EC9D31 Advanced Microeconomics
Final Exam 2022-23 - Section A
Questions and Answers

Question 1. Suppose preferences take the form:

$$u(x_1, x_2) = \alpha x_1 + (1-\alpha)\, x_2, \text{ with } \alpha \in (0,1)\,.$$

(a) Derive the Marshallian demands $x_i(p, m)$, $i = 1, 2$. Are the goods Marshallian complements or substitutes? **(4 marks)**

(b) Derive the indirect utility function $v(p, m)$. **(4 marks)**

(c) Derive the Hicksian demands $h_i(p, U)$, $i = 1, 2$. **(4 marks)**

(d) Derive the expenditure function $e(p, U)$. **(4 marks)**

(e) Suppose that a third good x_3 becomes available, such that preferences take now the form

$$u(x_1, x_2, x_3) = \alpha x_1 + (1-\alpha) \min\{x_2, x_3\}.$$

Derive the Marshallian demands, $x_i(p, m)$, $i = 1, 2, 3$, and the indirect utility function $v(p, m)$. **(9 marks)**

Answers to Q1 We proceed in sequence as follows.

(a) Because goods are perfect substitutes, the consumer either only buys good 1 (if the price of good 2 is high relative to its value), or only good 2. The optimal choice is $x_2 = m/p_2$ and $x_1 = 0$ if $p_2/p_1 < (1-\alpha)/\alpha$, $x_1 = m/p_1$ and $x_2 = 0$ if $p_2/p_1 > (1-\alpha)/\alpha$, and any allocation $(x_1, x_2) \geq 0$ such that $p_1x_1 + p_2x_2 = m$, when $p_2/p_1 = (1-\alpha)/\alpha$.

(b) Substituting the Marshallian demands into the utility formula:

$$v(p, m) = \max\{\alpha m/p_1, (1-\alpha)\, m/p_2\}.$$

(c) The optimal choice is $x_2 = u/(1-\alpha)$ and $x_1 = 0$ if $p_2/p_1 < (1-\alpha)/\alpha$, $x_1 = u/\alpha$ and $x_2 = 0$ if $p_2/p_1 > (1-\alpha)/\alpha$, and any allocation $(x_1, x_2) \geq 0$ such that $\alpha x_1 + (1-\alpha)\, x_2 = u$, when $p_2/p_1 = (1-\alpha)/\alpha$.

(d) Substituting the Hicksian demands into the utility formula: $e(p,u) = u\min\{p_1/\alpha, p_2/(1-\alpha)\}$.

(e) Now, the consumer either only buys good 1, or she buys goods 2 and 3 in equal amounts. Hence, the optimal choice is $x_1 = 0$ and $x_2 = x_3$ such that $p_2x_2 + p_3x_3 = m$, i.e., $x_2 = x_3 = m/(p_2+p_3)$, if $(p_2+p_3)/p_1 < (1-\alpha)/\alpha$, and $x_1 = m/p_1$ and $x_2 = x_3 = 0$ if $(p_2+p_3)/p_1 > (1-\alpha)/\alpha$, and finally any allocation $(x_1,x_2,x_3) \geq 0$ such that $x_2 = x_3$ and $p_1x_1 + p_2x_2 + p_3x_3 = m$, when $(p_2+p_3)/p_1 = (1-\alpha)/\alpha$. The indirect utility function is $v(p,m) = \max\{\alpha m/p_1, (1-\alpha)m/(p_2+p_3)\}$.

Question 2. There are two consumers A and B with the following utility functions and endowments, with $\omega_1 > 0$, $\omega_2 > 0$, $0 < \alpha < 1$ and $0 < \beta < 2\omega_1/\sqrt{\alpha\omega_2}$:

$$u_A = x_{1A}^{\alpha}x_{2A}^{1-\alpha}, \quad \boldsymbol{\omega}_A = (0,\omega_2)$$
$$u_B = x_{1B} + \beta\sqrt{x_{2B}}, \quad \boldsymbol{\omega}_B = (\omega_1, 0).$$

(a) Derive the Marshallian demands $x_i(p,m)$, $i = A, B$. **(7 marks)**

(b) Calculate the market clearing prices and the equilibrium allocations. **(5 marks)**

(c) Explain how the Walrasian equilibrium price of good 1 changes with α, β, ω_1 and ω_2. **(5 marks)**

(d) Explain how consumer A and B's demand for goods 1 and 2 changes with α, β, ω_1 and ω_2. **(8 marks)**

Answers to Question 2. We proceed in sequence as follows.

(a) Let p be the price of good 1 and normalize $p_2 = 1$. Given price p, consumer A chooses $\mathbf{x}_A$ so that

$$\max\{\alpha\ln x_{1A} + (1-\alpha)\ln x_{2A}\} \qquad s.t. \qquad px_{1A} + x_{2A} = \omega_2.$$

Hence,

$$\max\{\alpha\ln x_{1A} + (1-\alpha)\ln(\omega_2 - px_{1A})\},$$

first-order conditions are:

$$\frac{\alpha}{x_{1A}} = p\frac{1-\alpha}{\omega_2 - px_{1A}},$$

solving out, $x_{1A} = \alpha\omega_2/p$, substituting back, we obtain: $x_{2A} = \omega_2(1-\alpha)$.

Given price p, consumer B chooses $\mathbf{x}_B$ so that

$$\max\{x_{1B} + \beta\sqrt{x_{2B}}\} \qquad s.t. \qquad px_{1B} + x_{2B} = p\omega_1.$$

Hence,

$$\max\{\omega_1 - x_{2B}/p + \beta\sqrt{x_{2B}}\},$$

first-order conditions are:

$$-1/p + \beta/(2\sqrt{x_{2B}}) = 0,$$

solving out, $x_{2B} = p^2\beta^2/4$, substituting back, we obtain: $x_{1B} = \omega_1 - p\beta^2/4$.

(b) Market clearing condition, therefore, is:

$$x_{1A} + x_{1B} = \alpha\omega_2/p + \omega_1 - p\beta^2/4 = \omega_1,$$

which is satisfied only for:

$$p = 2\sqrt{\alpha\omega_2}/\beta,$$

excluding of course the negative solution. Hence, the equilibrium price is $p = 2\sqrt{\alpha\omega_2}/\beta$.

So, the equilibrium allocations are:

$$x_{1A} = \frac{\sqrt{\alpha\omega_2}}{2}\beta, \quad x_{2A} = \omega_2(1-\alpha),$$
$$x_{1B} = \omega_1 - \sqrt{\alpha\omega_2}\beta/2, \; x_{2B} = \alpha\omega_2.$$

(c) The price p of good 1 is:

$$p = \frac{2\sqrt{\alpha\omega_2}}{\beta},$$

differentiating with respect to $\alpha, \omega_1, \omega_2$, and β, I obtain:

$$\frac{\partial p}{\partial \alpha} = \frac{1}{\beta}\frac{\omega_2}{\sqrt{\alpha\omega_2}} > 0, \; \frac{\partial p}{\partial \beta} = -2\frac{\sqrt{\alpha\omega_2}}{\beta^2} < 0, \; \frac{\partial p}{\partial \omega_1} = 0, \; \frac{\partial p}{\partial \omega_2} = \frac{\alpha}{\beta\sqrt{\alpha\omega_2}} > 0$$

The equilibrium price of good 1 is constant in ω_1, increases in α and ω_2, and decreases in β.

(d) Differentiating x_{1A} and x_{2A} with respect to α, β, ω_1 and ω_2, I obtain:

$$\begin{aligned}
\frac{\partial x_{1A}}{\partial \alpha} &= \frac{1}{4}\beta\frac{\omega_2}{\sqrt{\alpha\omega_2}} > 0, \ \frac{\partial x_{1A}}{\partial \beta} = \frac{1}{2}\sqrt{\alpha\omega_2} > 0, \ \frac{\partial x_{1A}}{\partial \omega_1} = 0, \ \frac{\partial x_{1A}}{\partial \omega_2} = \frac{1}{4}\alpha\frac{\beta}{\sqrt{\alpha\omega_2}} > 0, \\
\frac{\partial x_{2A}}{\partial \alpha} &= -\omega_2 < 0, \ \frac{\partial x_{2A}}{\partial \beta} = 0, \ \frac{\partial x_{2A}}{\partial \omega_1} = 0, \ \frac{\partial x_{2A}}{\partial \omega_2} = (1-\alpha) > 0.
\end{aligned}$$

The demand x_{1A} increases in α, β and ω_2, and is constant in ω_1. The demand x_{2A} increases in ω_2, decreases in α, and is constant in β and ω_1.

Differentiating x_{1B} and x_{2B} with respect to α, β, ω_1 and ω_2, I obtain:

$$\begin{aligned}
\frac{\partial x_{1B}}{\partial \alpha} &= -\frac{1}{4}\beta\frac{\omega_2}{\sqrt{\alpha\omega_2}} < 0, \ \frac{\partial x_{1B}}{\partial \beta} = -\frac{1}{2}\sqrt{\alpha\omega_2} < 0, \ \frac{\partial x_{1B}}{\partial \omega_1} = 1, \ \frac{\partial x_{1B}}{\partial \omega_2} = -\frac{1}{4}\alpha\frac{\beta}{\sqrt{\alpha\omega_2}} < 0, \\
\frac{\partial x_{2B}}{\partial \alpha} &= \omega_2 > 0, \ \frac{\partial x_{2B}}{\partial \beta} = 0, \ \frac{\partial x_{2B}}{\partial \omega_1} = 0, \ \frac{\partial x_{2B}}{\partial \omega_2} = \alpha > 0.
\end{aligned}$$

The demand x_{1B} decreases in α, β, and ω_2, and increases in ω_1. The demand x_{2B} increases in α and ω_2, and is constant in β and ω_1.