The Banks Set and the Uncovered Set in Budget Allocation Problems

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Abstract

We examine how a society chooses to divide a given budget among various regions, projects or individuals. In particular, we characterize the Banks set and the Uncovered Set in such problems. We show that the two sets can be proper subsets of the set of all alternatives, and at times are very pointed in their predictions. This contrasts with well-known "chaos theorems," which suggest that majority voting does not lead to any meaningful predictions when the policy space is multidimensional.

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1 Introduction

McKelvey's celebrated theorems (1976, 1979) on the genericity of global cycles in majority voting are fundamental to our understanding of the potential outcomes of a social decision process. These results assert that if the set of social alternatives is a multidimensional Euclidean space, then under mild conditions on the profile of voters' preferences, there is a finite chain of alternatives starting at any given x and ending in any other y such that each alternative in the chain is preferred by a majority of voters to its predecessor. These results are often interpreted to show that in multidimensional policy spaces majority voting is chaotic or unstable since no alternative appears to dominate the others.

The general conclusions drawn from this interpretation of McKelvey's Chaos Theorems[1] have led many people to believe that it is impossible to predict the nature of social decision-making without a detailed specification of social institutions and rules. For instance, Stiglitz (1988) writes, "If, however, there are a variety of dimensions-some individuals are liberal on some issues and conservative on others-then the median voter is not well defined, and there may be no equilibrium to the political process." According to Persson and Tabellini (2000), "By the mid-to late 1970s, theorists had clearly demonstrated that searching for a universally applicable theory of political equilibrium is a futile exercise. Further, majority voting would generically lead to cycles, unless the voting agenda was restricted... The outlook of many researchers at the time was thus quite pessimistic: any positive theory of political choice - whether it was based on majority voting or not - seemingly had to rely on unattractive or arbitrary assumptions."

These beliefs have resulted in a growing body of literature that derives social equilibria conditional on explicit specifications of the institutional structure. Without denying the importance of that direction of investigation, one is left with the impression that except in exceptional cases, very little can be said about social choice outcomes that is *institution-free*. This is problematic if the equilibrium outcomes are highly sensitive to the fine details of the institutional process. For instance, if the "institution" resembles a bargaining game, then the equilibrium outcome will typically depend on the specific bargaining protocol. Therefore, it appears as if the unpredictability associated with the chaos theorems has been replaced by a predictability that may suffer from a lack of robustness.

The approach adopted in this paper, following another of McKelvey's (1986) influential papers, is to explore whether there are *some* predictions about majoritarian social choice

[1] A version of the Chaos Theorem for the finite unstructured setting has been proved by Bell (1981). We refer the reader to chapter 6 in Austen-Smith and Banks (1999) for an illuminating presentation of chaos results in the spatial model.

outcomes that are “relatively” institution-free. More precisely, is it possible to bound the set of “stable social outcomes” in the sense that all social equilibria must lie within these bounds under a wide variety of different institutional arrangements? In this paper, we focus on two such sets: the *Uncovered Set* of Miller (1980) and the *Banks Set* (Banks (1985)).

The two sets are arguably the most appropriate ones to bound the possible social outcomes. McKelvey (1986) demonstrated[2] that in any multidimensional setting where voters have quasi-concave preferences (the so-called *spatial setting*), the Uncovered Set contains the outcomes that would arise from equilibrium behavior under three different institutional frameworks : a two-candidate competition in a large electorate[3], cooperative behavior in small committees, and sophisticated behavior in a legislative environment. Hence, in a strong sense the Uncovered Set is a useful generalization of the notion of a Condorcet winner. The Banks Set is less general than the Uncovered Set in that it only applies to a specific institution. Nevertheless, that institution, voting by amendment agendas (also known as voting by successive elimination), is a very important one which is paradigmatic for most committee voting rules and is the procedure central to Roberts Parliamentary Rules of Order. It is often asserted that a chairman (or sub-committee) can manipulate the agenda so as to ensure the choice of an alternative which is in his (or its) interest. However, Miller and McKelvey showed that the set of sophisticated equilibrium outcomes corresponding to voting by successive elimination must lie in the Uncovered Set; and Banks (1985) provided a full characterization of this set of outcomes, which is the Banks Set. Thus, the Banks Set puts some bounds on the monopoly power of an agenda setter in the context of amendment agendas.[4]

The uncovered and Banks Sets have been investigated extensively in the case where the set of alternatives is finite and voters have no a priori structure to their preferences, and so some properties of these sets are now well-known.[5] However, the explicit computation of these sets is not easy, particularly when the feasible set of alternatives is some subset of multi-dimensional Euclidean space and one might expect some natural structure to voters' preferences. The only real analysis that has made any progress on that issue is in the case of purely distributive politics, where Epstein(1998) and Laslier and Picard(2002) have shown that the Uncovered Set is the entire set of alternatives;[6] and recent papers by Penn (2002)

[2]Miller (1980) had already shown that under a variety of institutional settings, game theoretic behavior by participants leads to outcomes in the Uncovered Set when the set of alternatives is finite.

[3]On this aspect, see also Banks, Duggan and Le Breton (2002).

[4]Miller, Grofman, and Feld (1990a), (1990b) argue that the interest for studying the Banks Set goes beyond this.

[5]See Laslier (1997) for a detailed description of this area of research.

[6]See also McKelvey (1986), Banks, et al (2003), Cox(1987), De Donder (2000), Feld, et al (1987), Fey

and Dutta, Jackson, and Le Breton (2002) who show that specific equilibrium notions of agenda formation can lead to some pointed predictions in some circumstances.

This paper furthers our understanding of the structure of the behavior of majority rule in spatial settings in two ways. The first is that we identify a broad class of social choice problems that includes many situations that have been analyzed separately in the literature. The common feature of these problems is that an alternative is a feasible allocation of a given budget across finitely many uses or projects. These projects could correspond to individuals if we model private transfers in a distributive politics environment, or to districts or regions or different types of public expenditures in the context of pork barrel politics and financing of local public goods (Ferejohn, Fiorina and McKelvey (1987), Lockwood (2002)). Other budget allocation problems covered by our framework include the mixed setting where some private projects compete with a global public project as in Lizzeri and Persico (2001). Our key assumption is that voters' preferences are linear; that is, indifference contour sets are parallel hyperplanes. While a special case of the spatial model, this is a rich setting that to the best of our knowledge has never been investigated in any generality before. Our second contribution is to provide some (partial) characterizations of the uncovered, Banks, and Top Cycle sets in this setting, in the context of some important special cases. In particular, we examine in some detail the case where each voter views each project as either being *good* or *bad*.

In the next section, we introduce and illustrate the linear setting and the main concepts which are used in this paper. In section 3, we introduce the main majoritarian sets which are examined in this paper. Then, in section 4-7, we focus on the special case where there are three projects and three voters. In section 4, we present the simple geometry of the majority relation in the linear setting. We show that the necessary and sufficient condition for the existence of a Condorcet winner in this setting is less stringent than the well-known condition of Plott (1967). We also offer a direct simple proof of McKelvey's chaos theorem on the Top Cycle. Then, we calculate explicitly the Uncovered Set for many important linear settings. We also calculate or describe with some accuracy the Banks Set to show how much it differs from the Uncovered Set. We conclude in section 8 with a discussion of the general case.

(2003), Hartley and Kilgour (1987), Koehler(1990).

2 A Model of Budget Allocation where Voters have Linear Preferences

Alternatives

The set of alternatives X consists of the set of feasible allocations of a given budget, denoted by M, among K distinct possible uses. The uses may be thought of, depending upon the context, as being districts, regions, individuals, public projects or other criteria. Generic elements of X are denoted x, y, and z, and are K-dimensional vectors.

If money is assumed to be perfectly divisible, then the set X is infinite. In this case, X is the simplex

$$\left\{ x \in I\!R_+^K : \sum_{k=1}^{K} x^k = M \right\}.$$

When $M = 1$, the set of social alternatives could be alternatively interpreted as the set of lotteries over an unstructured finite set of choices.

If, instead, money can only be divided into discrete units, with say 1 being the smallest unit of money, then the set X is finite. It is then defined as[7]

$$\left\{ x \in \{0, 1, \ldots, M\}^K : \sum_{k=1}^{K} x^k = M \right\}.$$

We shall alternate between the use of these two settings. While the infinitely divisible setting provides some technical advantages, we stick with the finite world in situations where we analyze the Banks Set, as an uncontroversial definition for the Banks Set has not been given for the case where X is infinite. We shall discuss the limit as the units become small (M becomes large), and this provides some predictions for the infinite case.

This also gives us some feel for the importance of divisibilities, as we shall see that at least in some cases the Banks Set changes as the units become relatively small, and the limit may have different features from situations with substantial indivisibilities.

We assume that $M \geq 4$ in the indivisible setting, as the case where $M \leq 3$ is an easily analyzed special case where the geometry of the problem degenerates.

Voters and Preferences

The committee or society of voters is described by the finite set $\mathcal{N} = \{1, ..., N\}$.

Voter i has preferences over the set of alternatives represented as follows. There exists a vector $u_i \in I\!R^K$ such that the utility to i of an alternative x is simply $u_i \cdot x$. Thus, i prefers

[7] We presume that the entire budget is allocated. This is in line with voters viewing at least one of the projects as not being objectionable.

an alternative x to an alternative y if and only if

$$u_i \cdot x > u_i \cdot y.$$

Thus, u_i^k denotes i's marginal valuation for project k. So, preferences are completely described by the matrix $u \equiv (u_1, ..., u_N) \in I\!R^{NK}$.

The linearity of indifference is obviously special; but, as we now illustrate, it is general enough to cover a large family of interesting problems.

Example **1** *Private Projects: Divide the Dollar*

This corresponds to the case where $K = N$ and the matrix u is equivalent to

$$u = \begin{array}{ccccc} 1 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 1 \end{array}$$

The conventional interpretation of this problem is that an alternative is a division of the amount M among the N voters, who are assumed to derive utility exclusively from the amount they receive, the larger the better. A second interpretation views the K dimensions as K public projects in competition and assuming that each voter cares exclusively about the amount allocated to a specific project, justifying the terminology "private projects" even if the projects have the features of public projects.

Example **2** *Goods and Bads*

Consider a world where each dimension is viewed by a voter as either a "good" or a "bad" project. Goods are equivalent in the voter's view, as are bads. To normalize things, goods have a marginal value of 1 and bads have a marginal value of 0. So each u_i is a vector of 0's and 1's.

A special case of this is the divide the dollar setting described in Example 1, where each player has a different dimension that is a good, and only one dimension, and where the number of dimensions is the number of voters $K = N$. Another special case is the private and public good example (Example 3, below), in the case where $b = 1$. More generally, the goods and bads model is one where K might differ from N, several voters might view any particular dimension as good, and players might consider several dimensions to be "goods."

Example 3 *Private Projects versus a Public Project*

This example, inspired by Lizzeri and Persico (2001), is a setting mixing the divide the dollar setting (Example 1) with an extra public project that is a pure public good. Here voters are bargaining between allocating resources to a common public good, and payments directly to the voters themselves. In particular, $K = N + 1$ and

$$u = \begin{array}{cccccc} 1 & 0 & \cdot & \cdot & 0 & b \\ 0 & 1 & \cdot & \cdot & 0 & b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & b \end{array}$$

where b is a positive parameter describing the common willingness to pay of each voter for the public project.

Example 4 *Choice Between Public Projects*

Consider a society allocating resources to any of a list of projects, which may have private and/or public components. In this case, there are no specific restrictions on the matrix u. The K dimensions are interpreted as K different potentially projects that are in competition for funding. The allocation x^k defines the scale of operation of project k (variations in costs can be incorporated into the marginal utilities). Certainly, voter i would like to see all the budget allocated to his or her "favorite projects" (projects k such that $u_i^k \geq u_i^{k'}$ for all k'). However, unlike the goods and the bads model, an agent is not systematically indifferent between projects that are identical in their allocation to the agent's most preferred projects.

When $K = N$, and the K dimensions are interpreted as districts or states in a federation or regions in a country, this model describes pork barrel politics with some form of externalities across projects. Suppose that region i derives a benefit equal to x_i from project i operated at the scale x_i, but also derives some benefits from projects implemented in other regions. The benefits resulting from these other projects are less important, the more "distant" is the region (where distance might or might not be a physical measure). Precisely, $u_i^k = 1 - \alpha d_{ik}$. Knowledge of the intensity α of the externality and the pattern describing the geographical network, is essential for understanding the voting behavior.[8]

[8]See Lockwood (2002) for an analysis of a model with externalities.

Example 5 *Criteria*

When bargaining over the split of a budget, it is often the case that the discussion takes place on various criteria that might be used to allocate the budget instead of directly in terms of the allocation itself.[9] For instance, consider a university deciding on how to allocate a budget among a set of departments. The decision might be based on a whole set of criteria including quality of research and teaching measured by various indicators, numbers of students, numbers of researchers, etc. Let K be the number of such criteria. With respect to these criteria, voter i (say department i) is described by the vector $\lambda_i = (\lambda_i^1, ..., \lambda_i^K)$ as to how "much" of each criterion voter i possesses. So, λ_i^1 might be a measure of department i's research output, λ_i^2 might be a measure of the number of students enrolled in the department i's courses, and so on.

Here, an alternative x is a decision on the relative weight of each criterion in allocating the budget. Given an x the allocation of the budget is such that voter (department) i receives

$$M \sum_{k=1}^{K} x^k \frac{\lambda_i^k}{\sum_{1 \leq j \leq N} \lambda_j^k}.$$

For instance, if $x^2 = 1/3$, then in the university example 1/3 of the budget will be allocated based on the number of students that a department has. There, $\frac{\lambda_i^2}{\sum_{1 \leq j \leq N} \lambda_j^2}$ would measure the fraction of all students that a given department has.

Presuming that voter i prefers to be allocated as much of the budget as possible, we end up with preferences for voter i described by

$$u_i^k = \frac{\lambda_i^k}{\sum_{1 \leq j \leq N} \lambda_j^k},$$

for each k.

3 Majority Voting and Tournaments

Let us now discuss how we model the choice of alternatives made by voters.

The strict *majority preference* induced by a profile u of preferences is denoted by $T(u)$ and defined over X as follows.

$$xT(u)y \ \Leftrightarrow \ \#\{i \in \mathcal{N} : u_i \cdot x > u_i \cdot y\} > \#\{i \in \mathcal{N} : u_i \cdot y > u_i \cdot x\}$$

[9]We thank Salvador Barbera for having suggested this problem.

If N is odd and individual preferences are strict, then $T = T(u)$ is complete.[10] Otherwise, ties may occur and this results in some freedom in how one defines the sets and procedures that we examine next.

For the following definitions, T may be an arbitrary asymmetric (and possibly incomplete) binary relation.

Condorcet Winners

An alternative x is a *Condorcet winner* if :

$$xTy \text{ for all } y \in X\backslash\{x\}.$$

An alternative x is a *weak Condorcet winner* if :

$$\text{not } [yTx] \text{ for any } y \in X.$$

Let $WC(T)$ denote the set of weak Condorcet winners associated with T.

In the case where T is complete, the two definitions coincide. In fact, it is easy to see that whenever there is a Condorcet winner then that alternative must also be the unique weak Condorcet winner. However, in cases where T is incomplete it is possible for there to exist many weak Condorcet winners, in which case there is no Condorcet winner.

The Top Cycle

As the majority preference is not necessarily transitive, it can have cycles. A prominent cycle that we refer to in the sequel is the Top Cycle associated with T.

Let a *weak T-chain* between alternatives x and y be a sequence of alternatives $x_1, \ldots, x_k$ such that $x_1 = x$, $x_k = y$, and not $x_{j+1}Tx_j$ for each $j = 1, \ldots, k-1$.

The *Top Cycle* of T, denoted by $TC(T)$ is the set[11]

$$TC(T) = \{x \mid \forall y \in X, \exists \text{ a weak T-chain between } x \text{ and } y\}$$

Thus, the Top Cycle is the set of alternatives that can reach any alternative in X via some weak T-chain.

The Uncovered Set

[10] A binary relation which is asymmetric and complete is called a tournament. See Laslier (1997) for an illuminating account of the principal results in the vast literature on tournaments and majority voting.

[11] When the majority preference is not complete, there are various possible definitions of the Top Cycle (see Schwartz (1972) and Duggan and Le Breton (2001)). All of these definitions coincide with the definition of TC considered in this paper when the majority preference is complete.

The *Uncovered Set* of T, denoted $UC(T)$, is the set of maximal elements of the *covering* relation $C(T)$ defined over X. Defining $C(T)$ by

$$xC(T)y \text{ if and only if } xTy \text{ and for all } z \in X : yTz \text{ implies } xTz,$$

$$UC(T) = \{x \mid \text{ not } yC(T)x \forall y \in X\}.$$

Again, it should be pointed out that when T is not complete, there are several possible of the Uncovered Set. The definition above, which is the most relevant for our subsequent analysis of the Banks Set, corresponds to UC_d in Banks and Bordes (1988), to F_d in Bordes (1983) and to Miller's subset in Bordes, Le Breton and Salles (1992). It does not correspond to the definition of the Uncovered Set which is found in Banks, Duggan and Le Breton (2002, 2003), Dutta and Laslier (1999) and McKelvey (1986). There, the Uncovered Set is defined as the set of maximal elements of the partial order $\overline{C}(T)$ defined over X by

$$x\overline{C}(T)y \text{ iff } xTy \text{ and for all } z \in X : [yTz \text{ implies } xTz] \text{ and } [zTx \text{ implies } zTy]$$

Let

$$\overline{UC}(T) = \{x \mid \text{ not } y\overline{C}(T)x \forall y \in X\}.$$

Since $\overline{C}(T)$ is a subrelation of $C(T)$, the Uncovered Set that we focus on in this paper $UC(T)$ is a subset of this other Uncovered Set $\overline{UC}(T)$.

Amendment Agendas and Voting by Successive Elimination

A prominent procedure that selects a single allocation out of the feasible set X is that based on amendment agendas, as central to Roberts Parliamentary Rules of Order. This procedure, is also often referred to as voting by successive elimination in the literature, and is defined as follows.

Consider an ordering σ of the finite set of alternatives X and let $\sigma = (x_1, \ldots, x_L)$ where L denotes the number of alternatives in X. A vote is first taken to eliminate either x_M or x_{M-1}. The 'winning' alternative from the first round is compared to x_{M-2}, and a vote is taken to eliminate either surviving alternative from the first vote or x_{M-2}, and so on. After $(M-1)$ comparisons, the last surviving alternative is declared to be the voting outcome.

At each stage, the elimination of one alternative is according to majority voting, or more generally according to the binary relation T. This is well-specified when T is complete. In cases where there are ties under the majority preference relation, or T is incomplete, the voting procedure needs to be more completely specified. We do so as follows. At each stage allow individuals to vote for one of the two alternatives or to abstain (in the case where they are indifferent). In case of a tie in the voting between alternatives x_l and $x_{l'}$, x_l is elected if and only if x_l comes before $x_{l'}$ in the ordering σ of voting; that is, $l < l'$. This favors alternatives proposed earlier in the agenda under ties, which is a natural way to break ties given that they have not already been broken under T.

In order to determine the eventual voting outcome, it is also necessary to describe how voters act. We consider the case where they vote strategically at each stage, and so focus on the sophisticated voting outcome of this binary voting procedure.[12] This is the outcome under the iterative elimination of weakly dominated strategies that has been well-studied. As demonstrated by Shepsle and Weingast (1984),[13] the sophisticated outcome induced by the ordering σ, denoted $S(\sigma, T)$, is equal to w_L^σ which is the last element of the finite sequence described by the following algorithm:

$$w_1^\sigma = x_1, \text{ and for all } l > 1 \; w_l^\sigma = \begin{cases} x_l \text{ if } x_l T w_{l'}^\sigma \text{ for all } l' < l, \text{ and} \\ w_{l-1}^\sigma, \text{ otherwise.} \end{cases}$$

The Banks Set

The *Banks Set* (Banks (1985)), denoted $B(T)$, is the subset of alternatives which are sophisticated outcomes for at least one ordering of X. Formally,

$$B(T) = \{x \in W : \exists \sigma \in \Sigma \text{ such that } x = S(\sigma, u)\},$$

where Σ denotes the set of permutations of X.

Let a T-*chain* between alternatives x and y be a sequence of alternatives $x_1, \ldots, x_k$ such that $x_1 = x$, $x_k = y$, and $x_j T x_{j+1}$ for each $j = 1, \ldots, k-1$.

Given an alternative $x \in X$, an x-*chain* of T is a chain H with $x \in H$ such that xTy for all $y \in H$. The set of all x-chains is denoted $H(x, T)$.

Thus, an x-chain is a chain where x beats all the other alternatives in the chain according to T.

The characterization provided by Banks (1985), stated to accommodate the possible incompleteness, can be stated as follows.

Proposition **1** *(Banks (1985))*

$$B(T) = \{x \mid \exists H \in H(x, T) \text{ s.t. } \forall y \notin H \; \exists z \in H \text{ s.t. not } yTz\}.$$

[12] For more on sophisticated voting, see Farquharson (1969) and McKelvey and Niemi (1978).

[13] The Shepsle-Weingast algorithm was defined for the case where T is complete. Our procedure of breaking possible ties in the majority preference relation coming *earlier* in the ordering σ ensures that the sophisticated outcome can be derived from a straightforward variation on the algorithm derived by Shepsle and Weingast, as shown, for instance, in Banks and Bordes (1988).

Thus, Banks showed that the outcomes found by varying the ordering (for a fixed tournament) of the amendment agenda when voting by successive elimination correspond to the endpoints of chains, where the chains are such that any alternative not included in the chain is beaten by something in the chain. The intuition behind the characterization is that the alternatives in the chain are those who temporarily "win" at some stage in the voting (the w_k's in the Shepsle-Weingast algorithm), and the remaining alternatives are those who are eliminated at their stages.

The following variation on well-known inclusions is helpful in what follows.

LEMMA 1 *If T is an asymmetric binary relation, then $WC(T) \subset B(T) \subset UC(T) \subset TC(T)$.*

The first inclusion is easily seen by noting that any weak Condorcet winner forms a maximal T-chain. This means that if the ordering is such that this weak Condorcet winner appears first in the order, then it will be the outcome of the amendment agenda, as no other alternative beats it. The second inclusion appears as theorem 4.1 in Banks and Bordes (1988). The third inclusion follows easily from the definitions.

In what follows we use the notation $WC(u), B(u), UC(u), TC(u)$ to denote the sets $WC(T(u)), B(T(u)), UC(T(u)), TC(T(u))$.

In the following sections, we examine the simplest framework for which the class of allocation problems described in the preceding section is not degenerate. If $K = 2$ or $N = 2$, there is always at least one weak Condorcet winner and all sets coincide with the set of weak Condorcet winners. When $K \geq 3$ and $N \geq 3$, the set of (weak) Condorcet winners is sometimes empty or some set of points that is not a singleton and not the whole set, and the determination of TC, UC, and B becomes more challenging and interesting as we have a true multidimensional problem. Thus, in what follows we restrict attention to the case of $K = N = 3$.

4 The Simple Geometry of the Majority Relation and the Top Cycle

In this section, we consider the continuous version of X and assume, without loss of generality, that $M = 1$. Under the assumption that $K = N = 3$, we are in position to use the simple geometry of the triangle to support our formal arguments. Given these dimensionality assumptions,

$$X = \left\{ x = (x_1, x_2) \in \mathbb{R}^2_+ : x_1 + x_2 \leq 1 \right\};$$

that is, X is the triangle represented on Figure 1. The three vertices of the triangle are denoted e^1, e^2 and e^3.

Insert Figure 1 about here

Given $u \in u^N$ and $x \in X$, let $U(u, x)$ be the set of alternatives that are considered strictly superior to x by a majority, the so-called *win set* of x and by $L(u, x)$ the set of alternatives that are considered strictly inferior to x by a majority of voters. When there is no Condorcet winner, these two sets are the union of three simple sets as pictured in Figure 2.

Insert Figure 2 about here

The following simple consequences of the linearity assumption on preferences will be very useful in the sequel.

(a) If $xT(u)y$, $z \in X$ and $\lambda > 0$, then $\lambda x + (1 - \lambda)z \; T(u) \; \lambda y + (1 - \lambda)z$.

(b) If $xT(u)y$ and $\lambda, \mu \in [0, 1]$, then $\lambda > \mu$ implies $\lambda x + (1 - \lambda)y \; T(u) \; \mu x + (1 - \mu)y$.

(c) An immediate consequence of (a) is that if $U(x, u) \neq \emptyset$, then $U(u, x)$ intersects the boundary of X; a similar observation applies to $L(u, x)$.

(d) We deduce from (b) that if x majority dominates y, then any point belonging to the line segment joining x and y majority dominates any other point of the segment which is farther away from x.

In the rest of this section, we rule out preference profiles which either offer little interest or will be examined in some subsequent sections. In particular, we assume that each voter has a unique ideal point. It is straightforward to see that the linearity assumption implies that this ideal point is necessarily a vertex of the triangle; i.e., the ideal point must be one of e^1, e^2 or e^3. We also assume that the three ideal points are all different, as otherwise at least two voters have the same ideal point, which is then the unique Condorcet winner. Without loss of generality, let e^i be the ideal point of voter i for all $i = 1, 2, 3$. Finally, we limit our attention to the generic case where a given voter is never indifferent between the ideal points of the two other voters. A profile of preferences u displaying these features is described by matrix with three degrees of freedom. A profile of preferences is completely described by a vector $u \equiv (v_1, v_2, v_3) \in (0, 1)^3$ where v_i denotes the intensity of the preference of voter i for his second best choice among the vertices. Within this class of linear preference profiles, two situations may appear:

(1) None of the vertices dominates the two other vertices. Up to a permutation of voters' labels, a profile of preferences u in this category is described by a matrix

$$u = \begin{array}{ccc} 1 & 0 & v_3 \\ v_1 & 1 & 0 \\ 0 & v_2 & 1 \end{array}$$

where $0 < v_1, v_2, v_3 < 1$.

(2) One of the vertices majority dominates the two other vertices. When this happens, we call such a vertex a *vertex Condorcet winner* as it would be the obvious winner if competition was limited to the finite set of vertices. Up to a permutation of voters' labels, a profile of preferences u in this category is described by a matrix

$$u = \begin{array}{ccc} 1 & v_2 & v_3 \\ v_1 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

where $0 < v_1, v_2, v_3 < 1$.

We first examine the conditions under which a Condorcet winner exists.

PROPOSITION **2** *Let $u \equiv (v_1, v_2, v_3) \in (0,1)^3$ be a profile of preferences as described in (1) and (2) above. x is a Condorcet winner for u, if and only if it is a vertex and u falls in category (2) and satisfies $v_2 + v_3 \geq 1$ (where up to a permutation of labels, vertex e^1 is the winner).*

Proposition 2 departs in a fundamental way from Plott's well-known necessary and sufficient conditions for the existence of a Condorcet winner in the spatial model. His result asserts that for some alternative x to be a Condorcet winner, it has to be that x is the ideal point of some voter i and for any other voter j, there exists a voter k such that the normalized gradients of the utility functions of j and k evaluated in x are exactly opposite. Since such symmetry conditions are not robust to perturbations of preferences, a corollary of Plott's result is that Condorcet winners do not exist generically. It is often forgotten that this applies only if x is in the interior or relative interior of the feasible set, which is vacuously true if X is the entire Euclidean space. If, instead, like here, X is a compact convex subset of the Euclidean space, then Plott's conditions do not apply to alternatives on the boundary.[14]

[14]Plott (1967) applies a budget constraint, but does not impose any nonnegativity constraints and so does not consider boundary issues in the manner considered here.

This applies systematically in our linear setting, since we just demonstrated that Condorcet winners, when they exist, are on the boundary. The necessary and sufficient condition stated in Proposition 2 is robust to perturbations.

Let us make a final comment on the existence of Condorcet winners. The linear setting is a natural generalization of the finite setting and will be at least as complicated as the finite setting in that the majority tournament limited to the set of vertices can take any form. But the linear setting is richer in that a vertex doing well when matched exclusively against the other vertices may be defeated by a majority when compromises are introduced. Suppose u displays the pattern

$$u = \begin{array}{ccc} 1 & v_2 & v_3 \\ v_1 & 1 & 0 \\ 0 & 0 & 1 \end{array} .$$

Then e^1 is a vertex Condorcet winner: voters 2 and 3 cannot agree on another vertex. Can they agree on something else? They can if the intensity of their preference for e^1 is not too large, as stated by the inequality $v_2+v_3 < 1$ in Proposition 2. The condition is fairly intuitive since if v_2 and v_3 are small enough then the gap between their second best and worst choices vanishes, and it becomes possible to find a compromise $\lambda e^2 + (1-\lambda)e^3$ preferred by both of them to e^1. Figures 3 and 4 illustrate the two conceivable situations.

Insert Figure 3 about here

Insert Figure 4 about here

The following proposition provides a complete description of $TC(u)$ when $u \equiv (v_1, v_2, v_3) \in (0,1)^3$ is a profile of preferences as described above.

Proposition **3** *Let u be a profile of preferences as described in (1) or (2) above. Then, either there is a Condorcet winner, or $TC(u) = X$, or $TC(u) = X \setminus \{e^i\}$ for some i.*

Proposition 3 is a version of McKelvey's chaos theorem in our linear setting. The proof offered in the appendix shows how a cycle connecting any two alternatives is constructed, and the problems raised by the existence of a boundary are addressed. In contrast to the conditions leading to the existence of a Condorcet winner, the boundary does not have much impact here, as the only departure from total chaos is the exclusion of a Condorcet loser, when there is one.

5 The Goods and Bads Model

In this section, we return to the discrete version of the problem, still keeping with $N = K = 3$. We focus on the goods and bads model of example 2 and characterize all of the sets, including the Banks and Uncovered Sets.

Let us first start with the analysis of a prominent case that falls in the goods and bads model: that of the divide the dollar model.

PROPOSITION 4 *Consider the* divide the dollar model *of Example 1. The set of weak Condorcet winners is empty, the Top Cycle is the whole set of alternatives, and the Uncovered Set is the set of alternatives excluding the vertices. The Banks Set includes every $x \in X$ such that $x_i < [(x_j + x_k)^2 + 5(x_j + x_k) - 4]/2$, for some i and distinct j, k . Thus, the size of the Banks Set converges to the size of the set of alternatives as the grid becomes finer* ($\lim_{M\to\infty} \frac{\#B(u)}{\#X} = 1$). *However, the Banks Set is a strict subset of the Uncovered Set for any* $M > 5$; *as* $(M - 1, 1, 0)$ *and permutations of these points are not in the Banks Set.*[15]

Proposition 4 provides a different view of the Banks Set than what is previously known. While in some finite settings with arbitrary preferences, one can find examples where the Banks Set is a strict subset of the Uncovered Set (see Banks (1985)), it was not known whether this was true in more naturally structured environments. Indeed, Penn (2003) shows that in an infinitely divisible version of a divide the dollar game with three players, the Banks Set and Uncovered Set coincide.[16] Here, in contrast, the Banks Set makes a selection from the Uncovered Set. As the indivisibilities disappear, the sets converge to each other, with the Banks Set always remaining a strict subset of the Uncovered Set.

Let us now return to the more general analysis of the goods and bads model, where players may agree on which dimensions are goods, and may like several dimensions.

Let $s^k = \sum_i u_i^k$ and $s = \sum_k s^k = \sum_i \sum_k u_i^k$. Note that $s \in \{0, 1, \ldots, 9\}$, and $s^k \in \{0, 1, 2, 3\}$.

Thus, s^k is the strength of the support for dimension k. The analysis of the various sets now depends on the relative strengths of the dimensions.

[15]For larger M, one can also check that $(M - 2, 2, 0)$ and $(M - 2, 1, 1)$ (and permutations) are not in the Banks Set, and so forth; but the proofs become increasingly tedious as the number of chains to be ruled out grow as we move away from the vertices.

[16]Penn's definition of the Banks Set in infinite settings is directly in terms of maximal chains rather than in terms of an agenda, and her tie-breaking rule is different from ours. It is not clear that there is an unambiguously appealing definition of the Banks Set in the infinite setting, as without some modifications of tie-breaking there does not exist any maximal chains.

Proposition 5 *Consider the goods and bads model from Example 2, and assume that at least one voter is not completely indifferent. Without loss of generality, label the dimensions so that* $s^1 \geq s^2 \geq s^3$*.*

(1) If one dimension dominates the others ($s^1 > s^2$)*, then the vertex corresponding to that dimension is a Condorcet winner and all the sets coincide* ($TC(u) = UC(u) = B(u) = WC(u) = \{(M, 0, 0)\}$)*.*

(2) If there are two dimensions that have the same strength and dominate the third ($s^1 = s^2 > s^3$)*, then there is no Condorcet winner and the sets include all points that allocate only to the first two dimensions* ($TC(u) = UC(u) = B(u) = WC(u) = \{x \mid x^3 = 0\}$)*.*

(3) In the case where the strength of support for the three dimensions is identical ($s^1 = s^2 = s^3$)*:*

(3a) If some voter is completely indifferent, then no alternative beats any other, and so $X = WC(u) = B(u) = UC(u) = TC(u)$*.*

(3b) If each voter views a different two dimensions as goods, then $WC(u) = \{(M, 0, 0), (0, M, 0), (0, 0,$ *while* $TC(u) = X$*, and* $B(u) = UC(u) = X \setminus \{(M-2, 1, 1), (1, M-2, 1), (1, 1, M-2)\}$*.*

(3c) If each voter views one dimension as a good then we are back in the divide the dollar game setting as characterized in Proposition 4.

Proposition 5 states that the analysis of the goods and bads model breaks into five cases, basically depending on how much agreement there is among the voters as to which dimensions are goods. When there is enough agreement (as in (1) or (2)), then the predictions are narrow, while when there is significant disagreement (as in (3a) (3b) and (3c)) then many voting cycles appear and the sets are nearer to the entire space. Interestingly, the only situation where something falls in between is in the divide the dollar game with smaller M (substantial indivisibilities) where the Banks Set is narrower than the Uncovered Set and Top Cycle.

More specifically, in the first case, there is some dimension that receives more support than any other, and then giving the full budget to this dimension is a Condorcet winner. In the second case, there are two dimensions that are viewed as goods by an equal number of voters and the third dimension is viewed as a good by a lesser number. Here, the set of weak Condorcet winners is the set of alternatives that give only to the two dimensions with broader support. In the third, fourth, and fifth cases, all of the dimensions have equal support. However, they behave quite differently. In the third case, no alternative beats any other, as the two voters who are not indifferent completely disagree on the goods and bads, and so all sets are the whole space. In the fourth case, the three vertices form the set of Weak Condorcet winners. The Top Cycle is the whole set X, while the Banks and Uncovered Sets are almost the entire set X. The fifth case refers to the divide the dollar game, as already discussed.

6 Beyond the Goods and Bads Model.

We have offered a complete description of $WC(u)$, $TC(u)$ and $UC(u)$, and some bounds on the description of $B(u)$, for the goods and bad model,. In this section, we return to the more general linear model. In Section 4, we analyzed that model in terms of understanding the Top Cycle. We now return to that analysis to see what we can say about the Uncovered Set.

Precisely, we focus on the generic case where there is not a Condorcet winner and the profile of preferences is described by the pattern

$$u = \begin{matrix} 1 & v_2 & v_3 \\ v_1 & 1 & 0 \\ 0 & 0 & 1 \end{matrix},$$

where $0 < v_1, v_2, v_3 < 1$. Thus, e^1 is a *vertex* Condorcet winner, as it beats the other vertices in a majority contest.

What does the Uncovered Set look like in such a setting? If e^1 is a Condorcet winner, then obviously $UC(u) = \{e^1\}$. So let us assume that e^1 is not a Condorcet winner. From Proposition 2, this holds true if and only if $v_2 + v_3 < 1$. In such a case, $TC(u)$ rules out the Condorcet loser e^3, but none of the points arbitrarily close to e^3. The following proposition demonstrates that there is a neighborhood of e^3 which is outside $UC(u)$. The proof technique is based on the following simple but useful lemma which follows immediately from the definition of covering ($C(T)$).

Lemma **2** *$x \in UC(u)$ if and only if for all $y \in Y$ either not yTx or there exists $z \in X$ such that xTz and not yTz.*

This lemma states a version of the two-step principle (a terminology due to Miller and McKelvey). Indeed, the lemma states that to be in the Uncovered Set an alternative x must weakly majority dominate any other alternative in either one step or two steps; and if there are two steps then the first component of the weak T-chain must be strict. Let $L^2(x)$ be the set $(X\backslash U(u,x)) \cup (\cup_{y\in L(u,x)}(X\backslash U(u,y)))$.[17] The lemma asserts that $x \in UC(u)$ if and only if $L^2(x) = X$.

PROPOSITION **6** *Let u be as discussed above and $x = (x_1, x_2) \in X$. Then, $x \in UC(u)$ if and only if*

$$\frac{x_1}{v_1} + \frac{x_2}{v_2} \geq 1$$

It is interesting to note that the condition in Proposition 6 does not involve v_3. If $v_1 = v_2 \equiv v$, then the condition in the Proposition is simply that :

$$x_1 + x_2 \geq v.$$

Obviously, from Proposition 6 it follows that the Uncovered Set rules out many points around e^3. This is a first step in an exploration of $UC(u)$. This provides the interesting conclusion that the Uncovered Set is a subset of the space of alternatives that depends in interesting ways on the utility profile.

Likely, a similar analysis can be conducted in the case where there is no vertex Condorcet winner.

7 The Mixed Private versus Public Goods Model

In this final section, we investigate the mixed private versus public goods model defined as Example 3. In this model, a profile of preferences is identified by the single positive parameter b describing the common willingness to pay of each voter for the public project. To emphasize this specificity, we use the notation $WC(b)$, $TC(b)$ and $UC(b)$ instead of $WC(u)$, $TC(u)$ and $UC(u)$. The following proposition describes the dependence of the three sets[18] upon the parameter b.

PROPOSITION **7** *Consider the mixed private versus public goods model from Example 3.*

(1) If the benefit from the public good is large ($b > \frac{1}{2}$), then allocating the entire budget to the public good is a Condorcet Winner (and thus, $WC(b) = UC(b) = TC(b) = (0, 0, 0, M)$).

(2) If the benefit from the public good is intermediate ($\frac{1}{3} < b < \frac{1}{2}$), then there are no weak Condorcet winners, the Top Cycle is the whole set of alternatives, and the Uncovered Set is the set of alternatives such that at most two voters get a positive amount of the private good and no voter gets the entire supply of the private good ($UC(b) = \Big\{x \in X : x_i = 0$ for at least one $i \in \{1, 2, 3\}$ an

[17] The notation L^2 is justified by the fact that when T is a tournament, $L^2(x) = L(u,x) \cup (\cup_{y\in L(u,x)} L(u,y))$.

[18] We have not calculated the Banks Set in this model.

(3) If the benefit from the public good is small ($b < \frac{1}{3}$), then the sets look like they do in the divide the dollar game ($WC(b) = \emptyset$, $UC(b) = \left\{x \in X : x_4 = 0 \text{ and } x^k \neq M \text{ for any } k\right\}$ and $TC(b) = X$).

Proposition 7 demonstrates that the presence of the public project has an impact on the distributive politics component of the budget allocation. If the benefit from the public good is sufficient, then it swamps the private allocation, as in (1). If it is too small, then the problem becomes similar to the divide the dollar game, as in (3). In the middle case, we see some interesting impact of the public good. One voter among the three should derive his payoff exclusively from public consumption. This is due to the fact that when $b > \frac{1}{3}$, the Bowen-Lindahl-Samuelson first order optimality condition rules out any interior allocation. Since the Uncovered Set is a subset of the Pareto set, this provides an upper bound. We prove that, in fact, the two sets coincide.

8 Concluding Remarks and Higher Dimensions

We have shown that it is possible to make predictions about the nature of voting equilibria under majoritarian rule that are not too sensitive to specific institutional details, even in multi-dimensional policy space. We did this by analyzing budget allocation problems where voters' preferences are linear.

Section 4 describes the geometric structure of the Top Cycle set. Proposition 2 showed that if there is a Condorcet winner, then it must be a vertex. We also found that the conditions under which a Condorcet winner exists extend Plott's analysis because they are applicable even when a Condorcet winner lies on the *boundary* of the feasible set - which is absent from his analysis. Having a boundary on the problem provides a different perspective than one gets from Plott's analysis, and the possibility of a Condorcet winner is no longer so extreme. Proposition 3 is the counterpart of Mckelvey's chaos theorems, and shows that if a Condorcet winner does not exist, then the Top Cycle set is virtually the entire set - at most it excludes the three vertices. So, while we still come to the conclusion that the Top Cycle is either a single point or the whole space, the conditions under which it is a single point are no longer so extreme.

We went on to consider the goods and bads model, where voters view each dimension as either a good or a bad. Proposition 5 demonstrates that there are cases where the Banks Set and Uncovered Sets are strict subsets of the feasible set, even in situations where no Condorcet winner exists. The Banks Set is generally a strict subset of the Uncovered Set, but the difference between the two sets disappears as the divisibility of the budget becomes finer.

In section 6, we returned to the more general linear preference framework of section 4, but restrict attention to the analysis of preference profiles which give rise to a vertex Condorcet winner. We characterize the Uncovered Set and show that it excludes a neighborhood of points close to the vertex Condorcet loser. This provides an interesting setting in which the Uncovered Set makes pointed predictions about the outcome of any majority rule based collective decision.

Finally, section 7 looks at the "mixed" public and private goods model (Example 3). Not surprisingly, voters' common willingness to pay for the public good turns out to be the crucial parameter in this model. If this willingness to pay is very high, then the entire budget will be spent in production of the public good under majoritarian rule. Conversely, if the willingness to pay is low, then the Uncovered Set excludes production of the public good. The interesting case is when the common willingness to pay takes on an intermediate value, and then the Uncovered Set predicts that at least one voter must be excluded from consumption of the private good.

The bulk of our analysis was in the special case where there are three projects and three voters, as that case is still tractable and yet introduces the full force of multi-dimensionality. Certainly, it is worthwhile to explore beyond this. While the extension to more than three projects and/or three voters does not raise conceptual difficulties, it is obviously much trickier. One reason is that the linear model is at least as difficult as the finite model and therefore moving to larger K complicates the combinatorics of the problem, as we know from the theory of majority tournaments. We have listed below several directions of investigation that seem promising to explore as a continuation of the analysis performed here.

• What happens to the goods and bads model in higher dimensions? The following conjecture might be considered.

Conjecture: Consider the goods and bads model, and a case where there is some dimension k that a strict majority of agents view as a good. Let J be the set of agents who think k is a good and let $A = \{k' : \text{such that } u_i^{k'} = 1 \text{ for some } i \in J\}$. If $x \in UC(u)$ and $\ell \notin A$, then $x^\ell < K$.

• It seems that Proposition 2 generalizes to higher dimensions. A Condorcet winner will have to be a vertex. What conditions ensure that this vertex Condorcet is a Condorcet winner? Using Farkas' Lemma, it seems that a complete characterization of preference profiles for which this holds is possible! Once again, this will depart from Plott's symmetry conditions.

• It seems that we can also generalize Proposition 3 to higher dimensions, as follows. Define the *vertex Top Cycle*, denoted $VTC(u)$, to be the subset of vertices that are in the

Top Cycle of the majority weak tournament restricted to the vertices and the *vertex bottom cycle*, denoted $VBC(u)$, to be the subset of vertices that are in the bottom cycle of the majority weak tournament restricted to the vertices. We conjecture that

$$TC(u) = \left\{ x \in X : x_k = 0 \text{ for all } e^k \in VBC(u) \right\}.$$

This implies that if a vertex is in the vertex top cycle, then it is in the Top Cycle, but the converse does not hold, as we know already from the case where $K = N = 3$.

• The computation of the Uncovered Set does not seem out of reach either. One preliminary question we may ask could be the following. Define the *Vertex Uncovered Set* to be the subset of vertices which are in the Uncovered Set of the majority weak tournament restricted to the vertices. Is it true that a vertex is in the Vertex Uncovered Set must also be in the Uncovered Set? We know that the converse does not hold from Proposition 4.

• Finally, a detailed exploration of Example 3 would be valuable. It is straightforward to check that if there is a Condorcet winner, it must give the whole allocation to the public project. Furthermore, this project is a Condorcet winner if and only if

$$b \geq \frac{1}{\left(\frac{N}{2}\right)^{-} + 1}.$$

The following conjecture, extending Proposition 7, could be considered.

Conjecture: Consider the private versus public goods model. If $b > \frac{1}{M}$ for some positive integer M, then $x \in UC(b) \Rightarrow \# \{i : x_i > 0\} < M$. Furthermore, if $b < \frac{1}{M-1}$, then $UC(b) = \{x \in X : x_i = 0 \text{ for at least } N - M + 1 \text{ voters } \}$.

Note that the first assertion is true from an analysis of the Pareto set. Only the second assertion remains to be proved.

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Appendix

Proof of Proposition 2: Let x be a Condorcet winner for u. We first show that x must be on the boundary of the triangle. Assume to the contrary that x is in the interior of X. Then, for all $i, j \in N$, the indifference lines of voters i and j passing through x must be identical, as otherwise, there would exist y in the neighborhood of x such that $u_i \cdot y > u_i \cdot x$ and $u_j \cdot y > u_j \cdot x$, contradicting our assumption that x is a Condorcet winner. This implies that the slopes of the indifference lines of voters i and j through x are the same, so that $v_i v_j = 1$. This cannot be, as there is no solution $(v_1, v_2, v_3) \in (0,1)^3$ to the system of equations

$$v_1 v_2 = 1, v_1 v_3 = 1 \text{ and } v_2 v_3 = 1.$$

So we have shown that a Condorcet winner must be on the boundary of X.

Next, we show that a Condorcet winner must be a vertex. We know from above that a Condorcet winner x can be written as $x = \lambda e^i + (1-\lambda)e^j$ for some $0 < \lambda < 1$. Then, either $u_k \cdot e^i > u_k \cdot e^j$, in which case e^i majority dominates e^j via the coalition $\{i, k\}$; or, $u_k \cdot e^i < u_k \cdot e^j$, in which case e^j majority dominates e^i via the coalition $\{j, k\}$. Therefore, either $\lambda = 0$ or $\lambda = 1$, and the Condorcet winner must be a vertex.

We complete the proof by showing that (2) must apply and that $v_j + v_k \geq 1$ for some j and k. Without loss of generality, let $x = e^1$. Then, u must fall in (2) and it must be that either

$$u = \begin{array}{ccc} 1 & v_2 & v_3 \\ v_1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \quad \text{or} \quad u = \begin{array}{ccc} 1 & v_2 & v_3 \\ 0 & 1 & 0 \\ v_1 & 0 & 1 \end{array}.$$

Indeed, since e^1 majority dominates e^2 and e^3, then either e^3 is the worst choice for 1 and 2 or e^2 is the worst choice for 1 and 3. Without loss of generality, consider the first case. For e^1 to be Condorcet winner it is necessary and sufficient that there not exist $(y_1, y_2) \in X$ such that

$$v_2 y_1 + y_2 > v_2 \text{ and } (v_3 - 1)y_1 - y_2 > v_3 - 1.$$

It is straightforward to check that this system of inequalities is consistent with $(y_1, y_2) \in X$ if and only if $v_2 + v_3 < 1$. ∎

Proof of Proposition 3 : Assume that there is no Condorcet winner. We distinguish two cases.

Case 1: There is no vertex Condorcet winner.

In this case, up to a permutation, $e^1 T(u) e^2 T(u) e^3 T(u) e^1$ and from (d) above, the cycle extends to the whole boundary of X: for any two points z and w on the boundary there is a weak T-chain between z and w. Now take x and y in X. Since there is no Condorcet winner, we deduce from (c) above that there exist z and w on the boundary of X such that $xT(u)z$ and $wT(u)y$. The existence of a weak T-chain between x and y follows from the juxtaposition of the three weak T-chains. This proves that $TC(u) = X$.

Case 2: There is a vertex Condorcet winner.

Without loss of generality, assume that e^1 is the vertex Condorcet winner. Since e^1 is not a Condorcet winner, there exists z on the segment $[e^2, e^3]$ such that $zT(u)e^1$. From (b) above, we deduce that $e^1T(u)e^2T(u)zT(u)e^1$ and from (d) above the cycle extends to the entire boundary of the triangle with vertices e^1, e^2 and z, as illustrated in Figure 5.

Insert Figure 5 about here

We first show that for all $x, y \in [e^1, e^2] \cup [e^2, e^3] \cup [e^2, e^3] \cup [e^1, z]$ such that $x \neq e^3$, there exists a weak T-chain from x to y. For any $x \in [e^1, e^2] \cup [e^2, \underline{z}] \cup [\underline{z}, e^1]$, the claim follows from the existence of a cycle as in claim. Consider now the case where $x \in [e^1, e^3] \cup [\underline{z}, e^3]$ with $x \neq e^3$. The idea is to construct a weak T-chain starting from x and ending in h belonging to the smaller triangle with vertices e^1, e^2 and z; once there, we just demonstrated that you can anywhere else on the boundary of X. The construction goes as follows.

First consider $f \in [\underline{z}, e^3]$ and let g be the intersection of $[e^1, e^3]$ with the indifference line of voter 2 passing through f. Given the slopes of the indifference line of voters 2 and 3, it is easy to see that this point is well defined and that $u_3 \cdot g > u_3 \cdot f$. Then, define h as being the intersection of the indifference line of voter 3 with either $[\underline{z}, e^1]$ or $[\underline{z}, e^3]$. Given the slopes of the indifference lines of voters 1 and 3, it is easy to see that this point is well defined and that $u_1 \cdot h > u_1 \cdot g$. We have obtained the short weak T-chain $fT(u)gT(u)h$. This is illustrated in Figure 6.

Insert Figure 6 about here

If $h \in [\underline{z}, e^1]$, we have completed the desired argument. If instead, $h \in [\underline{z}, e^3]$, it is easy to show that $\mid h - e^3 \mid > \mid f - e^3 \mid$. Starting from h, we repeat the argument above to obtain g' and $h' \in [\underline{z}, e^1] \cup [\underline{z}, e^3]$. If $h' \in [\underline{z}, e^1]$, we are done. Otherwise, we continue this process. After a finite number of steps, we will obtain a point in $[\underline{z}, e^1]$. This is illustrated in Figure 7.

Insert Figure 7 about here

The case where $f \in [e^1, e^3]$ follows from (a) above, since $f = \lambda e^1 + (1 - \lambda)e^3$ for some $\lambda \in]0, 1[$ and $e^1T(u)w$ for all $w \in]\underline{z}, e^3[$, we deduce that $fT(u)\lambda w + (1-\lambda)e^3$. The connection involving points in the interior of X is done as in case 1. This completes the proof of the claim that $TC(u) = X \backslash \{e^3\}$. ∎

Proof of Proposition 4: First, note that yTx implies that y exceeds x on exactly two dimensions.

From this it is clear that there are no weak Condorcet winners, as for any alternative there exists some other alternative that gives more to two of the dimensions (given that $M \geq 4$).

Next, let us check that the Uncovered Set is the set of all points less the vertices. Consider $x = (u, v, w)$ that is defeated by some $y = (a, b, c)$. Without loss of generality, let $a > u$, $b > v$, and $c \leq w - 2$. Consider $z = (M - c - 1, 0, c + 1)$. Here, provided $v > 0$, x beats z and yet z beats y. Thus, y cannot cover x. This implies that the only covered points could be the vertices. Indeed, the vertices never beat any point, and are beaten by any interior point, and so are covered.

To verify that the Top Cycle is X, we only need to check that the vertices are in the Top Cycle, as the other alternatives are all uncovered. We need to check that from any vertex, say $x = (M, 0, 0)$, and any other alternative y there is a weak T-chain. If y has a 0 in either of the last two dimensions, then x and y are non-comparable, and so there is a weak T-chain directly. Thus consider any alternative $y = (u, v, w)$, where $v > 0$ and $w > 0$. Let $z = (u + 1, v + w - 1, 0)$. Then x is non-comparable to z and z defeats y, so there is a weak T-chain from x to y. This completes the proof of the Top Cycle.

The claims about the Banks Set are established as follows. Let us identify a maximal T-chain with $x = (u, v, w)$, where $u \geq v \geq w$, at the end.

Consider the case where $u > v + w$. (u, v, w), $(u - 1, v + 2, w - 1)$, $(u - 4, v + 6, w - 2)$, $\ldots$ $(u - c_i, v + i + c_i, w - i)$, $\ldots$ ... where i is the index of the step until $w - i$ hits 0, then $(u - c_i, v - i + w, i + c_i)$ for the remaining steps until $v + w - i$ hits 0. Let us define c_i, and let i^* be the smallest i for which $u - (i^2 + 3i - 2)/2 \leq v + w - i$. Then for $i < i^*$ set $c_i = (i^2 + 3i - 2)/2$ For $i \geq i^*$ set $c_i = u - (v + w) + i$. Let us prove that this chain is maximal. Suppose that $y = (a, b, d)$ beats everything in the chain. It cannot be that $b \leq v$, as then there is some point in the chain with middle entry b. Similarly $d \leq w$ is not possible. So $b > v$ and $d > w$. Thus it must be that $a \leq u - 2$. It cannot be that $a \leq u - c_{i^*}$, as then there is some step with first entry a. So, it must be that $a > u - c_{i^*}$. Without loss of generality then, take $a = u - c_i + 1$, for some $i < i^*$. Then it must be that b and d beat all the second and third entries above this. This means that $b + d \geq v + w + c_{i-1} + (i - 1) + 1 + 1$ [either beating the highest second entry and $w + 1$, or the highest third entry and $v + 1$ if we are already in the second part of the algorithm] We also know from the value of a that

$v+w+c_i-1 \geq b+d$. This implies that $c_i \geq c_{i-1}+i+2$. This does not hold by the definition of c_i, which solving inductively amounts to $c_i \geq (i^2+5i-4)/2$, which cannot hold given that $u < ((v+w)^2+5(v+w)-4)/2$. So, we have reached a contradiction.

So, to complete the proof consider the case where $u \leq v+w$ and let us identify a maximal T-chain with $x=(u,v,w)$ at the end. (u,v,w), $(u-1,v+2,w-1)$, $(u-2,v+4,w-2)$, $\ldots$ $(u-i,v+2i,w-i)$, $\ldots$ ... where i is the index of the step until $w-i$ hits 0, then $(u-i,v+w-i,2i)$ for the remaining steps until $u-i$ hits 0. Note first that this chain of length $u+1$ is well defined; indeed, when $i=u$, $v+w-i \geq 0$. Let us prove that this chain is maximal. Assume on the contrary that $y=(a,b,d)$ beats everything in the chain. It cannot be that $a \leq u$, as then there is some point in the chain with first entry a. Similarly $d \leq w$ is not possible. The same reasoning show that b is such that either $b > v$ or $b < v+w-u$. The first case is not possible as it implies $a+b+d > u+v+w$ which is not possible. Consider the second case. Since (a,b,d) beats all alternatives in the chain, we deduce from $b < v+w-u$ that (a,b,d) is preferred by voters 1 and 3 to any alternative in the chain. This implies $a > u$ and $d > 2u$ and therefore, since $u \geq v \geq w$, $a+b+d \geq a+d > 3u \geq u+v+w$, which is not possible.

Finally let us show that for any $M > 5$, $(M-1,1,0)$ and its permutations are not in the Banks Set, and so $B(u) \neq UC(u)$. The only alternatives that this beats are $(k,0,M-k)$ for $k < M-1$. The only chains that could conceivably be maximal are then of the form $(k,0,M-k)$, $(M-1,1,0)$. If $k < M-3$, then the alternative $(M-3,2,1)$ beats both. If $k \geq M-3$, then $(0,2,M-2)$ beats both (provided $M-2 > M-(M-3) = 3$, so when $M > 5$). ∎

Proof of Proposition 5:

Case (1) is easily checked directly.

Let us check (2).

If $s^1 = s^2 = 3 > s^3$, then it must be that every voter weakly prefers any allocation x with $x^3 = 0$ to any allocation y with $y^3 \neq 0$, and some voter has a strict preference between any two such allocations. Moreover, all voters are indifferent between any two allocations that have $x^3 = 0$, and so the set of weak Condorcet winners is the set $\{x \mid x^3 = 0\}$. Since any allocation outside of this set is defeated by one inside this set, this is the Top Cycle. Also, since the set of weak Condorcet winners is a subset of the Banks Set, the claim follows from Lemma 1 noting that $\{x \mid x^3 = 0\} = WC(u) \subset B(u) \subset UC(u) \subset TC(u) = \{x \mid x^3 = 0\}$.

If $s^1 = s^2 = 2 > s^3$, then it can be checked that any allocation x with $x^3 = 0$ defeats any allocation y with $y^3 \neq 0$. [Such a y gets at most one vote versus such an x, and such an x always gets at least one vote versus such a y. For any configuration of preferences that fits in this case where such a y gets one vote, it must be that such an x gets two votes.] Also, there must be one voter who is indifferent between all allocations in $\{x \mid x^3 = 0\}$, while the other two agents split on it. Thus, again the set of weak Condorcet winners is the set $\{x \mid x^3 = 0\}$, and any allocation outside of this set is defeated by one inside this set. So the rest of the proof is as in the case above.

If $s^1 = s^2 = 1 > s^3 = 0$, then either there is one voter who thinks both dimensions 1 and 2 are goods, and other voters are completely indifferent, or there are two voters who each like one of the two dimensions, and the other voter is indifferent between all allocations. In either situation it is clear that the set of weak Condorcet winners is the set $\{x \mid x^3 = 0\}$, and any allocation outside of this set is defeated by one inside this set, as in the earlier cases.

Next, let us consider (3a). In this case, without loss of generality, suppose that voter 1 views dimension 1 as a good, voter 2 views dimensions 2 and 3 as goods, and voter 3 is completely indifferent. If we consider two alternatives that have the same allocation to dimension 1, then all voters are indifferent between these alternatives. If we consider two alternatives that have different allocations between dimension 1, then voters 1 and 2 will have opposing preferences over the alternatives. Thus, any two alternatives are non-comparable under $T(u)$.

Next, let us consider (3b). Note that no two voters agree on which dimensions are goods. There is one voter who likes dimensions 1 and 2, one who likes 2 and 3, and one who likes 1 and 3. One key observation is that if $yT(u)x$ in this case, it must be that y exceeds x on exactly one dimension and is less than x on the two remaining dimensions. (If it is the same on any dimension then they are non-comparable. If y exceeds x on two dimensions, then the sum of the remaining dimension together with either other dimension is greater under x, and x will win.) This results in the following observations about $T(u)$.

(a) Any two alternatives which agree on some dimension are non- comparable to each other.

(b) Any vertex will beat any alternative that is positive on the other two dimensions.

From (a) and (b) it follows that the vertices are not beaten by any alternative, and from (b), it follows that any other alternative is beaten by some vertex. Thus the set of weak Condorcet winners is exactly the set of vertices.

Next, let us show that $UC(u) = X \setminus \{(M-2,1,1),(1,M-2,1),(1,1,M-2)\} \equiv X^*$.

First, we show that $x = (M-2,1,1)$ is not in the Uncovered Set. Let $y = (M,0,0)$. Then, yTx. Suppose not yTz. Then, from (b), either $z_2 = 0$ or $z_3 = 0$. Without loss of generality, suppose $z_2 = 0$. In order for x to beat z, x has to be bigger than z in just one component, and smaller than z in the other two components. Since $z_2 = 0$, this means that $x_1 < z_1$ and $x_3 < z_3$. But this is not possible. So, xTz implies yTz. Hence, x is covered. Analogous arguments establish that $(1, M-2, 1)$ and $(1, 1, M-2)$ are covered.

Next, we show that no other element in X^* is covered.

Each vertex forms a maximal chain as a singleton and so is in the Banks Set and thus the Uncovered Set.

Next, consider an alternative $x \in X^*$ that has two dimensions positive and the other 0. Without loss of generality, say $x = (a, M-a, 0)$ where $a \geq M-a$. This alternative beats $(0, M-1, 1)$. Note also that any alternative y that has $y^3 \geq 1$ does not beat $(0, M-1, 1)$ (the voter who likes the last two dimensions is at best indifferent, and the voter who likes the first two dimensions prefers $(0, M-1, 1)$). Thus only alternatives with $y^3 = 0$ beat $(0, M-1, 1)$. Then, it follows from (a) that forming a chain of x and $(0, M-1, 1)$ is a maximal chain that results in x, and so x is in the Banks Set, and thus the Uncovered Set. Next, consider an interior alternative $x = (x^1, x^2, x^3) \in X^*$. Without loss of generality, assume that $x^1 \geq x^2 \geq x^3$. Note that since $x \in X^*$, $x^2 \geq 2$.

Suppose yTx and y is an interior alternative in X. Without loss of generality, let $y^i < x^i, y^j < x^j$ and $y^k > x^k$. Of course, such i, j, k must exist. So, $y^i \leq x^i - 1$, $y^j \leq x^j - 1$, $y^k \geq x^k + 2$. This must mean that $x^i \geq 2, x^j \geq 2$ since y is interior by assumption. Now, consider z such that $z^i = 0$, $z^j = x^i + x^j - 1$, and $z^k = x^k + 1$. Then, xTz, but zTy. So, x cannot be covered by y.

Next, suppose that yTx and y is not an interior point. If $y^1 = 0$, then choose $z = (0, x^2+1, x^3+x^1-1)$. Since $x^1 \geq 2$, we have xTz. But, by (a), y and z are non-comparable. So, y does not cover x.

If $y^2 = 0$, then choose $z = (x^1+1, 0, x^3+x^2-1)$. Note that since $x \in X^*$, $x^2 \geq 2$. Again, xTz, but y and z are non-comparable.

The last possibility is that $y^3 = 0$. Since yTx, there is i such that $y^i \geq x^i + 2$. Choose z such that $z^i = x^i + 1 < y^i$, $z^k = 0$ where $k \neq 0$, and $z^3 = M - x^i - 1$. Since $x^2 \geq 2$, check that $z^3 > x^3$. It follows that xTz and yTz. So, x is not covered.

Thus, we have shown that $UC(u) = X^*$, and so Lemma 1 implies that $X^* \subset TC(u)$. Now, consider $x = (M-2, 1, 1)$. We show that there is a weak T-chain connecting x to each of the vertices. Take $(M, 0, 0)$. Then, the weak T-chain is $(x, (M-2, 0, 2), (M, 0, 0))$. Weak T-chains to other vertices are obvious extensions of this weak T-chain. Similarly, there is a weak T-chain from x to any other point in X. Hence, $TC(u) = X$.

Next, let us identify the Banks Set. Our arguments above already show that the Banks Set includes all alternatives that are not in the interior. Consider $x = (x^1, x^2, x^3)$ in the interior. Without loss of generality, let $x^1 \geq x^2 \geq x^3 \geq 1$, and since $B(u) \subset UC(u) = X^*$, we know that $x^2 \geq 2$.

Let k^+ be the smallest integer greater than or equal to k, and k^- the greatest integer smaller than or equal to k.

Let us build a T-chain that ends in x and argue that it is maximal. This shows that x is in the Banks Set.

The first element in the chain is x. The next part of the sequence are the alternatives (x^1+1, x^2-2, x^3+1), (x^1+2, x^2-4, x^3+2), $\ldots$, $\left(x^1 + (\frac{x^2}{2})^-, 0, x^3 + (\frac{x^2}{2})^+\right)$.

Denote $x^1_M = x^1 + (\frac{x^2}{2})^-$, and $x^3_M = x^3 + (\frac{x^2}{2})^+$.

The last part of the sequence is $(x^1_M - 2, x^2+1, x^3_M+1)$, $\ldots$, $\left(0, x^2 + (\frac{x^1_M}{2})^-, x^3_M + (\frac{x^1_M}{2})^+\right)$.

It is easy to check that this is a chain. Let us show that it is maximal.

Suppose y beats everything in the chain. Consider the case where $y^1 > x^1$. The chain contains without any gap everything from x^1 to x^1_M. So, $y^1 > x^1_M$. But, then y cannot beat $(x^1_M, 0, x^3_M)$. The same argument rules out cases where $y^2 > x^2$. So we are left with the case $y^3 > x^3$. In the third dimension, the chain contains all consecutive elements from x^1 to $x^3_M + (\frac{x^1_M}{2})^+$ *except possibly* [19] $x^3_M - 1$ and $x^3_M + (\frac{x^1_M}{2})^+ - 1$.

Suppose $y^3 = x^3_M - 1$. Since $y^3 > x^3_M - 2$, we need $y^2 < 3$. But, then $y^1 \geq x^1_M - 1$. So, y does not beat $(x^1_M - 1, 3, x^3_M - 2)$, which is the element just before $(x^1_M, 0, x^3_M)$.

An analogous argument works if $y^3 = x^3_M + (\frac{x^1_M}{2})^+ - 1$.

(3c) follows from Proposition 4. ▮

Proof of Proposition 6: Let $x = (x_1, x_2) \in X$. We examine the indifferences lines of voters 1 and 2 through x. The four possible cases are:

· The indifference line of voter 1 through x intersects $[e^2, e^3]$ and the indifference line of voter 2 through x intersects $[e^1, e^3]$ (This is the case considered in Proposition 6)

· The indifference line of voter 1 through x intersects $[e^2, e^3]$ and the indifference line of voter 2 through x intersects $[e^1, e^2]$

· The indifference line of voter 1 through x intersects $[e^1, e^2]$ and the indifference line of voter 2 through x intersects $[e^1, e^3]$

· The indifference line of voter 1 through x intersects $[e^1, e^2]$ and the indifference line of voter 2 through x intersects $[e^1, e^2]$

This leads to a partition of the triangle X into four areas as indicated in Figure 8.

Insert Figure 8 about here

We can check that whenever x belongs to areas 2, 3 and 4, $L^2(x) = X$. From lemma 2, this implies that the union of these areas is included in the uncovered Set. Assume now

[19] There are no gaps if x^2 and x^1_M are even.

that x belongs to the first area. Then, $L(x)$ is the union of the quadrilateral xAe^3B and the two triangles xCD and xEF. This pattern is depicted in Figure 9 where the hatched area corresponds to $L(x)$.[20]

Insert Figure 9 about here

From Lemma 2, to test if $x \in UC(u)$, it is enough to calculate $L^2(x)$. From the geometry of the problem, it is straightforward to verify that $L^2(x)$ is the union of the two triangles e^3FG and e^3DH where G is the intersection of $[e^2, e^3]$ with the indifference line of voter 1 through F and H is the intersection of $[e^1, e^3]$ with the indifference line of voter 2 through D. Let $I \equiv (w_1, w_2)$ be the intersection of the lines GF and DH. Therefore, $L^2(x) = X$ iff $w_1 + w_2 < 1$. The rest of the proof amounts to simple calculus.

The first coordinate of F, say f_1, is solution of the equation

$$v_2 f_1 = v_2 x_1 + x_2$$

or

$$f_1 = x_1 + \frac{x_2}{v_2}.$$

Therefore, the line FG is described by the equation

$$y_1 + v_1 y_2 = x_1 + \frac{x_2}{v_2}$$

Similarly, the second coordinate of D, say d_2, is solution of the linear equation

$$v_1 d_2 = x_1 + v_1 x_2$$

or

$$d_2 = \frac{x_1}{v_1} + x_2.$$

Therefore, the line DH is described by the equation

$$v_2 y_1 + y_2 = \frac{x_1}{v_1} + x_2.$$

We deduce that

$$w_1 = \frac{x_2}{v_2} \text{ and } w_2 = \frac{x_1}{v_1},$$

which implies the conclusion. ∎

Proof of Proposition 7:

[20] Up to the exclusion of the boundaries.

(1) Let $b > \frac{1}{2}$ and $x \in X$ with $x_4 < 1$. Then, for at least two of the voters, say i and j, $x_i \leq \frac{1-x_4}{2}$ and $x_j \leq \frac{1-x_4}{2}$. Therefore, since $b > \frac{1}{2}$,

$$u_k \cdot x = x_k + bx_4 \leq \frac{1-x_4}{2} + bx_4 < b = u_k \cdot (0,0,0,1) \text{ for } k \in \{i,j\}.$$

We deduce that $(0,0,0,1)T(b)x$ and the conclusion follows.

(2) Let $\frac{1}{3} < b < \frac{1}{2}$. It is clear that $WC(b) = \emptyset$ as $(\frac{1}{2}, \frac{1}{2}, 0, 0)T(b)(0,0,0,1)$. Let us prove that $UC(b) = \{x \in X : x_i = 0 \text{ for at least one } i \in \{1,2,3\}, x_k \neq M \text{ for any } k\, \{1,2,3\}\}$.

(i) $UC(b) \subset \{x \in X : x_i = 0 \text{ for at least one } i \in \{1,2,3\}, x_k \neq M \text{ for any } k\, \{1,2,3\}\}$.

Let $x \in X$ with $x_i > 0$ for all $i \in \{1,2,3\}$. Let $y = (x_1 - \delta, x_2 - \delta, x_3 - \delta, x_4 + 3\delta)$ where $0 < \delta < Min\ (x_1, x_2, x_3)$. Since $b > \frac{1}{3}$, y Pareto dominates and therefore covers x. Next, suppose that $x_k = M$ for some $k \in \{1,2,3\}$. Then, x is covered by $(0,0,0,M)$, as x does not defeat any alternative.

(ii) $UC(b) \supset \Big\{x \in X : x_i = 0 \text{ for at least one } i \in \{1,2,3\} \text{ and } x^k \neq M \text{ for any } k \in \{1,2,3\}\Big\}$. Without loss of generality, consider the case where $x_3 = 0$. Suppose that, contrary to the assertion, x is covered by y. Since we can take y to be uncovered, from (i) we deduce that either $y = (y_1, y_2, 0, y_4)$ or $y = (y_1, 0, y_3, y_4)$ or $y = (0, y_2, y_3, y_4)$. Let us consider the case where $x_4 \neq 0$.

Case 1 : $y = (y_1, y_2, 0, y_4)$.

Subcase 1 : $y_4 \leq x_4$. Since yTx, we deduce that $y_4 \neq x_4$ and therefore $y_1 > x_1$ and $y_2 > x_2$. It follows that $(x_1, x_2, 0, x_4)T(y_1 + y_2, 0, 0, y_4)$ but not $[(y_1, y_2, 0, y_4)T(y_1 + y_2, 0, 0, y_4)]$. This shows that y does not cover x.

Subcase 2 : $y_4 > x_4$. The extra public good $y_4 - x_4$ is financed by voters 1 and 2. Without loss of generality, assume that voter 2 pays at least half of the cost i.e. $y_2 - x_2 \leq -\frac{y_4 - x_4}{2}$. Consider the vector $z \equiv (1 - y_2 - by_4 - \varepsilon, y_2 + by_4 + \epsilon, 0, 0)$ where ε is a small positive number. Note that for ε small enough, z is a feasible allocation as $y_2 + by_4 < y_2 + \frac{y_4}{2} < 1$ since $b < \frac{1}{2}$. Then, for ε small enough, xTz since $x_4 \neq 0$ while however Not $[yTz]$. This shows again that y does not cover x.

Case 2 : $y = (y_1, 0, y_3, y_4)$.

Subcase 1 : $y^4 \leq x^4$. Then, for sufficiently small but positive, ϵ, we have :

$$(x_1, x_2, 0, x_4)T(y_1 + y_3, \epsilon, 0, y_4 - \epsilon) \text{ but Not } [(y_1, 0, y_3, y_4)T(y_1 + y_3, \epsilon, 0, y_4 - \epsilon)]$$

This shows that y does not cover x.

Subcase 2 : $y_4 > x_4$. Clearly, 3 prefers y to x. Suppose first that 1 also prefers y to x (and therefore that 2 prefers x to y), and consider the vector $z \equiv (1 - by_4 - \varepsilon, by_4 + \varepsilon, 0, 0)$, where ε is a small positive number. Then, for small enough ε, z is a feasible allocation and

xTz while, not $[yTz]$. If instead 1 prefers x to y, and therefore 2 prefers y to x, consider the vector $z \equiv (y_2 + by_4 + \varepsilon, 0, 1 - y_2 - by_4 - \varepsilon, 0)$, where ε is a small positive number. Then, for small enough ε, z is a feasible allocation and xTz since $x_4 \neq 0$, while not $[yTz]$. This again shows that y does not cover x.

Case 3 : $y = (0, y_2, y_3, y_4)$. This case is similar to case 2.

Consider the situation where $x_4 = 0$. The analysis has to be changed slightly.

Case 1 : $y = (y_1, y_2, 0, y_4)$. Since yTx and $b < \frac{1}{2}$, it must be that $y_4 > 0$. Furthermore, either $y_1 + by_4 < x_1$ or $y_2 + by_4 < x_2$. Without loss of generality assume that the second inequality holds and let $z \equiv (0, y_2 + by_4 + \varepsilon, 1 - y_2 - by_4, 0)$, where ε is a small positive number. Then, for small enough ε, z is a feasible allocation and xTz while not $[yTz]$.

Case 2 : $y = (y_1, 0, y_3, y_4)$.

Subcase 1 : $y^4 = 0$. Since $x_1 \neq 0$ and $x_2 \neq 0$, we deduce from Proposition 4 that y cannot cover x.

Subcase 2 : $y_4 > 0$. Clearly, 3 prefers y to x. Suppose that 1 also prefers y to x (and therefore that 2 prefers x to y), and consider the vector $z \equiv (0, by_4 + \varepsilon, 1 - by_4 - \varepsilon, 0)$, where ε is a small positive number. Then, for small enough ε, z is a feasible allocation and xTz while not $[yTz]$. If on the other hand, 1 prefers x to y, and therefore 2 prefers y to x, consider the vector $z \equiv (y_1 + by_4 + \varepsilon, 0, 1 - y_1 - by_4 - \varepsilon, 0)$ where ε is a small positive number. Then, for small enough ε, z is a feasible allocation and xTz since $x_2 \neq 0$ while not $[yTz]$. This shows again that y does not cover x.

Case 3 : $y = (0, y_2, y_3, y_4)$. Similar to case 2.

(3) Let $b < \frac{1}{3}$. Then

$$UC(b) = \left\{ x \in X : x^4 = 0 \text{ and } x^k \neq M \text{ for any } k \in \{1, 2, 3\} \right\}$$

The inclusion $UC(b) \subset \{x \in X : x_4 = 0\}$ follows from the fact that if $b < \frac{1}{3}$, then any x such that $x_4 > 0$ is Pareto dominated and therefore covered. Since any alternative in $UC(b \mid PO(b))$[21] where $PO(b)$ denotes the set of Pareto undominated allocations is in $UC(b)$[22], we deduce from Proposition 4 that $UC(b)$ contains the set

$$\left\{ x \in X : x^4 = 0 \right\}.$$

[21]For any $A \subseteq X$, $UC(b \mid A)$ denotes the Uncovered Set when the set of alternatives is restricted to the subset A.

[22]We leave the proof of this simple claim to the reader. Note however that the reverse inclusion $UC \subseteq UC(PO)$ does not always hold i.e. while the deletion of Pareto undominated alternatives can never hurts an alternative already in the Uncovered Set, the consideration of such alternatives may help some other alternatives which would not be in otherwise !

We now only need to prove that the vertices are not in $UC(b)$. This follows from the fact that for any vertex x, there is no y such that xTy.

To complete the proof, it remains to be shown that if $b < \frac{1}{2}$, then $TC(b) = X$.

From (3c) in Proposition 2 we know that any alternative in the set

$\widetilde{X} \equiv \{x = (x_1, x_2, x_3, x_4) \in X : x_4 = 0\ \}$ is connected to any other alternative in that set (the weak T-chain is in $\widetilde{X}$). To conclude it remains to prove that $\widetilde{X}$ and $X \backslash \widetilde{X}$ are connected. Let $x \in X$ with $x_4 > 0$. Since $b < \frac{1}{2}$, $(x_1 + \frac{x_4}{2}, x_2 + \frac{x_4}{2}, x_3, 0)Tx$. Finally, observe that $xT(x_1 + x_4, x_2, x_3, 0)$. ∎

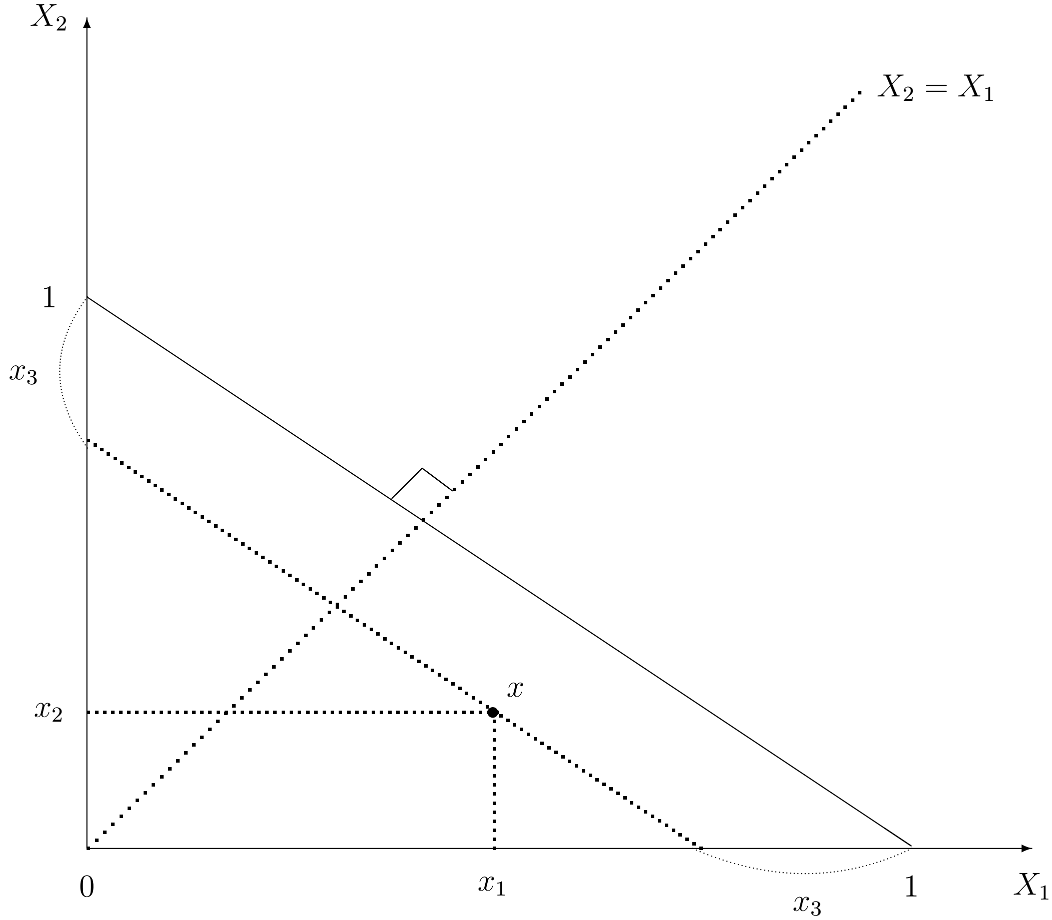

Figure 1
Representation of the alternatives

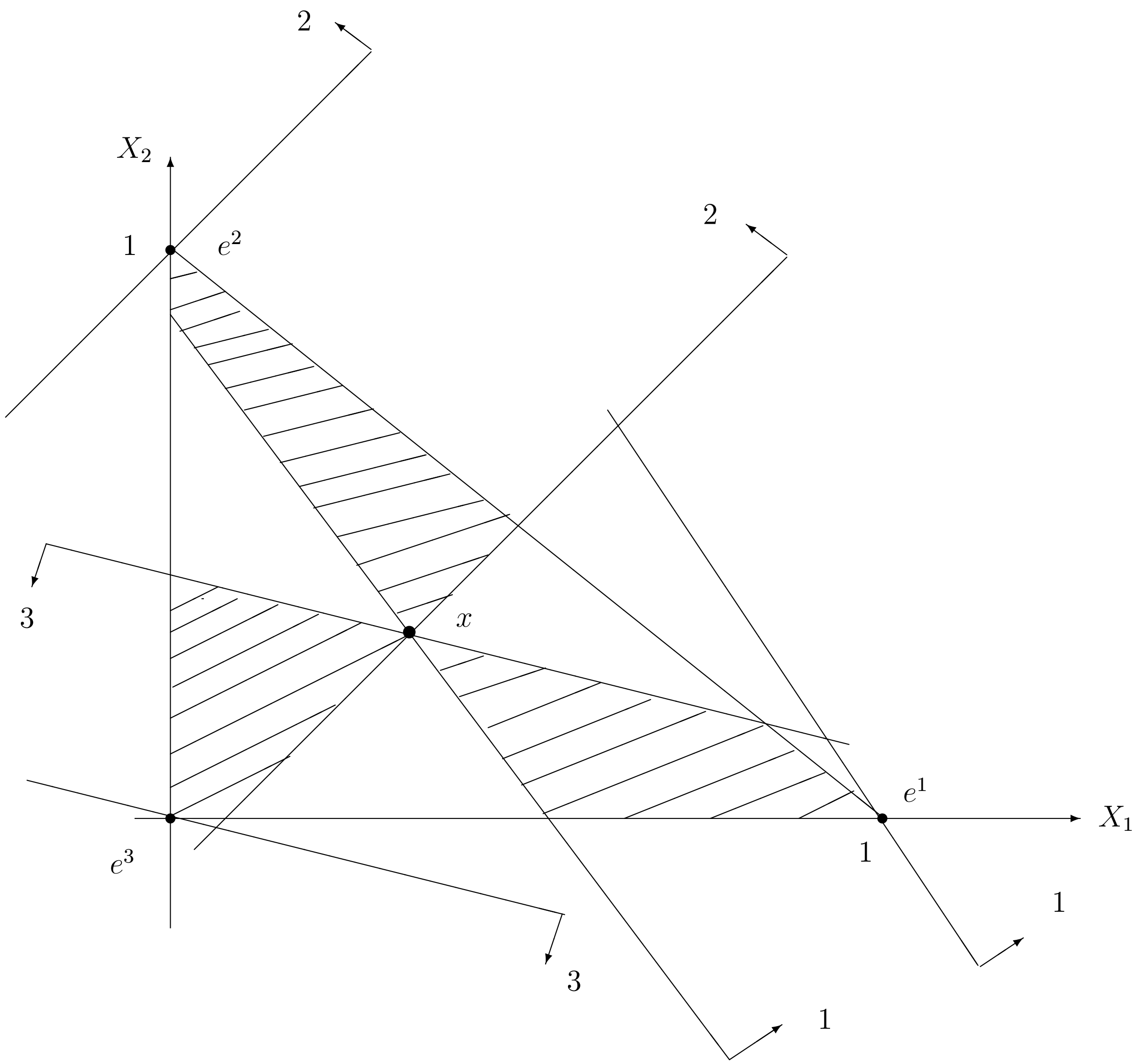

Figure 2
The Win Set of x

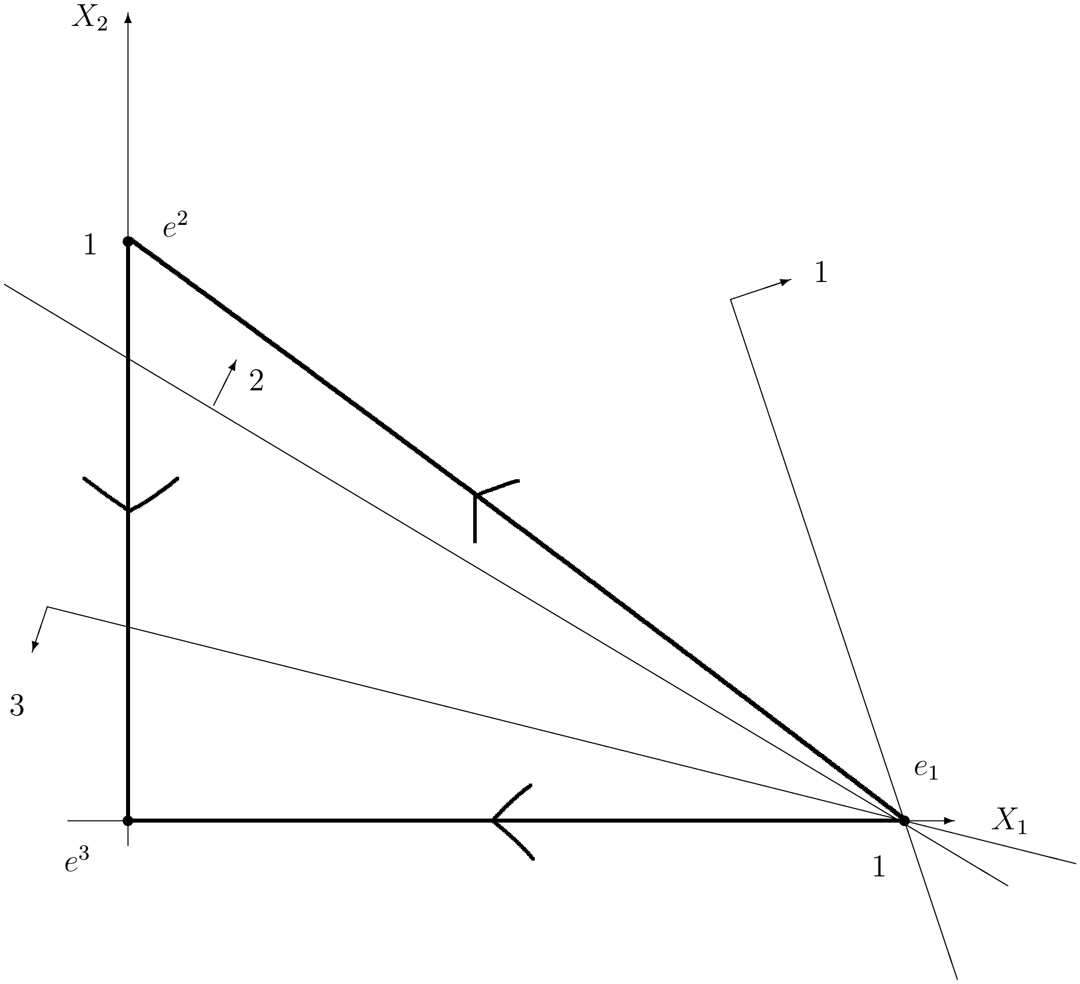

Figure 3
A Condorcet Winner

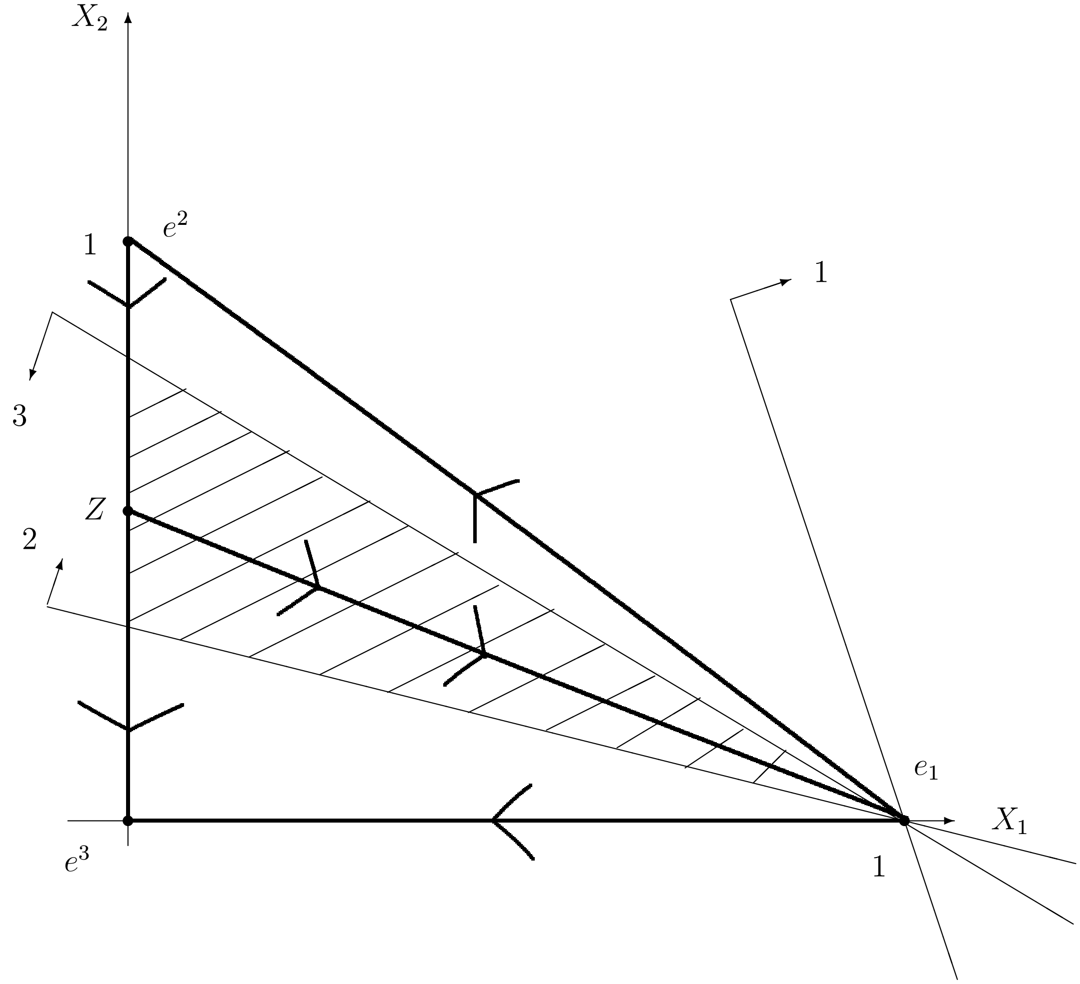

Figure 4
A “Vertex” Condorcet Winner
which is not a Condorcet Winner

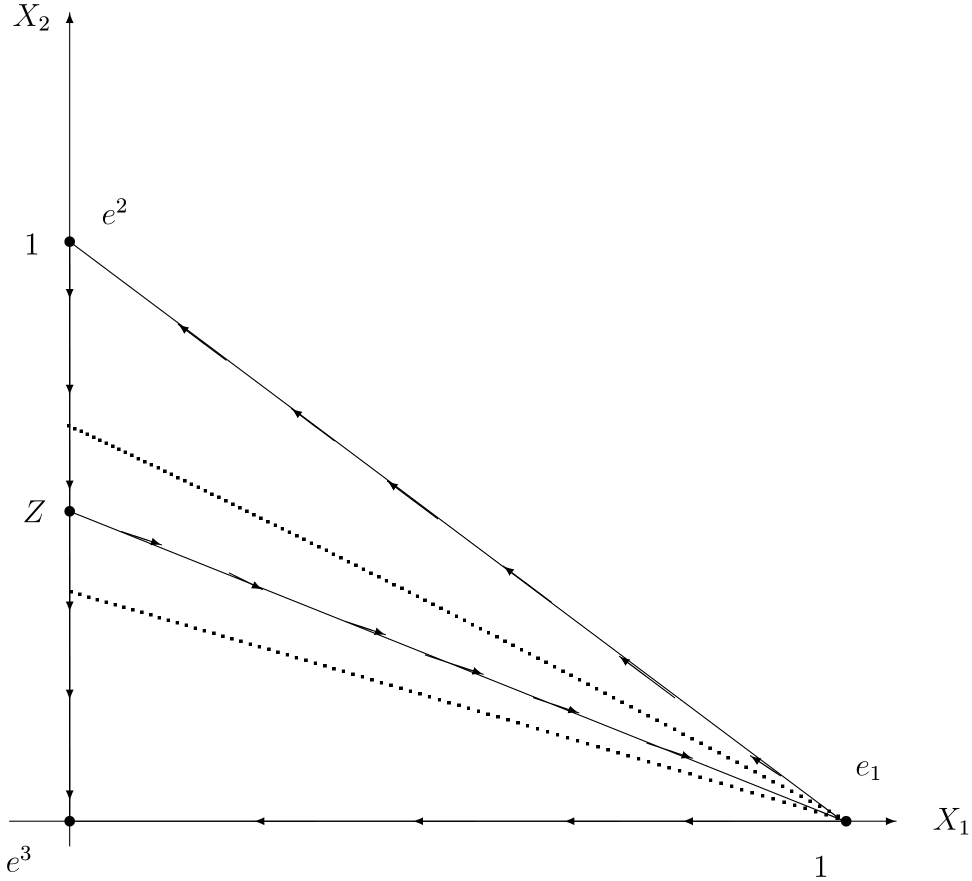

Figure 5
The "Cycles" on the boundary
of the triangle

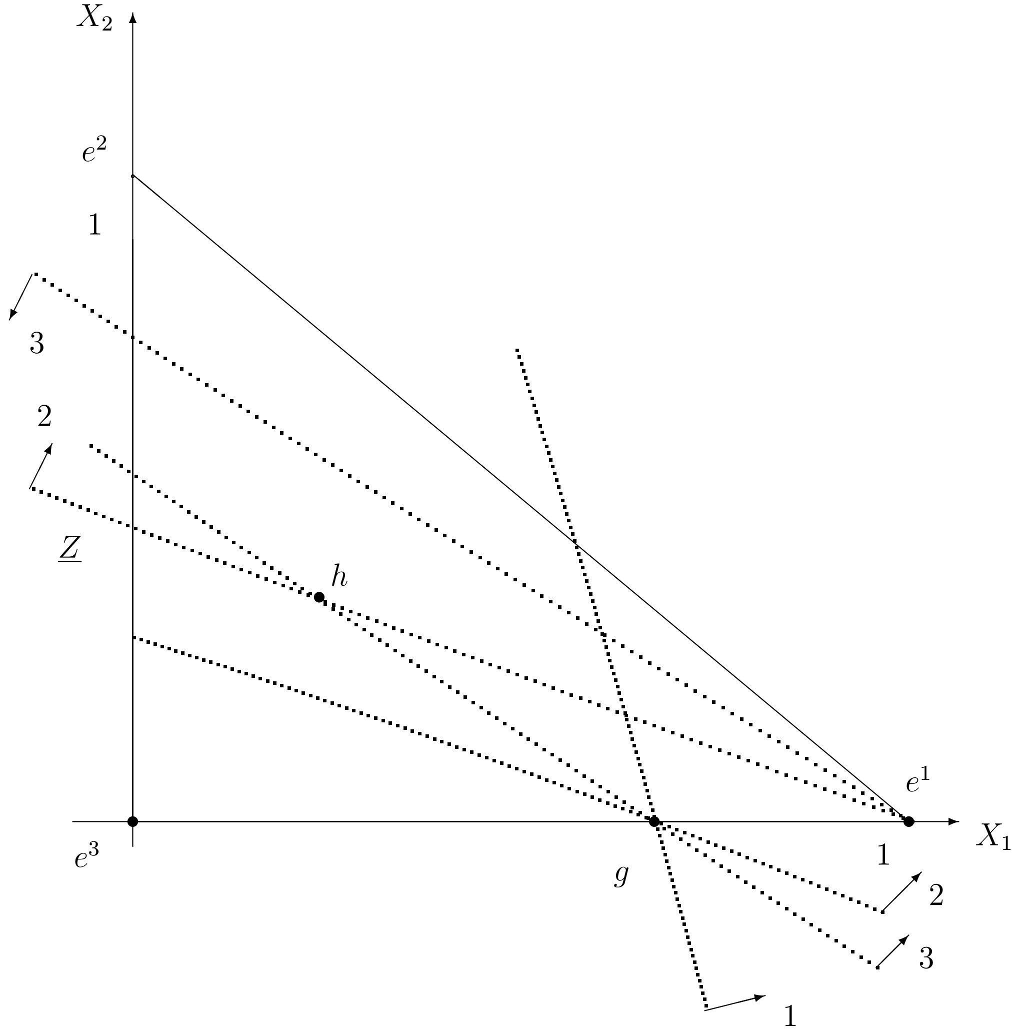

Figure 6
Construction of h

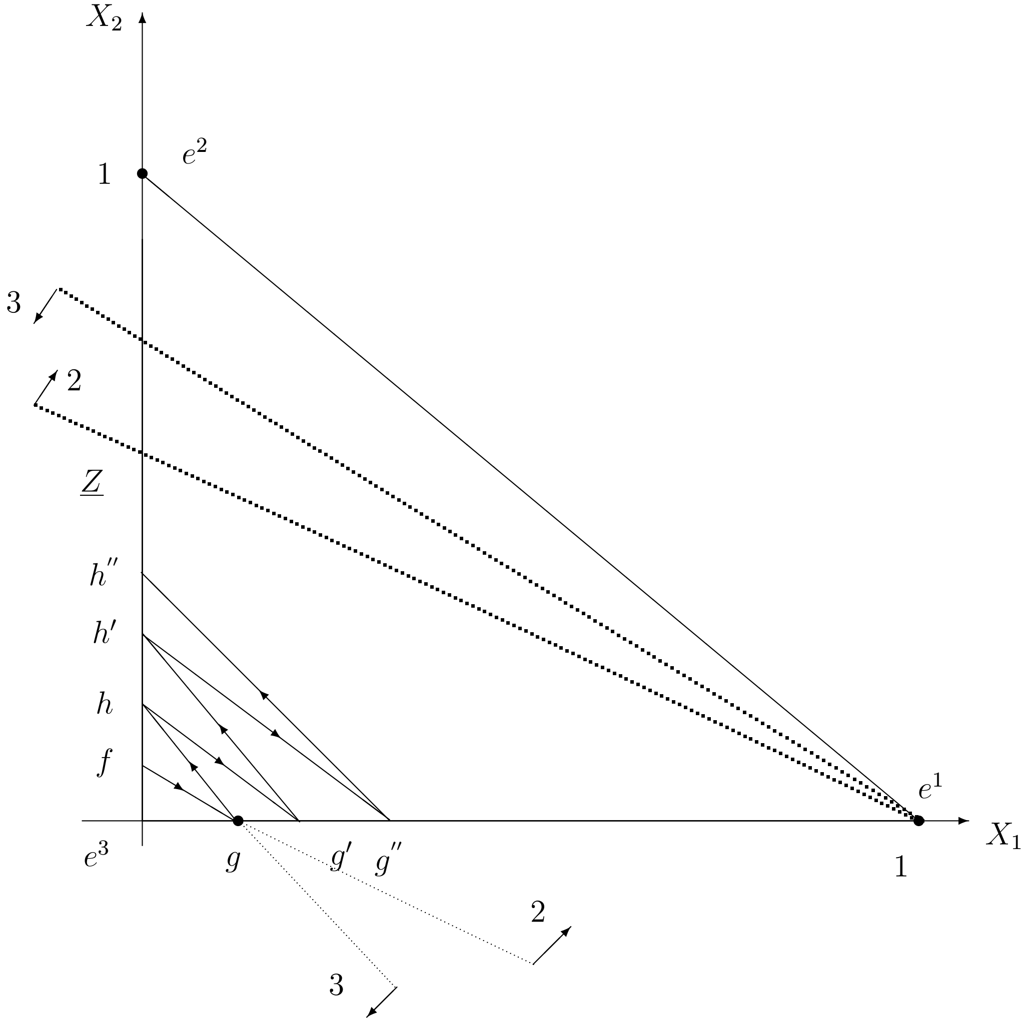

Figure 7
Connection to $[e^1, \underline{Z}]$

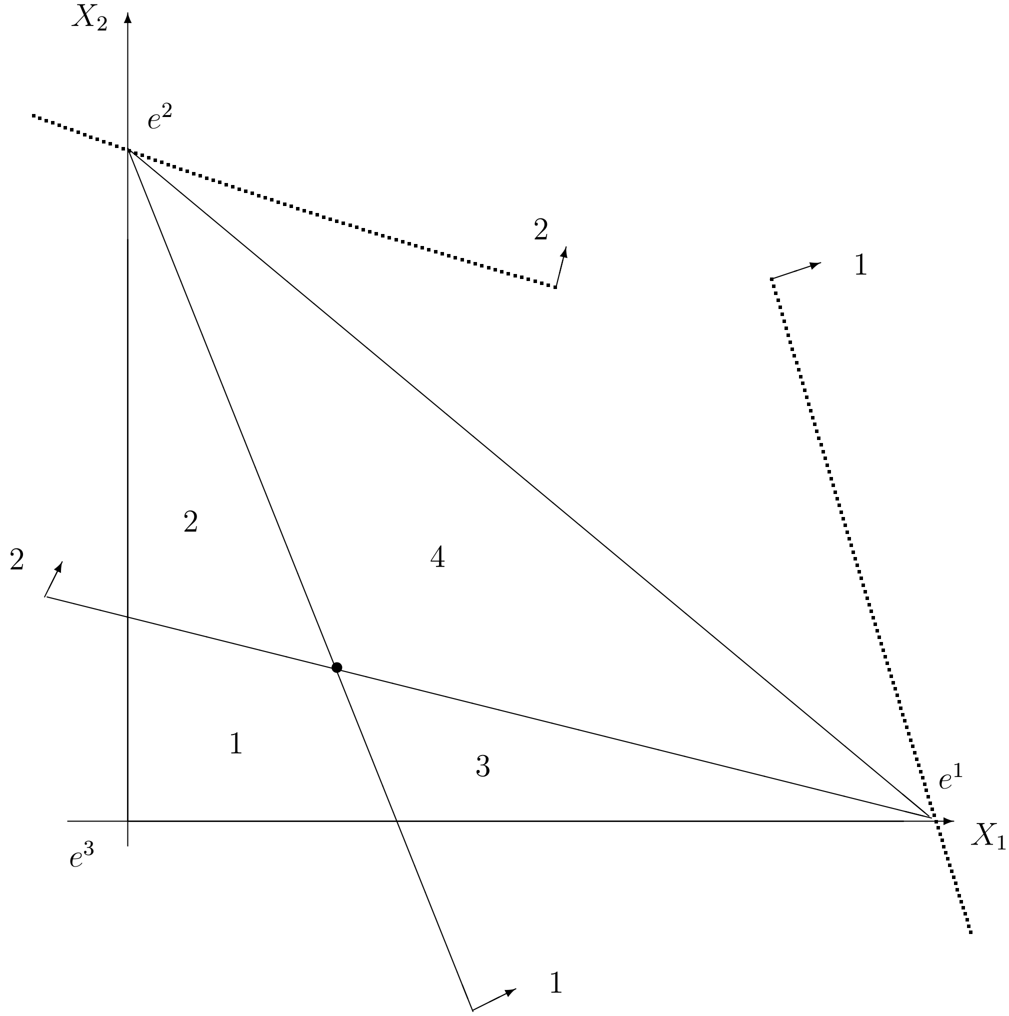

Figure 8
The Four Areas

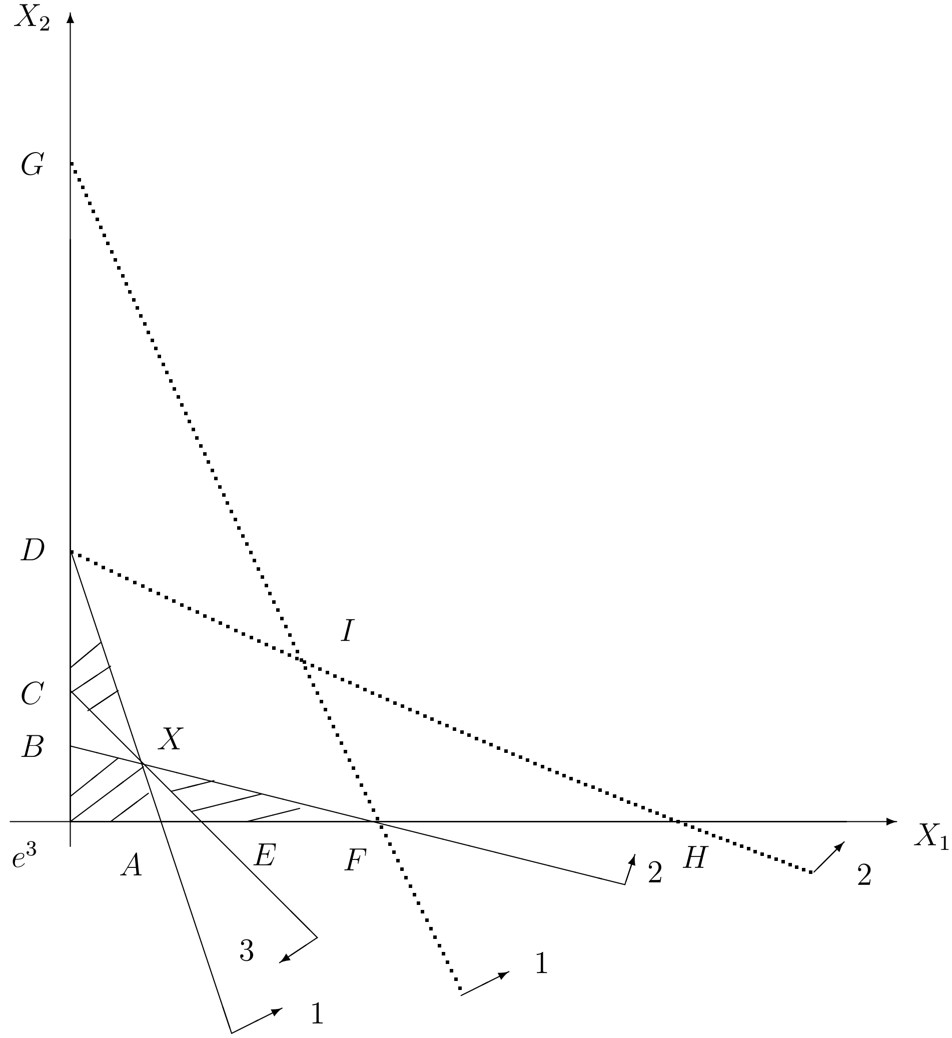

Figure 9
$L(X)$ and $L^2(X)$