EC9D3 Advanced Microeconomics, Part I: Lecture 2

Francesco Squintani

August, 2020

Budget Set

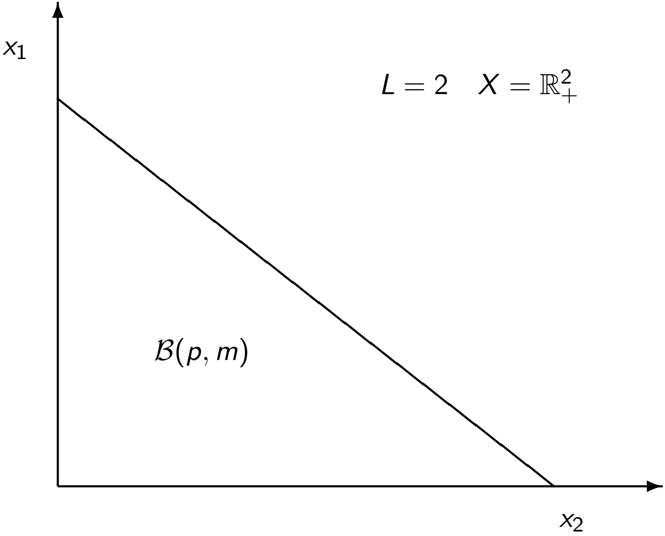
- Up to now we focused on how to represent the consumer's preferences.
- We shall now consider the sour note of the constraint that is imposed on such preferences.

Definition (Budget Set)

The consumer's budget set is:

$$\mathcal{B}(p, m) = \{x \mid (p\, x) \leq m, x \in X\}$$

Budget Set (2)



Income and Prices

- The two exogenous variables that characterize the consumer's budget set are:
  - the *level of income* m
  - the *vector of prices* $p = (p_1, \ldots, p_L)$.
- Often the budget set is characterized by a level of income represented by the value of the consumer's endowment x_0 (labour supply):

$$m = (p\, x_0)$$

Utility Maximization

The basic *consumer's problem* (with rational, continuous and monotonic preferences):

$$\max_{\{x\}} \quad u(x)$$
$$s.t. \quad x \in \mathcal{B}(p, m)$$

Result

If $p > 0$ and $u(\cdot)$ is continuous, then the utility maximization problem has a solution.

Proof: If $p > 0$ (i.e. $p_l > 0$, $\forall l = 1, \ldots, L$) the budget set is compact (closed, bounded) hence by Weierstrass theorem the maximization of a continuous function on a compact set has a solution. □

First Order Condition

Result

If $u(\cdot)$ is continuously differentiable, the solution $x^* = x(p, m)$ to the consumer's problem is characterized by the following necessary conditions. There exists a Lagrange multiplier λ such that:

$$\nabla u(x^*) \leq \lambda \, p$$

$$x^* \; [\nabla u(x^*) - \lambda \, p] = 0$$

$$p \, x^* \leq m$$

$$\lambda \, [p \, x^* - m] = 0.$$

where

$$\nabla u(x^*) = [u_1(x^*), \ldots, u_L(x^*)].$$

First Order Condition (2)

Meaning that $\forall l = 1, \ldots, L$:

$$u_l(x^*) \leq \lambda p_l$$

and

$$x_l^* \left[u_l(x^*) - \lambda p_l \right] = 0$$

That is if $x_l^* > 0$ then $u_l(x^*) = \lambda p_l$ while if $u_l(x^*) < \lambda p_l$ then $x_l^* = 0$.

Moreover

$$\sum_{l=1}^{L} p_l \, x_l^* \leq m, \qquad \text{and} \qquad \lambda \left[\sum_{l=1}^{L} p_l \, x_l^* - m \right] = 0$$

First Order Condition (3)

In other words:

- if $\lambda > 0$ then

$$\sum_{l=1}^{L} p_l \, x_l^* = m.$$

- if

$$\sum_{l=1}^{L} p_l \, x_l^* < m.$$

  then $\lambda = 0$

- If preferences are strongly monotonic (or locally non-satiated) then

$$\sum_{l=1}^{L} p_l \, x_l^* = m.$$

First Order Condition (4)

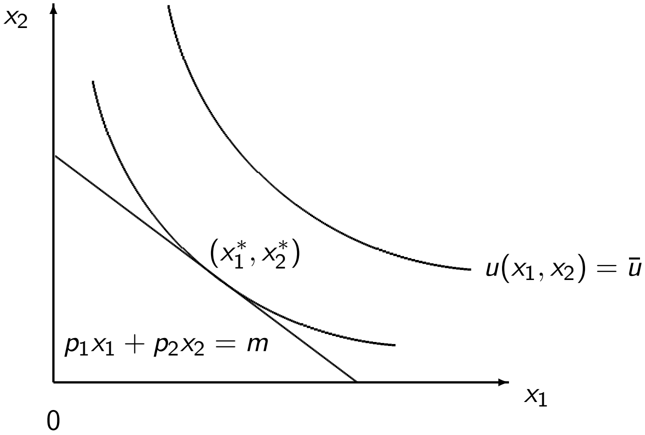
In the case $L = 2$ and $X = \mathbb{R}^2_+$ these conditions are:

$$\text{if} \quad x_1^* > 0 \ \text{ and } \ x_2^* > 0 \quad \text{then} \quad \frac{u_1}{u_2} = \frac{p_1}{p_2}$$

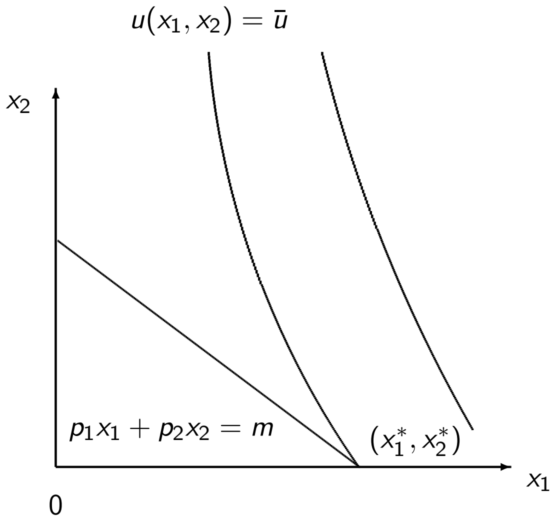
$$\text{if} \quad \frac{u_1}{u_2} < \frac{p_1}{p_2} \quad \text{then} \quad x_1^* = 0 \ \text{ and } \quad x_2^* > 0$$

$$\text{if} \quad \frac{u_1}{u_2} > \frac{p_1}{p_2} \quad \text{then} \quad x_1^* > 0 \quad \text{and} \quad x_2^* = 0$$

Interior Solution $L = 2$



Corner Solution $L = 2$



Sufficient Conditions

- The conditions we stated are merely necessary.
- What about sufficient conditions?

Result

If $u(\cdot)$ is quasi-concave and monotone,

$$\nabla u(x) \neq 0 \quad \textit{for all} \quad x \in X,$$

then the Kuhn-Tucker first order conditions are sufficient.

Sufficient Conditions (2)

Result

If $u(\cdot)$ is not quasi-concave then a $u(\cdot)$ locally quasi-concave at x^*, where x^* satisfies FOC, will suffice for a local maximum.

- Local (strict) quasi-concavity can be verified by checking whether *the determinants of the bordered leading principal minors of order*

$$r = 2, \ldots, L$$

of the Hessian matrix of $u(\cdot)$ at x^* have the sign of

$$(-1)^r.$$

Sufficient Conditions (3)

- The Hessian is:

$$H = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u}{\partial x_1 \partial x_L} & \cdots & \frac{\partial^2 u}{\partial x_L^2} \end{pmatrix}$$

- The bordered leading principal minor of order r of the Hessian is:

$$\begin{pmatrix} H_r & [\nabla u(x^*)]_r^T \\ [\nabla u(x^*)]_r & 0 \end{pmatrix}$$

H_r is the leading principal minor of order r of the Hessian matrix H and $[\nabla u(x^*)]_r$ is the vector of the first r elements of $\nabla u(x^*)$.

Marshallian Demands

Definition (Marshallian Demands)

The *Marshallian or uncompensated demand functions* are the solution to the utility maximization problem:

$$x = x(p, m) = \begin{pmatrix} x_1(p_1, \ldots, p_L, m) \\ \vdots \\ x_L(p_1, \ldots, p_L, m) \end{pmatrix}$$

Notice that strong monotonicity of preferences implies that the budget constraint will be *binding* when computed at the value of the Marshallian demands. *(building block of Walras Law)*

Indirect Utility Function

Definition

The function obtained by substituting the Marshallian demands in the consumer's utility function is the *indirect utility function:*

$$V(p, m) = u(x^*(p, m))$$

We derive next the properties of the indirect utility function and of the Marshallian demands.

Properties of the Indirect Utility Function

1. $\dfrac{\partial V}{\partial m} \geq 0$ and $\dfrac{\partial V}{\partial p_i} \leq 0$ for every $i = 1, \ldots, L$.

2. *$V(p, m)$ continuous in (p, m)*.

   It rules out situations in which the consumption feasible set is *non-convex* (e.g. indivisibility).

Properties of the Indirect Utility Function (2)

3. *$V(p, m)$ homogeneous of degree 0 in (p, m).*

Definition

$F(x)$ is *homogeneous of degree r* iff $F(k\ x) = k^r\ F(x) \quad \forall k \in \mathbb{R}_+$

Proof: Multiply both the vector of prices p and the level of income m by the same positive scalar $\alpha \in \mathbb{R}_+$ we obtain the budget set:

$$\mathcal{B}(\alpha\ p, \alpha\ m) = \{x \in X \mid \alpha\ p\ x \leq \alpha\ m\} = \mathcal{B}(p, m)$$

hence the indirect utility (and Marshallian demands) are the same. □

Properties of the Indirect Utility Function (3)

4. $V(p, m)$ is quasi-convex in p, that is:

$$\{p \mid V(p, m) \leq k\}$$

is a convex set for every $k \in \mathbb{R}$.

Proof: let p, m and p', be such that:

$$V(p, m) \leq k \qquad V(p', m) \leq k.$$

and $p'' = tp + (1 - t)p'$ for some $0 < t < 1$. We need to show that also $V(p'', m) \leq k$. Define:

$$\mathcal{B} = \{x \mid (p\ x) \leq m\}\ \mathcal{B}' = \left\{x \mid (p'\ x) \leq m\right\}\ \mathcal{B}'' = \left\{x \mid (p''\ x) \leq m\right\}$$

Properties of the Indirect Utility Function (4)

Claim

It is the case that:

$$\mathcal{B}'' \subseteq \mathcal{B} \cup \mathcal{B}'$$

Proof: Consider $x \in \mathcal{B}''$, then

$$\begin{aligned} p''x &= [tp + (1-t)p']\, x \\ &= t\,(p\,x) + (1-t)\,(p'\,x) \leq m \end{aligned}$$

which implies either $p\,x \leq m$ or/and $p'\,x \leq m$, or $x \in \mathcal{B} \cup \mathcal{B}'$. □

Properties of the Indirect Utility Function (5)

Now

$$\begin{aligned} V(p'', m) &= \max_{\{x\}} u(x) \quad s.t. \quad x \in \mathcal{B}'' \\ &\leq \max_{\{x\}} u(x) \quad s.t. \quad x \in \mathcal{B} \cup \mathcal{B}' \\ &= \max \left\{ V(p, m), V(p', m) \right\} \leq k \end{aligned}$$

Since by assumption: $V(p, m) \leq k$ and $V(p', m') \leq k$. □

Properties of the Marshallian Demand $x(p, m)$

1. $x(p, m)$ is continuous in (p, m), (consequence of the convexity of preferences).

2. $x_i(p, m)$ homogeneous of degree 0 in (p, m).

Proof: Once again if we multiply (p, m) by $\alpha > 0$:

$$\mathcal{B}(\alpha\ p, \alpha\ m) = \{x \in X \mid \alpha\ p\ x \leq \alpha\ m\} = \mathcal{B}(p, m)$$

the solution to the utility maximization problem is the same. □

Constrained Envelope Theorem

- Consider the problem:

$$\max_x f(x)$$
$$\text{s.t. } \; g(x,a) = 0$$

- The Lagrangian is: $\quad L(x,\lambda,a) = f(x) - \lambda \, g(x,a)$

- The necessary FOC are:

$$f'(x^*) - \lambda^* \frac{\partial g(x^*,a)}{\partial x} = 0$$

$$g(x^*(a),a) = 0$$

Constrained Envelope Theorem (2)

- Substituting $x^*(a)$ and $\lambda^*(a)$ in the Lagrangian we get:

$$\mathcal{L}(a) = f(x^*(a)) - \lambda^*(a)\, g(x^*(a), a)$$

- Differentiating, by the necessary FOC, we get:

$$\begin{aligned}\frac{d\mathcal{L}(a)}{d\ a} &= \left[f'(x^*) - \lambda^* \frac{\partial g(x^*, a)}{\partial x}\right] \frac{d\ x^*(a)}{d\ a} - \\ &\quad - g(x^*(a), a)\, \frac{d\lambda^*(a)}{da} - \lambda^*(a)\, \frac{\partial g(x^*, a)}{\partial a} \\ &= -\lambda^*(a)\, \frac{\partial g(x^*, a)}{\partial a}\end{aligned}$$

- In other words: *to the first order only the direct effect of a on the Lagrangian function matters.*

Properties of the Marshallian Demand $x(p, m)$ (2)

3. *Roy's identity:*

$$x_i(p, m) = -\frac{\partial V / \partial p_i}{\partial V / \partial m}$$

Proof: By the constrained envelope theorem and the observation:

$$V(p, m) = u(x(p, m)) - \lambda(p, m)\ [p\, x(p, m) - m]$$

we obtain:

$$\partial V / \partial p_i = -\lambda(p, m)\, x_i(p, m) \leq 0$$

and

$$\partial V / \partial m = \lambda(p, m) \geq 0$$

which is the marginal utility of income.

Properties of the Marshallian Demand $x(p, m)$ (3)

Notice: the sign of the two inequalities above prove the first property of the indirect utility function $V(p, m)$.

The proof follows from substituting

$$\partial V / \partial m = \lambda(p, m)$$

into

$$\partial V / \partial p_i = -\lambda(p, m)\, x_i(p, m)$$

and solving for $x_i(p, m)$. □

Properties of the Marshallian Demand $x(p, m)$ (4)

4. *Adding up results:* From the identity:

$$p\, x(p, m) = m \qquad \forall p, \qquad \forall m$$

Differentiating with respect to m gives:

$$\sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial m} = 1$$

while with respect to p_j gives:

$$x_j(p, m) + \sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial p_j} = 0$$

Properties of the Marshallian Demand $x(p, m)$ (5)

More informatively:

$$0 \geq \sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial p_h} = -x_h(p, m)$$

which means that at least one of the Marshallian demand function has to be *downward sloping in p_h*.

Consider, now, the effect of a change in income on the level of the Marshallian demand:

$$\frac{\partial x_l}{\partial m}$$

Properties of the Marshallian Demand $x(p, m)$ (6)

In the two commodities graph the set of tangency points for different values of m is known as the *income expansion path*.

In the commodity/income graph the set of optimal choices of the quantity of the commodity is known as *Engel curve*.

Income Effect

We shall classify commodities with respect to the effect of changes in income in:

- *normal goods:*

$$\frac{\partial x_l}{\partial m} > 0$$

- *neutral goods:*

$$\frac{\partial x_l}{\partial m} = 0$$

- *inferior goods:*

$$\frac{\partial x_l}{\partial m} < 0$$

Income Effect (2)

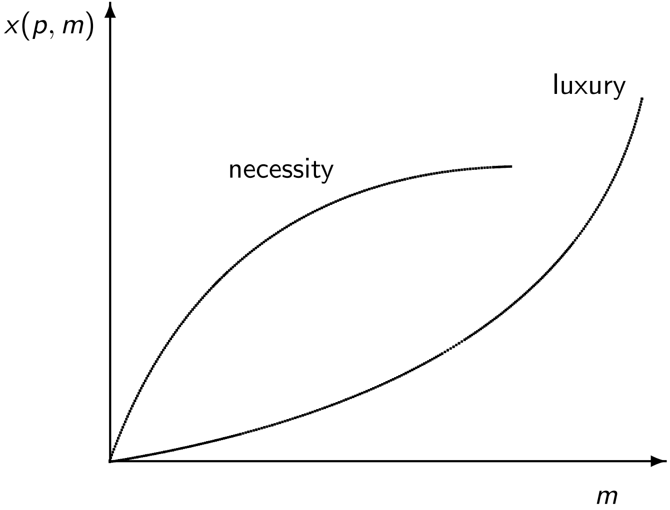
Notice that from the adding up results above for every level of income m at least one of the L commodities is normal:

$$\sum_{l=1}^{L} p_l \frac{\partial x_l}{\partial m} = 1$$

We also classify commodities depending on the curvature of the Engel curves:

- if the Engel curve is *convex* we are facing a *luxury good*
- If the Engel curve is *concave* we are facing a *necessity.*

Income Effect (3)



Expenditure Minimization Problem

- The *dual problem* of the consumer's utility maximization problem is the *expenditure minimization problem:*

$$\min_{\{x\}} \quad p\, x$$
$$\text{s.t.} \quad u(x) \geq U$$

- Define the solution as the *Hicksian (compensated) demand functions:*

$$x = h(p, U) = \begin{pmatrix} h_1(p_1, \ldots, p_L, U) \\ \vdots \\ h_L(p_1, \ldots, p_L, U) \end{pmatrix}$$

- We shall also define the *expenditure function* as:

$$e(p, U) = p\; h(p, U)$$

Properties of the Expenditure Function

1. $e(p, U)$ is continuous in p and U.

2. $\dfrac{\partial e}{\partial U} > 0$ and $\dfrac{\partial e}{\partial p_l} \geq 0$ for every $l = 1, \ldots, L$.

   Proof: $\dfrac{\partial e}{\partial U} > 0$. Suppose it does not hold.

   Then there exist $U' < U''$ such that (denote x' and x'' the, corresponding, solution to the e.m.p.)

   $$p\, x' \geq p\, x'' > 0$$

   If the latter inequality is strict we have an immediate contradiction of x' solving e.m.p.

Properties of the Expenditure Function (2)

If on the other hand

$$p\,x' = p\,x'' > 0$$

then by continuity and strict monotonicity of $u(\cdot)$ there exists $\alpha \in (0,1)$ close enough to 1 such that

$$u(\alpha\,x'') > U'$$

Moreover

$$p\,x' > p\,\alpha x''$$

which contradicts x' solving e.m.p. □

Properties of the Expenditure Function (3)

Proof: $\dfrac{\partial e}{\partial p_l} \geq 0$

Consider p' and p'' such that $p''_l \geq p'_l$ but $p''_k = p'_k$ for every $k \neq l$.

Let x'' and x' be the solutions to the e.m.p. with p'' and p' respectively.

Then by definition of $e(p, U)$

$$e(p'', U) = p''\, x'' \geq p'\, x'' \geq p'\, x' = e(p', U)$$

that concludes the proof. $\square$

Properties of the Expenditure Function (4)

3. $e(p, U)$ is homogeneous of degree 1 in p.

Proof: The feasible set of the e.m.p. does not change when prices are multiplied by the factor $k > 0$:

$$u(x) \geq U$$

Hence $\forall k > 0$, minimizing $(k\ p)\ x$ on this set leads to the same answer.

Let x^* be the solution, then:

$$e(k\ p, U) = (k\ p)\ x^* = k\ e(p, U)$$

that concludes the proof. □

Properties of the Expenditure Function (4)

4. *$e(p, U)$ is concave in p.*

Proof: Let $p'' = t\ p + (1-t)\ p'$ for $t \in [0,1]$.

Let x'' be the solution to e.m.p. for p''.

Then

$$e(p'', U) = p''\ x'' = t\ p\ x'' + (1-t)\ p'\ x''$$

$$\geq t\ e(p, U) + (1-t)\ e(p', U)$$

by definition of $e(p, U)$ and $e(p', U)$ and $u(x'') \geq U$. □

Properties of the Hicksian demand functions $h(p, U)$

1. *Shephard's Lemma.*

$$\frac{\partial e(p, U)}{\partial p_l} = h_l(p, U)$$

Proof: By constrained envelope theorem. □

2. *Homogeneity of degree 0 in p.*

Proof: By Shephard's lemma and the following theorem. □

Properties of the Hicksian demand functions $h(p, U)$ (2)

Theorem

If a function $F(x)$ is homogeneous of degree r in x then $(\partial F/\partial x_l)$ is homogeneous of degree $(r-1)$ in x for every $l = 1, \ldots, L$.

Proof: Differentiating with respect to x_l the identity, $F(k\, x) \equiv k^r\, F(x)$, we get:

$$k\, \frac{\partial F(k\, x)}{\partial x_l} = k^r \frac{\partial F(x)}{\partial x_l}$$

This is the definition of homogeneity of degree $(r-1)$:

$$\frac{\partial F(k\, x)}{\partial x_l} = k^{(r-1)}\, \frac{\partial F(x)}{\partial x_l}.$$

□

Euler Theorem

Theorem (Euler Theorem)

If a function $F(x)$ is homogeneous of degree r in x then:

$$r\ F(x) = \nabla F(x)\ x$$

Proof: Differentiating with respect to k the identity:

$$F(k\ x) \equiv k^r\ F(x)$$

we obtain:

$$\nabla F(kx)\ x = rk^{(r-1)}\ F(x)$$

for $k = 1$ we obtain:

$$\nabla F(x)\ x = r\ F(x).$$

□

Properties of the Hicksian demand functions $h(p, U)$ (3)

3. The matrix of own and cross-partial derivatives with respect to p *(Substitution matrix)*

$$S = \begin{pmatrix} \frac{\partial h_1}{\partial p_1} & \cdots & \frac{\partial h_1}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_L}{\partial p_1} & \cdots & \frac{\partial h_L}{\partial p_L} \end{pmatrix}$$

is *negative semi-definite and symmetric*.

Properties of the Hicksian demand functions $h(p, U)$ (4)

Proof: Simmetry follows from Shephard's lemma and Young Theorem:

$$\frac{\partial h_l}{\partial p_i} = \frac{\partial}{\partial p_i}\left(\frac{\partial e(p,U)}{\partial p_l}\right) = \frac{\partial}{\partial p_l}\left(\frac{\partial e(p,U)}{\partial p_i}\right) = \frac{\partial h_i}{\partial p_l}$$

Negative semi-definiteness follows from the concavity of $e(p, U)$ and the observation that S is the Hessian of the function $e(p, U)$. □

Identities

Since the expenditure minimization problem is the dual problem of the utility maximization problem the following identities hold:

$$
\begin{aligned}
V[p, e(p, U)] &\equiv U \\
e[p, V(p, m)] &\equiv m \\
x_l[p, e(p, U)] &\equiv h_l(p, U) \quad \forall l = 1, \ldots, L \\
h_l[p, V(p, m)] &\equiv x_l(p, m) \quad \forall l = 1, \ldots, L
\end{aligned}
$$

Slutsky Decomposition

Start from the identity

$$h_l(p, U) \equiv x_l[p, e(p, U)]$$

if the price p_i changes the effect is:

$$\frac{\partial h_l}{\partial p_i} = \frac{\partial x_l}{\partial p_i} + \frac{\partial x_l}{\partial m} \frac{\partial e}{\partial p_i}$$

Notice that by Shephard's lemma:

$$\frac{\partial e}{\partial p_i} = h_i(p, U) = x_i[p, e(p, U)]$$

you obtain the *Slutsky decomposition:*

$$\frac{\partial x_l}{\partial p_i} = \frac{\partial h_l}{\partial p_i} - \frac{\partial x_l}{\partial m} x_i.$$

Slutsky Equation

Own price effect gives *Slutsky equation:*

$$\frac{\partial x_l}{\partial p_l} = \frac{\partial h_l}{\partial p_l} - \frac{\partial x_l}{\partial m} x_l.$$

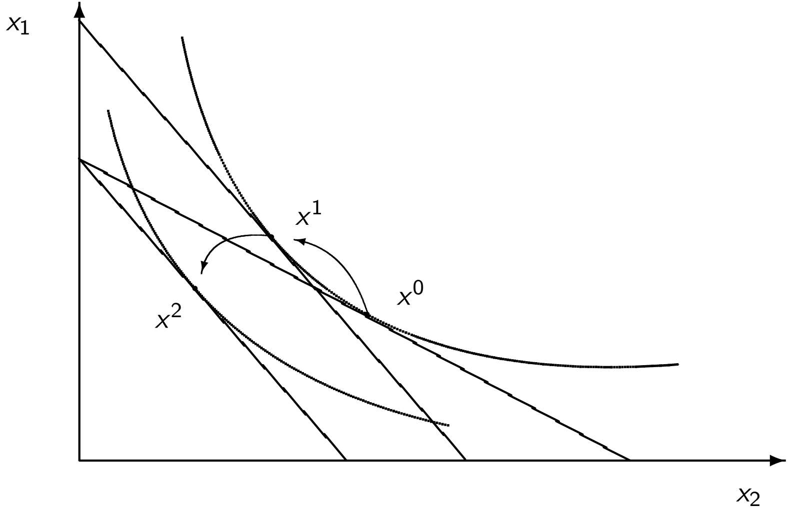
Substitution effect:

$$\frac{\partial h_l}{\partial p_l}$$

Income effect:

$$\frac{\partial x_l}{\partial m} x_l$$

Slutsky Equation (2)



Slutsky Equation (3)

We know the sign of the *substitution effect* it is *non-positive.*

The sign of the *income effect* depends on whether the good is *normal or inferior.*

In particular we conclude that the good is *Giffen* if

$$\frac{\partial x_l}{\partial p_l} > 0$$

This is not a realistic feature: inferior good with a big income effect.