Asset Prices, Market Selection and Belief Heterogeneity

Complete Markets with Heterogeneous Beliefs
The Market Selection Hypothesis

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References

- This lecture is based on:

1. Sandroni, A (2000): "Do markets favor agents able to make accurate predictions?", *Econometrica*, 68. 1303 -1342.
2. Blume, L. and D. Easley (2006): "If Youre So Smart, Why Arent You Rich? Belief Selection in Complete and Incomplete Markets. *Econometrica*, Vol. 74, No. 4 (July), 929 - 966.
3. Beker, P. and E. Espino (2011): "The Dynamics of Efficient Asset Trading with Heterogeneous Beliefs," *Journal of Economic Theory*, 146, 189 - 229.

Motivation

- A long-standing theory in economics is that agents who do not make rational decisions are driven out of the market (see Alchian (1950) and Friedman (1953)). Friedman writes:

  "Whenever this determinant (of business behavior) happens to lead to behavior consistent with rational and informed maximization of returns, the business will prosper and acquire resources with which to expand; whenever it does not the business will tend to lose resources and can be kept in existence only by the addition of resources from the outside. The process of natural selection thus helps to validate the hypothesis (of profit maximization) or, rather, given natural selection, acceptance of the hypothesis can be based largely on the judgment that it summarizes appropriately the conditions for survival.

Motivation

- This theory is extremely influential. It underlies the efficient-markets hypothesis and the use of rational expectations equilibrium as a solution concept in modern economics because it implies that asset prices will eventually reflect the beliefs of agents making accurate predictions.
- For example, Cootner (1967) writes:

  "Given the uncertainty of the real world, the many actual and virtual investors will have many, perhaps equally many, forecasts...If any group of investors was consistently better than average in forecasting stock prices, they would accumulate wealth and give their forecasts greater and greater weight. In this process, they would bring the present price closer to the true value."
- Surprisingly, reasonably general conditions under which this theory is true were not known until Sandroni's work (ECTA, 2000).
- What is the formalisation of the idea of "an investor being consistently better than average in forecasting" that has the property that leads to more wealth accumulation?

Results

- Sandroni says that an agent eventually makes accurate predictions if given enough data, the difference between the agent's beliefs over infinite horizon events and the true probability of those events becomes arbitrarily small in the sup norm.
- If markets are dynamically complete, agents are SEU maximisers, endowments are uniformly bounded and some agent makes accurate predictions, then the MSH is correct.
    - Among agents with equal discount factors, those eventually making accurate predictions accumulate more wealth.
    - Convergence to a Rational Expectations equilibria obtains as a result of the Market Selection process.
    - If agents have equal discount factors, all agents making accurate predictions survive. Preferences over risk do not matter!!!!

Results

- What if nobody is able to eventually make accurate predictions?
- Sandroni explains who survives even if nobody eventually makes accurate predictions.
    - The entropy of an agent is the "measure" of proximity between probabilities with the property that being closer to the truth implies accumulating more wealth.
    - Entropy depends only on exogenous parameters and do not depend on preferences.
- What if nobody make accurate predictions and several agents' belief have the same entropy?
    - Blume and Easley (ECTA, 2006) consider Bayesian learners with continuous densities.
    - They show that only agents whose support has the lower dimension survive.
    - They generalise Sandroni's result to any equilibrium (from a complete or incomplete markets economy) that is Pareto Optimal

Generality

- Is the MSH correct in any equilibrium that decentralise a PO allocation? That is, does Sandroni (ECTA, 2000) and Blume and Easley (ECTA, 2006) cover all type of priors?
- No, Beker and Espino (JET, 2011) show that even if one agent learns, an agent who never learns survives.
- However, now we have a good understanding of sufficient reasonable conditions under which the MSH holds in PO allocations.

Definitions

- Let P_{s^t} be the posterior probabilities of P (conditional on s^t).

$$P_{s^t}(A) = \frac{P(A_{s^t})}{P(C(s^t))} \text{ for every } A \in \Im,$$

where $A_{s^t} = \{s \in S^\infty \mathit{such\ that}\ s = (s^t, \tilde{s}),\ \tilde{s} \in A\}$.

- Given any probability measure Q on $(S^\infty, \Im)$

$$\begin{aligned} dQ_t(s) &\equiv Q(C(s^t)), dQ_0 \equiv 1 \\ Q_t(s) &\equiv \frac{dQ_t(s)}{dQ_{t-1}(s)} \end{aligned}$$

Notions of Accuracy

- Let Q and $\tilde{Q}$ be two probability measures on $(S^{\infty}, \Im)$.
  - $\left\|\tilde{Q} - Q\right\| \equiv \sup_{A \in \Im} \left|\tilde{Q}(A) - Q(A)\right|$
  - $d_l(Q, \tilde{Q}) \equiv \max_{A \in \Im_l} \left|Q(A) - \tilde{Q}(A)\right|$
- **Definition 2** Agent i eventually makes accurate predictions on a path $s \in S^{\infty}$, $s = (s^t, \ldots)$, *if* $\left\|P^i_{s^t} - P_{s^t}\right\| \to 0$ as $t \to \infty$.
- **Definition 3** Agent i eventually makes accurate next period predictions on a path $s \in S^{\infty}$, $s = (s^t, \ldots)$, *if* $d_1(P^i_{s^t}, P_{s^t}) \to 0$ as $t \to \infty$.
- **Definition 4** Agent i always makes inaccurate next period predictions on a path $s \in S^{\infty}$, $s = (s_t, \ldots)$, *if there is* $\varepsilon > 0$ *such that* $d_1(P^i_{s_t}, P_{s_t}) > \varepsilon$ *for all* $t \in N$.
- Agent i's beliefs (weakly) merge with the truth if, $P - a.s.$, agent i eventually makes accurate (next period) predictions.

Merging and weak-merging

- Merging implies weak merging but the converse is no true.
  - $S = \{H, T\}$ The true probability of H is 1 but the agent believes at period t the probability of H is $1 - \frac{1}{t}$
  - The agent eventually make next period accurate predictions.
  - But he does not eventually make accurate predictions as, in every period t, he believes the event "H happens in all remaining periods" has probability zero when the true is 1.'
- A Bayesian learner with the true parameter in the support of his prior, might not merge with the truth, i.e. might not eventually make accurate predictions, even though he learns the true parameter with probability one.
  - $S = \{H, T\}$. The true process is iid: H occurs with probability $\theta^* = 0.5$.
  - The agent does not know p and has a prior with uniform density on $(0, 1)$.
  - By bayesian consistency, if $\theta^* \in U$, $\pi(U|s^t) \to 1$ as $t \to \infty$.
  - But the agent, in every period t, believes the event "the fraction of times H occurs is $\frac{1}{2}$ has probability zero while it has probability 1 (by the SLLN).

$$P_{s^t}(A) = \int P^\theta(A)\pi(d\theta|s^t)$$

Rational Expectations Equilibrium

- **Definition 9** A rational expectations equilibrium *is a probability measure* $\hat{P}$*, share prices* $\hat{q} = (\hat{q}_t,\ t \geq 0)$, *dividends* $\hat{d} = \left(\hat{d}_t,\ t \geq 0\right)$ *and initial shares of the trees* $\hat{\theta}^i_{-1}$*, such that* $(\hat{c}^i_t, \hat{\theta}^i_t,\ t \geq 0)$ *maximizes*

$$E^{\hat{P}} \left\{ \sum_{t=0}^{\infty} \beta_i^t u^i(c_t^i) \right\},$$

subject to

$$c_t^i + \hat{q}_t \theta_t^i \leq (\hat{q}_t + \hat{d}_t) k_{t-1}^i,\ w_t^i \geqslant 0,\ c_t^i \geqslant 0,\ t \geq 0.$$

Moreover,

$$\sum_{i=1}^{I} \hat{c}_t^i = \sum_{j=1}^{J} \hat{d}_{j,t}, \text{ and } \sum_{i=1}^{I} \hat{\theta}_{j,t}^i = 1,\ j = 1, ..., J,\ t \geq 0.$$

Survival

- **Definition 1** *Agent* i is driven out of the market on a path $s \in S^{\infty}$ *if agent* i*'s wealth,* $w_t^i(s)$*, converges to zero as* t *goes to infinity. Agent* i survives on a path $s \in S^{\infty}$ *if agent* i *is not driven out of the market on* s*.*

- **Definition 6** The entropy of agent i's beliefs at period t, $\mathcal{E}_t^i$, *is given by*

$$\mathcal{E}_t^i \equiv E^P \left(\log \left(\frac{P_{t+1}^i}{P_{t+1}} \right) \mid \Im_t \right)$$

- **Definition 7** The entropy of agent i, $\mathcal{E}^i$, *is given by*

$$\mathcal{E}^i \equiv \log \beta_i + \lim_{t \to \infty} \frac{1}{t} \sum_{1 \leqslant k \leqslant t} \mathcal{E}_k^i .$$

Survival and Wealth

- **Lemma 1:** *In every equilibrium, for any path $s \in S^\infty$ and agent $i \in \{1, ..., I\}$,*
$$c_t^i(s) \underset{t\to\infty}{\longrightarrow} 0 \text{ if and only if } w_t^i(s) \underset{t\to\infty}{\longrightarrow} 0.$$
- **Lemma A.2** *In every equilibrium, for every agent $i = 1, ..., I$,*
$$\beta_i^t dP_t^i u_i'(c_t^i) = u_i'(c_0^i) p(s^t) \text{ for all } t \in N.$$
- **Lemma 2:** *Fix an agent $i \in \{1, ..., I\}$ and a path $s \in S^\infty$. In every equilibrium, if there exists an agent $j \in \{1, ..., I\}$ such that*
$$\frac{\beta_i^t dP_t^i(s)}{\beta_i^t dP_t^j(s)} \underset{t\to\infty}{\longrightarrow} 0$$
then agent i is driven out of the market on s. Moreover, if for all agents $j \in \{1, ..., I\}$ there exists $\varepsilon > 0$ such that
$$\frac{\beta_i^t dP_t^i(s)}{\beta_j^t dP_t^j(s)} > \varepsilon$$
then agent i survives on s.

Main Theorems I

- **Lemma 3:** *For every agent* $i \in \{1, ..., I\}$, $P - a.s.$,

$$\infty > \lim_{t \to \infty} \frac{dP_t^i}{dP_t} \geqslant 0.$$

Moreover, $P - a.s.$, *agent* $i \in \{1, ..., I\}$ *eventually makes accurate predictions on a path* $s \in S^\infty$ *iff*

$$\infty > \lim_{t \to \infty} \frac{dP_t^i(s)}{dP_t(s)} > 0.$$

Definition

The measure Q is absolutely continuous with respect to $\tilde{Q}$ (denoted $Q << \tilde{Q}$) if for every set $A \in \mathcal{F}$, $\tilde{Q}(A) = 0 \Rightarrow Q(A) = 0$.

- $P << P_i \Leftrightarrow \infty > \lim_{t \to \infty} \frac{dP_t^i(s)}{dP_t(s)} > 0.$

Main Theorems II

- **Proposition 2:** *Assume that all agents have the same intertemporal discount factor and some agents eventually make accurate predictions. Then, in every equilibrium, P-a.s.,*
  - 1. *Any agent who does not eventually make accurate predictions on a path $s \in S^\infty$ is driven out of the market on the path s.*
    2. *Any agent who eventually makes accurate predictions on a path $s \in S^\infty$ survives on s.*

Main Theorems III

- **Proposition 3** *Assume that the ratio of beliefs and true probabilities over states of nature in the next period is uniformly bounded away from zero and infinity. In every equilibrium, $P-a.s.$, if the entropy of agent i is strictly smaller than the entropy of agent j on a path $s \in S^{\infty}$, $\left(\mathcal{E}^i(s) < \mathcal{E}^j(s)\right)$, then agent i is driven out of the market on s.*

Convergence to Rational Expectations

- **Proposition 6** Assume that all agents have the same intertemporal discount factor and some agents eventually make accurate predictions. Then, in every equilibrium, $P - a.s.$, in every path $s \in S^\infty$ the economy weakly converges to an $\hat{I}(s)-$ *rational expectations equilibrium.* Moreover, the set $\hat{I}(s)$ is the non-empty set of agents who eventually make accurate predictions on s.

Blume and Easley (ECTA, 2006)

1. They show that Sandroni's convergence to rational expectations equilibria result extends to any Pareto optimal equilibrium regardless of whether markets are complete or not.
2. They also extend Sandroni's result to the case where nobody eventually makes accurate predictions and there are several agents with the largest entropy.
3. They provide an example where markets are incomplete and the agent with correct beliefs is driven out of the market.

Blume and Easley (ECTA, 2006)

Theorem Clarke and Barron

For all θ,

$$\left(\log \frac{dP_t^{\theta}(s)}{dP_{i,t}(s)}\right) - \left(\frac{d}{2}\log\frac{t}{2\pi} + \frac{1}{2}\log\ det\ I(\theta) - \log\pi(\theta)\right) \to \chi^2(d).$$

Theorem

Suppose agent j has a prior belief that has positive density on the open manifold of dimension d_j, denoted by Θ_j, and agent i has a prior belief that has positive density on the open manifold of dimension d_i, denoted by Θ_i. If $d_j < d_i$ and $\Theta_j \subset \Theta_i$, then for all $\theta \in \Theta_j$, agent i vanishes in probability.

$$\begin{aligned} plim_{t\to\infty} \log\frac{dP_{i,t}(s)}{dP_{j,t}(s)} &= plim_{t\to\infty}\left(\log\frac{dP_t^{\theta}(s)}{dP_{j,t}(s)} - \log\frac{dP_t^{\theta}(s)}{dP_{i,t}(s)}\right) \\ &= \lim_{t\to\infty}\frac{1}{2}\log\frac{t}{2\pi}(d_j - d_i) = -\infty \end{aligned}$$