Asset Prices, Market Selection and Belief Heterogeneity

Subjective Expected Utility and Bayesian Learning

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This lecture is based on "Market Selection and Asset Pricing," by Blume and Easley (2009).

Subject Expected Utility

- STATES: $s \in S$.
- PRIZES: C.
- ACTS: $f : S \to C$.
- SET OF ACTS: $\mathcal{L}$
- PREFERENCES: $\succsim$, a binary relation on $\mathcal{L}$.

Definition

The preference relation $\succsim$ on $\mathcal{L}$ has a subjective expected utility (SEU) representation if there is a function $u : C \to \Re$ and a probability distribution π on S such that

$$f \succsim g \Leftrightarrow \sum_{s \in S} \pi(s) u(f(s)) \geq \sum_{s \in S} \pi(s) u(g(s))$$

Bayesian Updating

- Let $A \subset S$ such that $\pi(A) > 0$.
- $\pi(B|A) = \frac{\pi(B \cap A)}{\pi(A)}$

SEU and Bayesian Updating

- Every expected utility maximiser is a Bayesian.
- For acts f and h, define:

$$f_A h(s) = \begin{cases} f(s) & \text{if } s \in A \\ h(s) & \text{if } s \notin A \end{cases}$$

Definition

f is at least as good as g given A, denoted $f \succsim_A g$, if for all acts $h \in \mathcal{L}$, $f_A h(s) \succsim g_A h(s)$.

Theorem

Suppose $\succsim$ has an SEU representation. Then,

$$f \succsim_A g \Leftrightarrow E_{\pi(\cdot|A)} u(f(s)) \geq E_{\pi(\cdot|A)} u(g(s)).$$

- PROOF: Let $h \in \mathcal{L}$.

$$f_A h \succsim g_A h$$

$$\Updownarrow$$

$$\sum_{s \in S} \pi(s) u(f_A h(s)) \geq \sum_{s \in S} \pi(s) u(g_A h(s))$$

$$\Updownarrow$$

$$\sum_{s \in A} \pi(s) u(f(s)) + \sum_{s \notin A} \pi(s) u(h(s)) \geq \sum_{s \in A} \pi(s) u(g(s)) + \sum_{s \notin A} \pi(s) u(h(s))$$

$$\Updownarrow$$

$$\sum_{s \in A} \pi(s) u(f(s)) \geq \sum_{s \in A} \pi(s) u(g(s))$$

$$\Updownarrow$$

$$\sum_{s \in A} \frac{\pi(s)}{\pi(A)} u(f(s)) \geq \sum_{s \in A} \frac{\pi(s)}{\pi(A)} u(g(s))$$

$$\Updownarrow$$

$$\sum_{s \in A} \pi(s|A) u(f(s)) \geq \sum_{s \in A} \pi(s|A) u(g(s))$$

Bayesian Learning: Simple Case

- An agent believes flips of a coin are i.i.d. and that the probability of H is either $p > \frac{1}{2}$ or $q < \frac{1}{2}$. He believes in p with probability α.
- $S = \{H, T\}$.
- $\Omega = S^{\infty}$.
- π are beliefs on $\omega \in \{p, q\} \times \Omega$

$$
\begin{aligned}
\pi(p) &= \alpha & \pi(q) = 1-\alpha. \\
\pi(H|p) &= p & \pi(H|q) = q \\
\pi(p,H) &= \alpha p & \pi(q,H) = (1-\alpha)q \\
\pi(H) &= \alpha p + (1-\alpha)q \\
\pi(p|H) &= \frac{\pi(p,H)}{\pi(H)} = \frac{\alpha p}{\alpha p + (1-\alpha)q} > \alpha = \pi(p)
\end{aligned}
$$

Bayesian Learning: Simple Case

$$\frac{\pi(p|H,T)}{\pi(q|H,T)} = \frac{\frac{\pi(p,H,T)}{\pi(H,T)}}{\frac{\pi(q,H,T)}{\pi(H,T)}} = \frac{\pi(p,H,T)}{\pi(q,H,T)}$$

$$= \frac{\pi(T|p,H)\pi(p|H)\pi(H)}{\pi(T|q,H)\pi(q|H)\pi(H)}$$

$$= \frac{\pi(T|p)\pi(p|H)}{\pi(T|q)\pi(q|H)} = \left(\frac{1-p}{1-q}\right)\frac{\pi(p|H)}{\pi(q|H)}$$

In general,

$$\frac{\pi(p|s^t)}{\pi(q|s^t)} = \frac{\pi(p|s^{t-1},s_t)}{\pi(q|s^{t-1},s_t)} = \frac{\pi(s_t|p)\pi(p|s^{t-1})}{\pi(s_t|q)\pi(q|s^{t-1})}$$

$$= \frac{p^{\mathbb{1}_H(s_t)}(1-p)^{\mathbb{1}_T(s_t)}}{q^{1_H(s_t)}(1-q)^{\mathbb{1}_T(s_t)}}\frac{\pi(p|s^{t-1})}{\pi(q|s^{t-1})}$$

$$= \left(\frac{p}{q}\right)^{\mathbb{1}_H(s_t)}\left(\frac{1-p}{1-q}\right)^{\mathbb{1}_T(s_t)}\frac{\pi(p|s^{t-1})}{\pi(q|s^{t-1})}$$

Bayesian Learning: Simple Case

$$
\begin{aligned}
\frac{\pi(p|s^t)}{\pi(q|s^t)} &= \left(\frac{p}{q}\right)^{\mathbb{1}_H(s_t)} \left(\frac{1-p}{1-q}\right)^{\mathbb{1}_T(s_t)} \frac{\pi(p|s^{t-1})}{\pi(q|s^{t-1})} \\
&= \left(\prod_{k=1}^{t} \left(\frac{p}{q}\right)^{\mathbb{1}_H(s_k)} \left(\frac{1-p}{1-q}\right)^{\mathbb{1}_T(s_k)}\right) \frac{\pi(p)}{\pi(q)} \\
&= \left(\frac{p}{q}\right)^{n_t^H} \left(\frac{1-p}{1-q}\right)^{n_t^T} \frac{\pi(p)}{\pi(q)}
\end{aligned}
$$

$$
\begin{aligned}
\log \frac{\pi(p|s^t)}{\pi(q|s^t)} &= \log \frac{\pi(p)}{\pi(q)} + \sum_{k=1}^{t} \mathbb{1}_H(s_k) \log \frac{p}{q} + \mathbb{1}_T(s_k) \log \frac{1-p}{1-q} \\
\frac{1}{t} \log \frac{\pi(p|s^t)}{\pi(q|s^t)} &= \frac{1}{t} \log \frac{\pi(p)}{\pi(q)} + \frac{1}{t} \left(\sum_{k=1}^{t} \mathbb{1}_H(s_k) \log \frac{p}{q} + \mathbb{1}_T(s_k) \log \frac{1-p}{1-q} \right) \\
&\to r \log \frac{p}{q} + (1-r) \log \frac{1-p}{1-q} \quad (SLLN).
\end{aligned}
$$

Relative Entropy

Definition

The relative entropy of p with respect to r is

$$I_r(p) = r \log \frac{r}{p} + (1-r) \log \frac{1-r}{1-p}$$

1. $I_r(p) \geq 0$.
2. $I_r(p) = 0 \Leftrightarrow p = r$

- Posteriors converge:

$$\frac{1}{t} \log \frac{\pi(p|s^t)}{\pi(q|s^t)} \to I_r(q) - I_r(p)$$

- Posteriors concentrate on the points of the support with lowest entropy.
  - $I_r(p) < I_r(q) \Rightarrow \pi(p|h^t) \to 1$.
  - $I_r(p) > I_r(q) \Rightarrow \pi(q|h^t) \to 1$
- If the true distribution is in the support, posteriors converge to the truth:
  - If $r = p$, $I_r(p) = 0 < I_r(q) \Rightarrow \pi(p|h^t) \to 1$.

General Setup

- X is a finite set and $(X, \mathcal{A})$ is a measurable space.
- Θ is the space of all probability measures on $(X, \mathcal{A})$.
- $\theta : \mathcal{A} \mapsto [0, 1]$ is a probability measure on X (an element of the $\#X - 1$ dimensional simplex).
- $\{X_n\}_{n=1}^{\infty}$ is a sequence of $X-$valued random variables that are iid as θ_o.
- $\Omega^n \equiv X^n$ and $\Omega \equiv X^{\infty}$.
- P_{θ}^{∞} is the i.i.d. product measure on $\mathcal{A}^{\infty}$.

Beliefs

- π is the prior, a probability measure on $(\Theta, \mathcal{B}(\Theta))$.
- λ is the joint distribution on $(\Theta \times \Omega, \mathcal{B}(\Theta) \times \mathcal{A}^\infty)$ given by:

$$\lambda(B \times A) = \int_B P_\theta^\infty(A) d\pi(\theta) \qquad \text{for all } B \times A \in \mathcal{B}(\Theta) \times \mathcal{A}^\infty.$$

- $\pi(\cdot | \cdot) : \mathcal{B}(\Theta) \times \Omega^n \mapsto [0, 1]$ is the posterior given X^n if

$$\pi(B | X^n) = \frac{\int_B \prod_{k=1}^n \theta(X_k) d\pi(\theta)}{\int_\Theta \prod_{k=1}^n \theta(X_k) d\pi(\theta)} \qquad \text{for all } B \in \mathcal{B}(\Theta).$$

Consistency

Definition

The sequence $\pi(\cdot|\, X^n)$ is consistent at θ_o if for every neighbourhood B of θ_o:

$$\pi\left(B|\, X^n(\omega)\right) \to 1, \qquad P_{\theta_o}^{\infty} - a.s.\ \omega$$

There are several consistency theorems due to:

- Doob (1949),
- Schwartz (1965),
- Wald (1949).

Theorem

Let X be a finite set and Π be a prior on Θ. Then the posterior is consistent at all points in the support of Π.

Sketch of Proof

$$\frac{\pi\left(B|X^T\right)}{\pi\left(B^c|X^T\right)} = \frac{\int_B \Pi_{t=0}^T \theta(X_t)\pi(d\theta)}{\int_{B^c} \Pi_{t=0}^T \theta(X_t)\pi(d\theta)} = \frac{\int_B \Pi_{t=0}^T \frac{\theta(X_t)}{\theta_o(X_t)}\pi(d\theta)}{\int_{B^c} \Pi_{t=0}^T \frac{\theta(X_t)}{\theta_o(X_t)}\pi(d\theta)} = \frac{\int_B e^{\log\left(\Pi_{t=0}^T \frac{\theta(X_t)}{\theta_o(X_t)}\right)}\pi(d\theta)}{\int_{B^c} e^{\log\left(\Pi_{t=0}^T \frac{\theta(X_t)}{\theta_o(X_t)}\right)}\pi(d\theta)}$$

$$= \frac{\int_B e^{\sum_{t=0}^T \log\left(\frac{\theta(X_t)}{\theta_o(X_t)}\right)}\pi(d\theta)}{\int_{B^c} e^{\sum_{t=0}^T \log\left(\frac{\theta(X_t)}{\theta_o(X_t)}\right)}\pi(d\theta)} = \frac{\int_B e^{-T\sum_{t=0}^T \frac{1}{T}\log\left(\frac{\theta_o(X_t)}{\theta(X_t)}\right)}\pi(d\theta)}{\int_{B^c} e^{-T\sum_{t=0}^T \frac{1}{T}\log\left(\frac{\theta_o(X_t)}{\theta(X_t)}\right)}\pi(d\theta)}$$

$$= \frac{\int_B e^{-T\sum_{x\in X} \frac{T_x}{T}\log\left(\frac{\theta_o(x)}{\theta(x)}\right)}\pi(d\theta)}{\int_{B^c} e^{-T\sum_{t=0}^T \frac{T_x}{T}\log\left(\frac{\theta_o(X_t)}{\theta(X_t)}\right)}\pi(d\theta)}$$

- Choose δ such that $\varepsilon_1 \equiv \sup_{\theta\in B_\delta} I_{\theta_o}(\theta) < \inf_{\theta\in B^c} I_{\theta_o}(\theta) \equiv \varepsilon_2$. Then,

$$\frac{\pi\left(B|X^T\right)}{\pi\left(B^c|X^T\right)} \geq \frac{\int_{B_\delta} e^{-T\sum_{x\in X} \frac{T_x}{T}\log\left(\frac{\theta_o(x)}{\theta(x)}\right)}\pi(d\theta)}{\int_{B^c} e^{-T\sum_{t=0}^T \frac{T_x}{T}\log\left(\frac{\theta_o(X_t)}{\theta(X_t)}\right)}\pi(d\theta)}$$

- Note that $\sum_{x\in X} \frac{T_x}{T}\log\left(\frac{\theta_o(x)}{\theta(x)}\right) \to I_{\theta_o}(\theta)$. Use the uniform SLLN to argue that:

$$\frac{\pi\left(B|X^T\right)}{\pi\left(B^c|X^T\right)} \geq \frac{\Pi(B_\delta)}{\Pi(B^c)}\frac{e^{-T\varepsilon_1}}{e^{-T\varepsilon_2}} = \frac{\Pi(B_\delta)}{\Pi(B^c)}e^{-T(\varepsilon_1-\varepsilon_2)} \to \infty$$