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Solutions to Assignment 4
EC9D3 Advanced Microeconomics

1. We answer the question in three steps.

(i) Individual i dictatorship satisfies Unrestricted Domain because i's preferences are complete. It satisfies the Weak Pareto Principle, because if $xP(j)y$ for all j, then of course $xP(i)y$. It satisfies Independence of Irrelevant Alternatives because if every individual j ranks x and y in the same way under $R(j)$ and $R'(j)$, then so does i.

(ii) The same arguments imply that lexicographic dictatorship satisfies UD, WP and IIA. To show that it differs from individual dictatorship suppose that $xI(1)y$ and $xP(2)y$. Under individual dictatorship xIy, under lexicographic dictatorship xPy.

(iii) Consider the following dictatorship rule: pick an alternative x, say that for any y, $xR(1)y$ if and only if xRy. For all other alternatives x' and y', say that if $x'R(2)y'$, then $x'Ry'$. The same arguments as in (a) imply this dictatorship rule satisfies UD, WP and IIA. This rule is of course different from individual and lexicographic dictatorship.

2. We proceed to solve the question considering its parts in sequence.

(i) The answer is no.

For example, assume that there is only one good and agents, $N = 2$, and that the feasible set is: $T = \{x \in \mathbb{R}^2_+ : x_1^2 + x_2^2 \leq 1\}$. Assume further that the functions ψ_i for $i = 1, 2$ take the form $\psi_i(u_i(x_i)) = b_i u_i(x_i)$ such that $b_i \geq 0$ for both $i = 1, 2$.

Then the allocation x^* maximizes $\sum_{i=1}^N \psi_i(u_i(x_i))$ over $x \in T$ if and only if $b_1 = b_2$. In fact, the solution of

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} b_1 u_1(x_1) + b_2 u_2(x_2) \quad \text{s.t.} \quad x_1^2 + x_2^2 \leq 1$$

is such that

$$\frac{b_1}{b_2}\frac{u_1'(x_1)}{u_2'(x_2)} = \frac{x_1}{x_2},$$

which clearly depends on b_1/b_2.

The reason why x^* need not maximise $\sum_{i=1}^N \psi_i(u_i(x_i))$ over $x \in T$ is because the transformations ψ_i differ across the individuals i and make utilities $\psi_i(u_i)$ non-comparable across individuals. The utilitarian welfare maximization requires comparison across individuals.

(ii) The answer is no again.

For example, assume that there is only one good and agents, $N = 2$, and that the feasible set is: $T = \{x \in \mathbb{R}^2_+ : x_1^2 + x_2^2 \leq 1\}$. Assume further that the utility functions take the forms: $u_1(x_1) = x_1$ and $u_2(x_2) = 2x_2$, and the function ψ takes the form $\psi(u_i(x_i)) = \ln(u_i(x_i))$.

Then, x^* is the solution of

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} x_1 + 2x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 \leq 1$$

and satisfies

$$\frac{1}{2} = \frac{x_1^*}{x_2^*}, \quad x_1^{*2} + x_2^{*2} = 1. \quad \text{i.e.} \quad x_1^* = \sqrt{5}/5, x_2^* = 2\sqrt{5}/5,$$

whereas the solution of

$$\max_{(x_1,x_2)\in\mathbb{R}^2_+} \ln x_1 + \ln 2x_2 \quad \text{s.t.} \quad x_1^2 + x_2^2 \leq 1$$

is such that

$$\frac{x_2}{x_1} = \frac{x_1}{x_2}, \quad x_1^2 + x_2^2 = 1, \quad \text{i.e.} \quad x_1 = \sqrt{2}/2, \quad x_2 = \sqrt{2}/2.$$

The reason why x^* need not maximise $\sum_{i=1}^N \psi(u_i(x_i))$ over $x \in T$ is that the transformation ψ is common across the individuals i but arbitrary. Utilitarian welfare maximization is not invariant to ordinal utility transformations.

(iii) The answer is no again. The same example as in part (i) apply.

(iv) The answer is yes, because the transformation ψ is linear and common across individuals. The solution of

$$\max_{x\in\mathbb{R}_+^n} \sum_{i=1}^{N} [a_i + bu_i(x_i)] \quad \text{s.t.} \quad x \in T$$

is such that, for all pairs of agents i, l and goods j, k,

$$\frac{b}{b}\frac{\partial u_i(x_i)/\partial x_{ji}}{\partial u_l(x_l)/\partial x_{kl}} = \frac{\partial \tilde{T}(x)/\partial x_{ji}}{\partial \tilde{T}(x)/\partial x_{kl}},$$

where the implicit function $\tilde{T}(x) = 0$ describes the "north-east" frontier of T in $\mathbb{R}_+^{nN}$, i.e. $\tilde{T} : \mathbb{R}_+^{nN} \to \mathbb{R}$ increasing in each argument x_{ji} and such that $\tilde{T}(x) = 0$ if and only if $x' \notin T$ for all $x' \geq x$. In the above formula, the coefficients b cancel out.

3. We solve the two parts of the question as follows.

(i) To show non manipulability, note first that, of course, 1 has no reason to misreport her preferences, as they are implemented by f. To see that 2 has no reason to misreport her preferences, note that her report matters only if 1 reports a tie, in which case 2 gets one of her top alternatives, among the tie reported by 1.

(ii) Consider the following pair of preference profiles R and $\tilde{R}$: xP_1yP_2z and yP_2zP_2x, so that $f(R) = x$, and $x\tilde{I}_1y\tilde{P}_2z$, $y\tilde{P}_2z\tilde{P}_2x$, so that $f(\tilde{R}) = y$.

The social function is not strongly monotonic, because $f(\tilde{R}) = y \neq x$ despite the fact that $f(R) = x$ and that if xR_iy then $x\tilde{R}_iy$. The last clause hold because the only individual i such that xR_iy is $i = 1$ and it is the case that $x\tilde{R}_1y$.

4. We answer the question in three parts.

(i) To show Pareto efficiency, suppose that xR_iy for all i, and that xP_iy for some i. Then, majority rules f selects x, and the argument is the same if interchanging y with x. So, f is Pareto efficient.

(ii) To see that f is not manipulable, suppose that an agent i misreports her preferences, and say that her true preferences are xP_iy.

If she reports $y\tilde{P}_ix$ instead of xP_iy, she may only turn the majority outcome from $f(R_i, R_{-i}) = x$ to $f(\tilde{R}_i, R_{-i}) = y$, when the other agents' reported preferences R_{-i} are such that the number of agents j who report xP_jy is equal to the number of agents who report yP_jx, or it is smaller by one. For all other reported preferences R_{-i}, the majority outcome is the same regardless of whether she reports $y\tilde{P}_ix$ or xP_iy.

If agent i reports $x\tilde{I}_iy$ instead of xP_iy, she may only turn the majority outcome from $f(R_i, R_{-i}) = x$ to $f(\tilde{R}_i, R_{-i}) = y$, when the other agents' reported preferences R_{-i} are such that the agents j who report yP_jx outnumber those who report xP_jy by one. For all other reported preferences R_{-i}, the majority outcome is the same regardless of whether she reports $y\tilde{I}_ix$ or xP_iy. In sum, agent i prefers not to misreport her preferences.

The case in which her true preferences are yP_ix is analogous, and what she reports when xI_iy is irrelevant for her.

(iii) To see that none of the agents i is dictatorial, note that if xP_iy and yP_jx for all the other agents j, then $f(R) = y$, and that if yP_ix and xP_jy for all the other agents j, then $f(R) = x$.