Theorem of the Maximum and Envelope Theorem

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1 The Theorem of the Maximum

Economic theory has many "comparative statics" results. These describe what happens to an optimal solution in response to changes in exogenous parameters such as prices. In particular, will small changes in these parameters lead to only small changes in the objective function? And to small changes in the optimal solution? The purpose of this section is to establish some of those results.

Example 1 (Utility Maximisation).

Example 2 (Profit Maximisation).

Let $X \subset \mathbb{R}^L$ and $Y \subset R^K$. The set X is the set of exogenous parameters of the problem live and the set Y is the set of choice variables. Suppose $f : X \times Y \mapsto \mathbb{R}$ is a function and $\Gamma : X \mapsto Y$ is a non-empty correspondence. We are interested in the following problem:

$$\sup_y f(x, y)$$
$$\text{s.t. } y \in \Gamma(x)$$

where the correspondence $\Gamma : X \mapsto Y$ describe the feasibility constraints.

If $\Gamma(x)$ is nonempty and compact valued, then Weirstrass theorem implies $v : X \mapsto \mathbb{R}$

$$v(x) \equiv \sup_{y\in\Gamma(x)} f(x, y) \tag{1}$$

is well defined. Moreover, $G : X \mapsto Y$ defined by

$$G(x) = \{y \in \Gamma(x) : f(x, y) = v(x)\} \tag{2}$$

is the set of values of y that solve the problem for each x.

In this section, we seek restrictions on the correspondence Γ and the objective function f which ensures that $v : X \mapsto \mathbb{R}$ is a continuous function of x and that the correspondence G varies continuously with x.

Definition 1. *A correspondence $\Gamma : X \mapsto Y$ is lower hemi-continuous (l.h.c.) at x if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \to x$ and for every $y \in \Gamma(x)$, there exists $N \geq 1$ and a sequence $\{y_n\}_{n=N}^{\infty}$ such that $y_n \to y$ and $y_n \in \Gamma(x_n)$, all $n \geq N$.*

Definition 2. *A correspondence $\Gamma : X \mapsto Y$ is upper hemi-continuous (u.h.c.) at x if $\Gamma(x)$ is nonempty and if, for every sequence $x_n \to x$ and every sequence $\{y_n\}_{n=1}^{\infty}$ such that $y_n \in \Gamma(x_n)$, all n, there exists a convergent subsequence of $\{y_n\}_{n=N}^{\infty}$ whose limit point y is in $\Gamma(x)$.*

Definition 3. *A correspondence $\Gamma : X \mapsto Y$ is continuous at $x \in X$ if it is both u.h.c. and l.h.c. at x.*

A correspondence $\Gamma : X \to Y$ is called l.h.c, u.h.c., or continuous if it has that property at every point $x \in X$.

In Figure 1, the correspondence is l.h.c but not u.h.c at x_1 and u.h.c but not l.h.c at x_2.

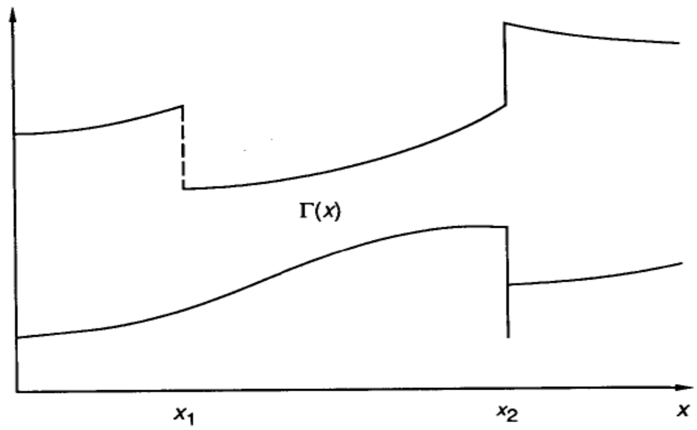

Figure 1: Lower- and hemi - continuity

EXERCISE 1. *Show that:*

a. *if Γ is single valued and u.h.c., then it is continuous.*

b. *if Γ is single valued and l.h.c., then it is continuous.*

The next exercise shows some of the relationship between constraints stated in terms of inequalities involving continuous functions and those stated in terms of continuous correspondences. These relationships are extremely important for many problems in economics where constraints are often stated in terms of productions functions, budget constraints, and so on.

EXERCISE 2.

a. *Let $\Gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be defined by $\Gamma(x) = [0, x]$. Show that Γ is continuous.*

b. *Let $f_i : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$, be a continuous functions and define the correspondence $\Gamma : \mathbb{R}^K_+ \mapsto \mathbb{R}_+$ by $\Gamma(x) = [0, f(x)]$. Show that Γ is continuous.*

We are now ready to answer under what conditions do the function $v(x)$ defined in (1) and the associated set of maximising values $G(x)$ defined in (2) varies continuously with x.

THEOREM 1 (Theorem of the Maximum). *Let $X \subset \mathbb{R}^L$ and $Y \subset R^K$, let $f : X \times Y \mapsto \mathbb{R}$ be a continuous function and $\Gamma : X \mapsto Y$ be a compact-valued and continuous correspondence. Then the function $v : X \mapsto \mathbb{R}$ defined in (1) is continuous, and the correspondence $G : X \mapsto Y$ defined in (2) is nonempty, compact valued, and u.h.c.*

Proof: *Q.E.D.*

EXAMPLE 3. *Let $X = \mathbb{R}$ and $\Gamma(x) = Y = [-1, 1]$, all $x \in X$. Define $f : X \times Y \mapsto \mathbb{R}$ by $f(x, y) = xy^2$. Then,*

$$G(x) = \begin{cases} \{-1, 1\} & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ \{0\} & \text{if } x < 0 \end{cases}$$

We show $G(x)$ is u.h.c. at $x = 0$. First note that $\Gamma(0)$ is nonempty and compact valued. Let $x_n \to 0$ be arbitrary. Let $y_n \in \Gamma(x_n)$. Suppose there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} < 0$ for all k. Then $y_{n_k} = 0$ for all k and so there exists a subsequence of $\{y_n\}$ that converges to $0 \in \Gamma(0)$. Suppose there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} > 0$ for all k. It follows that there exists a convergent subsequence of $\{y_{n_k}\}_{k=1}^{\infty}$ that converges to either $1 \in \Gamma(0)$ or $-1 \in \Gamma(0)$. We conclude $G(x)$ is u.h.c. at $x = 0$.

To see $G(x)$ is not l.h.c choose $y = 0.5 \in \Gamma(0)$. Let $x_n \to 0$ be a sequence such that $x_n < 0$ for all $n \in \mathbb{N}$. Hence, $y_n = 0$ for all $n \in \mathbb{N}$. Hence it cannot be the case that $y_n \to y = 0.5$.

2 Envelope Theorem

Suppose now that both sets $Y \subseteq \mathbb{R}^K$ and $X \subseteq \mathbb{R}^L$ are open. Suppose that $f : X \times Y \to \mathbb{R}$ and $g : X \times Y \to \mathbb{R}^J$, and consider the following (simplified) parametric problem: given $x \in X$, let

$$v(x) = \max_{y \in Y} f(x, y) : g(x, y) = 0.$$

Suppose that the differentiability and second-order conditions are given, so that a point y^* solves this maximisation problem if and only if there is a $\lambda^* \in \mathbb{R}^J$ such that $D\mathcal{L}(x, y^*, \lambda^*) = 0$.

Suppose furthermore that we can define functions $h : X \to Y$ and $\lambda : X \to \mathbb{R}^J$, given by the solution of the problem and the associated multiplier, for every x. Then, it follows directly from the Implicit Function Theorem that if, for a given $\bar{x} \in X$,

$$\operatorname{rank} \begin{pmatrix} 0_{J\times J} & D_x g(\bar{x}, y^*) \\ D_x g(\bar{x}, y^*)^\top & D^2_{x,x}\mathcal{L}(\bar{x}, y^*, \lambda^*) \end{pmatrix} = J + K,$$

then there exists some $\epsilon > 0$ such that on $B_\epsilon(\bar{x})$ the functions x and λ are differentiable and

$$\begin{pmatrix} D\lambda(\bar{x}) \\ Dx(\bar{x}) \end{pmatrix} = - \begin{pmatrix} 0_{J\times J} & D_x g(\bar{x}, y^*) \\ D_x g(\bar{x}, y^*)^\top & D^2_{x,x}\mathcal{L}(\bar{x}, y^*, \lambda^*) \end{pmatrix}^{-1} \begin{pmatrix} D_x g(\bar{x}, h(\bar{x})) \\ D^2_{x,x}\mathcal{L}(\bar{x}, h(\bar{x}), \lambda(x)) \end{pmatrix}.$$

It is then immediate that v is differentiable at $\bar{x}$ and

$$Dv(\bar{x}) = D_x f(\bar{x}, h(\bar{x})) Dx(\bar{x}).$$

A simpler method, however, is given by the following theorem

THEOREM 2 (The Envelope Theorem). *If v is continuously differentiable at $\bar{x}$, then $Dv(\bar{x}) = D_x\mathcal{L}(\bar{x}, h(\bar{x}), \lambda(\bar{x}))$.*

Proof: One just needs to use the Chain Rule: by assumption,

$$D_x f(x, h(x)) + D_x g(x, h(x))^\top \lambda(x) = 0,$$

whereas $g(x, h(x)) = 0$, so

$$D_x g(x, h(x)) Dh(x) + D_x g(x, h(x)) = 0;$$

meanwhile,

$$\begin{aligned} Dv(x) &= Dh(x)^\top D_x f(x, h(x)) + D_x f(x, h(x)) \\ &= -D_h(x)^\top D_x g(x, h(x))^\top \lambda(x) + D_x f(x, h(x)) \end{aligned}$$

and

$$\begin{aligned} D_x\mathcal{L}(x, h(x), \lambda(x)) &= D_x f(x, h(x)) + D_x g(x, h(x))^\top \lambda(x) \\ &= D_x f(x, h(x)) - D_h(x)^\top D_x g(x, h(x))^\top \lambda(x), \end{aligned}$$

which gives the result. *Q.E.D.*

EXERCISE 3. *Let $f : \mathbb{R}^K \to \mathbb{R}$, $g : \mathbb{R}^K \to \mathbb{R}^J \in \mathbf{C}^2$, with $J \leq K \in \mathbb{N}$. Suppose that for all $x \in \mathbb{R}^m$, the problem*

$$\max f(y) : g(y) = x$$

has a solution, which is characterised by the first order conditions of the Lagrangean defined by $\mathcal{L}(x, y, \lambda) = f(y) + \lambda \cdot (x - g(y))$. Suppose furthermore that these conditions define differentiable functions $h : \mathbb{R}^L \to \mathbb{R}^K$ and $\lambda : \mathbb{R}^L \to \mathbb{R}^L$. Prove that $Dv(x) = \lambda(x)$, for all x, where $v : \mathbb{R}^L \to \mathbb{R}$ is the value function of the problem.