jackpot winners.

We will first derive the expected value of a ticket from the perspective of a participant who has not yet made his/her choice of numbers but who will use the distribution π to do so. The value will depend on n_F and n_J (as well as on n), which are not known for certain when the participation decision is made. It will prove convenient to condition on the draw, $\sigma \in S$, and let $N_J(\sigma)$ and $N_F(\sigma)$ be random variables denoting the number of jackpot and fixed prize winners, respectively, among the other participants. If this purchaser wins the jackpot, the total number of jackpot winners is $N_J(\sigma) + 1$, so the expected prize per winner is

$$E\left[\frac{\rho\{(1-\tau)n - N_F(\sigma)a\} + R}{N_J(\sigma) + 1}\right] = \alpha, \text{ say.}$$

If the purchaser wins neither the jackpot nor a fixed prize, the expected value is

$$E\left[\frac{(1-\rho)\{(1-\tau)n - N_F(\sigma)a\}}{n - N_J(\sigma) - N_F(\sigma)}\right] = \beta, \text{ say.}$$

Hence, the expected value of a lottery ticket, conditional on σ, is

$$E[V|\sigma] = \alpha\pi_J(\sigma) + \beta(1 - \pi_J(\sigma) - \pi_F(\sigma)) + a\pi_F(\sigma)$$

because any given ticket may win at most one prize. Note that α and β will depend on σ. The random variables $N_J(\sigma)$ and $N_F(\sigma)$ follow a trinomial distribution, $T(n-1, \pi_J, \pi_F)$, given by

$$P(N_J = i, N_F = j) = \pi_J^i \pi_F^j (1 - \pi_J - \pi_F)^{n-1-i-j} \frac{(n-1)!}{i!j!(n-1-i-j)!}, \quad (1)$$

where we have suppressed the dependence on σ. This enables us to evaluate the expectation of the ratios of random variables in α and β (see App. A for details) and leads to Proposition 1.

Proposition 1. The expected value conditional on σ is given by the expression

$$E[V|\sigma] = \left[\rho(1-\tau) + \frac{R}{n}\right][1-(1-\pi_J)^n] + (1-\rho)(1-\tau)[1-(\pi_J+\pi_F)^n] + a\pi_F[\rho(1-\pi_J)^{n-1} + (1-\rho)(\pi_J+\pi_F)^{n-1}].$$

Because n is large and π_F, and especially π_J, are small, then $(\pi_J+\pi_F)^{n-1}$ effectively vanishes and $(1-\pi_J)^n$ and $(1-\pi_J)^{n-1}$ can both be closely approximated by $e^{-\pi_J n}$. Assuming that the lottery is fair, we obtain the following corollary.

Corollary 1. For large n, the expected value of a ticket is given by

$$V = \frac{1}{|S|}\sum_{\sigma\in S}\phi(\sigma),$$

where

$$\phi(\sigma) = 1-\tau-K(\sigma)e^{-\pi_J(\sigma)n} + \frac{R}{n}\left[1-e^{-\pi_J(\sigma)n}\right]$$

and $K(\sigma) = \rho[(1-\tau) - a\pi_F(\sigma)]$.

1.2 Comparative Statics

It is immediate, from Corollary 1, that $V_R > 0, V_\tau < 0$, where subscripts indicate partial differentiation: The interpretation of these results is obvious. Moreover, $V_a > 0$, which reflects the fact that fixed prizes, unlike the jackpot, are not subject to rollovers and so cannot be captured by participants in future draws: An increase in a reduces the size of any rollover. Finally, $V_\rho < 0$ provided that $K(\sigma) > 0$ for all σ. This inequality simply says that a is set at a level at which the prize fund is always large enough to pay the fixed prizes plus a positive jackpot and holds for the U.K. NL, both for the uniform case and for our estimated π. We have assumed that n is large enough for the law of large numbers to hold.

Comparative statics with respect to n (which we shall treat as continuous) are more involved than would be the case were conscious selection not a possibility [see Farrell and Walker (1999) for the simpler case]. The basic results for ϕ will be summarized later. Proofs of the less obvious assertions will be found in Appendix A.

Proposition 2.

1. As $n \to \infty, \phi(\sigma) \to 1-\tau$.
2. If $R = 0, \phi(\sigma)$ is a strictly concave and increasing function of n.
3. If $0 < R < \bar{R}(\sigma) = 2K(\sigma)/\pi_J(\sigma)$, then there is an $\hat{n} > 0$ such that $\phi(\sigma)$ is strictly increasing for $n < \hat{n}$ and strictly decreasing for $n > \hat{n}$.
4. If $R \geq \bar{R}(\sigma), \phi(\sigma)$ is strictly decreasing for all n.

The behavior of ϕ is determined by two countervailing effects. First, an additional participant increases the prize pool and decreases the probability that the jackpot is not won. When $R = 0$, this gives 2 (the concavity reflects the fact that the marginal benefit is decreasing). Clotfelter and Cook (1993) referred to this phenomenon as lotto's "peculiar economies of scale," referring to the fact that, for a given game design, the larger the sales the greater the expected return. When $R > 0$, there is a second effect: Additional participants dilute any individual's potential share of the fixed sum R. If $R \geq \bar{R}(\sigma)$, the marginal dilution effect outweighs the marginal benefit for all values of n. Otherwise this inequality is reversed for $n < \hat{n}$. In all cases, both effects vanish for large n and the entire prize pool is returned to participants; this is part 1.

Note that parts 1, 3, and 4 of Proposition 2 imply that, if $R > 0, \phi > 1-\tau$ for all large enough n, while part 4 implies that, for sufficiently large values of R, this inequality holds for all values of n. The important feature of these results for our empirical work is that they imply that when a rollover occurs sales will rise if demand is increasing in ϕ, but the rise in sales cannot be sufficient to arbitrage the rise in ϕ away (at least not for finite n).

If lottery participants select numbers according to a uniform distribution, then π_J and π_F are independent of σ. Hence V inherits all the properties of $\phi(.)$ described in Proposition 2. If the selection of numbers is not uniform, parts 1, 2, and 4 of Proposition 2 hold except that the formula for $\bar{R}$ is no longer valid in the case in which a is nonzero (and no simple alternative formula is available). Part 3 may fail, however, even when $a = 0$ and $\rho = 1$. A counterexample with these parameter values is given in Appendix A. Figure 1 illustrates these results for the 6/49 case used in the NL. The solid lines correspond to the case in which there is uniform selection and no rollover (i.e., $R = 0$), £15 million, and £40 million, which are typical for single and double rollovers (where the jackpot is rolled over for two consecutive weeks). In the nonrollover case, V approaches $(1-\tau)$ from below monotonically as n in-

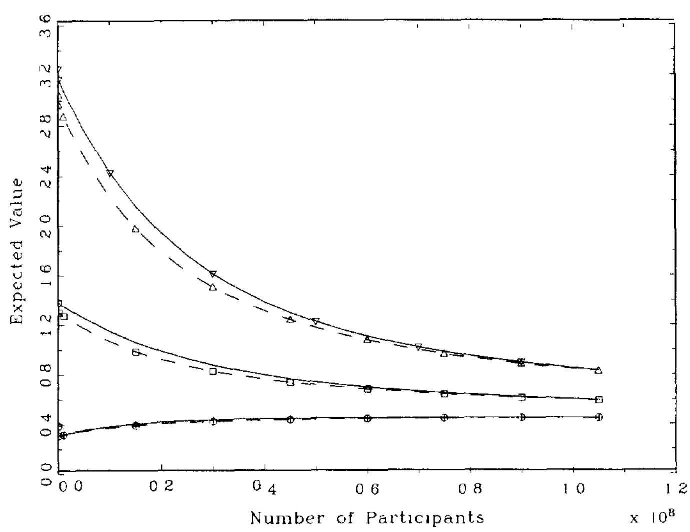

Figure 1. The Expected Value Function.

creases. In the case of a small rollover, V need no longer be monotonic and will eventually rise above $(1-\tau)$, which it then approaches from above as n tends to infinity. In the larger rollover cases illustrated here, V will be monotonically decreasing and again approach $(1-\tau)$ from above. The dashed lines are the corresponding V functions when the selection is made from the distribution of numbers estimated in Section 2.3. In this case V lies below the uniform case although the difference is small at large values of n because both the uniform and nonuniform cases are asymptotic to $(1-\tau)$.

1.3 The Effect of Nonuniform Choice

In the absence of fixed prizes $(a=0)$, V is a strictly concave function of the probability distribution $\pi(\sigma), \sigma \in S$. It follows from Jensen's inequality that V has a strict global maximum at $\pi = \pi_U$, where $\pi_U(\sigma) = 1/|S|$ for all $\sigma \in S$—that is, the uniform distribution. Thus, the expected value of tickets would increase for all participants if they adopted a uniform choice distribution. Whether this represented an improvement for these agents would depend on the extent to which the freedom to choose one's own numbers was a significant factor in the return to participating. It has been suggested that the original New Hampshire lotto game was unsuccessful because it did not permit players to select their own numbers and that when it did allow this facility it became successful. GTECH, a software company used by many lotteries around the world, suggests that players should not be allowed to select randomly until they have been denied this opportunity for some time so as to promote the "ownership" of number combinations by players. This, of course, leads to more rollovers, more revenue for the government, and more profits for the operator.

Most lotto games offer some form of "Lucky Dip" in which the uniformity required to implement π_U is performed by the game operator. It is important to note that we are not claiming that participants can maximize their individual expected returns by choosing π_U. Indeed, to do this one would need to select combinations that are chosen infrequently by others. It has even been claimed that this permits the expected value of a ticket to exceed the cost [e.g., see Haigh (1996) and Ziemba, Brumelle, Gauthier, and Schwartz (1986), and see MacLean, Ziemba, and Blazenko (1992) for strategies for maximizing expected income growth in gambling when there are such profitable opportunities].

When $a > 0$, the presence of fixed prizes makes the story more complicated. We show in Appendix A that, provided the rules for determining fixed prize winners award such a prize, if $q (< p)$ of the p numbers selected on the ticket are included in the p drawn (in the NL, $q=3$), then the Hessian of V regarded as a function of π is positive definite at π_U for sufficiently large values of n ($n > 14 \times 10^6$ would be sufficient). Appendix A shows that this result is true for much more general rules. Because the first-order conditions for a maximum of V [subject to the constraint $\sum_{\sigma \in S} \pi(\sigma) = 1$] are satisfied (trivially, by symmetry) at π_U, we deduce that π_U is a strict local maximum. Attempts to turn this into a global result encounter difficulties arising from the simplification gained from ignoring the possibility that the prize pool is inadequate for paying the fixed prizes: For π sufficiently remote from π_U, there could be $\sigma \in S$ for which the expected value of a fixed prize exceeds $1-\tau$, which, by the law of large numbers, leads to the prize pool being unable to fund the fixed prizes and drives the jackpot to 0. For our estimated distribution, however we still find that the expected ticket value is less than $E[V(\pi_U)]$.

2. NONUNIFORM CHOICES AND STATISTICAL INFERENCE

2.1 Modeling

The purpose of this section is to propose a model of the mechanism that individuals use to choose the numbers on which they bet and show how consistent estimation can be achieved. Similar issues in the literature in statistics have also been tackled by Johnson and Klotz (1993) and Finklestein (1995); for a survey, see Haigh (1996). Our interest in this is to compute the expected value allowing for the nonuniform choices of numbers. Thus, we need to estimate the extent to which choices are nonuniform. Our aim is to allow for a nonuniform distribution over the set of all possible combinations. The possible parameterizations we can consider are limited by the information available, at least as far as small samples are concerned. Because we cannot observe a sample of the actual combinations played and aggregate quantities are the only available substitute, we propose a parsimonious parameterization that can be identified from observations on the weekly number of players, the number of winners, and the weekly set of numbers drawn. Moreover, the approach we propose is easily extended to allow for the case in which a known proportion of the population of players draws uniformly among combinations.

The natural way to describe individual behavior is to assume that they draw without replacement 6 numbers out of 49 in the same way the lottery mechanism does. We assume, however, that the individual probability distribution does not give a uniform weight to every number. We take for granted that the actual distribution of numbers that are drawn is uniform, which is supported by tests reported by Haigh (1996). To formalize this idea, let us denote $\mathbf{p} = (p(1), p(2), \ldots, p(49))$ as the vector of probabilities $p(i)$ that number i is picked among 49 numbers. $p(i)$ is the probability that i is chosen first among the 49 numbers and should have the properties that $p(i) \geq 0$ for all $i \in \{1, \ldots, 49\}$ and $\sum_{i=1}^{49} p(i) = 1$. The lottery draw mechanism is unbiased so that all $p(i)$ are equal to 1/49. In contrast, the individual probabilities for the numbers that individual players actually choose could be such that some numbers have larger probability of being chosen than others.

The probability of drawing a given (ordered) arrangement of numbers $\mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6)$ is

$$p(\mathbf{a}; \mathbf{p}) = \frac{p(a_1)p(a_2)p(a_3)p(a_4)p(a_5)p(a_6)}{[(1-p(a_1))(1-p(a_1)-p(a_2))\ldots (1-p(a_1)-p(a_2)-p(a_3)-p(a_4)-p(a_5))]}, \quad (2)$$

where the expression between brackets exists because of the "without replacement" nature of the draw. Obviously the probability of choosing the selection $\sigma = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ in any order is as follows:

$$\pi(\sigma; \mathbf{p}) = \sum_{\rho \in P(\sigma)} \frac{p(a_1)p(a_2)p(a_3)p(a_4)p(a_5)p(a_6)}{[(1-p(\rho(a_1)))(1-p(\rho(a_1))-p(\rho(a_2))) \ldots (1-p(\rho(a_1))-p(\rho(a_2))-\ldots-p(\rho(a_5)))]} \quad (3)$$

where the summation is over all the elements, ρ, of the set of possible permutations of the elements of $\sigma, P(\sigma)$. Note that $p(i)$ is not the probability that i is chosen whatever its rank; that is, $p(i) \neq \sum_{\{i\} \in \sigma, \sigma \in S} \pi(\sigma)$. Moreover, note that $p(i)$ parameterizes the probability distribution on the set of draws of 6 out of 49 in a very restrictive manner because the less restrictive parameterization would be to consider each $\pi(\sigma)$ as a parameter (there are millions of these!). The latter would encapsulate any correlation structure between the numbers drawn. Clearly, the former imposes constraints on the correlation structure between two draws but one that is difficult to characterize.

The probability of choosing three numbers out of $\sigma = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ is slightly more involved because we have to consider all arrangements of six numbers that contain exactly three in σ. We can write down the probability of such an event in the following way:

$$\pi_F(\sigma; \mathbf{p}) = \sum_{(j_1,j_2,j_3,j_4,j_5,j_6) \in S(3,\sigma)} \frac{p(j_1)p(j_2)p(j_3)p(j_4)p(j_5)p(j_6)}{((1-p(j_1))(1-p(j_1)-p(j_2)) \ldots (1-p(j_1)-p(j_2)-\ldots-p(j_5)))} \quad (4)$$

where $S(3, \sigma)$ is the set of all (ordered) arrangements of six numbers such that three exactly are in σ and three are in the complement of σ in $\{1, \ldots, 49\}$.

Similar expressions can be obtained for the probability to choose four or five numbers among the six in σ. Because of the large number of elements (around $6.4E+9$) in a set like $S(3, \sigma)$, direct evaluation is not possible, but modern Monte Carlo techniques can be easily adapted to provide a convenient method to evaluate $\pi_F(\sigma, \mathbf{p})$ with controlled accuracy.

2.2 Estimation

Assume that we observe D draws of 6 numbers out of 49; for each draw we observe the 6 numbers drawn, $\sigma_d, d = 1, \ldots, D$, the numbers of players (or equivalently the number of tickets sold), N_d, and the number of winners of each type—that is, with three correct numbers $n_{3,d}$, with four correct numbers $n_{4,d}$, and so forth. Assume, moreover, that the players follow our model of individual choice of numbers, independently and with identical parameters $\mathbf{p}$. Identification, of the parameter $\mathbf{p}$ could be achieved, in principle, as soon as each number has been drawn at least once by the lottery. It is then possible to obtain consistent estimates of the parameters through the maximization of the likelihood. Although it is possible to write down the likelihood for the observation of $(\sigma_d, N_d, n_{3,d}, n_{4,d}, n_{5,d}, n_{5+1,d} n_{6,d})_{d=1\ldots D}$, in practice it is easier to consider only a restricted information set. For instance, suppose the only information to be used is just $(\sigma_d, N_d, n_{6,d})_{d=1\ldots D}$, then the likelihood is

$$\ln L(\mathbf{p}; (\sigma_d, N_d, n_{6,d})_{d=1\ldots D}) \propto \sum_{d=1}^{D} [n_{6,d} \ln(\pi(\sigma_d; \mathbf{p})) + (N_d - n_{6,d}) \ln(1 - \pi(\sigma_d; \mathbf{p}))]. \quad (5)$$

Following the same idea, we can extend the information set to the number of winners with three correct numbers, we have $\ln L(\mathbf{p}; (\sigma_d, N_d, n_{3,d}, n_{6,d})_{d=1\ldots D})$, which is proportional to

$$\sum_{d=1}^{D} [n_{6,d} \ln(\pi(\sigma_d; \mathbf{p})) + n_{3,d} \ln(\pi_F(\sigma_d; \mathbf{p})) + (N_d - n_{6,d} - n_{3,d}) \ln(1 - \pi(\sigma_d; \mathbf{p}) - \pi_F(\sigma_d; \mathbf{p}))]. \quad (6)$$

Extending this to cover other possible prize pools is tedious and computationally demanding but conceptually trivial. Because of the difficulty of evaluating $\pi_F(\sigma; \mathbf{p})$ directly, we substitute a smoothed unbiased simulated version of it [see McFadden (1989) and Monfort and van Dijk (1995) for an introduction to recent developments). Although there are several ways to carry out the simulation, the simplest is to apply the technique of importance sampling. This is described in Appendix B.

We can easily extend the preceding approach to account for a known proportion of players choosing their combination uniformly (i.e., players using the Lucky Dip option). In the simple case, the information set becomes $(\sigma_d, N_d, n_{6,d}, \delta_d)_{d=1\ldots D}$, where δ_d is the proportion of uniform players in draw d. The likelihood (5) then becomes $\ln L(\mathbf{p}; (\sigma_d, N_d, n_{6,d}, \delta_d)_{d=1\ldots |D})$ which is proportional to

$$\sum_{d=1}^{D} \left[n_{6,d} \ln \left(\delta_d \frac{1}{|S|} + (1-\delta_d)\pi(\sigma_d; \mathbf{p}) \right) + (N_d - n_{6.d}) \ln \left(1 - \delta_d \frac{1}{|S|} - (1-\delta_d)\pi(\sigma_d; \mathbf{p}) \right) \right].$$

An expression similar to this can be derived to extend the likelihood in (6).

2.3 Data and Estimates of the Distribution

Our data are the first 116 weekly draws from the U.K. online lottery. The operators, Camelot PLC, are obliged to publish the level of sales, the winning numbers, and the number of winners of each prize pool. They are not obliged to publish the distribution of numbers actually chosen by players, and they regard this as commercially confidential information. This information is only published rarely. To our knowledge, only a lottery in Switzerland and one in Canada publish this information. Our data records aggregate weekly ticket sales for the online game only starting from the first week of the lottery's operation in November 1994 until February 1996 (after which the operator introduced a second draw each week). The level of sales is given in Figure 2, and it is obvious where the rollovers and double rollovers occur. The average sales are approximately £65 million in regular weeks, approximately £75 million in the 15 rollover weeks, and the two double rollover weeks have sales in excess of £100 million (a further two weeks were "superdraws" in which the operator guaranteed a minimum size of jackpot, but this guarantee did not prove to be a binding constraint in either week—that is, a sufficient number of tickets were sold for the operator not to have to top up the jackpot prize pool). The proportion of draws that resulted in rollovers is .2361—that is, there were more than 20 times more rollovers than we would have expected.

Figure 3 shows the results of estimating the unknown distribution of numbers chosen by players derived using the number of winners of the six-ball (i.e., jackpot) and three-ball prize pools, the level of sales, and the published number of tickets that were bought using the Lucky Dip facility. There is obviously considerable nonuniformity in the selection of numbers by players. The likelihood ratio test of uniformity is highly significant. The difference between the likelihoods with and without allowing for Lucky Dip is 924.936, which shows that allowing for the Lucky Dip is important. Nevertheless the estimates are similar; see Figure 3.

The popular conception that numbers above 31 are less popular (because players are thought to choose their birth dates) appears to be broadly true, although precisely why 33 appears to be popular and 1, 21, and 26 so unpopular is unclear. This estimated distribution allows us to predict the number of winners of each prize pool for any draw of numbers. Thus, in Figures 4 and 5, we show the predicted and actual proportion of winners of the six (i.e., jackpot) and three-ball match prize pools. Our estimated distribution of numbers selected clearly does much better at predicting the number of winners than the uniform distribution does, although there are some outliers that suggest we have not captured all aspects of the distribution. One particular outlier illustrates the problem. In draw 10 there were 133 jackpot winners (each won a little more than £100,000—compared

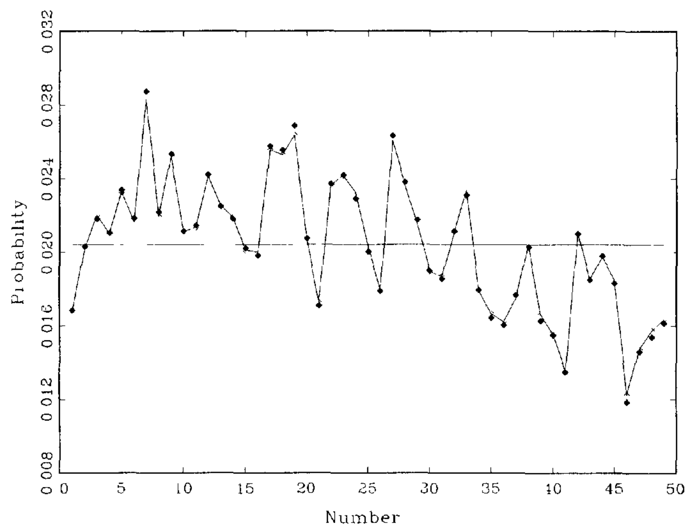

Figure 3. Estimated Distribution of Numbers Selected.

with a typical jackpot winner who would expect to receive approximately £2 million) and in draw 69 there were 53. The mean number of jackpot winners when ticket sales are £65 million per draw should be 4.65 with a standard deviation of 2.16 when players choose uniformly. The assumption that the numbers chosen by players are independent of each other would not allow us to capture combinations that form patterns, and hence our estimated distribution does not capture this phenomenon. It would be extremely difficult to generalize our methodology to capture such behavior [but see Cox, Daniell, and Nicole (1998) for similar results using an alternative methodology that does attempt to capture this behavior]. Notice that outliers are less of a problem with the £10 prize (three-ball matches) and our estimates of the proportion of winners are very close to the actual proportions. In draw 10, the winning combination made a pattern on the ticket panel. Matching "patterns" are less of an issue when only three numbers are involved.

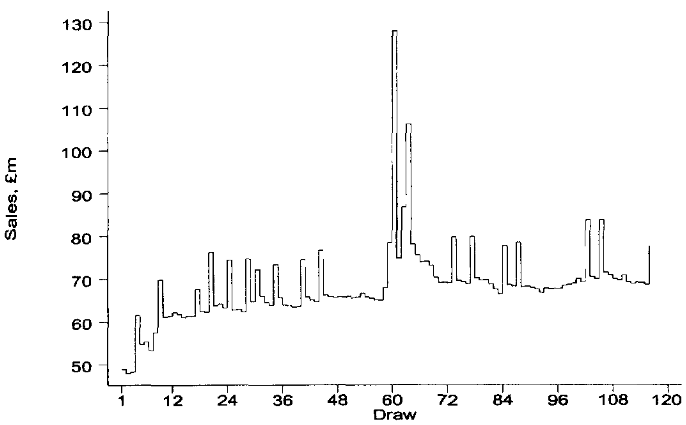

Figure 2. National Lottery Weekly Sales.

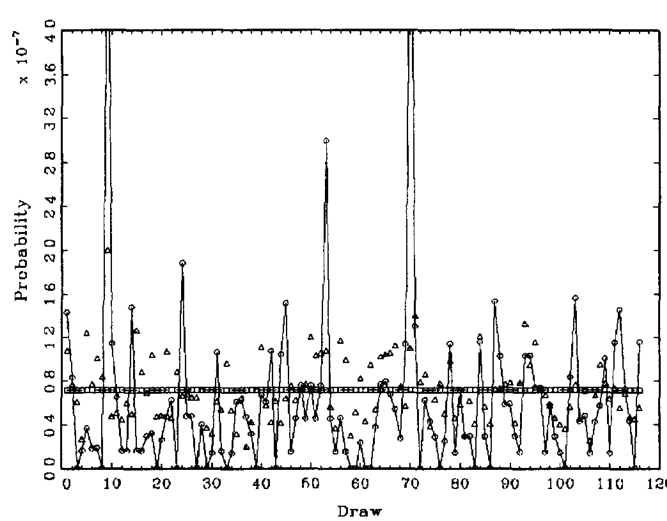

Figure 4. Predicted and Actual Winners—Jackpot Prize. Week 10 is so much of an outlier (the proportion of winners was 1.904×10^{-6}) that to include it would so distort the scale that little other variation would be seen.

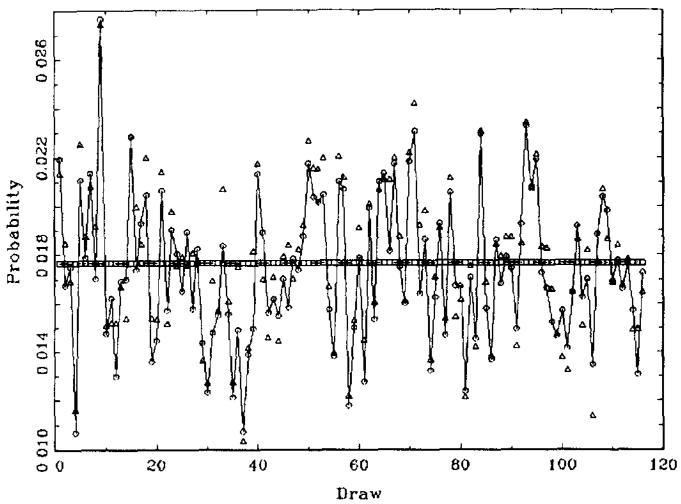

Figure 5. Predicted and Actual Winners —— £10 Prize.

Our estimated distribution should allow us to achieve a higher return on a ticket. The return from exploiting the estimated distribution (i.e., the actual return on average) is just .44 in regular weeks when the average sales are £65 million. In a double-rollover week, when sales are £110 million and the rollover is more than £20 million so that the jackpot is approximately £40 million, it rises to .65. In contrast, random choice (i.e., using the operator's Lucky Dip facility) would yield expected returns of .45 and .67 in the regular and double-rollover cases, respectively. Figure 1 plots the relationship between expected value and sales for both the uniform and the estimated distribution cases. The differences are small when the returns are evaluated at the observed, very high, levels of sales. The six least popular numbers (46, 41, 47, 48, 40, 36) would, in a typical nonrollover week, yield a return of .9533 (i.e., −4.63% per week) whereas the six most popular numbers (7, 19, 27, 17, 18, 9) would yield a return of only .2303 (i.e., −76.34% per week)—assuming that the distribution remains unchanged. In Figure 6 we plot the distribution of expected values implied by our estimates. Of course, in a rollover the expected values of all tickets are raised and then a knowledge of the popular combinations might give one sufficient "edge" to make investing in this asset a profitable activity, although it would be an extremely risky investment.

Our finding, shown earlier in Figure 1, that the expected value seems to be relatively insensitive to conscious selection explains why players choose popular numbers: There is some utility gain associated with playing one's "lucky" combination, and this is sufficient to outweigh the small loss in expected value associated with the higher probability of sharing the jackpot in the remote event that one wins. What the source of this utility gain might be is unclear: There is some literature that suggests that players suffer from an "illusion of control," but it is also possible that choosing one's own numbers might endow one with "bragging rights" in the event of winning.

3. ESTIMATING THE DEMAND ELASTICITY

The previous sections have developed the relevant theory of how the expected value of holding a ticket is determined and estimated the relevant parameters to enable these expected values to be computed. This section computes those expected values and shows how they vary with sales and with rollovers. The aim is to estimate the "price" elasticity of demand for lottery tickets with respect to the expected value of a ticket and compare this in the case in which the expected value allows for conscious selection with the uniform selection case. Figure 7 plots the relationship between sales (in £ million) and the expected value (assuming the uniform case). It clearly shows the distinction between regular draws with no rollover from the rollover and two double-rollover cases.

The essence of our empirical model is formed by confronting some aggregate demand specification, $D(V)$, with a mechanism by which individuals form their expectations about the expected value that will reign when the draw takes place. We make the rational-expectations assumption that individuals forecast sales correctly so that their expectation of the expected value at the time of the draw is given by

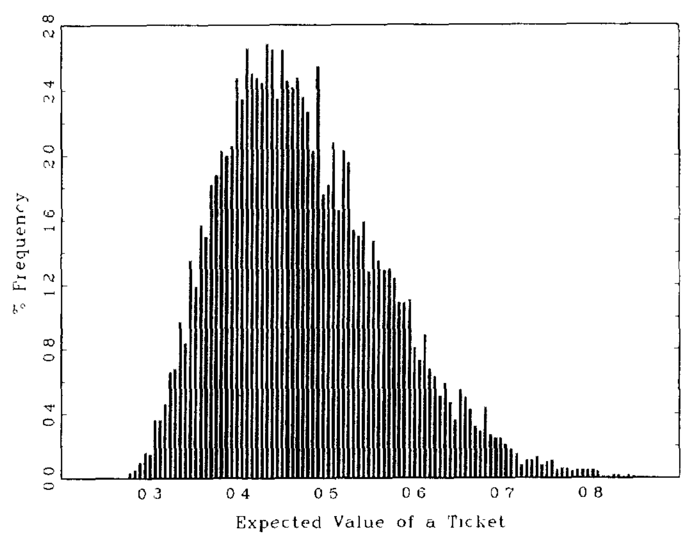

Figure 6. Estimated Distribution of Expected Value.

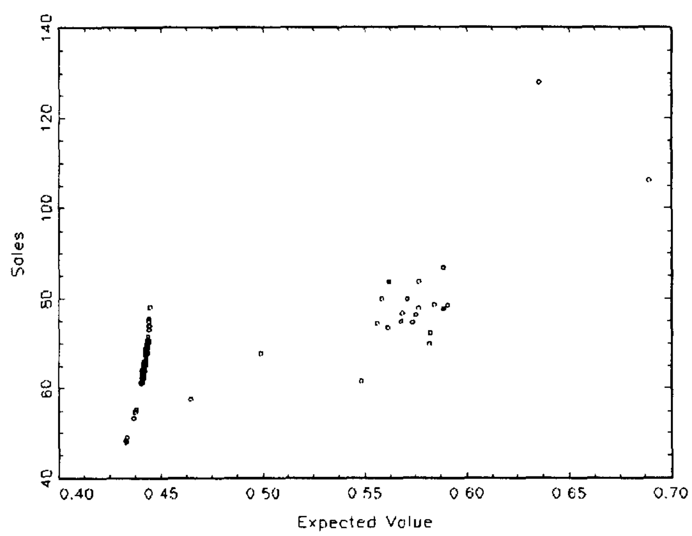

Figure 7. Sales and Expected Value.

the actual expected value given the aggregate sales for that draw. The majority of sales occur in the 36 hours immediately prior to the draw after the operator has announced its forecast of the likely size of the jackpot, which has typically been very accurate. Moreover, we assume that players take account of the extent of conscious selection that occurs. The assumption that demand responds to the expected value and not to higher moments of the distribution can be defended on the grounds that the wager is typically very small and the assumption that agents are risk neutral can be thought of as a good local approximation for such small wagers. That is, we assume that

$$V = V(R, n^S), \quad n^D = D(V) + \varepsilon, \quad n^D \equiv n^S = n, \quad (7)$$

where ε is a random disturbance uncorrelated with V and V is the expected value that depends on R, the size of the rollover, and the level of sales. The demand for tickets depends on V. We do not consider other regressors because rollovers are random events uncorrelated with other systematic variables so that there is no question of inducing omitted variable bias. Including these variables would only serve to improve the fit of the model. The resulting estimating equation is

$$n = D(V(R, n)) + \varepsilon, \qquad \varepsilon \sim \mathrm{N}(0, \sigma_\varepsilon^2), \quad (8)$$

which cannot be solved explicitly for n because the inverse supply function cannot be inverted. Thus, it is not appropriate to estimate (8) simply by least squares. Clotfelter and Cook (1993) used instrumental variables (IV) with R employed as an instrument for V, but they also included the size of the jackpot as well as V in the demand equation itself. Because V is a linear function of $J + R/n$, where J is the jackpot, there is a perfect collinearity problem, so their results are unsatisfactory at this point.

It might be argued that intertemporal substitution is an important consideration when the expected value varies from draw to draw as a result of the random incidence of rollovers. Thus, one might be tempted to include the expected value of V_{t+1} in $D(.)$. This expected future value can be expressed, however, as the weighted sum of the value when a rollover occurs and the value when no rollover occurs, where the weights are the probabilities of a rollover occurring and not occurring. That is $V_{t+1}^e = \xi(R_{t-1}).V(R_t, n_{t+1}(R_t)) + (1 - \xi(R_{t-1})).V(0, n_{t+1}(0))$, where $\xi(.)$ is the rollover probability. This can be rewritten as $V_{t+1}^e = V(0, n_{t+1}(0)) + \xi(R_{t-1}).[V(R_t, n_{t+1}(R_t)) - V(0, n_{t+1}(0))]$. The first term is constant across draws because it is simply the value of a ticket when there is no rollover. The second term does vary across draws depending on whether there is a rollover or not, but it is the product of two small numbers, the probability of a further rollover and the effect of that rollover on the expected value. We ignore this complication here and intend to return to the issue when we have suitable panel microdata available to us.

We adopt two approaches to estimation of our simple framework. One can either estimate the elasticity using IV or one can explicitly estimate Equation (8) by nonlinear methods. In Table 1 we present estimates in which we use the size of the rollover as our instrument and a dummy variable for a rollover having occurred, as well as full information maximum likelihood (FIML), nonlinear least squares, and ordinary least squares (OLS). The estimates presented in Table 1 are based on a double log specification, although other specifications yield similar elasticities. It is difficult to be more general than this because we have essentially just two observations, the mean of sales and V in regular weeks and the means in rollover weeks (with just two double-rollover weeks). Thus, our estimates should be regarded as a local approximation to the demand curve—we do not have sufficient information to allow for greater flexibility and are unlikely to ever have more because we cannot expect to have many triple rollovers given the game design and size of sales. The results are a significant improvement on those of Clotfelter and Cook (1993): Here we find a positive relationship between the expected value of a ticket and the level of sales—that is, a negative "price" elasticity—whereas Clotfelter and Cook found a negative relationship. The effects of expected value are highly significant, and the generalized R^2 shows a respectable degree of fit considering that there is a single explanatory variable [see Pesaran and Smith (1994), whose criterion we use for the OLS and IV cases, and we calculate an equivalent statistic for the FIML and nonlinear least squares cases]. Adding other explanatory variables, to capture the time trend and some of the dynamics, improves the fit, but they have little effect on the estimated V elasticity, as one would expect because these variables are orthogonal to rollovers.

Although instrumenting with a dummy variable equal to unity when there is a rollover results in a smaller elasticity than instrumenting with the size of the rollover, neither of these elasticities are significantly different from the nonlinear FIML estimates. Moreover, the least squares results are essentially identical. Thus, the results suggest that the endogeneity of V does not give rise to large bias in simple least squares estimates. This arises because the U.K. game operates at a high level of sales in all draws—that is, close to the asymptote where the expected value is essentially independent of the level of sales so that the endogeneity suggested by the theory is empirically unimportant. Inspection of Figure 7 shows that there is essentially no variation in the expected value in regular weeks; That is, the sales levels are so high that the V function is close to being flat so that all of the variation in the expected value comes from the effect of rollovers. It could be that, in different lottery designs in which the V function is not so flat at observed

Table 1. Estimated V Elasticities: Nonuniform Case

Estimation method	*V elasticity*	*Generalized R^2*
FIML	.772 (.121)	.457
OLS	768 (.079)	453
IV (rollover size)	.857 (.081)	421
IV (rollover dummy)	.654 (.083)	422

NOTE Number of observations is 116 Standard errors are in parentheses levels of play, there would be some more pronounced differences across estimation methods.

Clotfelter and Cook (1993) also made the assumption of uniform selection and so did not take account of the possible effects of nonuniform selection on this estimated relationship. When we estimate (8) using expected values generated under the assumption of uniformity, however, we find results (not presented here) that are not significantly different from those in Table 1. This is for the same reasons as IV and OLS being little different: The U.K. game operates at very high levels of sales given its 6/49 design. Again, however, it would be unwise to generalize from this to other games that have not been designed to operate at high sales.

Relative to games elsewhere, the U.K. game is relatively easy to win because the jackpot odds are just 1 in 14 million and yet the population is 58 million. In contrast, several U.S. games operate at levels of V considerably below their asymptotic expected values because the game is designed to be difficult to win—examples are Powerball (which has a jackpot probability of approximately 1 in 80 million) and the Californian State game (which has a probability of approximately 1 in 15 million but a somewhat smaller population than the U.K.). The large number of rollovers in the U.K. occurs mainly because of the extent of conscious selection, not just because of the underlying 6/49 design. Thus, we would not expect the results here to carry over to other games.

Note that the "price" elasticity is the elasticity with respect to V, reported in Table 1, multiplied by $V/(1-V)$ so that the price elasticity ranges from $-.80$ to -1.06 and is never significantly different from unity for any of the estimates. Thus, we conclude that the elasticity is very close to unity and quite precisely estimated. We would normally expect a revenue-maximizing monopolist to price output where the demand elasticity is unity, but this product is more complex because there is an intertemporal effect from current pricing decisions. That is, raising the price from the unit elasticity price will reduce demand in regular weeks but increase the frequency of rollovers that would tend to counteract the first effect on sales. Because the U.K. game operates at such a high level of sales, however, we can safely ignore this complication.

4. IMPLICATIONS AND CONCLUSION

This article is concerned with the elasticity of demand for lottery tickets. Earlier work has typically exploited differences in the terms of lotteries across time or across states to estimate the elasticity and found elasticities greater than unity. Our analysis shows that rollovers produce changes in the expected value of holding a ticket that cannot be arbitraged away, and we produce estimates that are based on these exogenous variations in the expected value of a ticket arising from rollovers. That is, we derive the inverse supply curve for the industry and exploit the large number of rollovers that have occurred in our time series data that shift the supply curve to facilitate the identification of the price elasticity of demand.

Indeed, we found many more rollovers than we would have expected from the design of the lottery. The reason for this is that participants do not choose their numbers uniformly—that is, randomly from a uniform distribution. This results in a smaller number of combinations of numbers being sold than would otherwise have been the case. The result is an increase in the variance in the number of jackpot winners (and of other prize pools). Thus, there are more occasions when there are many jackpot winners and more occasions when there are no winners; that is, there are more rollovers than one would otherwise expect. Our estimates of the actual probability distribution for the numbers chosen by players shows significant departures from uniformity and a corresponding increase in the frequency of rollovers.

We used time series data from the U.K. National Lottery lotto game to illustrate the methodology. Having derived the inverse supply curve for the market, we note that it cannot be inverted analytically, so we cannot obtain a linear reduced form for the model. Thus, we apply nonlinear estimation methods. We find that the elasticity of demand with respect to rollover-induced changes in the expected value of a ticket in a simple demand specification is close to unity. Because the objective of the government in initiating the lottery was to maximize "good-causes" revenue, our results suggest that the take-out rate is set at about the right level. That is, good-causes revenue could not be raised by either raising or lowering the take-out rate (i.e., increasing or reducing the amount returned as prizes).

Our estimates assume, however, that there are no omitted variables that are correlated with rollovers, and there are two reasons to doubt this that require further investigation. First, when large jackpots occur, there is additional publicity in the media that may directly affect demand, and this would imply that the estimates were not appropriate for analyzing the effect of changes in "price" arising from factors not accompanied by additional publicity—such as design changes. That is, even though players have no preference over the variance or skewness in returns, in aggregate, changes in higher moments of the prize distribution may affect the publicity that the game gets and hence sales. This might suggest that a rollover-induced change in expected value might have a greater effect on sales than one induced by the same change in expected value from a change in the take-out rate, in which case our estimated price elasticity would be biased upward.

Second, there may be systematic differences in players across rollover and regular draws. For example, if the rich face higher transaction costs than the poor, they may be more likely to choose to play only during rollover weeks and, because income may have a direct effect on demand, the composition of players would change systematically with rollovers. In this case our estimated price elasticity would be biased downward/upward if lottery tickets had a positive/negative income elasticity.

Because this article has established that conscious selection of numbers by players does not affect the estimated demand elasticity, we will rely on these findings in subsequent work that addresses these complexities.

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APPENDIX A: PROOFS OF THE PROPOSITIONS

Proposition A.1. If N_J and N_F are distributed as $T(n-1, \pi_J, \pi_F)$, then

$$\text{(a)}\quad E\left(\frac{1}{N_J+1}\right) = \frac{1-(1-\pi_J)^n}{n\pi_J},$$

$$\text{(b)}\quad E\left(\frac{N_F}{N_J+1}\right) = \frac{\pi_F[1-(1-\pi_J)^{n-1}]}{\pi_J},$$

$$\text{(c)}\quad E\left(\frac{1}{n-N_J-N_F}\right) = \frac{1-(\pi_J+\pi_F)^n}{n(1-\pi_J-\pi_F)},$$

and

$$\text{(d)}\quad E\left(\frac{N_F}{n-N_J-N_F}\right) = \frac{\pi_F[1-(\pi_J+\pi_F)^{n-1}]}{1-\pi_J-\pi_F}.$$

Proof.

(a) It is a standard result that $N_J \sim B(n-1, \pi_J)$. Hence,

$$E\left(\frac{1}{N_J+1}\right) = \sum_{i=0}^{n-1} \frac{1}{i+1} \pi_J^i (1-\pi_J)^{n-1-i} \binom{n-1}{i}$$
$$= \frac{1}{\pi_J n} \sum_{i=0}^{n-1} \pi_J^{i+1} (1-\pi_J)^{n-1-i} \binom{n}{i+1}.$$

The result follows from the binomial theorem.

$$\text{(b)}\quad E\left(\frac{N_F}{N_J+1}\right) = \sum_{\substack{i,j=0\\ i+j\le n-1}}^{n-1} \frac{j}{i+1} \pi_J^i \pi_F^j \times (1-\pi_J-\pi_F)^{n-1-i-j} \times \left(\begin{array}{c} n-1 \\ i+1 \quad j-1 \quad n-1-i-j \end{array}\right)$$

$$= \frac{\pi_F}{\pi_J} \sum_{\substack{i=0,j=1\\ i+j\le n-1}}^{n-1} \pi_J^{i+1} \pi_F^{j-1} \times (1-\pi_J-\pi_F)^{n-1-i-j} \times \left(\begin{array}{c} n-1 \\ i+1 \quad j-1 \quad n-1-i-j \end{array}\right)$$

$$= \frac{\pi_F}{\pi_J} \sum_{i'=1}^{n-1} \pi_J^{i'} \binom{n-1}{i'} \sum_{j'=0}^{n-1-i'} \pi_F^{j'} \times (1-\pi_J-\pi_F)^{n-1-i'-j'} \times \binom{n-1-i'}{j'},$$

where we have set $i' = i+1, j' = j-1$. The inner sum is $(1-\pi_J)^{n-1-i'}$ by the binomial theorem, and the result follows from a second application of this theorem.

(c) We have

$$E\left(\frac{1}{n-N_J-N_F}\right) = \sum_{\substack{i,j=0\\ i+j\le n-1}}^{n-1} \frac{1}{n-i-j} \pi_J^i \pi_F^j \times (1-\pi_J-\pi_F)^{n-1-i-j} \times \left(\begin{array}{c} n-1 \\ i \quad j \quad n-1-i-j \end{array}\right)$$

$$= \frac{1}{n(1-\pi_J-\pi_F)} \sum_{\substack{i,j=0\\ i+j\le n-1}}^{n-1} \pi_J^i \pi_F^j \times (1-\pi_J-\pi_F)^{n-i-j} \times \left(\begin{array}{c} n \\ i \quad j \quad n-i-j \end{array}\right)$$

$$= \frac{1}{n(1-\pi_J-\pi_F)} \times \left[1 - \sum_{\substack{i,j=0\\ i+j=n}}^{n} \pi_J^i \pi_F^j \left(\begin{array}{c} n \\ i \quad j \quad 0 \end{array}\right)\right]$$

using the multinomial theorem. The result follows from the binomial theorem.

(d) We have

$$E\left(\frac{N_F}{n-N_J-N_F}\right) = \sum_{\substack{i,j=0\\ i+j\le n-1}}^{n-1} \frac{jn}{n-i-j} \pi_J^i \pi_F^j \times (1-\pi_J-\pi_F)^{n-1-i-j} \times \left(\begin{array}{c} n-1 \\ i \quad j \quad n-1-i-j \end{array}\right)$$

$$= \frac{\pi_F}{1 - \pi_J - \pi_F} \sum_{\substack{i,j'=0 \\ i+j' \leq n-2}}^{n-2} \pi_J^i \pi_F^{j'}$$
$$\times (1 - \pi_J - \pi_F)^{n-1-i-j'}$$
$$\times \binom{n-1}{i \quad j' \quad n-1-i-j'},$$

where $j' = j - 1$. The proof is completed as in (c).

Proof of Proposition 2. We now justify the comparative statics results of Section 3. The proofs of parts 1 and 2 of Proposition 2 are obvious. Now observe that ϕ_n, the partial derivative of ϕ with respect to n, can be written in the form

$$\phi_n = \pi_J e^{-\pi_J n}\{K - R\pi_J \Psi(\pi_J n)\},$$

where $\Psi(x) = (e^x - 1 - x)/x^2$ and we have suppressed the dependence on σ. The series expansion of the exponential function can be readily used to show that (a) Ψ is strictly increasing for $x \geq 0$, (b) $\Psi(x) \to \infty$ as $x \to \infty$, and (c) $\Psi(x) \to \frac{1}{2}$ as $x \to 0$. Proposition 2, part 4, follows from (a) and (c). To establish Proposition 2, part 3, observe that any stationary point $\hat{n}$ of ϕ satisfies $\hat{n} = \hat{x}/\pi_J$, where $\Psi(\hat{x}) = K/R\pi_J = C$, say, which can be rearranged as

$$e^{\hat{x}} = 1 + \hat{x} + C\hat{x}^2. \tag{A.1}$$

Because an exponential curve can intersect a parabola at, at most, two points and $\hat{x} = 0$ satisfies (A.1), there is, at most, one stationary point of ϕ in $n > 0$. It also follows from (b) and (c) that, if $R < \bar{R}(\sigma)$, such a stationary point must be a maximum. This proves Proposition 2, part 3. We also note that, as R increases, C in (A.1) decreases and hence so does the right side of (A.1). Because e^x is increasing, the positive root of (A.1) decreases. This implies that the locus of maxima of ϕ as R varies is downward sloping, asymptotic to the horizontal line cutting the vertical axis at $(1 - \tau)$, and intersects the vertical axis at $1 - \tau + K(\sigma) + \pi_J(\sigma)\bar{R}(\sigma) = 1 - \tau + K(\sigma)$.

It is clear that the results of parts 1, 2, and 4 of Proposition 2 are preserved by the addition operation in the definition of V (with $\bar{R} = \max_{\sigma \in S} \bar{R}(\sigma)$ for Proposition 2, part 4). This is not true for the assertion in Proposition 2, part 3, however, as the following example shows. Let $\tau = .5, \rho = 1, a = 0$. That is, the only prize is the jackpot so that $\pi_F(\sigma)$ disappears from the expression for ϕ. The distribution π on the set S of six subsets of $\{1, 2, \ldots 49\}$ is described by partitioning S into two arbitrary subsets S_1 and S_2 with $|S_1| = 9{,}413{,}941$ and $|S_2| = 4{,}569{,}875$ and setting

$$\pi_J(\sigma) = \begin{cases} 9.6825 \times 10^{-8} & \text{for } \sigma \in S_1 \\ 1.9365 \times 10^{-8} & \text{for } \sigma \in S_2 \end{cases}.$$

It is readily verified that $\sum_{\sigma \in S} \pi_J(\sigma) = 1$, and numerical evaluation with $R = 5.164 \times 10^6$ shows a (shallow) local minimum at $n = 63.5 \times 10^6$. Indeed, Table A.1 gives values of $\phi_1 = \phi(\sigma)$ for $\sigma \in S_1, \phi_2 = \phi(\sigma)$ for $\sigma \in S_2$, and V at five different values of n in millions (the second and fourth corresponding approximately to the two local maxima of V). Note that the maximum of ϕ_1 is approximately at 20×10^6, but that of ϕ_2 lies above 200×10^6.

Table A.1. Two Local Maxima of V

n	20	40	63 5	110	200
ϕ_1	648	.6160	.5800	.5469	.5258
ϕ_2	.2435	.3391	4113	.4819	.5149
V	.5164	5255	.5244	.5257	5222

Properties of V as a Function of π

To derive properties of V as a function of $\pi(= \pi_J)$, we must express π_F in terms of π. We assume that for each draw $\sigma \in S$ a fixed prize is won by any participant whose selection lies in the set $S(\sigma)$, where $\sigma \notin S(\sigma)$. Then the expansion for V in Corollary 1 can be rearranged as

$$V(\pi) = \text{constant} + \sum_{\sigma \in S} \left[B \sum_{\sigma' \in S(\sigma)} \pi(\sigma') - D \right] e^{-\pi(\sigma)n}, \tag{A.2}$$

where $B = a\rho/|S|, D = [\rho(1-\tau) + R/n]/|S|$, and the "constant" refers to terms independent of π.

We need to assume that $S(\sigma)$ is sufficiently regular in the sense that, for all $\sigma, \sigma' \in S$,

$$|S(\sigma)| = |S(\sigma')| \tag{A.3}$$

and

$$\sigma' \in S(\sigma) \Leftrightarrow \sigma \in S(\sigma'). \tag{A.4}$$

For many lotteries, σ is a selection of p numbers and $S(\sigma) = \{\sigma' \in S: |\sigma \cap \sigma'| = q\}$ for some $q < p$. In such a case (A.3) and (A.4) are clearly satisfied.

Proposition A.2. Suppose (A.3) and (A.4) hold for all $\sigma, \sigma' \in S$, and let $\bar{S}$ be the common value of $|S(\sigma)|$. If

$$\bar{S}a < (1 - \tau)|S|, \tag{A.5}$$

then the Hessian matrix of $V(\pi)$ is negative definite for $n > \bar{n}$, where

$$\bar{n} = \frac{2a\bar{S}|S|}{[(1 - \tau)|S| - \bar{S}a]}. \tag{A.6}$$

Remark. Because $\bar{S}/|S|$ is the probability of winning a fixed prize, (A.5) asserts that the expected value of a fixed prize to an individual is less than the total expected value of prizes for that individual. If (A.5) fails, it follows from the law of large numbers that we are unlikely to be able to pay fixed prizes out of the prize pool for large n; practical lottery designs must satisfy (A.5). For the U.K. lottery, (A.6) is also satisfied because we have, approximately, $\bar{n} = 14 \times 10^6$ and $n > \bar{n}$ is always satisfied at observed levels of participation.

Proof of Proposition A.2.

A straightforward calculation applied to (A.2) and using

(A.4) shows that

$$\frac{\partial^2 V}{\partial\pi(\sigma)\partial\pi(\sigma')} = \begin{cases} n^2\left(B\sum_{\sigma''\in S(\sigma)}\pi(\sigma'')-D\right)e^{-\pi(\sigma)n} & \text{if } \sigma'=\sigma \\ -nB(e^{-\pi(\sigma)n}+e^{-\pi(\sigma')n}) & \text{if } \sigma'\in S(\sigma) \\ 0 & \text{otherwise.}\end{cases}$$

Writing $\mathbf{H}$ for the Hessian matrix evaluated at $\pi(\sigma) = \pi_U(\sigma) = 1/|S|$ for all $\sigma \in S$ and letting $\mathbf{x}$ be a vector with S as index, set

$$\mathbf{x}^T\mathbf{H}\mathbf{x} = n^2(B\bar{S}/|S| - D)e^{-n/|S|}\sum_{\sigma\in S}x(\sigma)^2 + k,$$

where

$$\begin{aligned} k &= -2nB\sum_{\sigma}\sum_{\sigma'\in S(\sigma)} e^{-n/|S|}x(\sigma)x(\sigma') \\ &\leq 2nBe^{-n/|S|}\sqrt{\sum_{\sigma}\sum_{\sigma'\in S(\sigma)}x(\sigma)^2\sum_{\sigma}\sum_{\sigma'\in S(\sigma)}x(\sigma')^2} \\ &= 2nB\bar{S}e^{-n/|S|}\sum_{\sigma}x(\sigma)^2 \end{aligned}$$

and where the second line applies the Cauchy–Schwarz inequality and the subsequent equation exploits (A.3) and (A.4). It follows that $\mathbf{H}$ is negative definite provided $n^2(B\bar{S}/|S| - D) + 2nB\bar{S} < 0$, and this can be rewritten as

$$n > \left(2a\bar{S} - \frac{R}{\rho}\right)\Big/\left[1 - \tau - a\frac{\bar{S}}{|S|}\right]. \qquad \text{(A.7)}$$

When $R = 0$, this reduces to $n > \bar{n}$. If $R > 0$, then $n > \bar{n}$ is sufficient for (A.7); indeed, for large enough R any positive n satisfies (A.7).

APPENDIX B: DESCRIPTION OF THE SIMULATOR

In our case, a convenient way to simulate $\pi_F(\sigma;\mathbf{p})$ is to consider the following quantity:

$$\tilde{\pi}_F(\sigma;\mathbf{p},\mathbf{U}) = \frac{1}{H}\sum_{h=1}^{H}\frac{p(\mathbf{j}(u_h);\mathbf{p})}{q(\mathbf{j}(u_h))}, \qquad \text{(B.1)}$$

where $q(\mathbf{j})$ is a probability function that supports $S(3,\sigma), \mathbf{j}$ is an element of $S(3,\sigma), p(\mathbf{j};\mathbf{p})$ is the probability that the arrangement $\mathbf{j}$ occurs (in that order) when the probability distribution of the first number drawn is $\mathbf{p}$ [Eq. (4)], and H is the number of simulations. $\mathbf{U}$ is an $(H \times 9)$ matrix of realizations of uniform deviates that are the basis for the generation of the H elements $\mathbf{j}(u_h)$ of $S(3,\mathbf{a})$. For example, the realizations $\mathbf{j}(u_h)$ can be obtained in the following stages:

1. Using the first three columns of $\mathbf{U}$, draw H times without replacement three numbers out of σ; note that this implies an ordering.
2. Using the next three columns of $\mathbf{U}$, draw H times without replacement three numbers out of the complement of σ in $\{1,\ldots,49\}$.
3. Using the last three columns of $\mathbf{U}$, draw H times one number out of $\{1, 2, 3, 4\}$, say $i_1(\mathbf{U})$, one number out of $\{i_1(\mathbf{U})+1,\ldots,5\}$, say $i_2(\mathbf{U})$, and one number out of $\{i_2(\mathbf{U})+1,\ldots,6\}$, say $i_3(\mathbf{U})$. This defines the positions of the three numbers drawn out of σ among the 6.

This last operation, together with the draws without replacement, defines an arrangement of numbers such that exactly 3 are in σ. These operations define $q(\mathbf{j}(u))$ such that

$$q(\mathbf{j}(u)) = \frac{3!}{6!}\frac{40!}{43!}\frac{1}{4}\frac{1}{5-i_1(u)}\frac{1}{6-i_2(u)}, \qquad \text{(B.2)}$$

where u is a (1×9) vector of realizations of uniform variables and $i(u)$ are as defined previously. By construction, $\tilde{\pi}_F(\sigma,\mathbf{p},\mathbf{U})$ is unbiased for $\pi_F(\sigma;\mathbf{p})$; moreover, it is continuous with respect to each element of $\mathbf{p}$.

The precision of $\tilde{\pi}_F(\sigma,\mathbf{p},\mathbf{U})$ can be further improved by using an antithetic variance-reduction argument (see Davidson and MacKinnon 1993). The simplest way to do so is to calculate $\tilde{\pi}_F(\sigma,\mathbf{p},\mathbf{U})$ and $\tilde{\pi}_F(\sigma,\mathbf{p},1-\mathbf{U})$, for the same $\mathbf{U}$, and then consider $\bar{\pi}_F(\sigma,\mathbf{p},\mathbf{U}) = \frac{1}{2}(\tilde{\pi}_F(\sigma,\mathbf{p},\mathbf{U}) + \tilde{\pi}_F(\sigma,\mathbf{p},1-\mathbf{U}))$. The improvement comes because $\bar{\pi}_F(\sigma,\mathbf{p},\mathbf{U})$ has a smaller variance than $\tilde{\pi}_F(\sigma,\mathbf{p},\mathbf{U})$. The extent of the improvement is difficult to measure a priori and depends on the negative correlation between $\tilde{\pi}_F(\sigma,\mathbf{p},\mathbf{U})$ and $\tilde{\pi}_F(\sigma,\mathbf{p},1-\mathbf{U})$.

It can be shown that if H and D tend to infinity in such a way that $\sqrt{D}/H \to 0$, the maximization of the simulated maximum likelihood gives asymptotically consistent estimates (Monfort and van Dijk 1995).

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