Asset Prices, Market Selection and Belief Heterogeneity

Arrow-Debreu and Sequential Markets

Pablo F. Beker

Department of Economics
University of Warwick

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This lecture is based on "Implementing Arrow-Debreu equilibria by trading infinitely-lived securities." by Huang and Werner, Economic Theory, 24, 2004.

The Economy

- There is a single perishable consumption good every period.
- A consumption plan is a sequence $\{c_t\}_{t=0}^{\infty}$ such that $c_0 \in \mathbb{R}_+$ and $c_t : S^{\infty} \to \mathbb{R}_+$ is $\mathcal{F}_t-$measurable for all $t \geq 1$ and $\sup_{(t,s)} c_t(s) < \infty$.
- Let $c(s^t) \equiv c_t(s)$ for any t and $s^t \in S^t$.
- Given s_0, $\mathbb{C}(s_0)$ denotes the set of all consumption plans.

- The economy is populated by I (types of) infinitely-lived agents where $i \in \mathcal{I} = \{1, ..., I\}$ denotes an agent's name.
- Agent i is endowed with initial endowment $\omega_i \in \mathbb{C}(s_0)$
- The aggregate endowment $\overline{\omega} \equiv \sum_i \omega_i$.

- An allocation is a collection of plans $\{c_i\}_{i \in I}$.
- An allocation is feasible if $\sum_i c_i(s^t) \leq \overline{\omega}, \forall s^t, \forall t$.

Arrow-Debreu Markets

- There exists a market at the initial date 0 for consumption at date t conditional on event s^t, for every date t and every event s^t.
- Prices are described by a *pricing functional*, that is, a linear functional P which is positive and well-defined (finitely valued) on each consumer's initial endowment.
- It follows that a pricing functional is well-defined on the aggregate endowment $\overline{\omega}$ and, therefore, on each feasible allocation. It may or may not be well-defined on the entire consumption set $\mathbb{C}(s_0)$
- The price of one unit of consumption in event s^t under pricing functional P is $p(s^t) \equiv P(e(s^t))$, where $e(s^t)$ denotes the consumption plan equal to 1 in event s^t at date t and zero in all other events and all other dates.
- A pricing functional P is countably additive if and only if $P(c) = \sum_t \sum_{s^t} p(s^t)c(s^t)$ for every c for which $P(c)$ is well-defined.

Arrow Debreu Budget Set

- Trades occur only at date zero.
- Agent i can only choose a consumption plan such that the value of consumption does not exceed the value of agent i's endowment.
- The agent chooses a plan on the budget set $B_{AD}(P, \omega_i)$ where:

$$\begin{aligned} B_{AD}(P, \omega_i) &\equiv \{c \in \mathbb{C}(s_0) : P(c) \leq P(\omega_i)\} \\ &= \left\{ c \in \mathbb{C}(s_0) : \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t) c(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t \in S^t} p(s^t) \omega_i(s^t) \right\} \end{aligned}$$

- Consumer i's problem is to choose a consumption plan $c_i \in \mathbb{C}(s_0)$ such that

$$c_i \succsim_i c, \forall c \in B_{AD}(P, \omega_i)$$

Arrow Debreu Equilibrium

- Agents trade at date zero under a single budget constraint.

Definition

An Arrow-Debreu equilibrium is a pricing functional P and a consumption allocation $\{c^i\}_{i=1}^{I}$ such that c^i solves consumer i's problem and markets clear.

- The Arrow-Debreu model of contingent commodity markets is hardly realistic.
- Yet, it serves as an important tool for the analysis of infinite-time security markets.
- This is because one can show that Arrow-Debreu equilibria and equilibria in sequential security markets with debt constraints have the same consumption allocations when markets are dynamically complete and debt bounds are nonbinding.

Sequential Markets

- There are $J \geq S$ infinitely-lived securities traded at every date.
- Each security j is specified by a dividend process d_j which is adapted to $\{\mathcal{F}_t\}_{t=0}^{\infty}$ and nonnegative.
- The ex-dividend price of security j in event s^t is denoted by $q_j(s^t)$, and q_j is the price process of security j.
- Portfolio strategy θ specifies a portfolio of J securities $\theta(s^t)$ held after trade in each event s^t.
- The payoff of portfolio strategy θ in event s^t at a price process q is

$$z(q,\theta)(s^t) \equiv \underbrace{\left[q(s^t) + d(s^t)\right]}_{r(s^t)} \theta(s^{t-1}) - q(s^t)\theta(s^t)$$

Definition

Security price process q is one-period-arbitrage free in event s^t if there does not exist a portfolio $\theta\left(s^t\right)$ such that:

$$\left[q(s^t, s_{t+1}) + d(s^t, s_{t+1})\right]\theta(s^t) \geq 0 \text{ for all } s \text{ and } q(s^t)\theta(s^t) \leq 0,$$

with at least one strict inequality.

No arbitrage

- If q is arbitrage free in every event, then there exists a sequence of strictly positive state prices $\left\{ \{\pi_q(s^t)\}_{s^t \in S^t} \right\}_{t=0}^{\infty}$ with $\pi_q(s^0) = 1$ such that

$$\pi_q(s^t) q_j(s^t) = \sum_{s_{t+1} \in S} \pi_q(s^t, s_{t+1}) \left[q_j(s^t, s_{t+1}) + d_j(s^t, s_{t+1}) \right] \quad \forall s^t, \forall j$$

Definition

Security markets are one-period complete in event s^t at prices q if the one-period payoff matrix $\left[q(s^t, s_{t+1}) + d(s^t, s_{t+1}) \right]_{s_{t+1} \in S}$ has rank S. Security markets are complete at q if they are one-period complete at every event.

- Suppose the security prices q are one-period arbitrage free and that markets are complete at q. Then, the fundamental value of security j at s^t is defined using the unique state prices as

$$\frac{1}{\pi_q(s^t)} \sum_{\tau=1}^{\infty} \sum_{s^\tau \in S^\tau} \pi_q(s^t, s^\tau) d_j(s^t, s^\tau) \tag{1}$$

Sequential Markets

- Each agent i has an initial portfolio $\alpha_i \in \Re^J$ at date 0.
- The dividend stream $\alpha_i d$ on initial portfolio constitutes one part of consumer i's endowment. The rest is $y_i \in \mathbb{C}(s_0)$ and becomes available to the consumer at each date in every event. Thus,

$$\omega_i(s_t) = y_i(s^t) + \alpha_i d(s^t), \qquad \forall s^t \in S^t$$

- The supply of securities is $\overline{\alpha} = \sum_i \alpha_i$.
- The adjusted aggregate endowment is $\overline{y} = \sum_i y_i$. Let's assume $\overline{\alpha} \geq 0$.

Sequential Budget Set

- θ_i supports c_i at (q, y_i) if

$$c_{i,0} + q(s^0)\theta(s^0) \leq y_i(s_0) + q(s_0)\alpha_i$$

$$c_i(s^t) + q(s^t)\theta(s^t) \leq y_i(s^t) + \left[q(s^t) + d(s^t)\right]\theta(s^{t-1}), \quad \forall s^t \neq s_0$$

- Consumers must also face constraints in their portfolio strategies for otherwise they would use Ponzi schemes. There is a set Θ_i of feasible supporting portfolios.

- The sequential budget set is:

$$B(q; y_i) \equiv \Big\{c_i \in \mathbb{C}(s_0) : \exists \theta_i \in \Theta_i \ni c_i(s^t) + q(s^t) \cdot \theta_i(s^t) \leq y_i(s^t) + r(s^t) \cdot \theta_i(s^{t-1}), \forall\, s \in S^{\infty}, \forall\, t \geq 0.\Big\}$$

The Wealth Constraint

- A frequently used portfolio constraint is the so-called wealth constraint. It prohibits a consumer from borrowing more than the present value of his future endowment. Formally,

Definition

Portfolio θ satisfies the wealth constraint if

$$q(s^t)\theta(s^t) \geq -\sum_{\tau=1}^{\infty}\sum_{s^\tau \in S^\tau}\frac{\pi_q(s^t, s^\tau)}{\pi_q(s^t)}y(s^t, s^\tau)$$

- The set of Arrow-Debreu equilibrium allocations is the same as the set of Sequential Markets equilibrium allocations under the wealth constraint with no bubbles.
- There always exist a sequential equilibria with price bubbles under the wealth constraint if some securities are in zero net supply.

Essentially Bounded Portfolios

- A portfolio constraint for which neither price bubbles nor negative security prices arise in equilibrium and AD equilibria can be implemented in sequential markets.

Definition

A portfolio θ is bounded from below if $\min_j \inf_{(t,s^t)} \theta_j(s^t) > -\infty$

Definition

A portfolio strategy θ is essentially bounded from below at q if there is a bounded from below portfolio strategy b s.t. $q(s^t)\theta(s^t) \geq q(s^t)b(s^t)\ \forall s^t$.

Proposition

If security price vector $q(s^t)$ is positive and nonzero for every partial history s^t, then portfolio θ is essentially bounded if and only if $\inf_{s^t} \frac{q(s^t)}{\sum_j q_j(s^t)}\theta(s^t) > -\infty$.

Euler Equations

- We say that c_i satisfies the *Euler* equation at the price process q if

$$u_i'(c_{i,t}(s))q_{j,t}(s) = \beta_i \cdot E_{P_i}[r_{j,t+1} \cdot u_i'(c_{i,t+1})|\mathcal{F}_t](s)\ \forall j \in J, \forall s \in S^\infty, \forall t \geq 0.$$

- ASSUMPTION $\mathcal{U}$: $u_i : R_{++} \to R$ is
  (i) strictly increasing, strictly concave, C^1 & $u_i(0) \equiv \lim_{c \to 0^+} u_i(c)$
  (ii) $\beta_i \in (0, 1)$.

Definition

For i, c_i is a maximizer given q if

1. $c_i \in B(q; y_i)$ and
2. there is no $\tilde{c}_i \in B(q; y_i)$ for which

$$\lim_{T \to +\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(\tilde{c}_{i,t})] > \lim_{T \to +\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(c_{i,t})].$$

Necessary Condition

- Suppose the investor can freely buy or sell as much of asset j as she wishes at a price $q_{j,t}$.
- Denote by c_i the optimal consumption plan.
- She can alter her consumption plan as follows:

$$\widetilde{c}_{i,t} = c_t - q_{j,t} \cdot \xi_t \qquad \widetilde{c}_{i,t+1} = c_{t+1} + r_{j,t+1} \cdot \xi_t$$

- If c_i maximises the consumer's utility, then

$$q_{j,t} \cdot u_i'(c_{i,t}) \quad = \quad E_{P_i}\left[\beta_i \cdot u_i'(c_{i,t+1}) \cdot r_{j,t+1} \middle| \mathcal{F}_t\right],$$

Sufficient Conditions

Theorem

Suppose Assumption $\mathcal{U}$. Given (q, y_i), let $c_i \in B(q; y_i)$ be such that

1. $\lim_{T\to+\infty} \sum_{t=0}^{T} \beta_i^t E_{P_i}[u_i(c_{i,t})] > -\infty$,
2. *satisfies the Euler equation at the price process q,*
3. *for every $\tilde{\theta}_i$ that supports a $\tilde{c}_i \in B(q; y_i)$ the transversality condition at date 0 holds,*

$$\lim_{T\to+\infty} \beta_i^T E_{P_i}\left[u_i'(c_{i,T}) \cdot q_T \cdot \left(\tilde{\theta}_{i,T} - \theta_{i,T}\right)\right] \geq 0.$$

where θ_i supports c_i at (q, y_i). Then c_i is the maximiser on $B(q; y_i)$.

Bubbles

- If the fundamental value (1) is finite, the price bubble $\sigma_{qj}(s^t)$ is

$$\sigma_{qj}(s^t) \equiv q_j(s^t) - \frac{1}{\pi_q(s^t)} \sum_{\tau=1}^{\infty} \sum_{s^\tau \in S^\tau} \pi_q(s^t, s^\tau) d_j(s^t, s^\tau) \qquad (2)$$

Proposition

If the price of security j is nonnegative in every event, then the fundamental value of security j is finite and does not exceed the price of security j, i.e. $0 \leq \sigma_{qj}(s^t) \leq q_j(s^t)$ for every s^t. If the fundamental value of security j is finite and $\sigma_{qj}(s^t) \geq 0$ for every s^t, then $q_j(s^t) \geq 0$ for every s^t.

- Note that (1) and (2) implies that

$$\sigma_{qj}(s^t) = \frac{1}{\pi_q(s^t)} \sum_{s_{t+1} \in S} \pi_q(s^t, s_{t+1}) \sigma_{qj}(s^t, s_{t+1})$$

$$\sigma_{qj}(s^t) = \lim_{T \to \infty} \frac{1}{\pi_q(s^t)} \sum_{s^T \in S^T} \pi_q(s^t, s^T) q_j(s^t, s^T)$$

Bubbles under the Wealth Constraint

- A representative agent economy without uncertainty: $y_t = y$ for all $t \geq 0$.
- A consol pays $d_t = d < y$, is in zero net-supply and trades at price q_t^c.
- In any equilibrium $c_t = y$ and $\theta_t = 0$ for all $t \geq 0$.
- $q_t^c = \frac{\beta}{1-\beta} d + \varepsilon_t$ where $\varepsilon_t = \varepsilon_0 \left(\frac{1}{\beta}\right)^t$. Hence, $\pi_q(s^t) = \beta^t$.
- The wealth constraint: $q^c(s^t)\theta(s^t) \geq -\sum_{\tau=1}^{\infty} \frac{\pi_q(s^t, s^\tau)}{\pi_q(s^t)} y(s^t, s^\tau) = -y\frac{\beta}{1-\beta}$
- c satisfies the Euler equation:

$$q_t^c \;\; = \;\; \beta \left(\frac{1}{1-\beta} d + \varepsilon_0 \left(\frac{1}{\beta}\right)^{t+1} \right) = \beta \left(\frac{\beta}{1-\beta} d + d + \varepsilon_{t+1} \right) = \beta \left(q_{t+1}^c + d \right)$$

- c satisfies the TC:

$$\lim_{T\to\infty} \beta^T q_T^c \left(\tilde{\theta}_T - \theta_T\right) \;\; = \;\; \lim_{T\to\infty} \beta^T q_T^c \tilde{\theta}_T \geq \lim_{T\to\infty} \beta^T \left(-y\frac{\beta}{1-\beta}\right) = 0.$$

Bubbles under the Wealth Constraint

- Suppose there is a risk-free bond with price q_t^b. Clearly, $q_t^b = \beta$.
- Let $q_t = \left(q_t^c, q_t^b\right)$.
- Suppose the agent shorts the consol in one unit and invest $\frac{\beta}{1-\beta}d$ units of the bond at zero to meet the consol payments?
  - $\tilde{\theta}_t = \left(-1, \frac{\beta}{1-\beta}\frac{d}{\beta}\right)$ for all $t \geq 0$.
  - $q_0\tilde{\theta}_0 = -q_0^c + \frac{\beta}{1-\beta}d = \varepsilon_0 > 0$.
  - $\left[\left(q_t^c + d\right)\tilde{\theta}_{t-1}^c + \tilde{\theta}_{t-1}^b\right] - \left[q_t^c\tilde{\theta}_t^c + q_t^b\tilde{\theta}_t^b\right] = -d + \frac{\beta}{1-\beta}d + \beta\frac{\beta}{1-\beta}\frac{d}{\beta} = 0$
  - $\tilde{\theta}_t$ supports $\tilde{c}_0 = c_0 + \varepsilon, \tilde{c}_t = c_t$.
- How is this compatible with c being optimal?
- The key is that $\tilde{\theta}$ violates the wealth constraint and so $\tilde{c} \notin \mathcal{B}(q, y)$.
  - $q_t\tilde{\theta}_t = -q_t^c + q_t^b\frac{\beta}{1-\beta}\frac{d}{\beta} = -\left(\frac{\beta}{1-\beta}d + \varepsilon_t\right) + \frac{\beta}{1-\beta}d = -\varepsilon_0\left(\frac{1}{\beta}\right)^t \to -\infty$.

No Bubbles with Essentially Bounded Portfolios

Theorem

If q is an equilibrium price process such that θ is essentially bounded and security markets are complete at q, then $q(s^t) \geq 0$ and $\sigma_{qj}(s^t) = 0$ for every s^t.

Equivalence I

Theorem

Let allocation $\{c_i\}_{i=1}^{I}$ *and pricing functional* P *be an Arrow-Debreu equilibrium. If* P *is countably additive,* $P(d_j) < \infty$ *for each* j*, security markets are complete at prices* q *given by*

$$q_j(s^t) = \frac{1}{p(s^t)} \sum_{\tau=t+1}^{\infty} p(s^\tau) d_j(s^\tau) \qquad \forall s^t, \forall j \tag{3}$$

and there exists an essentially bounded portfolio strategy η *such that*

$$-\frac{1}{p(s^t)} \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} p(s^\tau) \overline{y}(s^\tau) \geq q(s^t)\eta(s^t) \qquad \forall s^t, \forall j \tag{4}$$

then there exists a portfolio allocation $\{\theta_i\}_{i=1}^{I}$ *such that* q *and the allocation* $\{c_i, \theta_i\}_{i=1}^{I}$ *are a sequential equilibrium with essentially bounded portfolios.*

Equivalence II

Theorem

Let security prices q and $\{c_i, \theta_i\}_{i=1}^{I}$ be a sequential equilibrium with essentially bounded portfoilios. If security markets are complete at q and there exists an essentially bounded portfolio strategy η such that

$$-\frac{1}{\pi_q(s^t)} \sum_{\tau=t+1}^{\infty} \sum_{s^\tau \in S^\tau} \pi_q(s^\tau)\overline{y}(s^\tau) \geq q(s^t)\eta(s^t) \qquad \forall s^t, \tag{5}$$

then $\{c_i, \theta_i\}_{i=1}^{I}$ and pricing functional P given by

$$P(c) = \sum_t \sum_{s^t \in S^t} \pi_q(s^t) c(s^t) \tag{6}$$

are an Arrow-Debreu equilibrium.