EC9A0: Pre-sessional Advanced Mathematics Course

Real Analysis

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Slides Outline

Sets

- A set is a collection of (finitely or infinitely many) objects.
  - For any set A, we use the notation $x \in A$ to indicate that "x is an element of A" ("or belongs to A" or "is a member of A").
  - Two sets A and B are equal ($A = B$) if they have the same elements.
  - The *empty set*, $\emptyset$, is the only set with no elements at all.
  - $\mathbb{N} := \{1, 2, \ldots\}$ denotes the (countably infinite) set of *natural numbers*
  - $\mathbb{R}$ denotes the (uncountable) set of *real numbers*.
  - For any sets A and B, the cartesian product $A \times B$ is the set $\{(a_1, b_1), (a_2, b_2), \ldots\}$ where $a_i \in A$ and $b_i \in B$ for all i.
- For any $K \in \mathbb{N}$, the K-dimensional real (Euclidean) space is the K-fold Cartesian product of $\mathbb{R}$, denoted by $\mathbb{R}^K$.
  - $x \in \mathbb{R}^K \implies x = (x_1\ x_2 \ldots x_K)$.

The Euclidean Space

- The *origin* of $\mathbb{R}^K$ is the vector zero given by $(0, 0, ..., 0)$.
- Given any pair $x, y \in \mathbb{R}^K$ where $\#K \geq 2$,
  1. $x \gg y$ iff $x_i > y_i$ for all $i \in K$;
  2. $x > y$ iff $x \neq y$ and $x_i \geq y_i$ for all $i \in K$;
  3. $x \geqq y$ iff $x_i \geq y_i$ for all $i \in K$.
- The *non-negative orthant* of $\mathbb{R}^K$ is $\mathbb{R}^K_+ := \{x \in \mathbb{R}^K \mid x \geqq 0\}$;
- The *positive orthant* of $\mathbb{R}^K$ is $\mathbb{R}^K_{++} := \{x \in \mathbb{R}^K \mid x \gg 0\}$;
- No special notation for the set $\mathbb{R}^K_+ \setminus \{0\} = \{x \in \mathbb{R}^K \mid x > 0\}$;
- Define vector addition by $x + y = (x_1 + y_1 \; x_2 + y_2 \ldots x_K + y_K)$;
- Define scalar multiplication by $\alpha x = (\alpha x_1 \; \alpha x_2 \ldots \alpha x_K)$.

Correspondences and Functions

Definition

A *correspondence* f from a set $X \neq \emptyset$ into a set $Y \neq \emptyset$, denoted $f : X \to Y$, is a rule that assigns to each $x \in X$ a set $f(x) \subset Y$

Definition

A *function* f from a set $X \neq \emptyset$ into a set $Y \neq \emptyset$, denoted $f : X \to Y$, is a rule that assigns to each $x \in X$ a unique $f(x) \in Y$

- X is said to be the *domain* of f. Y its *target set* or *co-domain*.
- If $f : X \to Y$ and $A \subseteq X$, the *image of A under f*, denoted by $f[A]$, is the set

$$f[A] = \{y \in Y | \exists x \in A : f(x) = y\}.$$

- The image $f[X]$ of the whole domain is called the *range* of f.
- If $f : X \to Y$, and $B \subseteq Y$, the *inverse image of B under f*, denoted $f^{-1}[B]$, is the set

$$f^{-1}[B] = \{x \in X | f(x) \in B\}.$$

Properties of Functions

Definition

Function $f : X \to Y$ is said to be:

- *Onto*, or *surjective*, if $f[X] = Y$;
- *One-to-one*, or *injective*, if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$;
- *Bijective*, if it is both onto and one-to-one.

Examples

- $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $f(x_1, x_2) = x_1^2 + x_2^2$ is neither one-to-one nor onto.
- $f : \mathbb{R} \setminus \{0\} \mapsto \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is one-to-one but not onto.
- $f : \mathbb{R}^2 \mapsto \mathbb{R}$ defined by $f(x_1, x_2) = x_1 + x_2$ is onto but not one-to-one.
- $f : \mathbb{R} \mapsto \mathbb{R}$ defined by $f(x) = x$ is one-to-one and onto.

Inverse Function

Definition

If $f : X \to Y$ is a one-to-one function, the *inverse function* $f^{-1} : f[Y] \to X$ is implicitly defined by $f^{-1}(y) = f^{-1}[\{y\}]$.

Theorem

The function $f : X \to Y$ is onto iff for all non-empty $B \subseteq Y$ one has $f^{-1}[B] \neq \varnothing$.

Fields

Definition

A set $\mathbb{F}$ is said to be a field if there are two binary operations $(x, y) \mapsto x \oplus y$ from $\mathbb{F} \times \mathbb{F}$ to $\mathbb{F}$ and $(x, y) \mapsto x \otimes y$ from $\mathbb{F} \times \mathbb{F}$ to $\mathbb{F}$ called addition and multiplication, respectively, such that for all $x, y, z \in \mathbb{F}$:

1. $x \oplus y = y \oplus x$ (addition commutes);
2. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ (addition is associative);
3. There exists an element $0 \in \mathbb{F}$, such that $x \oplus 0 = x$ (additive identity);
4. For each $x \in \mathbb{F}$, there is a unique *inverse* element in $\mathbb{F}$, denoted -x such that $x \oplus (-x) = 0$;
5. $x \otimes y = y \otimes x$ (multiplication is commutative);
6. $(x \otimes y) \otimes z = z \otimes (y \otimes z)$ (multiplication is associative);
7. There exists and element $1 \in \mathbb{F}$ such that $1 \neq 0$ and $1 \otimes x = x$;
8. If $x \in \mathbb{F}$ and $x \neq 0$, there is an element $\frac{1}{x} \in \mathbb{F}$ such that $x \otimes (\frac{1}{x}) = 1$
9. $x \otimes (y \oplus z) = x \otimes y \;\oplus\; x \otimes z$ (distributive law);

Vector Spaces

Definition

A set L is said to be a vector (or linear) space over the scalar field $\mathbb{F}$ if there are two binary operations $(x, y) \mapsto x \oplus y$ from $L \times L$ to L and $(\lambda, x) \mapsto \lambda \otimes x$ from $\mathbb{F} \times L$ to L called addition and scalar multiplication, respectively, and a unique *null vector* $\theta \in L$, such that for all $x, y, z \in L$ and $\lambda, \mu \in \mathbb{F}$:

1. $x \oplus y = y \oplus x$ (addition commutes);
2. $(x \oplus y) \oplus z = x + (y \oplus z)$ (addition is associative);
3. $x \oplus \theta = x$ (additive identity);
4. for each $x \in L$, there is a unique *inverse* $-x$ such that $x \oplus (-x) = \theta$;
5. $\lambda \otimes (\mu \otimes x) = (\lambda \cdot \mu) \otimes x$ (scalar multip. is associative);
6. $0 \otimes x = \theta$;
7. $1 \otimes x = x$;
8. $(\lambda + \mu) \otimes x = \lambda \otimes x \;\oplus\; \mu \otimes x$ (first distributive law);
9. $\lambda \otimes (x \oplus y) = \lambda \otimes x \;\oplus\; \lambda \otimes y$ (second distributive law).

Vector Spaces: Examples

Examples

- $\mathbb{R}^K$ is a vector space.
- The set $\mathbb{R}^\infty$ consisting of all infinite sequences $\{x_0, x_1, x_2 \ldots\}$ is a vector space.
- The unit circle in $\mathbb{R}^2$ is not a vector space.
- The set of all nonnegative functions on [a, b] is not a vector space.

Distance Function

Definition

Given any set X, the function $d : X \times X \to \mathbb{R}$ is a *metric or distance function* on X if the following properties hold:

- *Positivity:* $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0$ iff $x = y$.
- *Symmetry:* $d(x, y) = d(y, x)$.
- *Triangle Inequality:* $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in X$.

Example

Euclidean distance: $d(x, y) = \left(\sum_{i \in K} (x_i - y_i)^2\right)^{1/2}$.

Example

Let $p \in \mathbb{R}_+$. Define $d_p : \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ by $d_p(x, y) = \sum_{i \in K} |x_i - y_i|^p$.

- d_p is a distance iff $p \geq 1$.

Metric Spaces

Definition

A metric space is a pair (X, d) where X is a set and $d : X \times X \to \mathbb{R}$ is a metric.

Examples

1. the set of integers with $d(x, y) = |x - y|$.
2. the set of integers with
$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$
3. the set of all continuous, strictly increasing functions on $[a, b]$, with
$$d(x, y) = max_{a \leq t \leq b}|x(t) - y(t)|.$$
4. $\mathbb{R}$ with $d(x, y) = f(|x - y|)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, and strictly concave, with $f(0) = 0$.

Norms

Definition

Given any vector space X, a norm on X is a function $\|\cdot\| : X \mapsto \mathbb{R}$ such that for all $x, y \in X$ and $\alpha \in \mathbb{R}$:

1. $\|x\| \geq 0$, with equality if and only if $x = \theta$;
2. $\|\alpha x\| = |\alpha| \, \|x\|$; *and*
3. $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality)

- In order to measure how far from 0 an element x of $\mathbb{R}^K$ is, we use the *Euclidean norm* which is defined as

$$\|x\| = \left(\sum_{k=1}^{K} x_k^2 \right)^{1/2}.$$

Normed Vector Spaces

Definition

A normed vector space is a pair $(X, \|\cdot\|)$ where X is a vector space and $\|\cdot\| : X \mapsto \mathbb{R}$ is a norm.

- It is standard to view any normed vector space $(X, \|\cdot\|)$ as a metric space where the metric $d(x, y) = \|x - y\|$ for all $x, y \in X$.

Examples

1. $X = \mathbb{R}^K$, with $\|x\| = \left[\sum_{k=1}^{K} x_k^2\right]^{\frac{1}{2}}$ (Euclidean Space)
2. $X = \mathbb{R}^K$, with $\|x\| = max_i |x_i|$.
3. $X = \mathbb{R}^K$, with $\|x\| = \sum_{k=1}^{K} |x_k|$.
4. X is the set of all bounded infinite sequences $\{x_k\}_{k=1}^{\infty}$ with $\|x\| = \sup_k |x_k|$. (This space is called l_∞)

Sequences in $\mathbb{R}^K$

Definition

A sequence in $\mathbb{R}^K$ is a function $f : \mathbb{N} \to \mathbb{R}^K$.

- $(a_1, a_2, \dots)$ or $(a_n)_{n=1}^{\infty}$, where $a_n = f(n)$, for $n \in \mathbb{N}$.
- $(a_n)_{n=1}^{\infty}$ is
  - *nondecreasing* (*increasing*) if $a_{n+1} \geq (>) a_n$ for all $n \in \mathbb{N}$;
  - *nonincreasing* (*decreasing*) if $a_{n+1} \leq (<) a_n$ for all $n \in \mathbb{N}$;
  - *bounded above* if there exists $\bar{a} \in \mathbb{R}^K$ such that $a_n \leq \bar{a}$ for all n;
  - *bounded below* if there exists $\underline{a} \in \mathbb{R}^K$ such that $a_n \geq \underline{a}$ for all n;
  - *bounded* if it is bounded both above and below.

Definition

Given a sequence $(a_n)_{n=1}^{\infty}$, a sequence $(b_m)_{m=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$ if there exists an increasing sequence $(n_m)_{m=1}^{\infty}$ such that $n_m \in \mathbb{N}$ and $b_m = a_{n_m}$ for all $m \in \mathbb{N}$.

Example

$(1/\sqrt{2n+5})_{n=1}^{\infty}$ is a subsequence of $(1/\sqrt{n})_{n=1}^{\infty}$ for $(n_m)_{m=1}^{\infty} = (2m+5)_{m=1}^{\infty}$.

Limits of Sequences

Definition

A sequence $(a_n)_{n=1}^{\infty}$ converges to $a \in \mathbb{R}^K$ (written $a_n \to a$), if for each $\varepsilon > 0$ there exists some $N_\varepsilon \in \mathbb{N}$ such that

$$d(a_n, a) < \varepsilon \text{ for all } n \geq N_\varepsilon.$$

Theorem

Let d be the Euclidean distance. Then, $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}^K$ converges to a if and only if $(a_{k,n})_{n=1}^{\infty}$ in $\mathbb{R}$ converges to a_k for all $k = 1, \ldots, K$.

Theorem

Sequence $(a_n)_{n=1}^{\infty}$ converges to $a \in \mathbb{R}^K$ if and only if every subsequence of $(a_n)_{n=1}^{\infty}$ converges to a.

Limits of Sequences

Definition

For a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$, we say that $\lim_{n\to\infty} a_n = \infty$ if for all $\Delta > 0$ there exists some $n^* \in \mathbb{N}$ such that $a_n > \Delta$ for all $n \geq n^*$. We say that $\lim_{n\to\infty} a_n = -\infty$ when $\lim_{n\to\infty}(-a_n) = \infty$. We say that a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ diverges to ∞ $(-\infty)$ if $\lim_{n\to\infty} a_n = \infty(-\infty)$.

Examples

1. Does $((-1)^n)_{n=1}^{\infty}$ converge? Does $(-1/n)_{n=1}^{\infty}$?
2. Does the sequence $(\frac{3n}{\sqrt{n}})_{n=1}^{\infty}$ have a limit? Does it converge?

Limits of Sequences: Properties I

Theorem

If $a_n \to x$ and $a_n \to y$, then $x = y$.

Theorem

For sequences $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $a_n > 0$ for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Limits of Sequences: Properties II

Theorem (Arithmetic of Limits)

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences in $\mathbb{R}$. Suppose that $a, b \in \mathbb{R}$, we have that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,

1. $\lim_{n\to\infty}(a_n + b_n) = a + b$;
2. $\lim_{n\to\infty}(\alpha a_n) = \alpha a$, *for all* $\alpha \in \mathbb{R}$;
3. $\lim_{n\to\infty}(a_n b_n) = ab$;
4. *if* $b \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty}(a_n / b_n) = a/b$.

Theorem (Weak Inequalities are Preserved under Sequential Limits)

If $a_n \leq \alpha$, for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = a$, then $a \leq \alpha$.

- Can we strengthen the last Theorem to strict inequalities?

Limits of Sequences: Properties III

Theorem

Every sequence $(a_n)_{n=1}^{\infty}$ has a monotone subsequence.

Theorem

If sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ is convergent, then it is bounded.

Theorem

If a sequence $(a_n)_{n=1}^{\infty}$ is monotone and bounded, then it is convergent.

Theorem (Bolzano-Weierstrass)

If sequence $(a_n)_{n=1}^{\infty}$ is bounded, then it has a convergent subsequence.

Cauchy Sequences

Definition

A sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence (or satisfies the Cauchy criterion) if for each $\varepsilon > 0$, there exists N_ε such that

$$d(a_n, a_m) < \varepsilon, \text{ for all } n, m \geq N_\varepsilon.$$

Example

Is the sequence $(1/\sqrt{n})_{n=1}^{\infty}$ Cauchy?

Theorem

1. *If a sequence is convergent, then it is a Cauchy sequence.*
2. *If a sequence is Cauchy, then it is bounded.*

Complete Metric Space

Definition

A metric space (X, d) is complete if every Cauchy sequence in X converges to an element of X.

Fact: $\mathbb{R}$ with $d(x, y) = |x - y|$ is a complete metric space.

Exercise: Show that:

1. The set of integers with $d(x, y) = |x - y|$ is a complete metric space.
2. The set of continuous, strictly increasing functions on $[a, b]$, with
$$d(x, y) = max_{a \leq t \leq b}|x(t) - y(t)|.$$
is not a complete metric space.
3. The set of continuous functions on $[0, 1]$ with
$$d(f, g) = \int_0^1 |f(x) - g(x)|dx$$
is not a complete metric space. What if $d(f, g) = sup_{x \in [0,1]}|f(x) - g(x)|$?

Limit Points

Definition

Let $x \in \mathbb{R}^K$ and $\delta > 0$. The open ball of radius δ around x, denoted $B_\delta(x)$, is the set

$$B_\delta(x) = \{y \in \mathbb{R} : d(y, x) < \delta\}.$$

Definition

The punctured open ball of radius δ around x, denoted $B'_\delta(x)$, is the set $B'_\delta(x) = B_\delta(x) \setminus \{x\}$.

Definition

A point $\bar{x} \in \mathbb{R}^K$ is a limit point of $X \subseteq \mathbb{R}^K$ if for all $\varepsilon > 0$, $B'_\varepsilon(\bar{x}) \cap X \neq \varnothing$

Limits of Functions in $\mathbb{R}$

Definition

Consider $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^K$. Suppose that $\bar{x} \in \mathbb{R}^K$ is a limit point of X and that $\bar{y} \in \mathbb{R}$. We say that $\lim_{x \to \bar{x}} f(x) = \bar{y}$ when for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), \bar{y}) < \varepsilon$ for all $x \in B'_\delta(\bar{x}) \cap X$.

Definition

Consider $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^K$. Suppose that $\bar{x} \in \mathbb{R}^K$ is a limit point of X. We say that $\lim_{x \to \bar{x}} f(x) = \infty$ when for all $\Delta > 0$, there exists $\delta > 0$ such that $f(x) \geq \Delta$ for all $x \in B'_\delta(\bar{x}) \cap X$. We say that $\lim_{x \to \bar{x}} f(x) = -\infty$ when $\lim_{x \to \bar{x}} (-f)(x) = \infty$.

Limits of Functions: Examples

Example

Suppose that $X = \mathbb{R}$ and $f : X \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1/x, & \text{if } x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

What is $\lim_{x\to 5} f(x)$? What is $\lim_{x\to 0} f(x)$?

Example

Let $X = \mathbb{R} \setminus \{0\}$ and $f : X \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{otherwise.} \end{cases}$$

In this case, we claim that $\lim_{x\to 0} f(x)$ does not exist.

Limits of Functions and Sequences

Theorem

Consider a function $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^K$. Suppose that $\bar{x} \in \mathbb{R}^K$ is a limit point of X and that $\bar{y} \in \mathbb{R}$. Then, $\lim_{x\to\bar{x}} f(x) = \bar{y}$ if and only if for every $(x_n)_{n=1}^{\infty} \in X \setminus \{\bar{x}\}$ that converges to $\bar{x}$, $\lim_{n\to\infty} f(x_n) = \bar{y}$.

Limits of Functions: Properties I

Define:

- $(f+g) : X \to \mathbb{R}$ by $(f+g)(x) = f(x) + g(x)$.
- $(\alpha f) : X \times \mathbb{R} \to \mathbb{R}$ by $(\alpha f)(x) = \alpha f(x)$.
- $(f \cdot g) : X \to \mathbb{R}$ by $(f \cdot g)(x) = f(x)g(x)$
- $(\frac{f}{g}) : X_g^* \to \mathbb{R}$ by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, where $X_g^* = \{x \in X | g(x) \neq 0\}$.

Theorem

Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. Let $\bar{x}$ be a limit point of X. Suppose that $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$ and that $\lim_{x\to\bar{x}} f(x) = \bar{y}_1$ and $\lim_{x\to\bar{x}} g(x) = \bar{y}_2$.

1. $\lim_{x\to\bar{x}}(f+g)(x) = \bar{y}_1 + \bar{y}_2$;
2. $\lim_{x\to\bar{x}}(\alpha f)(x) = \alpha\bar{y}_1$, *for all* $\alpha \in \mathbb{R}$;
3. $\lim_{x\to\bar{x}}(f \cdot g)(x) = \bar{y}_1 \cdot \bar{y}_2$;
4. *if* $\bar{y}_2 \neq 0$, *then* $\lim_{x\to\bar{x}}(f/g)(x) = \bar{y}_1/\bar{y}_2$.

Limits of Functions: Properties II

Theorem

Consider $f : X \to \mathbb{R}$ *and* $\bar{y} \in \mathbb{R}$*, and let* $\bar{x} \in \mathbb{R}^K$ *be a limit point of* X*. If* $f(x) \leq \gamma$ *for all* $x \in X$*, and* $\lim_{x \to \bar{x}} f(x) = \bar{y}$*, then* $\bar{y} \leq \gamma$*.*

Corollary

Consider $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$*, let* $\bar{y}_1, \bar{y}_2 \in \mathbb{R}$*, and let* $\bar{x} \in \mathbb{R}^K$ *be a limit point of* X*. If* $f(x) \geq g(x)$*, for all* $x \in X$*,* $\lim_{x \to \bar{x}} f(x) = \bar{y}_1$ *and* $\lim_{x \to \bar{x}} g(x) = \bar{y}_2$*, then* $\bar{y}_1 \geq \bar{y}_2$*.*

Open Sets

Definition

Set X is *open* if for all $x \in X$, there is some $\varepsilon > 0$ for which $B_\varepsilon(x) \subseteq X$.

Theorem

The empty set, the open intervals in $\mathbb{R}$ and $\mathbb{R}^K$ are open.

Theorem

The union of any collection of open sets is an open set. The intersection of any finite collection of open sets is an open set.

Exercise

1. Prove the following: "If $x \in int(X)$, then x is a limit point of X."
2. Do we really need finiteness in the second part of the last Theorem?
   Consider $I_n = (-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$. Find the intersection of all those intervals, denoted $\cap_{n=1}^{\infty} I_n$. Is it an open set?

Closed Sets

Definition

Set $X \subset \mathbb{R}^K$ is *closed* if for every sequence $(x_n)_{n=1}^{\infty} \in X$ that converges to $\bar{x}$, then $\bar{x} \in X$.

Theorem

The empty set, the closed intervals in $\mathbb{R}$ and $\mathbb{R}^K$ are closed.

Theorem

A set X is closed if and only if X^c is open.

Theorem

The intersection of any collection of closed sets is closed. The union of any finite collection of closed sets is closed.

Compact Sets

Definition

A set $X \subseteq \mathbb{R}^K$ is said to be *bounded above* if there exists $\alpha \in \mathbb{R}^K$ such that $x \leq \alpha$ for all $x \in X$; it is said to be *bounded below* if for some $\beta \in \mathbb{R}^K$ one has that $x \geq \beta$ is true for all $x \in X$; and it is said to be *bounded* if it is bounded above and below.

Definition

A set $X \subseteq \mathbb{R}^K$ is said to be *compact* if it is closed and bounded.

Exercise

Prove the following statement: if $(x_n)_{n=1}^{\infty}$ is a sequence defined on a compact set X, then it has a subsequence that converges to a point in X.

Continuity of Functions

Definition

Function $f : X \to \mathbb{R}$ is *continuous at $\bar{x} \in X$* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(\bar{x})| < \varepsilon$ for all $x \in B_\delta(\bar{x}) \cap X$. It is *continuous* if it is continuous at all $\bar{x} \in X$.

Theorem

Suppose that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous at $\bar{x} \in X$, and let $\alpha \in \mathbb{R}$. Then, the functions $f + g$, αf and $f \cdot g$ are continuous at $\bar{x}$. Moreover, if $g(\bar{x}) \neq 0$, then $\frac{f}{g}$ is continuous at $\bar{x}$.

Properties of Continuous Functions

Theorem

The image of a compact set under a continuous function is compact.

Theorem

Function $f : \mathbb{R}^K \to \mathbb{R}$ is continuous if and only if for every open set $U \subseteq \mathbb{R}$ the set $f^{-1}[U]$ is open.

Theorem (The Intermediate Value Theorem in $\mathbb{R}$)

If function $f : [a, b] \to \mathbb{R}$ is continuous, then for every number γ between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ for which $f(x) = \gamma$.

Left- and Right- Continuity

Definition

One says that $\lim_{x \searrow \bar{x}} f(x) = \ell$, if for every $\varepsilon > 0$ there is a number $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $x \in X \cap B_\delta(\bar{x})$ and $x > \bar{x}$. In such case, function f is said to converge to ℓ as x tends to $\bar{x}$ from above. Similarly, $\lim_{x \nearrow \bar{x}} f(x) = \ell$, when for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in X \cap B_\delta(\bar{x})$ satisfying that $x < \bar{x}$. In this case, f is said to converge to ℓ as x tends to $\bar{x}$ from below.

Definition

Function $f : X \to \mathbb{R}$ is *right-continuous at $\bar{x} \in X$*, where $\bar{x}$ is a limit point of X, if $\lim_{x \searrow \bar{x}} f(x) = f(\bar{x})$. It is *right-continuous* if it is right-continuous at every $\bar{x} \in X$ that is a limit point of X. Similarly, $f : X \to \mathbb{R}$ is *left-continuous at $\bar{x}$* if $\lim_{x \nearrow \bar{x}} f(x) = f(\bar{x})$, and one says that f is *left-continuous* if it is left-continuous at all limit point $\bar{x} \in X$.

Differentiability

Definition

Let $f : \mathbb{R} \mapsto R$ be a function defined in a neighbourhood of x_0. Then f is said to be differentiable at x_0 with derivative equal to the real number $f'(x_0)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \varepsilon$$

- Since $x - x_0 \neq 0$, multiply the inequality above by $|x - x_0|$ to obtain

$$\left| f(x) - f(x_0) - f'(x_0)(x - x_0) \right| \leq \varepsilon |x - x_0|$$

  to see that $|f(x) - f(x_0) - f'(x_0)(x - x_0)|$ goes to zero faster than $|x - x_0|$.

Mean Value Theorem and Taylor's Theorem

Theorem (Mean Value Theorem)

Let f be a continuous function on $[a, b]$ that is differentiable in (a, b). Then there exists $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Theorem (Taylor's Theorem)

Let f be $\mathbb{C}^n$ in a neighborhood of x_0, and let

$$T_n(x_0, x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + ... + \frac{1}{n!} f^n(x_0)(x - x_0)^n.$$

Then for any $\varepsilon > 0$, there exists δ such that $|x - x_0| \leq \delta$ implies

$$|f(x) - T_n(x_0, x)| \leq \varepsilon |x - x_0|^n .$$

Theorem (Lagrange Remainder Theorem)

Suppose f is $\mathbb{C}^{n+1}$ in a neighborhood of x_0. Then for every x in the neighbourhood there exists x_1 between x_0 and x such that

$$f(x) = T_n(x_0, x) + \frac{1}{(n+1)!} f^{n+1}(x_1)(x - x_0)^{n+1}$$