University of Warwick, Department of Economics — Spring 2013
EC941: Game Theory — Prof. Francesco Squintani

Final Exam

Answer TWO questions. All questions carry equal weight. Time allowed 2 hours.

1. Consider the following game.

a. Find all strictly dominated strategies.

b. Find all rationalizable strategies.

c. Find all mixed-strategy Nash Equilibria.

	A	B	C	D
X	-1,1	2,-2	2,-1	1,-2
Y	2,-2	-3,3	2,0	0,1
W	1,1	-2,2	1,1	4,-2
Z	0,3	0,1	1,4	0,0

2. Consider the following variant of Bertrand's duopoly game. Costs and demand are linear, but the firms' unit costs are different, equal to c_1 and c_2, where $c_1 < c_2$. Let S_i be the share of consumers purchasing from firm i when prices set by the firms are equal (note that $S_i + S_j = 1$). Hence the profit of firm $i = 1, 2$ is $\pi_i(p_i, p_j) = (p_i - c_i)(\alpha - p_i)$ if $p_i < p_j$, $\pi_i(p_i, p_j) = S_i(p_i - c_i)(\alpha - p_i)$ if $p_i = p_j$, and $\pi_i(p_i, p_j) = 0$ if $p_i > p_j$. Let p_1^m be the price that maximizes $(p_1 - c_1)(\alpha - p_1)$ and assume that $c_2 < p_m^1$.

a. Suppose that $S_2 = 0$. Find all pure-strategy Nash equilibria.

b. Suppose that $S_2 = 0$. Find all pure-strategy Nash equilibria without weakly dominated strategies.

c. Suppose that $S_2 > 0$. Find all pure-strategy Nash equilibria.

3. An incumbent in an industry faces the possibility of entry by a challenger. First the challenger chooses whether or not to enter. If it does not enter, neither firm has any further action; the incumbent obtains the profit M in each of the following $T \geq 1$ periods) and the challenger's payoff is 0. If the challenger enters, it pays the entry cost $f > 0$, and in each of T periods the incumbent first commits to fight or cooperatewith the challenger in that period, then the challenger chooses whether to stay in the industry or to exit. If, in any period, the challenger stays in, each firm obtains in that period the profit $-F < 0$ if the incumbent fights and $C > \max\{F, f\}$ if it cooperates. If, in any period, the challenger exits, both firms obtain the profit zero in that period (regardless of the incumbent's action); the incumbent obtains the profit $M > 2C$ and the challenger the profit 0 in every subsequent period. Once the challenger exits, it cannot subsequently re-enter. Each firm cares about the sum of its profits.

a. Find the subgame perfect equilibria of the extensive game representing the above problem.

b. Consider a variant of the situation, in which the challenger is constrained by its financial war chest, which allows it to survive at most $T - 2$ fights. Specifically, consider the game that differs from the previou one in that the histories in which (i) at the start of the game the challenger enters and (ii) the incumbent fights in $T - 1$ periods, are terminal histories (the challenger has to exit). For the terminal history in which the incumbent fights in the first $T - 1$ periods the incumbent's payoff is $M - (T - 2)F$ and the challenger's payoff is $-f - (T - 2)F$ (in period $T - 1$ the incumbent's payoff is 0, and in the last period its payoff is M). For the terminal history in which the incumbent cooperates in one of the first $T - 1$ periods and fights in the remainder of these periods and in the last period, the incumbent's payoff is $C - (T - 2)F$ and the challenger's payoff is $-f + C - (T - 2)F$. Find the subgame perfect equilibria of this game.

4. Consider the model of jury presented in Lecture 4, and described in Section 9.7 of the textbook.

a. Find conditions under which the game, for an arbitrary number of jurors, has a Nash equilibrium in which every juror votes for acquittal regardless of her signal.

b. Find conditions under which the game, for an arbitrary number of jurors, has a Nash equilibrium in which every every juror votes for conviction regardless of her signal.

5. Suppose that each of the games below is finitely repeated for finite number of $T > 3$ periods with discount factor δ.

a. Find all subgame-perfect equilibria of game (a).

b. Describe a strategy by which the players achieve payoff of $(3,3)$ in all but the last 2 periods in game (b).

c. For which discount factors, is such a strategy a subgame-perfect equilibrium?

(a)	A	B	C	D
X	-1,3	0,4	2,-1	1,-2
Y	2,-2	1,2	2,0	0,1
W	1,1	-2,2	1,1	4,-2
Z	0,3	0,1	1,4	0,0

(b)	D	E	F
A	0,0	1,-1	4,1
B	1,4	0,2	3,3
C	0,0	-1,1	0,0

Answers

1a. At the first round of iterated deletion of strictly dominated strategies, Z is strictly dominated by the mixed strategy σ_1 such that $\sigma_1(W) \in (3/4, 1)$ and $\sigma_1(Y) = 1 - \sigma_1(W)$; and D is strictly dominated by the mixed strategy σ_2 such that $\sigma_2(C) \in (0, 2/3)$ and $\sigma_2(B) = 1 - \sigma_2(C)$.

1b. At the second round, C is strictly dominated by the mixed strategy σ_2 such that $\sigma_2(A) \in (1/3, 3/5)$ and $\sigma_2(B) = 1 - \sigma_2(B)$; and W is strictly dominated by the mixed strategy σ_1 such that $\sigma_1(X) \in (1/5, 1/3)$ and $\sigma_1(Y) = 1 - \sigma_1(X)$. The set of rationalizable strategies is $\{X, Y\} \times \{A, B\}$.

1c. The unique Nash equilibrium is $\sigma_1(X) = 5/8$, $\sigma_1(Y) = 1 - \sigma_1(X)$, $\sigma_2(A) = 5/8$, $\sigma_2(B) = 1 - \sigma_2(A)$.

2a. If all consumers buy from firm 1 when both firms charge the price c_2, then any profile (p_1, p_2) such that $p_1 = p_2 \in [c_1, c_2]$ is a Nash equilibrium by the following argument. Firm 1's profit is weakly positive, while firm 2's profit is zero (since it serves no customers).

- If firm 1 increases its price, its profit is zero.
- If firm 1 reduces its price to say p, then its profit is reduced from $(p_1 - c_1)(\alpha - p)$ to $(p - c_1)(\alpha - p)$.
- If firm 2 increases its price, its profit remains zero.
- If firm 2 decreases its price, its profit becomes negative (since its price is less than its unit cost).

Under this rule no other pair of prices is a Nash equilibrium, by the following argument.

- If $p_i < c_1$ for $i = 1, 2$ then the firm with the lower price (or either firm, if the prices are the same) can increase its profit to zero by raising its price above that of the other firm.
- If $p_1 > p_2 \geq c_2$ then firm 2 can increase its profit by raising its price a little.
- If $p_2 > p_1 \geq c_1$ then firm 1 can increase its profit by raising its price a little.
- If $p_1 = p_2 > c_2$ then at least one of the firms is not receiving all of the demand, and that firm can increase its profit by lowering its price a little.

2b. When $S_2 = 0$, any strategy $p_2 < c_2$ by firm 2 is weakly dominated by the strategy of setting the price to c_2, because for any $p_1 \leq p_2$ they both yield zero profit, whereas for any $p_1 > p_2$, firm 2 makes zero profit by setting the price equal to c_2 and strictly negative profits by choosing $p_2 < c_2$. Hence, the unique weakly undominated Nash equilibrium is (p_1, p_2) such that $p_1 = p_2 = c_2$.

2c. Now suppose that the rule for splitting up the customers when the prices are equal specifies that firm 2 receives some customers when both prices are c_2. By the argument for part a, the only possible Nash equilibrium is $(p_1, p_2) = (c_2, c_2)$. [The argument in part a that every other pair of prices is not a Nash equilibrium does not use the fact that customers are split equally when $(p_1, p_2) = (c_2, c_2)$.] But if $(p_1, p_2) = (c_2, c_2)$ and firm 2 receives some customers, firm 1 can increase its profit by reducing its price a little and capturing the entire market.

3a. Consider the last period, after any history. If the incumbent chooses to fight, the challenger's best action is to exit, in which case both firms obtain the profit zero. If the incumbent chooses to cooperate, the challenger's best action is to stay in, in which case both firms obtain the profit $C > 0$. Thus the incumbent's best action at the start of the period is to cooperate. Now consider period $T - 1$. Regardless of the outcome in this period, the incumbent will cooperate in the last period, and the challenger will stay in (as we have just argued). Thus each player's action in the period affects its payoff only because it affects its profit in the period. Thus by the same argument as for the last period, in period $T - 1$ the incumbent optimally cooperates, and the challenger optimally stays in if the incumbent cooperates. If, in period $T - 1$, the incumbent fights, then the challenger also optimally stays in, because in the last period it obtains $C > F$. Working back to the start of the game, using the same argument in each period, we conclude that in every period before the last the incumbent cooperates and the challenger stays in regardless of the incumbent's action. Given $C > f$, the challenger optimally enters at the start of the game. That is, the game has a unique subgame perfect equilibrium, in which:

- the challenger enters at the start of the game, exits in the last period if the challenger fights in that period, and stays in after every other history after which it moves
- the incumbent cooperates after every history after which it moves.

The incumbent's payoff in this equilibrium is TC and the challenger's payoff is $TC - f$.

3b. First consider the incumbent's action after the history in which the challenger enters, the incumbent fights in the first $T-2$ periods, and in each of these periods the challenger stays in. Denote this history h^{T-2}. If the incumbent fights after h^{T-2}, the challenger exits (it has no alternative), and the incumbent's total profit in the last two periods is M. If the incumbent cooperates after h^{T-2} then by the argument for the game in part (a), the challenger stays in, and in the last period the incumbent also cooperates and the challenger stays in. Thus the incumbent's payoff in the last two periods if it cooperates after the history h^{T-2} is $2C$. Because $M > 2C$, we conclude that the incumbent fights after the history h^{T-2}. Now consider the incumbent's action after the history in which the challenger enters, the incumbent fights in the first $T-3$ periods, and in each period the challenger stays in. Denote this history h^{T-3}. If the incumbent fights after h^{T-3}, we know, by the previous paragraph, that if the challenger stays in then the incumbent will fight in the next period, driving the challenger out. Thus the challenger will obtain an additional profit of $-F$ if it stays in and 0 if it exits. Consequently the challenger exits if the incumbent fights after h^{T-3}, making a fight by the incumbent optimal (it yields the incumbent the additional profit $2M$). Working back to the first period we conclude that the incumbent fights and the challenger exits. Thus the challenger's optimal action at the start of the game is to stay out.

In summary, the game has a unique subgame perfect equilibrium, in which:

- the challenger stays out at the start of the game, exits after any history in which the incumbent fought in every period, exits in the last period if the incumbent fights in that period, and stays in after every other history.
- the incumbent fights after the challenger enters and after any history in which it has fought in every period, and cooperates after every other history.

The incumbent's payoff in this equilibrium is TM and the challenger's payoff is 0.

4a. There is always an equilibrium in which all jurors vote acquittal regardless of their signal. In fact, if all other jurors vote A, no juror is ever pivotal, and hence each juror is indifferent between voting C or A.

4b. Suppose that all jurors vote C regardless of their signal. Then, each juror is always pivotal, and there is no information in this. Hence, a type b juror votes for C if and only if:

$$\begin{aligned} z &\leq \Pr(G|b) = \frac{\Pr(b|G)\Pr(b)}{\Pr(b|G)\Pr(G) + \Pr(b|I)\Pr(I)} \\ &= \frac{\pi(1-p)}{\pi(1-p) + (1-\pi)\,q}. \end{aligned}$$

This condition is also sufficient, because a type g juror votes for C for higher values of z.

5a. The unique stage-game Nash equilibrium of (a) is (Y, B). Because the game is only finitely repeated, the repetition of (Y, B) is the unique subgame-perfect equilibrium.

(b)	D	E	F
A	0,0	1,-1	4,1
B	1,4	0,2	3,3
C	0,0	-1,1	0,0

5b. One such strategy is a grim trigger strategy whereby the deviating player is punished by triggering play of her worse-preferred stage-game Nash Equilibrium. Specifically: for all $t < T - 2$,

$$s_1\left(h^t\right) = \begin{cases} B & \text{if } h^t = a^0 \text{ or } a_2^\tau = F \text{ for all } \tau < t, \\ A & \text{otherwise,} \end{cases}$$

$$s_2\left(h^t\right) = \begin{cases} F & \text{if } h^t = a^0 \text{ or } a_1^\tau = B \text{ for all } \tau < t, \\ D & \text{otherwise,} \end{cases}$$

For $t = T - 1$,

$$s_1\left(h^t\right) = \begin{cases} B & \text{if } a_2^\tau = F \text{ for all } \tau < t, \\ A & \text{otherwise,} \end{cases}$$

$$s_2\left(h^t\right) = \begin{cases} D & \text{if } a_2^\tau = F \text{ for all } \tau < t, \\ F & \text{otherwise,} \end{cases}$$

For $t = T$,

$$s_1\left(h^t\right) = \begin{cases} A & \text{if } a_1^\tau = B \text{ for all } \tau < t - 1, \\ B & \text{otherwise,} \end{cases}$$

$$s_2\left(h^t\right) = \begin{cases} F & \text{if } a_1^\tau = B \text{ for all } \tau < t - 1, \\ D & \text{otherwise.} \end{cases}$$

5c. To find out under which conditions, the above profile is a subgame perfect equilibrium, we only need to check deviation on path at time $t = T - 2$. This is because the strategies prescribe that Nash equilibria be played at times $T - 1$ and T, and because the triggered Nash Equilibrium punishment in case of deviation is the same for any $t < T - 1$, but the length of the punishment is the smallest for $t = T - 2$. Further, because the payoffs are symmetric, and player 1 obtains her preferred equilibrium on path later than player 2, she is the one with the biggest incentive to deviate. So, we do not need to consider one-shot deviations by player 2. By not deviating, player 1's payoff is:

$$u_1\left(s_1, s_2\right) = (1 - \delta)\left(3 + \delta + 4\delta^2\right).$$

By adopting the one-shot deviation r_1, player 1's payoff becomes:

$$u_1(r_1, s_2) = (1-\delta)(4+\delta+\delta^2).$$

Hence, the profile (s_1, s_2) is a subgame perfect equilibrium when $\delta \geq 1/\sqrt{3}$, so that the condition $u_1(s_1, s_2) \geq u_1(r_1, s_2)$ is satisfied.