EC9A0: Pre-sessional Advanced Mathematics Course

Fixed Point Theorems

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Lecture Outline

Definition of Contraction

Definition

Let (X, d) be a metric space and $f : X \mapsto X$. f is a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$, $d(f(x), f(y)) \leq \beta d(x, y)$, $\forall x, y \in X$.

Example

Let $a, b \in \mathbb{R}$ with $a < b$, $X = [a, b]$ and $d(x, y) = |x - y|$. Then f is a contraction if for some $\beta \in (0, 1)$,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \beta < 1, \text{ for all } x, y \in X \text{ with } x \neq y$$

That is, f is a contraction mapping if it is a function with slope uniformly less than one in absolute value.

Blackwell's sufficient conditions

Theorem : Blackwell's sufficient conditions for a contraction

Let $X \subset \mathbb{R}^K$, *and let* $B(X)$ *be a space of bounded functions* $f : X \mapsto \mathbb{R}$ *with the sup norm. Let* $T : B(X) \mapsto B(X)$ *satisfy*

1. *(monotonicity)* $f, g \in B(X)$ *and* $f(x) \leq g(x)$, *for all* $x \in X$, *implies* $(Tf)(x) \leq (Tg)(x)$, *for all* $x \in X$;
2. *(discounting) there exists some* $\beta \in (0, 1)$ *such that*
$$[T(f + a)](x) \leq (Tf)(x) + \beta a, \text{ for all } f \in B(X), a \geq 0, x \in X$$
where $(f + a)(x)$ *is the function defined by* $(f + a)(x) = f(x) + a$.

Then T *is a contraction with modulus* β

Contraction Mapping Theorem

Theorem

If (X, d) is a complete metric space and $T : X \mapsto X$ is a contraction mapping with modulus β, then

1. *T has exactly one fixed point $x \in X$, and*
2. *for any $x_0 \in X$, $d(T^n x_0, x) \leq \beta^n d(x_0, x), n = 0, 1, 2, \ldots$*

Application I: Neoclassical Growth Model

Example

In the one sector optimal growth problem, an operator T is defined by

$$(Tv)(x) = \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta v(y)\}$$

- If $v(y) \leq w(y)$ for all y, then $Tw \geq TV$ and so monotonicity holds.
- To show discounting note that:

$$\begin{aligned} T(v+a)(k) &= \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta[v(y) + a]\} \\ &= \max_{0 \leq y \leq f(x)} \{U[f(x) - y] + \beta v(y)\} + \beta a \\ &= (Tv)(x) + \beta a \end{aligned}$$

Application II: Differential Equations

Example

Consider the differential equation and boundary condition $\frac{dx(s)}{ds} = f[x(s)]$, all $s \geq 0$, with $x(0) = c \in \mathbb{R}$. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is continuous, and for some $B > 0$ satisfies the Lipschitz condition $|f(a) - f(b)| \leq B|a - b|$, all $a, b \in \mathbb{R}$. For any $t > 0$, consider $C[0, t]$, the space of bounded continuous functions on $[0, t]$, with the sup norm.

1. Show that the operator T defined by

$$(Tv)(s) = c + \int_0^s f[v(s)]dz, 0 \leq s \leq t$$

   maps $C[0, t]$ into itself.

2. Show that for some $\tau > 0$, T is a contraction on $C[0, \tau]$.
3. Show that the unique fixed point of T on $C[0, \tau]$ is a differentiable function, and hence that it is the unique solution on $[0, \tau]$ to the given differential equation.

Definitions

- f maps the set $X \subset \mathbb{R}^K$ into itself if $f(x) \in X$ for all $x \in X$.
- We would like to find conditions ensuring that any continuous function mapping X into itself has a fixed point.
- The following example shows that some restrictions must be placed on X:
  - $f(x) = x + 1$ maps $\mathbb{R}$ into itself.
  - $f(x)$ has no fixed point since $f(x) = x$ implies $1 + x = x$, an absurd.

Brouwer's Fixed Point Theorem

Theorem L.E.J. Brouwer's fixed point theorem

Let X be a nonempty compact convex set in $\mathbb{R}^K$, and f be a continuous function mapping X into itself. Then f has a fixed point x^*.

- For $X = \mathbb{R}$, a nonempty compact convex set is a closed interval $[a, b]$.
- A continuous function $f : [a, b] \mapsto [a, b]$ must have a fixed point.
- This follows from the IVT:
  - Define $g(x) = f(x) - x$.
  - x is a fixed point of f if and only if $g(x) = 0$.
  - Since $g(a) \geq 0$ and $g(b) \leq 0$, there is $x^* \in [a, b]$ such that $g(x^*) = 0$.
- We use Brouwer's fixed point Theorem to prove existence of equilibrium in a pure exchange economy.

Kakutani's Fixed Point Theorem

- Brouwer's Theorem deals with fixed points of continuous functions.
- Kakutani's theorem generalises the theorem to correspondences.

Definition

An element $x \in X$ is a fixed point of a correspondence $F : X \mapsto X$ if $x \in F(x)$.

Theorem Kakutani's Fixed Point Theorem

Let X be a nonempty compact convex set in $\mathbb{R}^K$ and $F : X \mapsto X$ be a correspondence. Suppose that:

1. *$F(x)$ is a nonempty convex set in X for each $x \in X$*
2. *F is upper hemicontinuous.*

Then F has a fixed point x^* in X

- We use Kakutani's Fixed Point Theorem to prove existence of a Mixed Strategy Nash Equilibrium in an N-player game with finite (pure) strategy sets.