Fixed Point Theorems

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1 Fixed Point Theorems

Definition 1. *An element $x \in X$ is a fixed point of $f : X \mapsto X$ if $f(x) = x$.*

1.1 Contraction Mapping Theorem

The Contraction Mapping Theorem (CMT hereafter) applies to complete metric spaces. The following theorem shows that the set of bounded continuous functions with the sup norm is a complete metric space.

Theorem 1. *Let $X \subset \mathbb{R}^K$, and let $C(X)$ be the set of bounded continuous functions $f : X \mapsto \mathbb{R}$ with the sup norm $\|f\| = \sup_{x \in X} |f(x)|$. Then $C(X)$ is a complete normed vector space.*

The CMT applies to a certain class of functions. The following definition makes precise the property those functions need to satisfy.

Definition 2. *Let (X, d) be a metric space and $f : X \mapsto X$. f is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$, $d(f(x), f(y)) \leq \beta d(x, y)$, for all $x, y \in X$.*

The following example illustrate the definition of contraction mapping.

Example 1. *Let $a, b \in \mathbb{R}$ with $a < b$, $X = [a, b]$ and $d(x, y) = |x - y|$. Then f is a contraction if for some $\beta \in (0, 1)$,*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \beta < 1, \textit{for all } x, y \in X \textit{ with } x \neq y$$

That is, f is a contraction mapping if it is a function with slope uniformly less than one in absolute value.

Finally, we have the CMT:

Theorem 2. *If (X, d) is a complete metric space and $f : X \mapsto X$ is a contraction mapping with modulus β, then*

a. f has exactly one fixed point $x \in X$, and

b. for any $x_0 \in X$, $d(f^n x, x) \leq \beta^n d(x_0, x), n = 0, 1, 2, ...$

Proof: To prove (a) we must find a candidate v, show that it satisfies $Tv = v$ and no other element $\hat{v} \in X$ does. Define $\{T^n\}_{t=0}^n$ by $T^0 x = x$ and $T^n x = T(T^{n-1} x)$, $n = 1, 2, ...$

- STEP 1: Let $v_0 \in X$, $\{v_n\}_{n=0}^{\infty}$ by $v_{n+1} = Tv_n$ so that $v_n = T^n v_0$. By the contraction property:

$$d(v_2, v_1) = d(Tv_1, Tv_0) \leq \beta d(v_1, v_0)$$

$$d(v_{n+1}, v_n) \leq \beta^n d(v_1, v_0), \quad n = 1, 2, ...$$

Let $m > n$,

$$\begin{aligned} d(v_m, v_n) &\leq d(v_m, v_{m-1}) + ... + d(v_{n+2}, v_{n+1}) + d(v_{n+1}, v_n) \\ &\leq \left[\beta^{m-1} + ... + \beta^{n+1} + \beta^n\right] d(v_1, v_0) \\ &= \beta^n \left[\beta^{m-n-1} + ... + \beta + 1\right] d(v_1, v_0) \\ &\leq \frac{\beta^n}{1-\beta} d(v_1, v_0), \end{aligned}$$

Thus $\{v_n\}_{n=0}^{\infty}$ is Cauchy. Since X is complete, $v_n \to v \in X$.

- STEP 2: To show that $Tv = v$, note that $\forall n$ and $\forall v_0 \in X$,

$$\begin{aligned} d(Tv, v) &\leq d(Tv, T^n v_0) + d(T^n v_0, v) \\ &\leq \beta \underbrace{d(v, T^{n-1} v_0)}_{\to 0} + \underbrace{d(T^n v_0, v)}_{\to 0} \to 0 \end{aligned}$$

- STEP 3: Suppose $\exists \hat{v} \neq v$ such that $T\hat{v} = \hat{v}$. Then,

$$0 < d(\hat{v}, v) = d(T\hat{v}, Tv) \leq \beta d(\hat{v}, v) < d(\hat{v}, v).$$

To prove (b), note that for any $n \geq 1$:

$$d(T^n v_0, v) = d(T\left(T^{n-1} v_0, Tv\right)) \leq \beta d(T^{n-1} v_0, v)$$

Q.E.D.

EXERCISE 1. *Consider the differential equation and boundary condition $\frac{dx(s)}{ds} = f[x(s)]$, all $s \geq 0$, with $x(0) = c \in \mathbb{R}$. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous, and for some $B > 0$ satisfies the Lipschitz condition $|f(a) - f(b)| \leq B|a - b|$, all $a, b \in \mathbb{R}$. For any $t > 0$, consider $C[0, t]$, the space of bounded continuous functions on $[0, t]$, with the sup norm. Recall from Theorem 1 that this space is complete.*

a. *Show that the operator T defined by*

$$(Tv)(s) = c + \int_0^s f[v(s)]dz, 0 \leq s \leq t$$

maps $C[0, t]$ into itself.

b. *Show that for some $\tau > 0$, T is a contraction on $C[0, \tau]$.*

c. *Show that the unique fixed point of T on $C[0, \tau]$ is a differentiable function, and hence that it is the unique solution on $[0, \tau]$ to the given differential equation.*

The following Theorem due to Blackwell gives a useful route to verify that an operator is a contraction.

THEOREM 3 (Blackwell's sufficient conditions for a contraction). *Let $X \subset \mathbb{R}^K$, and let $B(X)$ be a space of bounded functions $f : X \mapsto \mathbb{R}$ with the sup norm. Let $T : B(X) \mapsto B(X)$ satisfy*

a. *(monotonicity) $f, g \in B(X)$ and $f(x) \leq g(x)$, for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$;*

b. *(discounting) there exists some $\beta \in (0, 1)$ such that*

$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ for all } f \in B(X), a \geq 0, x \in X$$

where $(f + a)(x)$ is the function defined by $(f + a)(x) = f(x) + a$.

Then T is a contraction with modulus β

Proof: For any $f, g \in B(X)$,

$$f(x) - g(x) \leq \|f - g\|$$

Then properties (1) (a) and (b) imply that:

$$Tf(x) \leq T(g + \|f - g\|)(x) \leq Tg(x) + \beta\|f - g\|$$

Reversing the roles of f and g we obtain

$$Tg(x) \leq Tf(x) + \beta\|f - g\|.$$

Combining the two inequalities we get that $\|Tf - Tg\| \leq \beta\|f = g\|$, as desired. *Q.E.D.*

In many economic applications the two hypothesis of Blackwell's Theorem can be verified at a glance.

EXAMPLE 2. *In the one sector optimal growth problem, an operator T is defined by*

$$(Tv)(x) = \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta v(y)\}$$

If $v(y) \leq w(y)$ for all y, then $Tw \geq TV$ and so monotonicity holds. To show discounting note that:

$$\begin{aligned} T(v + a)(k) &= \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta[v(y) + a]\} \\ &= \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta v(y)\} + \beta a \\ &= (Tv)(x) + \beta a \end{aligned}$$

1.2 Brouwer's Fixed Point Theorem

Consider a function f that maps each point x of a set $X \subset \mathbb{R}^K$ to a point $f(x) \in X$. We say that f maps the set X into itself. We would like to find conditions ensuring that any continuous function mapping X into itself has a fixed point. The following example shows that some restrictions must be placed on X: $f(x) = x + 1$ maps $\mathbb{R}$ into itself but has no fixed point since $f(x) = x$ implies $1 + x = x$, an absurd.

The following result, due to L.E.J. Brouwer yields sufficient conditions for the existence of a fixed point:

Theorem 4 (Brouwer's fixed point theorem). *Let X be a nonempty compact convex set in $\mathbb{R}^K$, and f be a continuous function mapping X into itself. Then f has a fixed point x^*.*

For $X = \mathbb{R}$, a nonempty compact convex set is a closed interval $[a, b]$ or a single point. So Brouwer's Theorem asserts that a continuous function $f : [a, b] \mapsto [a, b]$ must have a fixed point. But this follows from the Intermediate Value Theorem. Indeed, define $g(x) = f(x) - x$ and note that x is a fixed point of f if and only if $g(x) = 0$. Since $g(a) \geq 0$ and $g(b) \leq 0$, then there is some $x^* \in [a, b]$ such that $g(x^*) = 0$.

We use Brouwer's fixed point Theorem, for example, to prove existence of equilibrium in a pure exchange economy.

1.3 Kakutani's Fixed Point Theorem

Brouwer's Theorem deals with fixed points of continuous functions. Kakutani's theorem generalises the theorem to correspondences.

Definition 3. *An element $x \in X$ is a fixed point of a correspondence $F : X \mapsto X$ if $x \in F(x)$*

Theorem 5 (Kakutani's Fixed Point Theorem). *Let X be a nonempty compact convex set in $\mathbb{R}^K$ and $F : X \mapsto X$ be a correspondence. Suppose that:*

a. $F(x)$ is a nonempty convex set in X for each $x \in X$

b. F is upper hemicontinuous.

Then F has a fixed point x^* in X

We use Kakutani's Fixed Point Theorem, for example, to prove existence of a Mixed Strategy Nash Equilibrium in an N-player game with finite (pure) strategy sets.