University of Warwick, Department of Economics — Spring 2015
EC941: Game Theory — Dr. Jonathan Cave and Prof. Francesco Squintani

Final Exam

1. Consider the following 2X2 games in normal (strategic) form:

G1	L	R
T	$100, 100$	$120, 50$
B	$50, 120$	$110, 110$

G2	L	R
T	$6, 4$	$3, 5$
B	$1, 2$	$4, 6$

G3	L	R
T	$10, 10$	$0, 5$
B	$5, 0$	$1, 1$

a. [10 marks] For each game find and sketch (in a graph with the Row and Column player's payoffs on the vertical and horizontal axes, resp.) all the Nash equilibrium payoffs in a one-shot version of the game.

b. [20 marks] For each game find and sketch (in a graph with the Row and Column player's payoffs on the vertical and horizontal axes, resp.) all the subgame perfect Nash equilibrium average payoffs in an undiscounted infinitely repeated version of the game.

c. [20 marks] Now consider discounted repeated versions of each game, where the two players have the same discount factor δ. Define the cooperative payoff to be the highest attainable symmetric payoff. For games G1 and G3, this can be obtained by following a pure strategy. For game G2, this involves a combination of two Pareto Optimal strategies with equal weights; you may assume that a referee flips a fair coin to choose which to play in each period; the players can then choose whether or not to follow the referee's recommendation. For each game, find a condition on that allows the players to get the cooperative payoff by following subgame perfect equilibrium strategies and specify strategies that achieve this.

2. There are I firms in an industry. Each can try to convince the Government to give the industry a subsidy. Let h_i denote the number of hours of effort put in by firm i, and let $c_i(h_i) = (h_i)^2 w_i$ be the cost of this effort to firm i, where w_i is a positive constant. When the effort levels of the firms are $(h_1, ..., h_I)$, the value of the subsidy that gets approved to firm i is $\alpha \sum_{i=1}^{I} h_i + \beta \Pi_{i=1}^{I} h_i$, where $\alpha > 0$ and $\beta \geq 0$ are constants. The I firms decide simultaneously and independently how many hours they will each devote to this effort.

a. [15 marks] Prove that each firm has a strictly dominant strategy if and only if $\beta = 0$, and derive each firm's dominant strategy in this case.

b. [10 marks] When $\beta > 0$, are firms' efforts strategic substitutes or strategic complements? Explain.

c. [25 marks] When $\beta > 0$, how does the symmetric equilibrium level of effort relate to the symmetric, Pareto-efficient level of effort (where efficiency is defined taking only the firms' benefits into account)—that is, the level of effort that would be best for all firms, if all firms chose it?

3. Person 1 owns an indivisible financial asset that is potentially more valuable to Person 2. Person i, $i = 1, 2$, has a privately observed signal y_i—a noisy but unbiased estimate—of the asset's value x, and cares only about the expected value of her/his financial worth (the asset value net of its price for the buyer, the price net of the asset value for the seller). Persons 1 and 2 can trade the asset only at price p, and will trade only if both think they are strictly better off trading. Trade is governed by the following rules: Each player i simultaneously observes his y_i, and then simultaneously says Trade or No trade. Trade takes place, at price p, only if both say Trade; otherwise there is no trade.

a. [15 marks] Formulate this interaction as a Bayesian game.

b. [20 marks] Prove that in any pure-strategy equilibrium of this trading game, there is no trading, regardless of p.

c. [15 marks] Prove that in any mixed-strategy equilibrium of this trading game, the probability of trade is zero, regardless of p.

4. Consider 2 firms competing in a Cournot Duopoly market. They face the inverse demand curve $P = 1 - q_1 - q_2$. Firm 1 produces at 0 marginal cost; firm 2 produces at constant marginal cost c, where $c < 0.5$.

Suppose first that the two firms bargain over the profits that they can jointly obtain according to the Nash Bargaining Solution, where the threat point for each firm is its payoff in the noncooperative Cournot-Nash equilibrium. Assume that the firms can make side-payments, so that the agreements involve sharing the maximum profit they can produce if they coordinate their production.

a. [5 marks] Identify the set of agreement profits.

b. [15 marks] Find the Nash bargaining Solution (in terms of profit shares and quantities produced).

c. [10 marks] How much would the high-cost firm be willing to pay to reduce its cost to 0?

Now suppose that the two firms have identical marginal costs (c) but different discount factors ($\delta_1 > \delta_2$) and bargain over the outcome using the Rubinstein bargaining solution, with firm 1 going first.

d. [15 marks] What outputs would they choose and what would the resulting payoffs be?
e. [5 marks] How much would firm 2 be willing to pay to go first and would firm 1 accept the offer?

5. Consider the following game of incomplete information. There are two players, Xavier (X) and Yolande (Y). Nature chooses Xavier's type, which can be Creative (C) or Diligent (D) – Xavier knows which type he is, but Yolande does not and thinks that both types are equally likely. After learning his type, Xavier offers Yolande either a Triple choice (Yolande can then pick Up, Middle or Down or a Double choice between Top or Bottom. The payoffs are seen in the extensive game tree below:

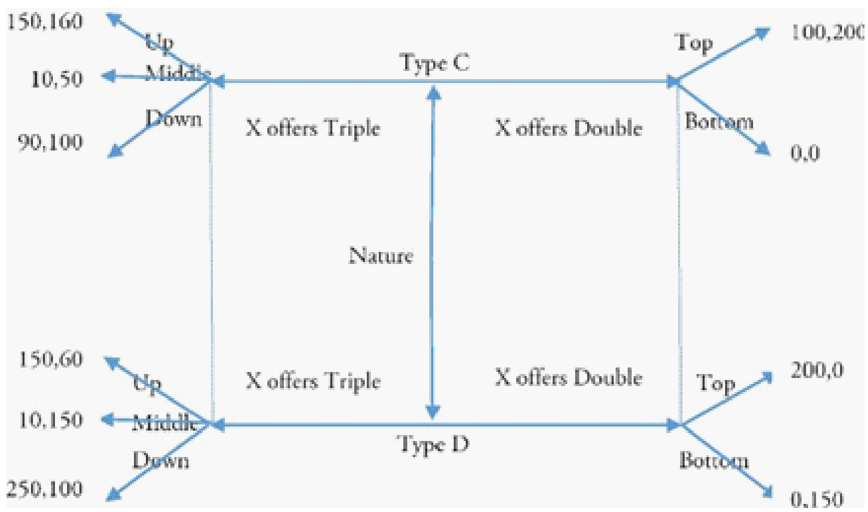

Figure 1: Game of Question 5.

a. [10 marks] What are Xavier's and Yolande's pure strategy spaces?
b. [40 marks] Find all pure strategy Perfect Bayesian Equilibria (PBE's) of this game.

Answers

1.a. The Nash Equilibria are (T, L) in G1, (B, R) in G2, and in G3 they are (T, L), (B, R) and the mixed strategy $\sigma_T = \sigma_L = 1/6$.

G1	L	R
T	$\mathbf{100}, \mathbf{100}$	$\mathbf{120}, 50$
B	$50, \mathbf{120}$	$110, 110$

G2	L	R
T	$\mathbf{6}, 4$	$3, \mathbf{5}$
B	$1, 2$	$\mathbf{4}, \mathbf{6}$

G3	L	R
T	$\mathbf{10}, \mathbf{10}$	$0, 5$
B	$5, 0$	$\mathbf{1}, \mathbf{1}$

b. The $\min\max$ strategies are (T, L) in G1, (B, R) for player 1 and (T, R) for player 2 in G2, and (B, R) in G3. By the folk theorem, every feasible payoff pair strictly above the associated payoffs can be sustained as a SPE.

c. The highest symmetric payoff in G1 is $(110, 110)$, associated with (B, R), whereas the highest symmetric payoff in G3 is $(10, 10)$ associated with the Nash Equilibrium (T, B). The latter can be achieved with the pure strategy pair: play T at every history h^t, play L at every history h^t. The former can be achieved with the pure strategy pair: start by playing B and play B as long as (B, R) was always played in the past, else play T, start by playing R and play R as long as (B, R) was always played in the past, else play L. The one-shot deviation property is satisfied for $110 \geq 120(1-\delta) + 100\delta$ or $\delta \geq 1/2$. The highest symmetric payoff in game G2 is $(5, 5)$ achieved by mixing between (T, L) and (B, R) with equal probability. It cannot be achieved with any $\delta < 1$, because of player 2's deviation.

2.a. Each firm i's payoff is $u_i(h_1, ..., h_I) = \alpha \sum_{i=1}^{I} h_i + \beta \Pi_{i=1}^{I} h_i - (h_i)^2 w_i$.

If $\beta = 0$, then the dominant strategy of firm i is $h_i = \frac{1}{2}\frac{\alpha}{w_i}$. The reason is that, when $\beta = 0$, firm i's payoff is $\alpha \sum_{j \neq i} h_j + \alpha h_i - (h_i)^2 w_i$. Taking the FOC and solving, we find that the optimal strategy is $h_i = \frac{1}{2}\frac{\alpha}{w_i}$ regardless of h_{-i}.

If $\beta \neq 0$, then taking the FOC we find

$$\frac{\partial u_i(h_1, ..., h_I)}{\partial h_i} = \alpha - 2w_i h_i + \beta \Pi_{j \neq i} h_j = 0,$$

so that firm i's best response is $BR_i(h_{-i}) = \frac{\alpha + \beta \Pi_{j \neq i} h_j}{2w_i}$ which depends on h_{-i}, so that there cannot be a strategy h_i that maximizes $u_i(h_1, ..., h_I)$ for all h_{-i}.

b. The definition of strategic substitute (complements) is that each player i's best response $BR_i(h_{-i})$ decreases (increases) in the opponents' strategies h_{-i}. So, the firms strategic efforts are strategic complements.

c. The firms' symmetric equilibrium level h^E solves

$$h^E = \frac{\alpha + \beta \Pi_{j \neq i} h^E}{2w},$$

or,

$$\alpha + \beta \left(h^E\right)^{I-1} = 2h^E w.$$

The firms' symmetric, Pareto efficient level of effort is the unique h^P that maximizes:

$$\sum_{i=1}^{I} \left[\alpha \sum_{j=1}^{I} h^P + \beta \Pi_{j=1}^{I} h^P - \left(h^P\right)^2 w \right] = I \left(\alpha I \left(h^P\right) + \beta \left(h^P\right)^I - \left(h^P\right)^2 w \right).$$

Taking the FOC, we obtain:

$$I \left(\alpha + \beta \left(h^P\right)^{I-1} \right) = 2 \left(h^P\right) w.$$

Comparing the two expressions, I conclude that $h^P > h^E$.

3.a. There are 2 players, $i = 1$ and 2. The state of the world is denoted by $(x, y_1, y_2) \in \mathbb{R}^+$. The common prior is denoted by π. A type of player $i = 1, 2$ is denoted by $\{y_i\} \times \{y_j \in \mathbb{R}^+\} \times \{x \in \mathbb{R}^+\}$, for $j \neq i$. I will denote types as y_i for brevity, later. The strategy space of each type of each player i is $S_i = \{T, NT\}$. The payoffs are $u_1(s, x, y_1, y_2) = p$ and $u_2(s, x, y_1, y_2) = x - p$ if $s = (T, T)$, and $u_1(s, x, y_1, y_2) = x$ and $u_2(s, x, y_1, y_2) = 0$, otherwise.

b. Trading requires that both players choose to trade for some pairs (y_1, y_2). Fix a strategy $s_2 : \mathbb{R}^+ \to \{T, N\}$ of player 2, that maps each type y_2 into a decision T or N. Type y_1 of player 1 chooses to trade if and only if $p > E_1 [x | s_2 = T, y_1] = \int_0^\infty \hat{x} \cdot d \Pr(x \leq \hat{x} | s_2 = T, y_1)$. So, there exists a threshold $\bar{y}_1$, function of s_2, such that player 1 chooses to trade if and only if $y_1 < \bar{y}_1$. Now consider the best response of player 2 to this strategy. Player 2 chooses to trade if and only if $p < E_2 [x | s_1 = T, y_2] = \int_0^\infty \hat{x} \cdot d \Pr(x \leq \hat{x} | s_1 = T, y_2) = \int_0^\infty \hat{x} p(x \leq \hat{x} | y_1 < \bar{y}_1, y_2) d\hat{x}$. So, there is a threshold $\bar{y}_2$, function of $\bar{y}_1$, such that player 2 chooses to trade if and only if $y_2 > \bar{y}_2$.

So, to prove the result is enough to show that there does not exist a pair $(\bar{y}_1, \bar{y}_2)$ such that the strategies s_1 and s_2, such that $s_1(y_1) = T$ iff $y_1 < \bar{y}_1$ and $s_1(y_2) = T$ iff $y_2 > \bar{y}_2$ are best responses to each other. For this to be the case, in fact, it would need to be the case that $\int_0^\infty \hat{x}p(x \leq \hat{x}|y_1 < \bar{y}_1, y_2 > \bar{y}_2)d\hat{x}$ is simultaneously strictly larger and strictly smaller than p, which is impossible.

c. The same argument can be repeated starting with a mixed strategy in the argument above. As the best responses to mixed strategies are, again, startegies characterized by thresholds, the same argument can be used to prove the result.

4.a. The joint profits are maximized by having only firm 1 produce, as it has the lowest marginal cost. The associated monopoly profit is:

$$\max_{q_1} (1 - q_1) q_1.$$

The first-order condition yields $q_1^M = 1/2$ and hence $\pi_1^M = 1/4$. Ruling out negative profits, the set of agreement profit is $\Pi = \left\{(\pi_1, \pi_2) : \pi_1 \geq 0,\ \pi_2 \geq 0,\ \pi_1 + \pi_2 \leq \pi_1^M = 1/4\right\}$.

b. The Nash bargaining Solution is identified by $\max_{\pi_1,\pi_2\in\Pi} \left(\pi_1 - \pi_1^C\right)\left(\pi_2 - \pi_2^C\right)$, where $\left(\pi_1^C, \pi_2^C\right)$ are the Cournot oligopoly profits. To calculate them, the best response functions are: $b_1(q_2) = (1 - q_2)/2$ and $b_2(q_1) = (1 - q_1 - c)/2$, so that $q_1^C = (c+1)/3$, $q_2^C = (1-2c)/3$, and $\pi_1^C = \pi_1\left(q_1^C, q_2^C\right) = (1 - (c+1)/3 - (1-2c)/3)(c+1)/3 = (c+1)^2/9$ and $\pi_2^C = \pi_2\left(q_1^C, q_2^C\right) = (1 - (c+1)/3 - (1-2c)/3)(1-2c)/3 - c(1-2c)/3 = (2c-1)^2/9$.

Substituting the expressions in the Nash solution program:

$$\max_{\pi_1+\pi_2\leq 1/4} \left(\pi_1 - (c+1)^2/9\right)\left(\pi_2 - (2c-1)^2/9\right),$$

the FOC

$$D_{\pi_1}\left(\left(\pi_1 - (c+1)^2/9\right)\left(1/4 - \pi_1 - (2c-1)^2/9\right)\right) = 0,$$

yields the solution $\pi_1^N = \frac{1}{8} + \frac{1}{6}c(2-c)$ and $\pi_2^N = \frac{1}{8} - \frac{1}{6}c(2-c)$.

c. If reducing the cost to zero, the high-cost firm obtains half of the joint collusive profit. So, it is willing to pay $1/8 - \pi_2^N(c) = \frac{1}{6}c(2-c)$ to reduce its cost to zero.

d. If the two firms have identical marginal costs (c), the joint collusive outcome is obtained with any sum of quantities $q = q_1 + q_2$ that maximizes joint profit:

$$(1-q)q_1 + (1-q)q_2 - cq_1 - cq_2 = (1-q)q - cq$$

Hence, the first order condition $1-2q-c=0$, yields joint production $q=(1-c)/2$ and joint profit $\Pi=(1-(1-c)/2)(1-c)/2-c(1-c)/2=(c-1)^2/4$. The Rubinstein bargaining solution is $\left(\Pi\frac{1-\delta_2}{1-\delta_1\delta_2},\Pi\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)=\left(\frac{(c-1)^2}{4}\frac{1-\delta_2}{1-\delta_1\delta_2},\frac{(c-1)^2}{4}\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$

e. The difference between the Rubinstein's payoff of firm 1 and 2 is: $\frac{(c-1)^2}{4}\left(\frac{1-\delta_2}{1-\delta_1\delta_2}-\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)=\frac{(c-1)^2}{4}\frac{1+\delta_1\delta_2-2\delta_2}{1-\delta_1\delta_2}$. Firm 2 would be willing to pay firm 1 this amount to go first.

5.a. The pure strategy space of Xavier is $\{TT,TD,DT,DD\}$, for offer triple and offer double, depending on its type C or D. Yolande's pure strategy space is $\{UT,UB,MT,MB,DT,DB\}$ for accepting up, middle or down, and top or bottom, depending on whether she is presented with the triple or double choice.

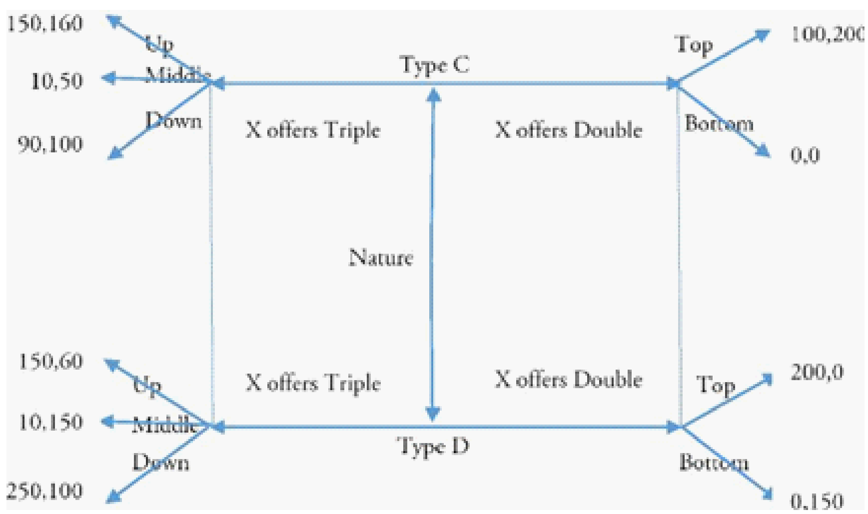

Figure 2: Game of Question 5.

b. Let Yolande's belief that Xavier's type is C be γ and τ if offered double or triple. Yolande plays T if $200\gamma\geq150(1-\gamma)$, or $\gamma\geq3/7$, else she plays B. Yolande plays U if $160\tau+60(1-\tau)\geq50\tau+150(1-\tau)$, or $\tau\geq9/20$ and $160\tau+60(1-\tau)\geq100$, or $\tau\geq2/5$, i.e., if $\tau\geq9/20$. Yolande plays M if $\tau\leq9/20$ and $50\tau+150(1-\tau)\geq100$, or $\tau\leq1/2$, i.e., $\tau\leq9/20$. Yolande never plays D.

If Xavier plays TT, then Yolande best responds either UT or UB, so (TT,UB) is a PBE, but (TT,UT) is not, as Xavier would deviate to TD. If Xavier plays DD, then Yolande best responds either UT or MT. So (DD,MT) is a PBE, but (DD,UT) is not as Xavier would deviate to TD. If Xavier plays TD, then Yolande best responds UB, and this cannot be a PBE, as Xavier would deviate to TT. If Xavier plays DT, then Yolande best responds MT, and this cannot be a PBE, as Xavier would deviate to DD.