Consumption Dynamics in General Equilibrium: A Characterization When Markets Are Incomplete

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Introduction

- **The asymptotic properties of consumption and asset prices when markets are complete are well understood.**
  - If all agents have the same discount factor and beliefs then, regardless of agents' levels of risk aversion, the consumption and the wealth of any agent remains bounded away from zero in all periods.
  - If all agents have the same discount factor then those with less accurate beliefs will be driven out of the market.
- **We consider an infinite horizon economy with one good, two agents, one inside asset and (potentially) dynamically incomplete markets**.
  - If all agents have the same discount factor and beliefs, there is an endowment distribution such that the consumption of one agent vanishes with probability one.
  - In any equilibrium where *markets are incomplete forever* either (i) the consumption of one agent converges to zero or (ii) the consumption of both agents approaches zero infinitely often.

Introduction

Complete Markets and Homogeneous Beliefs

$$\left(\frac{\beta_2}{\beta_1}\right)^T \cdot \frac{u_2'(c_{2,T}(\omega))}{u_1'(c_{1,T}(\omega))} = \frac{u_2'(c_{2,0}(\omega))}{u_1'(c_{1,0}(\omega))}$$

(P.1) Equal discount rates: Regardless of the agents' level of risk aversion, the consumption and the wealth of any agent remains bounded away from zero all periods. Hence, asset prices reflect heterogeneous degrees of risk aversion.

$$\frac{u_2'(c_{2,T}(\omega))}{u_1'(c_{1,T}(\omega))} = \frac{u_2'(c_{2,0}(\omega))}{u_1'(c_{1,0}(\omega))}$$

(P.2) Different discount rates: Only the most patient survives

- Ramsey (EJ, 28), Becker (QJE, 80), Bewley (JME, 82), Rader (JET, 81).

$$\left(\frac{\beta_2}{\beta_1}\right)^T \cdot \frac{u_2'(c_{2,T}(\omega))}{u_1'(c_{1,T}(\omega))} = \frac{u_2'(c_{2,0}(\omega))}{u_1'(c_{1,0}(\omega))}$$

Introduction

- **Complete Markets and Heterogeneous Beliefs**:
  - The Market Selection Hypothesis holds (MSH) for a wide class of priors.
  - Sandroni (ECTA, 2000), Blume & Easley (ECTA, 2006), Beker & Espino (JET, 2011).

Definition: Agent i eventually makes accurate predictions on $\omega \in \Omega$ if

$$\lim_{t \to \infty} \left\| P^i_{s^t(\omega)} - P_{s^t(\omega)} \right\| = 0.$$

If $P \overset{loc}{\ll} P^i$, then, $P - a.s.\ \omega$,

$$\lim_{t \to \infty} \left\| P^i_{s^t(\omega)} - P_{s^t(\omega)} \right\| = 0 \text{ iff } 0 < \lim_{t \to \infty} \frac{P^i\left(s^t\left(\omega\right)\right)}{P\left(s^t\left(\omega\right)\right)} < \infty.$$

(P.3) On any path ω where some agent eventually makes accurate predictions, an agent survives if and only if she also makes accurate predictions.

$$\frac{P_2\left(s^T\left(\omega\right)\right)}{P_1\left(s^T\left(\omega\right)\right)} \cdot \frac{u_2'(c_{2,T}(\omega))}{u_1'(c_{1,T}(\omega))} = \frac{u_2'(c_{2,0}(\omega))}{u_1'(c_{1,0}(\omega))}$$

Introduction

- **BUT** even Bayesian agents (with diffuse priors), cannot make accurate predictions! (Blume and Easley)

- **What happens when no agent eventually makes accurate predictions?**
  - If on ω the entropy of agent i's beliefs is smaller than the entropy of agent j's beliefs, agent i vanishes on ω. (Sandroni)
  - Among Bayesian agents with equal discount rate and priors with support containing the true parameter, only those with the lowest dimensional support survive. (Blume & Easley)

If markets are complete, for a wide class of priors only the accuracy of beliefs or discount rates matter for asset pricing.

- Efficiency or Perfect Competition? → Incomplete Markets
  - Blume and Easley (Econometrica, 2006)

Beker and Chattopdhyay (JET, 2010)

- **Leading example.**
  - The factors determining survival with complete markets have little relevance when markets are dynamically incomplete.
  - Neither of the properties **(P1) – (P3)** hold in one good, two agents infinite horizon incomplete markets economy.
- **On almost every path where market incompleteness is *effective forever,***
  - some agent consumes arbitrarily close to zero infinitely often (Theorem 1)
  - no agent vanishes in a finite state time homogeneous Markov economy where individual endowments are bounded away from 0, agents have identical discount rates and beliefs. (Theorem 2)
  - in general, it is possible to find endowment distributions such that the consumption of one agent converges to zero. (Theorem 4)
- **Methodological Contribution**.

Leading Example

Time: $t = 1, 2, ...$

States: $Z \in \{\underline{z}, \bar{z}\}$ with prob. $p \in (0,1)$ and $(1-p)$.

Endowments: $z_1 = (0, Z_1, Z_2, ...), z_2 = (Z_0, 0, 0, ...)$ $Z_t \sim Z$ $\forall t \geq 1$.

Assets: $r_t(\omega) = Z_t(\omega)$.

Preferences: $u_1(x) = \frac{x^{1-\sigma}}{1-\sigma}$ $(\sigma > 0, \sigma \neq 0)$ $u_2(x) = \log x$.

Beliefs: $p_i \in (0,1)$

IDC Budget Set

$$BC_i(q) = \left\{ c_i \in \Psi_+ : \begin{array}{l} c_{i,t}(\omega) + q_t(\omega)\theta_{i,t}(\omega) \leq z_{i,t}(\omega) + Z_t(\omega)\theta_{i,t-1}(\omega) \\ \sup_{t \geq 0, \omega \in \Omega} |q_t(\omega) \cdot \theta_{i,t}(\omega)| < \infty \end{array} \right\}$$

IDC Maximizer

c_i is an *IDC maximizer* given q if

(i) $c_i \in BC_i(q)$.

(ii) there is no $\tilde{c}_i \in BC_i(q)$, with supporting portfolio $\tilde{\theta}_i$, for which

$$\lim_{T \to +\infty} \sum_{t=0}^{T} \beta_i^t E[u_i(\tilde{c}_{i,t})|\mathcal{F}_0] > \lim_{T \to +\infty} \sum_{t=0}^{T} \beta_i^t E[u_i(c_{i,t})|\mathcal{F}_0]$$

IDC Equilibrium

An *IDC equilibrium* $(c_1^*, c_2^*, \theta_1^*, \theta_2^*, q^*)$ is a market clearing allocation such that, at the prices q^*, c_i^*, with supporting portfolio θ_i^*, is an IDC maximizer for $i = 1, 2$.

CONSUMPTION PATHS

$$w_{2,t}(\omega) = Z_t(\omega) \cdot \theta_{2,t-1}(\omega),\ t \geq 1 \qquad w_{2,0} = Z_0$$

$$c_{2,t}(\omega) = (1-\beta_2) \cdot w_{2,t}(\omega) = (1-\beta_2) \cdot Z_t(\omega) \cdot \theta_{2,t-1}(\omega)$$

$$c_{1,t}(\omega) = Z_t(\omega)\left[1 - (1-\beta_2) \cdot \theta_{2,t-1}(\omega)\right]$$

$$q_t(\omega) = \beta_2 \cdot \frac{w_{2,t}(\omega)}{\theta_{2,t}(\omega)} = \beta_2 \cdot \frac{Z_t(\omega) \cdot \theta_{2,t-1}(\omega)}{\theta_{2,t}(\omega)}$$

EULER EQUATION FOR AGENT 1

$$E_1\left[\beta_1 \cdot Z_t \cdot \left(\frac{c_{1,t}}{c_{1,t+1}}\right)^{\sigma} | \mathcal{F}_t\right](\omega) = q_t(\omega)$$

$$\beta_1 \cdot E_1\left[Z_{t+1} \cdot \left(\frac{Z_t \cdot [1-(1-\beta_2)\cdot\theta_{2,t-1}]}{Z_{t+1}\cdot[1-(1-\beta_2)\cdot\theta_{2,t}]}\right)^{\sigma} | \mathcal{F}_t\right](\omega) = \beta_2 \cdot \frac{Z_t(\omega)\cdot\theta_{2,t-1}(\omega)}{\theta_{2,t}(\omega)}$$

$$\frac{Z_{t+1}(\omega)\ (1-\beta_2)\ \theta_{2,t}(\omega)}{Z_{t+1}^{\sigma}(\omega)\ [1-(1-\beta_2)\ \theta_{2,t}(\omega)]^{\sigma}} = \frac{\beta_2}{\beta_1} \frac{Z_{t+1}^{1-\sigma}(\omega)}{E_1[Z_{t+1}^{1-\sigma}|\mathcal{F}_t](\omega)} \frac{Z_t^{1}(\omega)\ (1-\beta_2)\ \theta_{2,t-1}(\omega)}{Z_t^{\sigma}(\omega)\ [1-(1-\beta_2)\ \theta_{2,t-1}(\omega)]^{\sigma}}$$

$$\frac{Z_{T+1}(\omega)\cdot(1-\beta_2)\cdot\theta_{2,T}(\omega)}{Z_{T+1}^{\sigma}(\omega)\cdot[1-(1-\beta_2)\cdot\theta_{2,T}(\omega)]^{\sigma}} = \left(\frac{\beta_2}{\beta_1}\right)^{T}\cdot\frac{\Pi_{t=1}^{T}Z_{t+1}^{1-\sigma}(\omega)}{(E_1[Z^{1-\sigma}])^{T}}\cdot\frac{Z_1(\omega)\cdot(1-\beta_2)\cdot\theta_{2,0}(\omega)}{Z_1^{\sigma}(\omega)\cdot[1-(1-\beta_2)\cdot\theta_{2,0}(\omega)]^{\sigma}}$$

$$\frac{1}{T}\log\left(\frac{Z_{T+1}(\omega)\cdot(1-\beta_2)\cdot\theta_{2,T}(\omega)}{Z_{T+1}^{\sigma}(\omega)\cdot[1-(1-\beta_2)\cdot\theta_{2,T}(\omega)]^{\sigma}}\right) = \log\left(\frac{\beta_2}{\beta_1}\right) + \underbrace{\left(\frac{1}{T}\sum_{t=1}^{T}\log[Z_{t+1}(\omega)]^{1-\sigma}\right) - \log E_1[Z^{1-\sigma}]}_{<0}$$

$$+\underbrace{\frac{1}{T}\cdot\log\left(\frac{Z_1(\omega)\cdot(1-\beta_2)\cdot\theta_{2,0}(\omega)}{Z_1^{\sigma}(\omega)\cdot[1-(1-\beta_2)\cdot\theta_{2,0}(\omega)]^{\sigma}}\right)}_{\longrightarrow\ 0}$$

$$\frac{1}{T}\sum_{t=1}^{T}\log[Z_{t+1}(\omega)]^{1-\sigma}\to E[\log Z^{1-\sigma}]\quad P-\text{a.s.}$$

$$\lim_{T\to\infty}\left(\frac{1}{T}\sum_{t=1}^{T}\log[Z_{t+1}(\omega)]^{1-\sigma}\right) - \log E[Z^{1-\sigma}] < 0 \quad P-\text{a.s.}$$

Suppose $p_1 = p$ and $\beta_1 = \beta_2 = \beta$.

$$\operatorname{Log}\left(\frac{Z_{T+1}(\omega)\cdot(1-\beta_2)\cdot\theta_{2,T}(\omega)}{Z_{T+1}^{\sigma}(\omega)\cdot[1-(1-\beta_2)\cdot\theta_{2,T}(\omega)]^{\sigma}}\right)\to-\infty\quad P-\text{a.s}$$

$$\Leftrightarrow\frac{Z_{T+1}(\omega)\cdot(1-\beta_2)\cdot\theta_{2,T}(\omega)}{Z_{T+1}^{\sigma}(\omega)\cdot[1-(1-\beta_2)\cdot\theta_{2,T}(\omega)]^{\sigma}}\to 0\Leftrightarrow\theta_{2,T}(\omega)\to 0\Leftrightarrow c_{2,T}(\omega)\to 0\quad P-\text{a.s}$$

IDC Equilibrium

$$0 < \theta^*_{2,t}(\omega) < \frac{1}{1-\beta_2}$$

$$c^*_i = \left\{c^*_{i,t}\right\}_{t=0}^{\infty} \text{ where} \begin{cases} c^*_{1,t} = Z_t(\omega) \cdot \left[1 - (1-\beta_2) \cdot \theta^*_{2,t-1}(\omega)\right] \\ c^*_{2,t} = Z_t(\omega) \cdot (1-\beta_2) \cdot \theta^*_{2,t-1}(\omega) \end{cases}$$

$$q^*_t(\omega) \cdot \theta^*_{2,t}(\omega) = \beta_2 \cdot w_{2,t}(\omega) = \beta_2 \cdot Z_t(\omega) \cdot \theta^*_{2,t-1}(\omega) < \beta_2 \cdot \overline{z} \cdot \frac{1}{1-\beta_2}$$

$(c^*_1, c^*_2, \theta^*_1, \theta^*_2, q^*)$ is an *IDC equilibrium in which* $c^*_{2,t}(\omega) \to 0 \quad P-$a.s.

- **(P1) does not hold when markets are incomplete.**

Remark: $\exists\, p_1 \neq p$ and $\beta_1 < \beta_2$ such that $(c^*_1, c^*_2, \theta^*_1, \theta^*_2, q^*)$ is an *IDC equilibrium in which* $c^*_{2,t}(\omega) \to 0 \quad P-$a.s.

- **Neither (P2) nor (P3) hold when markets are incomplete:**

  A marginally more impatient agent with incorrect beliefs drives the agent with correct beliefs out of the market !!!

- **In every equilibrium of the example,**

$$\beta_2 \cdot \frac{u_2'\left(c^*_{2,t+1}(\omega)\right)}{u_2'\left(c^*_{2,t}(\omega)\right)} \cdot \frac{r_{t+1}(\omega)}{q_t(\omega)} = \beta_2 \cdot \frac{c^*_{2,t}(\omega)}{c^*_{2,t+1}(\omega)} \cdot \frac{Z_{t+1}(\omega)}{q_t(\omega)} = 1$$

$$\beta_2^T \cdot \prod_{t=0}^{T-1} \frac{r_{t+1}(\omega)}{q_t(\omega)} = \prod_{t=0}^{T-1} \beta_2 \cdot \frac{Z_{t+1}(\omega)}{q_t(\omega)} = \frac{Z_T(\omega)}{Z_0(\omega)} \cdot \theta_{T-1}(\omega) \to 0$$

- **The example imposes a lot of structure on preferences, endowments, etc**
  - The result does not depend on the structure.
  - The structure guarantees that agent 2 vanishes in *every* equilibrium.

- **In every equilibrium of the example,**

$$\beta^T \cdot \frac{u_2'(c_{2,T})}{u_2'(c_{2,0})} \cdot \prod_{k=0}^{T-1} \frac{r_{k+1}(\omega)}{q_k(\omega)} = 1 \quad \forall \omega, \forall T \qquad \textbf{(E}_1\textbf{)}$$

$$\beta^T \cdot \prod_{k=0}^{T-1} \frac{r_{k+1}(\omega)}{q_k(\omega)} \to 0 \qquad \forall \omega \qquad \textbf{(E}_2\textbf{)}$$

- **(E_1) and (E_2) imply that $c_{2,T}(\omega) \longrightarrow 0$.**

- **In general, if markets are incomplete forever then (E_1) implies (E_2).**
  - $E_P\left[\beta \cdot \frac{u_1'(c_{1,t+1})}{u_1'(c_{1,t})} \cdot \frac{r_{t+1}}{q_t} \middle| \mathcal{F}_t\right](\omega) = 1 \ \forall t \geq 0$
  - $\beta^T \cdot \frac{u_1'(c_{1,T}(\omega))}{u_1'(c_{1,0}(\omega))} \cdot \prod_{k=0}^{T-1} \frac{r_{k+1}(\omega)}{q_k(\omega)} = \prod_{k=0}^{T-1} \beta \cdot \frac{u_1'(c_{1,k+1}(\omega))}{u_1'(c_{1,k}(\omega))} \cdot \frac{r_{k+1}(\omega)}{q_k(\omega)}$
  - **It converges to zero if there is ε > 0 such that**

$$\limsup_t var\left[\log\left(\frac{u_1'(c_{1,t+1})}{u_1'(c_{1,t})} \cdot \frac{r_{t+1}}{q_t}\right) \middle| \mathcal{F}_t\right](\omega) > \epsilon$$

The General Case: The Economy

Time: $t = 1, 2, ...$

Agents: $\mathcal{I} := \{1, 2\}$

States: $\mathcal{S} := \{1, 2, \cdots, S\}$.

$\Omega := \mathcal{S}^\infty$ with typical element $\omega = (s_1, s_2, \cdots)$.

$\Omega(s^t) := \{\omega \in \Omega \colon \omega = (s^t, s_{t+1}, \cdots)\}$ is a *cylinder*, $s^t \in \mathcal{S}^t$.

$\{\mathcal{F}_t\}_{t=1}^\infty$ where $\mathcal{F}_{t-1} \subset \mathcal{F}_t$ for all $t \geq 1$.

$dQ_t(\omega) := Q(\Omega(s^t(\omega)))$ is $\mathcal{F}_t-$measurable.

$Q_t(\omega) := \frac{dQ_t(\omega)}{dQ_{t-1}(\omega)}$ is $\mathcal{F}_t-$measurable.

$(\Omega, \mathcal{F}, P)$, $(\Omega, \mathcal{F}, P_i)$, $i = 1, 2$ $\quad \underline{p} := \inf_{t \geq 0, \omega \in \Omega} P_t(\omega)$.

The General Case: Assumptions

Beliefs: A.1: $0 < \underline{p} \leq \inf_{t \geq 0, \omega \in \Omega} P_{i,t}(\omega)$.

Endowments: A.2: $Z_t(\omega) \in [\underline{z}, \bar{z}] \subset R_{++}\ \forall t \geq 0\ \&\ \forall \omega \in \Omega$.

Preferences: A.3: $u_i : R_+ \rightarrow R$ satisfies Inada conditions.

Asset Return: A.4: $r_t(\omega) \in [\underline{r}, \bar{r}] \subset R_{++}$.

Survival

$$y_t(\omega) := \frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))}$$ → This is the process we need to study!!!

Agent 2 survives on $\omega \Leftrightarrow 0 \leq \liminf_t y_t(\omega) < \infty$.

Agent 2 vanishes on $\omega \Leftrightarrow \lim_t y_t(\omega) = \infty$.

Consumption Dynamics I

- Introduce one asset (assume identical discount rates)

$$\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_{t+1}(\omega)}{y_t(\omega)} = \frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{\frac{u_2'(c_{2,t+1}(\omega))}{u_1'(c_{1,t+1}(\omega))}}{\frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))}} = \frac{P_{2,t}(\omega) \cdot \frac{u_2'(c_{2,t+1}(\omega))}{u_2'(c_{2,t}(\omega))}}{P_{1,t}(\omega) \cdot \frac{u_1'(c_{1,t+1}(\omega))}{u_1'(c_{1,t}(\omega))}}$$

If $\frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_{t+1}(\omega)}{y_t(\omega)} < 1$ with conditional probability 1

$$\Rightarrow P_{2,t}(\omega) \cdot \frac{u_2'(c_{2,t+1}(\omega))}{u_2'(c_{2,t}(\omega))} < P_{1,t}(\omega) \cdot \frac{u_1'(c_{1,t+1}(\omega))}{u_1'(c_{1,t}(\omega))}$$ with conditional probability1

This contradicts $q_t(\omega) = \beta \cdot E_{P_i}\left[r_{t+1} \cdot \frac{u_i'(c_{i,t+1})}{u_i'(c_{i,t})} \middle| \mathcal{F}_t\right](\omega)$

Trading one assset implies

$$\frac{dP_{2,t}(\omega)}{dP_{1,t}(\omega)} \cdot y_t(\omega) := \frac{dP_{2,t}(\omega)}{dP_{1,t}(\omega)} \cdot \frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))}$$

grows (weakly) with positive conditional probability.

Consumption Dynamics II

- If $var\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1}\right](\omega) \geq \epsilon > 0$ and $y_{t-1}(\omega) > \underline{y}$,

$$\exists\gamma > 0 \text{ such that } \begin{cases} P\left[1-\gamma \geq \frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0 \\ \text{and} \\ P\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \geq 1+\gamma \middle| \mathcal{F}_{t-1}\right](\omega) \geq \underline{p} > 0. \end{cases}$$

- $V_0 \equiv \left\{\omega \in \Omega : \lim_t \text{var}\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1}\right](\omega) = 0\right\}$
- $V_\epsilon \equiv \left\{\omega \in \Omega : \limsup_t \text{var}\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1}\right](\omega) \geq \epsilon\right\}$

$$\Delta_t^\epsilon(\omega) := \inf\left\{k \geq 1 : \text{var}\left[\frac{P_{2,t+k}}{P_{1,t+k}} \frac{y_{t+k}}{y_{t+k-1}} \middle| \mathcal{F}_{t+k-1}\right](\omega) \geq \epsilon\right\}.$$

$$V_0 = \cup_{\epsilon>0} V_\epsilon.$$

- $V_{T,\epsilon} \equiv \left\{\omega \in V_\epsilon : \sup_t \Delta_t^\epsilon(\omega) = T\right\}$

Theorem I

- $V_{T,\epsilon}$ is the set of paths where the variable $\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}}$ displays conditional variability at least ϵ at least once in every span of T periods.
- $L_{\underline{\lambda},\overline{\lambda}} \equiv \{\omega \in \Omega : \underline{\lambda} < \liminf \frac{dP_{j,t}(\omega)}{dP_{i,t}(\omega)} \leq \limsup \frac{dP_{j,t}(\omega)}{dP_{i,t}(\omega)} < \overline{\lambda}\}$

THEOREM 1: Consider an IDC equilibrium. Assume $\beta_1 = \beta_2$ and A.1, A.2, A.3, and A.4. Then,

(i) $\lim_t \frac{P_{2,t}(\omega)}{P_{1,t}(\omega)} \cdot \frac{y_t(\omega)}{y_{t-1}(\omega)} = 1$ P-a.s. $\omega \in V_0$.

(ii) For every $T < \infty$, $\epsilon > 0$, $n \geq 1$, $\underline{\lambda} > 0$ and $\overline{\lambda} < \infty$,

$\limsup_t c_{i,t}(\omega) \leq 1/n \; P - a.s. \; \omega \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \{\omega : \liminf_t c_{j,t}(\omega) > 1/n\}$.

If markets are incomplete forever either
(a) the consumption of both agents is arbitrarily close to zero infinitely often or
(b) the consumption of one agent converges to zero.

Proof of Theorem 1 (ii)

(ii) For every $T < \infty$, $\epsilon > 0$, $n \geq 1$, $\underline{\lambda} > 0$ and $\overline{\lambda} < \infty$,
$\limsup_t c_{1,t}(\omega) \leq 1/n \;\; P-a.s.\; \omega \in V_{T,\epsilon} \cap L_{\underline{\lambda},\overline{\lambda}} \cap \{\omega : \liminf_t c_{2,t}(\omega) > 1/n\}$.

Sketch of Proof of Theorem 1(ii)

$\liminf_t c_{2,t}(\omega) > 1/n$

$$\Rightarrow \; y_t \leq \overline{y}_n = \frac{u_2'(1/n)}{u_1'(\underline{z} - 1/n)}$$

Suppose $c_{1,t}(\omega) > 1/n$

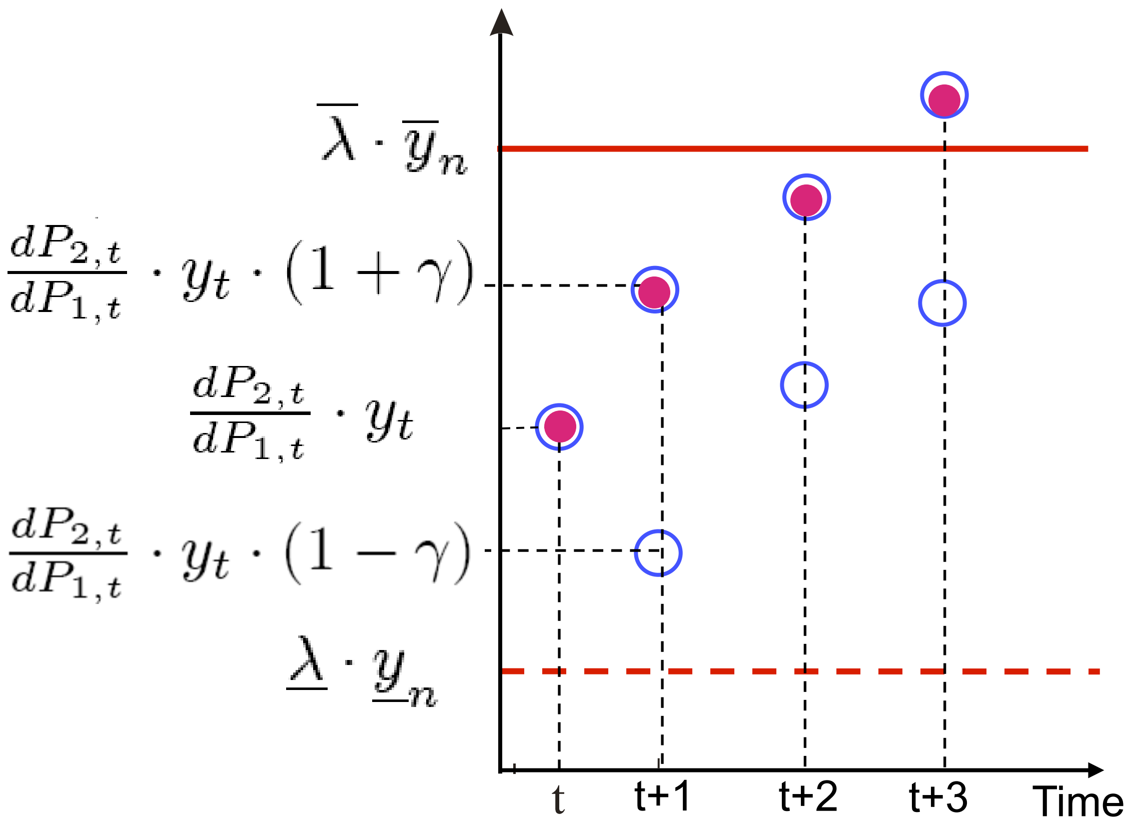
$$\underline{y}_n = \frac{u_2'(\overline{z} - 1/n)}{u_1'(1/n)}$$

To simplify, suppose $\forall t$:

$$\mathrm{var}\left[\frac{P_{2,t}}{P_{1,t}} \cdot \frac{y_t}{y_{t-1}} \middle| \mathcal{F}_{t-1} \right](\omega) \geq \epsilon$$

$T_n(\gamma)$ satisfies

$$\underline{\lambda} \cdot \underline{y}_n \cdot (1+\gamma)^{T_n(\gamma)} > \overline{\lambda} \cdot \overline{y}_n$$



When will no one Vanish?- Markov Economies

ASSUMPTION A.5: Suppose $Z_t(\omega)$ is a first-order time homogenous Markov process. The individual endowment and the asset payoff are also time homogeneous Markov processes defined as $z_{i,t}(\omega) \equiv z_i(Z_t(\omega)) \in [\underline{z}_i, \overline{z}_i]$ and $r_t(\omega) \equiv r(Z_t(\omega))$.

Properties of the economy where only j consumes and $P_j = P$.

- $q_j(z) \equiv \beta \cdot E_P\left(\frac{u_j'(Z_{t+1})}{u_j'(Z_t)} \cdot r(Z_{t+1}) \middle| Z_t = z \right).$
- $E_P\left(\frac{u_j'(Z_t)}{u_j'(Z_{t-n})} \cdot \prod_{\tau=t-n}^{t-1} \frac{r(Z_{\tau+1})}{q_j(Z_\tau)} \middle| Z_{t-n} = z \right)(\omega) = \left(\frac{1}{\beta}\right)^n \quad \forall z \in \mathcal{S},\ \forall n \geq 1.$
- $\prod_{\tau=t-n}^{t-1} \frac{r(Z_{\tau+1}(\omega))}{q_j(Z_\tau(\omega))} \geq \frac{u_j'(\overline{z})}{u_j'(\underline{z})} \cdot \left(\frac{1}{\beta}\right)^n$ infinitely often $P - a.s.$.
- $B_{i,t}(\omega) \equiv q_t(\omega) \cdot \theta_{i,t}(\omega).$

THEOREM 2: Consider an IDC equilibrium. Assume A.1-A.5 and that $\underline{z}_i > 0$. Suppose that for every $z' \in S$ there is $z \in S$ such that

$$\frac{r(z)}{q_j(z')} \cdot B\left[z', z'\right] + z_i(z) \neq B(z,z) \quad \text{where} \quad B(z,z') \equiv -\frac{z_i(z')}{\frac{r(z')}{q_j(z)} - 1}.$$

Then, the set $A \equiv \{\omega : c_{i,t}(\omega) \to 0\}$ has $P-$measure zero.

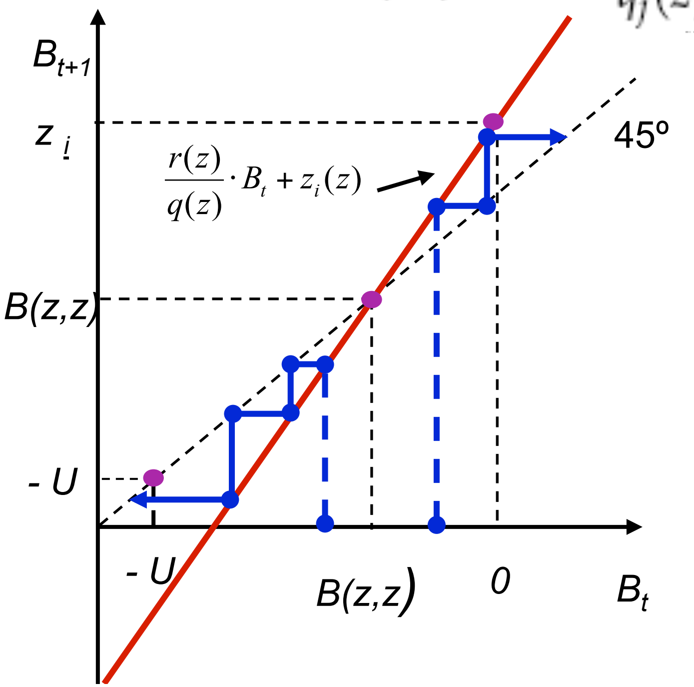
Sketch of Proof

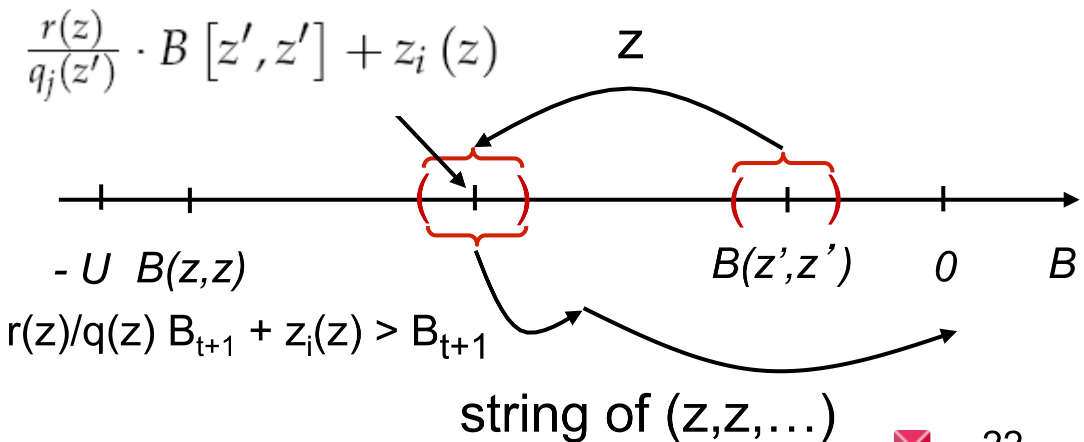
STEP 1: $q_t(\omega) \to q_j(Z_t(\omega))$, $P - a.s.\ \omega \in A$.

STEP 2: $\liminf B_{i,t}(\omega) < 0$, $P-$ a.s. $\omega \in A$.

STEP 3: $\limsup B_{i,t}(\omega) \leq \overline{B} < 0 \quad P-$ a.s. $\omega \in A$.

STEP 4: If $P(A) > 0$, $\frac{r(z)}{q_j(z)} > 1$ for all $z \in \mathcal{S}$. $\frac{\ldots B_{i,t+s}(\omega)}{\prod_{\tau=t}^{t+s-1} \frac{r_{\tau+1}(\omega)}{q_\tau(\omega)}} \Bigg) + \frac{B_{i,t+T}(\omega)}{\prod_{\tau=t}^{t+T-1} \frac{r_{\tau+1}(\omega)}{q_\tau(\omega)}}$





The Dynamics of the Ratio of Marginal Utilities

- $\hat{r}_{i,t}(\omega) := \dfrac{r_t(\omega) \cdot u_i'(c_{i,t}(\omega))}{E_{P_i}[r_t \cdot u_i'(c_{i,t}) | \mathcal{F}_{t-1}](\omega)} \quad E_{P_i}[\hat{r}_{i,t} | \mathcal{F}_{t-1}](\omega) = 1$

$$\frac{u_2'(c_{2,t-1}(\omega))}{u_1'(c_{1,t-1}(\omega))} = \frac{\beta_2}{\beta_1} \cdot \frac{E[r_t \cdot u_2'(c_{2,t}) | \mathcal{F}_{t-1}](\omega)}{E[r_t \cdot u_1'(c_{1,t}) | \mathcal{F}_{t-1}](\omega)}$$

$$= \frac{\beta_2}{\beta_1} \cdot \frac{r_t\,(\omega) \cdot u_1'(c_{1,t}(\omega))}{E[r_t \cdot u_1'(c_{1,t}) | \mathcal{F}_{t-1}](\omega)} \cdot \frac{E[r_t \cdot u_2'(c_{2,t}) | \mathcal{F}_{t-1}](\omega)}{r_t\,(\omega) \cdot u_2'(c_{2,t}\,(\omega))} \cdot \frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))}$$

$$\frac{u_2'(c_{2,t-1}(\omega))}{u_1'(c_{1,t-1}(\omega))} = \frac{\beta_2}{\beta_1} \cdot \frac{\hat{r}_{1,t}(\omega)}{\hat{r}_{2,t}(\omega)} \cdot \frac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))}$$

- It follows that $\dfrac{u_2'(c_{2,t}(\omega))}{u_1'(c_{1,t}(\omega))} = \dfrac{\beta_1}{\beta_2} \cdot \dfrac{\hat{r}_{2,t}(\omega)}{\hat{r}_{1,t}(\omega)} \cdot \dfrac{u_2'(c_{2,t-1}(\omega))}{u_1'(c_{1,t-1}(\omega))}$

Economies with Equilibria where Someone Vanishes

- In the example,

$$\hat{r}_{2,t}(\omega) \quad = \quad \frac{Z_t(\omega) \cdot u_2'(c_{2,t}(\omega))}{E[Z_t \cdot u_2'(c_{2,t})|\mathcal{F}_{t-1}](\omega)} = 1$$

$$\hat{r}_{1,t}(\omega) \quad = \quad \frac{Z_t(\omega) \cdot u_1'(c_{1,t}(\omega))}{E[Z_t \cdot u_1'(c_{1,t})|\mathcal{F}_{t-1}](\omega)} \to \frac{Z_t(\omega) \cdot u_1'(Z_t(\omega))}{E[Z_t \cdot u_1'(Z_t)|\mathcal{F}_{t-1}](\omega)}$$

where the limit is not degenerate, so Jensen's inequality applies.

$$\frac{u_2'(Z_t - c_{1,t}(\omega))}{u_1'(c_{1,t}(\omega))} = \frac{\beta_1}{\beta_2} \cdot \frac{1}{\hat{r}_{1,t}(\omega)} \cdot \frac{u_2'(Z_{t-1} - c_{1,t-1}(\omega))}{u_1'(c_{1,t-1}(\omega))} \qquad (1)$$

$$\frac{u_2'(Z_t(\omega) - c_{1,t}(\omega))}{u_1'(c_{1,t}(\omega))} = \left(\frac{\beta_1}{\beta_2}\right)^T \cdot \left(\prod_{t=1}^{T} \frac{1}{\hat{r}_{1,t}(\omega)}\right) \cdot \frac{u_2'(Z_0 - c_{1,0})}{u_1'(c_{1,0})} \qquad (2)$$

The Construction

- We construct <u>feasible</u> consumption processes $\widehat{c}_1$ & $\widehat{c}_2$ satisfying:

$$\beta_1 \cdot \frac{E[r_t \cdot u_1'(Z_t - c_{2,t})|\mathcal{F}_{t-1}](\omega)}{u_1'(Z_{t-1}(\omega) - c_{2,t-1}(\omega))} = \beta_2 \cdot \frac{r_t(\omega) \cdot u_2'(c_{2,t}(\omega))}{u_2'(c_{2,t-1}(\omega))} \qquad (1)$$

- Notice that (1) holds iff

$$\frac{\beta_1}{\beta_2} \cdot E[r_t \cdot u_1'(Z_t - c_{2,t})|\mathcal{F}_{t-1}](\omega) = \frac{r_t(\omega) \cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)}$$

- Let $\lambda \equiv \frac{r \cdot u_2'(c_2)}{y}$. A solution to (1) exists iff there is λ^* such that

$$f_{t,\omega,y}(\lambda^*) \equiv \frac{\beta_1}{\beta_2} \cdot E[r_t \cdot u_1'(Z_t - (u_2')^{-1}\left(\frac{y \cdot \lambda^*}{r_t}\right))|\mathcal{F}_{t-1}](\omega) = \lambda^*$$

- We show the function above has a fixed point

Theorem 4

- Define the (personalized) Arrow-Debreu prices as

$$p_{i,t}^{\widehat{c}_i} \equiv \beta_i^t \cdot \frac{u_i'\left(\widehat{c}_{i,t}\left(\omega\right)\right)}{u_i'\left(\widehat{c}_{i,0}\left(\omega\right)\right)}$$

- The obtained feasible consumption processes $\widehat{c}_1$ & $\widehat{c}_2$ satisfy:

  Euler equations and $\sum_{t=0}^{\infty} E_P\left[p_{i,t}^{\widehat{c}_i} | \mathcal{F}_0\right](\omega) < \infty$ (under A.6).

  A.6: For $i \in \mathcal{I}$, $\beta_i < 1/M$, $M \equiv \max\left\{\frac{\bar{r}\cdot u_2'(\underline{z}/2)}{\underline{r}\cdot u_2'(\bar{z})}; \frac{\beta_1}{\beta_2}\frac{\bar{r}\cdot u_1'(\underline{z}/2)}{\underline{r}\cdot u_1'(\bar{z})}\right\}$.

- $(\widehat{c}_1, \widehat{c}_2, 0, 0, \widehat{q})$ is a no trade IDC equilibrium.

When Does Agent 2 Vanishes?

- If the asset is a real bond, $r_t(\omega) = 1$, and there is aggregate aggregate risk, $\operatorname{var}(Z_t | \mathcal{F}_{t-1}) > \epsilon > 0$, then

$$u_2'(\widehat{c}_{2,t}(\omega)) \text{ is } \mathcal{F}_{t-1} - \text{measurable}$$

$$\operatorname{var}\left[\log \underbrace{\left(\frac{u_1'(Z_t - \widehat{c}_{2,t})}{E_P\left[u_1'(Z_t - \widehat{c}_{2,t}) | \mathcal{F}_{t-1}\right]}\right)}_{r_{1,t}(\omega)} \middle| \mathcal{F}_{t-1}\right](\omega) > \epsilon' > 0$$

- $\frac{1}{T}\sum_{t=1}^{T} \log \hat{r}_{1,t}(\omega) \to \frac{1}{T}\sum_{t=1}^{T} E_{P_1}[\log \hat{r}_{1,t} | \mathcal{F}_{t-1}](\omega) < 0$

$$\prod_{t=1}^{T} \hat{r}_{1,t}(\omega) \to 0 \quad P - \text{a.s.}$$

- $(\widehat{c}_1, \widehat{c}_2, 0, 0, \widehat{q})$ is an *IDC* equilibrium where $\widehat{c}_{2,t}(\omega) \to 0$, $P -$ a.s.

Robustness I

Can we have an IDC equilibrium where agent 2's endowment is bounded away from zero and agent 2 vanishes?

- Consider a no trade IDC equilibrium where $\widehat{c}_{2,t} \to 0$.
- Two properties of our construction:

1. $\frac{r_{t+1}(\omega)}{q_t^*(\omega)} = \frac{u_2'\left(c_{2,t+1}^*(\omega)\right)}{\beta_2 u_2'\left(c_{2,t}^*(\omega)\right)} > \frac{1}{\beta_2 M} > 1, \forall t \geq 0$ and $\omega \in \Omega$..
2. There is $\overline{M} < +\infty$ such that $\frac{r_{t+1}(\omega)}{q_t^*(\omega)} \leq \overline{M}, \forall t \geq 0$ and $\omega \in \Omega$.

- Since $z_{2,t}(\omega) \to 0$, there is $\widetilde{T}(\omega) \geq 1$ such that

$$0 < z_{2,t}(\omega) \leq \frac{Z_t(\omega)}{2} \forall t \geq \widetilde{T}(\omega).$$

- Pick $0 < \overline{\varepsilon}(\omega) < z_{2,\widetilde{T}(\omega)}(\omega)$ so that $\overline{\varepsilon}(\omega) \cdot (\overline{M} - 1) \leq \frac{z}{2}$.

Robustness II

- Define agent 2's endowment, $\widetilde{z}_2$, as follows:

$$\widetilde{z}_{2,t}(\omega) = \begin{cases} z_{2,t}(\omega) & \text{if } t < \widetilde{T}(\omega) \\ z_{2,\widetilde{T}(\omega)}(\omega) - \overline{\varepsilon}(\omega) & \text{if } t = \widetilde{T}(\omega) \\ z_{2,t}(\omega) + \overline{\varepsilon}(\omega) \cdot \left[\frac{r_t(\omega)}{\widehat{q}_{t-1}(\omega)} - 1 \right] & \text{otherwise} \end{cases}$$

and agent 1's endowment, $\widetilde{z}_1$, as $Z_t(\omega) - \widetilde{z}_{2,t}(\omega)$.

- Now, consider portfolios $\left(\widetilde{\theta}_1, \widetilde{\theta}_2\right)$ where $\widetilde{\theta}_1 = -\widetilde{\theta}_2$ and

$$\widetilde{\theta}_{2,t}(\omega) = \begin{cases} 0 & \text{if } t \leq \widetilde{T}(\omega) - 1 \\ -\frac{\overline{\varepsilon}(\omega)}{\widehat{q}_t(\omega)} & \text{if } t \geq \widetilde{T}(\omega) \end{cases}$$

- $\widetilde{\theta}_i$ supports consumption process $\widehat{c}_i$ at prices $\widehat{q}$
- $\left|\widehat{q}_t(\omega) \cdot \widetilde{\theta}_{i,t}(\omega)\right| \leq \overline{\varepsilon}(\omega) \leq \frac{\overline{z}}{2}$ for all $t \geq 0$ and $\omega \in \Omega$.

$(\widehat{c}_1, \widehat{c}_2, \widetilde{\theta}_1, \widetilde{\theta}_2, \widehat{q})$ is an *IDC* equilibrium in which $\widehat{c}_{2,t}(\omega) \to 0 \quad P -$ a.s.

Conclusions

- **On any path on which market incompleteness is effective forever, some agent's consumption is arbitrarily close to zero infinitely often and it might even be the case that one of the agents eventually consumes zero.**

- **No agent vanishes in a Markovian economy where individual endowments are bounded away from zero and agents have homogeneous beliefs and discount rates.**

- **In general, homogeneous beliefs and discount rates are not incompatible with vanishing consumption. Hence, both the Ramsey Conjecture as well as the MSH may fail under incomplete markets. Extreme consumption (and wealth) inequality is shown to be compatible with perfect competition.**

- **The role played by the heterogeneity of beliefs in determining the survival prospects of an agent is disentangled from that played by the market structure.**

- **A Methodologial Innovation is introduced: we show how to construct equilibria under incomplete markets and to characterize its long run properties. (Theorems 2 – 4)**

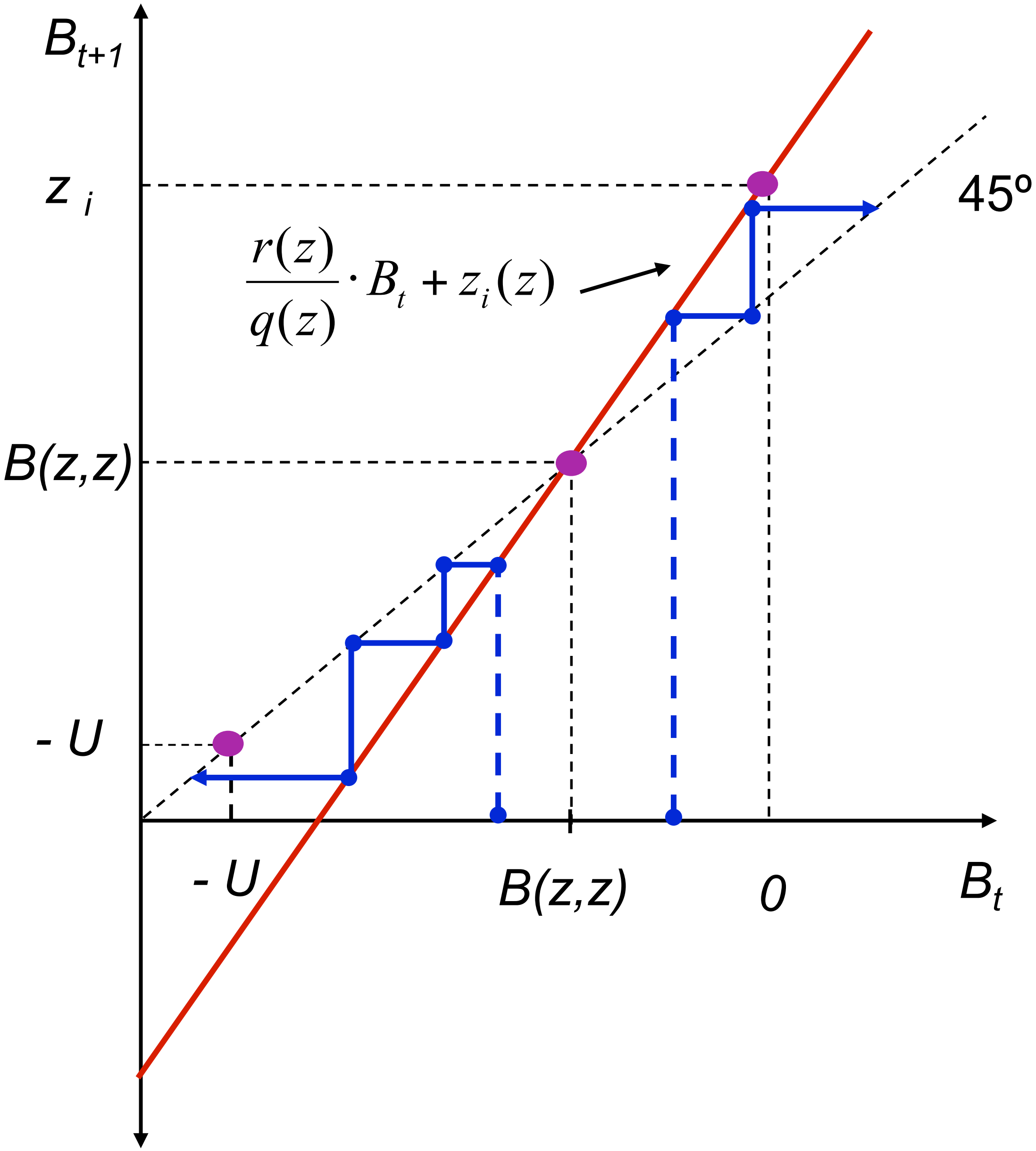
THE END

MANY THANKS!!

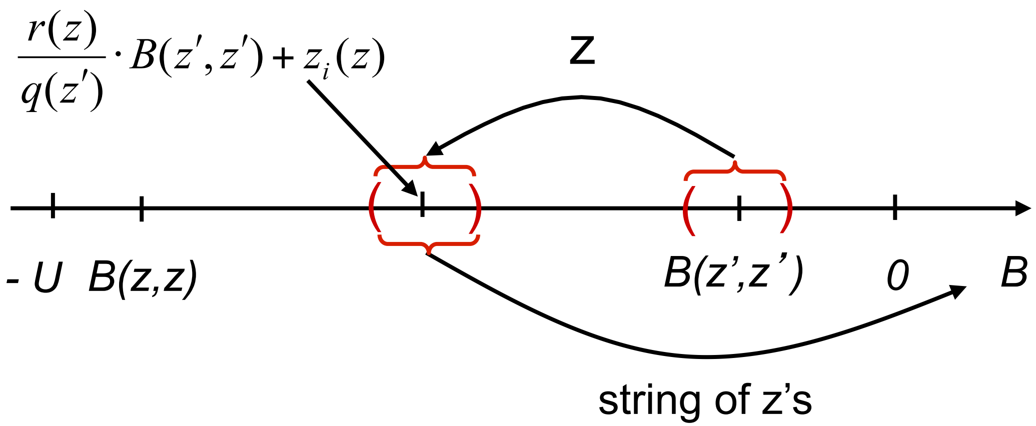
Blume and Easley's Example vs This Paper

- **They have an example where an agent with incorrect beliefs drives out of the market an agent with correct beliefs.**
  - If agents had homogeneous beliefs, markets would be complete in their example. So, their example does not say anything about survival in incomplete market economies when agents have homogeneous beliefs.
  - Our example works even if agents have homogeneous beliefs.

- **We have some general results for incomplete markets economies**.
  - Our Theorem 1 characterizes consumption behavior in one good two agent economies with incomplete markets.
  - Methodological Innovation: useful to construct equilibria and to analyze the asymptotic behavior of consumption in these economies. It generalizes the leading example and applies even when equilibrium saving rates are not ordered. (Theorems 4)



B_{t+1}
z_i
$B(z,z)$
- U
$\frac{r(z)}{q(z)} \cdot B_t + z_i(z)$
45º
- U
B(z,z)
0
B_t



$\frac{r(z)}{q(z')} \cdot B(z', z') + z_i(z)$
z
- U
B(z,z)
B(z',z')
0
B
string of z's

- We construct <u>feasible</u> consumption processes $\widehat{c}_1$ & $\widehat{c}_2$ satisfying (1). (Prop. 4)

$$\beta_1 \cdot \frac{E[r_t \cdot u_1'(Z_t - c_{2,t})|\mathcal{F}_{t-1}](\omega)}{u_1'(Z_{t-1}(\omega) - c_{2,t-1}(\omega))} = \beta_2 \cdot \frac{r_t(\omega) \cdot u_2'(c_{2,t}(\omega))}{u_2'(c_{2,t-1}(\omega))} \qquad (3)$$

Notice that (3) holds iff $\dfrac{\beta_1}{\beta_2} \cdot E[r_t \cdot u_1'(Z_t - c_{2,t})|\mathcal{F}_{t-1}](\omega) = \dfrac{r_t(\omega) \cdot u_2'(c_{2,t}(\omega))}{y_{t-1}(\omega)}$

Let $\lambda \equiv \frac{r \cdot u_2'(c_2)}{y}$. A solution to (3) exists iff there exists λ^* such that

$$\frac{\beta_1}{\beta_2} \cdot E[r_t \cdot u_1'(Z_t - (u_2')^{-1}\left(\frac{y \cdot \lambda^*}{r_t}\right))|\mathcal{F}_{t-1}](\omega) = \lambda^*$$

LEMMA 9: Assume A.2, A.3, and A.5. For $t \geq 1$ and $\omega \in \Omega$, and $y > 0$, let $\underline{\lambda}(t-1, \omega, y) \equiv \frac{r_t(\widetilde{\omega}) \cdot u_2'(Z_t(\widetilde{\omega}))}{y}$ for some $\widetilde{\omega} \in \Omega\left(s^{t-1}(\omega)\right)$ such that $Z_t(\widetilde{\omega}) \leq Z_t(\omega')$ $\forall \omega' \in \Omega\left(s^{t-1}(\omega)\right)$. Consider the function $f_{t-1,\omega,y} : [\underline{\lambda}(t-1, \omega, y), +\infty) \rightarrow [(\beta_1/\beta_2) \cdot \underline{r} \cdot u_1'(\bar{z}), +\infty)$ in the variable λ defined by

$$f_{t-1,\omega,y}(\lambda) := (\beta_1/\beta_2) \cdot E\left[r_t \cdot u_1'\left(Z_t - (u_2')^{-1}\left(\frac{y \cdot \lambda}{r_t}\right)\right)\middle|\mathcal{F}_{t-1}\right](\omega).$$

Then (i) $f_{t-1,\omega,y}$ has a unique fixed point denoted $\lambda^*(t-1, \omega, y)$.