# Four Characterizations of Constrained Pareto Efficiency in Continuum Economies with Widespread Externalities

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**Abstract**: In continuum economies, widespread externalities are those over which each individual has negligible control. Nash–Walrasian equilibria with lump-sum transfers are defined, and their existence proved. They are then characterized by the property of "$f$-constrained Pareto efficiency" for finite coalitions. More general "private good" Nash–Walrasian equilibria are characterized as private good constrained Pareto efficient. Introducing complete Pigou taxes or subsidies leads to equilibria that are characterized by constrained efficiency and $f$-constrained efficiency for given levels of the widespread externalities. But full efficiency requires resolving the public good problem of determining those aggregate externalities or, equivalently, of setting appropriate Pigou prices.

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**Widespread Externalities**

## 1. Introduction and Outline

In economies with a continuum of individual agents, the papers by Kaneko and Wooders (1986, 1989) and by Hammond, Kaneko and Wooders (1989, henceforth referred to HKW) describe an externality as "widespread" if each individual agent has negligible power to influence its effect on other individuals. Obvious examples include familiar externalities such as the emission of methane, carbon dioxide, or other greenhouse gases into the atmosphere, as well as gases causing ozone depletion, and farming practices which eventually add to pesticide residues in aquifers or in the oceans. But probably a majority of externalities can be thought of as widespread. For example, traffic congestion in one city, or even in one part of a city, appears to meet the definition. Clearly excluded, however, are "narrow" externalities arising from small group interaction, such as within families or local clubs, as well as those arising from (very) local public goods specific to a village or small neighbourhood.

In addition to the generally recognized forms of widespread externality discussed above, similar issues arise whenever there are public goods whose provision depends on individual action, or which are financed by well-defined fiscal rules that depend on the statistical distribution of private actions. Also, widespread externalities are effectively created by production decisions in the presence of fixed costs or other non-convexities that remain significant even in the aggregate economy (see Dierker, 1986 and Vega-Redondo, 1987). Basic research and technical knowledge are, of course, prime examples of global public goods (see Nelson, 1959 and Arrow, 1962), and the widespread externalities arising in their production lie at the heart of many recent models of "endogenous growth". As a final class of examples, suppose that the economic system pools risks in order to provide individual-specific benefits such as health care, or compensation for victims of floods (Kydland and Prescott, 1977) and earthquakes. Then widespread externalities are likely to arise in the future from current "morally hazardous" activities such as smoking, or building houses in areas subject to flooding or earthquakes. This is true whether the risk pooling is achieved through private insurance markets or as a result of public policy. In this sense, even history, or at least the statistical distribution of economically relevant personal histories, gives rise to a widespread externality (Hammond, 1993b).

Concepts similar to widespread externalities have been considered formally by Bergstrom (1976, p. 123), less formally by many writers, including Hardin (1968), Schelling (1978), and also Baumol and Oates (1988, especially the discussion on pp. 9–10). None of these consider a continuum economy, however. Yet, just as the concept of a continuum economy is needed to formalize the idea that individuals have no power to influence market prices, so is it also needed to formalize the corresponding idea that individuals have no power to affect widespread externalities.

The three earlier papers on widespread externalities cited above were mostly concerned with a particular form of core which we called the "$f$-core". This consists of allocations to a continuum of agents that (except for agents in a null set) could not be blocked by any finite coalition. In particular, we were able to demonstrate equivalence between the $f$-core and a form of "Nash–Walrasian" equilibrium in which private goods were bought and sold on complete perfectly competitive markets, but externalities were determined by individuals' optimizing decisions taking as given the externalities created by all other individuals.[1]

Here, however, the focus will be on the welfare properties of such Nash–Walrasian equilibria, as well as of alternative equilibria which arise when there is an attempt to reduce or remove efficiency losses by setting up markets for the rights to create externalities. It will turn out that such markets are equivalent to introducing Pigou taxes (or subsidies) on the externalities. And that, contrary to how many economists seem to have interpreted such standard works as Baumol and Oates (1973; 1988, especially ch. 3), in fact there remains an important public good problem that has to be solved in other ways.[2] At least

[1] In fact, we chose to consider only a special case in which the aggregate widespread externality could be expressed as the distribution generated by the allocation of private goods. This corresponds to the special situation considered in Section 8 of this paper, where individual rights for all externalities are traded at Pigou prices. Sometimes, however, such a restricted model can be derived as a reduced form of the general model considered in this paper. This derivation would leave externality vectors implicit while postulating that individuals choose them optimally given their own respective net trade vectors and the aggregate externalities. This reduction will work provided that the obvious mapping from each individual's net trade vector and the aggregate externalities to the optimal individual externality vector allows the aggregate externalities to be expressed in equivalent form as the distribution generated by the allocation of private goods. Precise conditions for this to be valid remain to be investigated, however.

[2] Of course, the analysis by Baumol and Oates is formally correct. Under their assumptions (which implicitly include differentiability and some sort of constraint qualification guaranteeing existence of Kuhn–Tucker prices), Pareto efficiency is achievable, as they claim, by a combination of competitive markets for private goods and appropriate Pigou taxes and subsidies for externalities.

Bergstrom (1976, p. 124) and Starrett (1988, especially pp. 69–70) bring out this point fairly clearly. See also the discussion by Arrow (1969), Starrett (1972), and Dasgupta and Heal (1979, ch. 3) of the case when anybody's consumption can create externalities for all other individuals. And also Klevorick and Kramer's (1973) interesting analysis of a particular institutional approach to the public good problem for the case of water pollution in the Ruhr region of North-West Germany. However, even these works do not explain how to separate Pigou taxes from Lindahl prices in the way Section 6 suggests (see also Hammond, 1994).

Specifically, this paper considers different concepts of constrained Pareto efficiency that characterize Nash–Walrasian equilibria, with or without Pigou taxes and subsidies, and then with or without Lindahl pricing of public goods in the case when appropriate Pigou taxes are instituted. These characterization results are somewhat similar to those of Grossman (1977) and Repullo (1988) for equilibrium with incomplete markets.

In what follows, Section 2 will specify the basic general equilibrium model of an economy with a continuum of agents, together with arbitrary finite numbers of exchangeable private goods and widespread externalities. A Nash–Walrasian equilibrium with lump-sum transfers (NWELT) will then be defined in Section 3. So will the less restrictive concept of private good Walrasian equilibrium with lump-sum transfers (PNWELT), which differs from a NWELT in that each agent's choice of private good vector is not required to be coordinated with the choice of externality vector.

Next, in Section 4 the set of PNWELT allocations will be characterized as "private good" constrained Pareto efficient. In other words, the allocation of private goods will be Pareto efficient, given all agents' choices of what externalities to create. Thus, Pareto improvements will not be possible unless inefficiencies arising from externality creation are reduced. Conversely, take virtually any private good constrained Pareto efficient allocation in which, given their choices of private good net trade vectors, individuals' choices of ex-

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But the Pigou taxes and subsidies have to be set at the right level. Or equivalently, the aggregate supply of rights or duties to create externalities must be determined appropriately before being sold on a competitive market. In fn. 17 on p. 45 of the second edition they do notice that a widespread externality "is, essentially, a public good". Also, on p. 131 they do briefly acknowledge that there is likely to be a collective decision problem. Yet here they ascribe this, quite incorrectly, to the presence of non-convexities, and also appear to confuse the first and second efficiency theorems of welfare economics.

ternalities are in Nash equilibrium. Generally, such an allocation is decentralizable by a private good price vector and lump-sum transfers which make that allocation a PNWELT.

The form of constrained Pareto efficiency discussed in Section 4 has an obvious counterpart even in economies with a finite number of agents. But for continuum economies, Section 5 offers a different characterization of NWELTs. In fact, they correspond to allocations that are "$f$-constrained" Pareto efficient, in the sense that almost no finite coalition can find an "$f$-constrained Pareto improvement", in which its members reallocate their private goods, while also changing the externalities they create as they please. In particular, $f$-constrained Pareto efficient allocations are precisely those which cannot be improved by Coasian bargaining within any finite coalition. Of course it is crucial that, in a continuum economy, no finite coalition can affect the aggregate widespread externalities.

Then Section 6 turns briefly toward a first-best remedy for the special case when the widespread externalities depend on the mean vector of externality creation levels in the population. Then, given complete information and enough convexity, a suitable combination of Pigou taxes or subsidies with Lindahl prices is able to solve not only the problem of distributing aggregate externalities between agents, but also, and simultaneously, the public good problem posed by the need to have the appropriate levels of the aggregate externalities.

The first-best solution of Section 6 is not truly feasible when the benefits and costs of widespread externalities remain known only to the affected individuals. Because of the free-rider problem, the incentive constraints created by such privacy of information will typically prevent the attainment of almost all first-best allocations. As implied by the discussion in Hammond (1979), the only exceptions are those allocations for which it can be arranged that individuals' allowable total net expenditures on private goods and on Pigou taxes combined are independent of their private information. For example, it would be incentive compatible and first-best Pareto efficient to have public goods financed by poll taxes, provided that it was commonly known that everybody really could afford to pay the required poll tax. But even then it would not generally be distributively just. Nor is an adequate poll tax feasible in practice, given that it is probably false, and is certainly not common knowledge, that everybody really can afford it. What is more, as Starrett (1972) and Starrett and Zeckhauser (1974) in particular have pointed out, externalities can easily give rise to fundamental aggregate non-convexities. These create additional difficulties in using Lindahl pricing schemes for solving the associated public good problem.[3]

Thus, private information and aggregate non-convexities confront us with insuperable incentive and other organizational constraints which, except in rare special cases, prevent any satisfactory first-best Pareto efficient resolution of the public good problem. Accordingly, for the case when the effect of externalities depends on the mean externality vector in the population, Section 7 turns to alternative second-best "Pigou–Walras" allocations. These result from competitive markets for both private goods and the rights (or duties) to create widespread externalities. Such allocations are externality constrained Pareto efficient in the sense that no Pareto improvement can result from simply reallocating either private goods or the rights to create externalities; some changes in the aggregate levels of the various widespread externalities are required. Pigou–Walras allocations are also characterized by the corresponding property for finite coalitions of $f$-externality constrained Pareto efficiency. Finally, it is argued that Pigou prices on their own work equally well (or badly) whether or not aggregate externalities are a source of non-convexity.

An appendix presents proofs of the "second $f$-efficiency" theorem and of the existence of Nash–Walrasian equilibria.

## 2. A Continuum Economy with Widespread Externalities

### *2.1. Agents, Goods, and Widespread Externalities*

Throughout the paper the setting will be an economy with a continuum of agents, as formulated by Aumann (1964, 1966) and Hildenbrand (1974). In fact, the space of agents will be $(I, \mathcal{B}, \alpha)$ where $I$ is the unit interval $[0, 1]$ of the real line, the family $\mathcal{B}$ of measurable sets is the Borel $\sigma$-algebra generated by the open sets, and $\alpha$ is Lebesgue measure defined on those sets.

It will be assumed that there is a finite set $G$ of exchangeable private goods, and a disjoint finite set $E$ of externalities. Each individual $i \in I$ will have a net trade vector $x^i \in \Re^G$ of exchangeable goods, and create an externality vector $e^i \in \Re^E$. In addition,

[3] As Baumol and Oates (1988, ch. 8) mention, others had also noticed difficulties with the usual second-order conditions for local efficiency. Starrett's key demonstration, however, is both simpler and, because it goes beyond mere local second-order conditions, much more damaging.

each agent $i \in I$ in common is affected by everybody's externality vectors. There are at least three ways of representing this common effect. The first and most general way is to consider the entire integrable or measurable mapping $\mathbf{e} : I \to \Re^E$ defined by the profile $e^i$ $(i \in I)$ of externality vectors. This is the case of *general widespread externalities* that was treated in Hammond, Kaneko, and Wooders (1989). Somewhat less general but simpler is the case of *distributional widespread externalities*, where only the statistical distribution or measure on $\Re^E$ induced by $\mathbf{e}$ is relevant. This case is considered in Kaneko and Wooders (1989).

Very often a third representation of the aggregate externality is possible. This case of *mean widespread externalities*, the simplest of the three, occurs when measuring each component of each individual's typical externality vector $e^i \in \Re^E$ in suitable units allows $\mathbf{e}$ to be summarized by its mean, which is the vector in $\Re^E$ defined by the integral $z = \bar{e} = \int_I e^i \, d\alpha$. For example, if the externality is a greenhouse gas such as methane, a very relevant measure of methane emissions is the aggregate quantity expressed in kilograms per head of world population. Other examples are presented in Kaneko and Wooders (1994). This is the only case considered in this paper. But if, as in Hammond (1993b), the widespread externality arises from the statistical distribution of personal histories, then this representation by the mean is generally impossible. Or at least, it is not generally possible in any finite-dimensional Euclidean space.

### 2.2. *Individual Feasible Sets and Preferences*

Each individual $i \in I$ is assumed to have a feasible set $F^i \subset \Re^G \times \Re^E \times \Re^E$ of triples $(x^i, e^i, z)$, over which is defined a (complete and transitive) preference ordering $R^i$.

Note that $(x^i, e^i, z) \; R^i \; (x^i, e^i, z) \iff (x^i, e^i, z) \in F^i$. It follows that $F^i$ and $R^i$ can both be represented simultaneously by the graph $G^i$ of the complete ordering $R^i$ in the Cartesian product set $(\Re^G \times \Re^E \times \Re^E)^2$. For technical reasons, it will be assumed that the correspondence $i \mapsto\!\!\!\to G^i$ from the space of agents $I$ to the space of preference graphs itself has a graph which is measurable subset of $I \times (\Re^G \times \Re^E \times \Re^E)^2$ when this space is given its usual product $\sigma$-algebra.

*2.3. Feasible and Pareto Superior Allocations*

In this economy, a *feasible allocation* $(\mathbf{x}, \mathbf{e}, z)$ consists of measurable mappings $\mathbf{x} : I \to \Re^G$ and $\mathbf{e} : I \to \Re^E$ such that:

(i) $(x^i, e^i, z) \in F^i$ for almost all $i \in I$;

(ii) $z = \int_I e^i \, d\alpha$;

(iii) $\int_I x^i \, d\alpha = 0$.

Note especially how (iii) excludes free disposal at the aggregate level, as seems appropriate when externalities are being discussed. In proving existence, however, free disposability by individuals of private goods or even of their own externalities will be assumed.

Given two feasible allocations $(\mathbf{x}, \mathbf{e}, z)$ and $(\tilde{\mathbf{x}}, \tilde{\mathbf{e}}, \tilde{z})$, say that the first *strictly Pareto dominates* the second if $(x^i, e^i, z) \; P^i \; (\tilde{x}^i, \tilde{e}^i, \tilde{z})$ for almost all $i \in I$; and that the first *Pareto dominates* the second if $(x^i, e^i, z) \; R^i \; (\tilde{x}^i, \tilde{e}^i, \tilde{z})$ for almost all $i \in I$, but for a set of agents of positive $\alpha$-measure, one has $(x^i, e^i, z) \; P^i \; (\tilde{x}^i, \tilde{e}^i, \tilde{z})$.

A feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ is *fully* or *first-best weakly Pareto efficient* if it is not strictly Pareto dominated by any other feasible allocation; it is *fully* or *first-best Pareto efficient* if it is not Pareto dominated by any other feasible allocation.

*2.4. Assumptions*

The rest of the paper will report four different versions, or rather applications in different contexts, of the two fundamental efficiency theorems of welfare economics, or else of the theorems on the $f$-core presented in HKW. For the different versions of the first efficiency theorem, on the weak Pareto efficiency of Walrasian equilibrium, no further assumptions are needed. Ordinary Pareto efficiency, however, will not be true in general without the additional assumption that preferences are *locally non-satiated in private goods* — i.e., that given any point of an individual's feasible set, any neighbourhood of that point contains a preferred point in which only the vector of net trades in private goods is different. Following Arrow (1951), and then Hildenbrand's (1969, 1974) later results for continuum economies, the proof of this first theorem is standard and elementary.

The second efficiency theorem claims that any Pareto efficient allocation can be decentralized as a Walrasian equilibrium with lump-sum transfers. For this, it will be assumed as usual that preferences are not only locally non-satiated in private goods, but also continuous. In addition, when discussing $f$-constrained Pareto efficient allocations in Sections 5 and 7, it will be assumed that preferences are locally non-satiated even when particular changes in vectors are restricted to have all their coordinates be rational numbers. Specifically, if $Q^G$ denotes the subset of $\Re^G$ whose members are vectors with all rational coordinates, it is required that, given any pair $x^i, \hat{x}^i \in \Re^G$ such that $(x^i, e^i, z), (\hat{x}^i, e^i, z) \in F^i$, as well as any neighbourhood $N$ of $x^i$ in $\Re^G$, there should exist $\tilde{x}^i \in N$ for which $x^i - \hat{x}^i \in Q^G$ and $(\tilde{x}^i, e^i, z) \in F^i$ with $(\tilde{x}^i, e^i, z)\ P^i\ (x^i, e^i, z)$. This assumption will be called *local non-satiation w.r.t. rational vectors*. It is satisfied whenever there is free disposal of private goods and preferences for private goods are monotone, but is much more general.

In fact, for some results it will be assumed that each individual's preference ordering $R^i$ on $F^i$ can be represented by an ordinal utility function $U^i(x^i, e^i, z)$ that is continuous as a function of $(x^i, e^i, z)$ and measurable as a function of $(i, x^i, e^i, z)$ all together. Furthermore, one particular relevant conditional feasible set for each individual should be convex. And, as the first of two assumptions needed to exclude the "exceptional case" first discussed by Arrow (1951), the integral of the consumers' conditional feasible sets of private good net trades should have the vector 0 as an interior point. These are *general assumptions* under which the second efficiency theorem holds.

Given these general assumptions, appropriate versions of the second efficiency theorem will then be true for particular *non-oligarchic* allocations, as defined in Hammond (1993a). These have the property that there is no (extreme) "oligarchy" in the form of a set $A$ of agents, with measure strictly between 0 and 1, who are so well off already that they could not all be made even better off if given unrestricted access to all the private goods that agents outside $A$ could possibly supply. Thus, non-oligarchic allocations have the property that, no matter how $I$ is partitioned into two disjoint non-null measurable subsets, the agents in each subset have something desirable to offer the agents in the other subset. This is the second of the two assumptions needed to exclude Arrow's exceptional case.

The existence proof in Section A.2 of the appendix will introduce some additional general assumptions. These will concern the integrable boundedness of the individuals' feasible and budget sets, the continuity of correspondences based on appropriate sections of the feasible set, the existence of cheaper points within almost all individuals' budget sets, and the impossibility of having markets for private goods clear unless all prices are positive. These more technical assumptions will be discussed only in the appendix.

## 3. Nash–Walrasian Equilibrium with Lump-Sum Transfers

### *3.1. Nash Equilibrium in a Generalized Game*

Consider a non-zero *price vector* $p \in \Re^G$ at which exchangeable goods are traded, together with a collection of *lump-sum transfers* represented by a measurable mapping $\mathbf{m} : I \to \Re$ satisfying $\int_I m^i \, d\alpha = 0$. The reason for introducing lump-sum transfers is so that most points of each relevant constrained Pareto frontier can emerge as possible equilibria. Consider too a given mean externality vector $z \in \Re^E$. Then each individual $i \in I$ is forced to take this price vector, $i$'s own transfer $m^i$, and $z$ all as given. Thus, $i$ faces the *conditional budget set*

$$B_C^i(p, m^i; z) := \{ (x^i, e^i) \in F_C^i(z) \mid p\, x^i \leq m^i \} \tag{1}$$

where $F_C^i(z) := \{ (x^i, e^i) \in \Re^G \times \Re^E \mid (x^i, e^i, z) \in F^i \}$ is obvious notation for the set of pairs that are conditionally feasible for $i$, given the mean externality $z$.

A *Nash–Walrasian Equilibrium with lump-sum transfers* (or NWELT) is then a collection $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, \mathbf{m})$ consisting of a feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$, a price vector $p$, and transfers $\mathbf{m}$, such that for almost all $i \in I$, one has $(\hat{x}^i, \hat{e}^i) \in B_C^i(p, m^i; \hat{z})$ and also $p\, x^i > m^i$ whenever $(x^i, e^i, \hat{z})\ P^i\ (\hat{x}^i, \hat{e}^i, \hat{z})$. Note that feasibility implies market clearing for private goods. Also, the last part of the definition is equivalent to preference maximization over the budget set $B_C^i(p, m^i; \hat{z})$. Furthermore, each individual $i$ chooses $e^i$ without any restriction beyond individual feasibility, conditional on $\hat{z}$ and the choice of an $x^i$ satisfying the budget constraint $p\, x^i \leq m^i$. In this sense, there is a Nash equilibrium in the (generalized) game where each player $i$ chooses $(x^i, e^i)$ subject to $(x^i, e^i) \in B_C^i(p, m^i; \hat{z})$.[4]

[4] Following Debreu (1952), a *generalized game* is one with the property that some players' strategy choices may constrain what strategy choices are feasible for other players. It should be noted as well that requiring $\mathbf{x}$ and $\mathbf{e}$ to be measurable also imposes restrictions on what groups of players can choose, given the choices of neighbouring players, as has been pointed out by Greenberg, Monderer, and Shitovitz (1993). One way to resolve this issue would be to consider joint distributions over players' labels $i \in I$ and their strategies, as Mas-Colell (1984) has done, for instance. But then the definition of Pareto improvement has to consider suitable joint distributions of allocations before and after any reallocation, which also affects the definition of Pareto efficiency. In any case, formally one can instead follow Schmeidler (1973), Dubey and Kaneko (1984) and Green (1984) in considering *continuum games* where strategy choices are restricted to be measurable functions.

### 3.2. *Dominant Strategy Demand Revelation*

Actually, there is a rather different game of *demand revelation* in which each individual has a (weakly) dominant strategy leading to the equilibrium feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$.[5]

Note that, though free disposal has not been assumed at the aggregate level, in the existence theorem it has been at the individual level. In this case, any equilibrium price vector must obviously be semi-positive. For the moment, assume that the same is true in other cases as well. So let $P$ denote the unit simplex in $\Re^G_+$. Its members are *normalized* price vectors, in effect.

In the demand revelation game, each player $i \in I$ chooses a strategy in the form of a *revealed demand correspondence* $d^i : P \times \Re \times \Re^E \twoheadrightarrow \Re^G \times \Re^E$. Whenever the conditional budget set $B^i_C(p, m^i; z)$ is non-empty, this correspondence must select a non-empty set $d^i(p, m^i; z) \subset B^i_C(p, m^i; z)$ for each $i \in I$, and have a closed graph. In fact, this is equivalent to having each individual $i \in I$ choose as a graph of some demand correspondence $d^i$ a closed set $D^i$. Moreover, $D^i$ must be a subset of the graph of the conditional budget correspondence $B^i_C : P \times \Re \times \Re^E \twoheadrightarrow \Re^G \times \Re^E$, so a closed subset of $P \times \Re \times \Re^E \times \Re^G \times \Re^E$, with the property that the projection of $D^i$ onto $P \times \Re^E$ is equal to the projection of the graph of $B^i_C$. Furthermore, the correspondence $\mathbf{D} : I \twoheadrightarrow P \times \Re^E \times \Re^G \times \Re^E$ from individuals to the graphs $D^i$ of their respective demand correspondences must have a measurable graph.

The demand revelation game will also involve a *lump-sum transfer system* in the form of functions $m^i : P \times \Re^E \to \Re$ making each individual $i$'s transfer depend on both prices and the mean externality vector. It is assumed that each individual $i$'s function $m^i(p, z)$ is continuous in $(p, z)$, and that the mapping $(i, p, z) \mapsto m^i(p, z)$ is measurable, with $\int_I m^i(p, z)\, d\alpha = 0$. In addition, the functions $m^i(p, z)$ must make the resulting *reduced budget correspondences* $\hat{B}^i(p, z) := B^i_C(p, m^i(p, z); z)$ non-empty valued for almost all $i \in I$, and in fact they should be what Grandmont and McFadden (1972) called *sagacious* in ensuring that there always exists some $(x^i, e^i, z) \in F^i$ with $p\, x^i < m^i$.[6]

[5] The highly appropriate term "demand revelation" was introduced by Tideman and Tullock (1976, p. 1146) in a closely related context.

[6] This will be possible under the assumption that the set $\int_I X^i_G(z) d\alpha$ has 0 as an interior point, where $X^i_G(z)$ is defined in (7) of Section 5.

The *outcome* of this game is defined to be any Nash–Walrasian equilibrium

$$(\hat{\mathbf{x}}(\mathbf{D}), \hat{\mathbf{e}}(\mathbf{D}), \hat{z}(\mathbf{D}), p(\mathbf{D}), \mathbf{m}(p(\mathbf{D}), z(\mathbf{D}))) \tag{2}$$

in what would be the true economy if $\mathbf{D}$ were the true profile of demand correspondences. Furthermore, this outcome should be the same for the two profiles $\mathbf{D}$ and $\mathbf{D}'$ whenever $\mathbf{D} = \mathbf{D}'$ for almost all $i \in I$. In this demand revelation game, it is then a weakly dominant strategy for each individual $i$ to announce the true preference maximizing demand correspondence $d^i$. For announcing any other demand correspondence $\tilde{d}^i$ would make no difference to the profile $\mathbf{D}$, and so no difference to the induced budget set $B^i(\mathbf{D}) := \hat{B}^i(p(\mathbf{D}), \hat{z}(\mathbf{D}))$ that individual $i$ faces. Of course, because the outcome of the game is a NWELT, the pair $(x^i(\mathbf{D}), e^i(\mathbf{D}))$ must be one of the best points of the budget set $B^i(\mathbf{D})$. But announcing any other $\tilde{d}^i$ would lead to $i$ being forced to have some member of the set $\tilde{d}^i(p(\mathbf{D}), \hat{z}(\mathbf{D}))$. In general, members of this set are not optimal in $B^i(\delta)$. This justifies the claim that announcing the true preference maximizing $d^i$ is at least a weakly dominant strategy.

Of course, for the game to be well-defined, a NWELT should exist for every allowable profile $\mathbf{D}$ of demand correspondences. This is an implication, however, of the existence theorem presented in the appendix.

### *3.3. Private Good Nash–Walrasian Equilibrium*

Section 4 below will make use of a different and weaker concept of Nash–Walrasian equilibrium. This has individuals act as if they are unable to coordinate their choices of private good net trade and externality vectors with each other. Formally, given any pair $(e^i, z) \in \Re^E \times \Re^E$, define the *private good budget set* as

$$B^i_P(p, m^i; e^i, z) := \{\, x^i \in \Re^G \mid (x^i, e^i, z) \in F^i \quad \text{and} \quad p\, x^i \leq m^i \,\}. \tag{3}$$

Then a *private good Nash–Walrasian Equilibrium with lump-sum transfers* (or PNWELT) is a collection consisting of a feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$, a price vector $p \neq 0$, and transfers $\mathbf{m}$ such that, for almost all $i \in I$:

(i) $\hat{x}^i \in B^i_P(p, m^i; \hat{e}^i, \hat{z})$ and also $p\, x^i > m^i$ whenever $(x^i, \hat{e}^i, \hat{z})\ P^i\ (\hat{x}^i, \hat{e}^i, \hat{z})$;

(ii) $(\hat{x}^i, \hat{e}^i, \hat{z})\ R^i\ (\hat{x}^i, e^i, \hat{z})$ for all $e^i$ satisfying $(\hat{x}^i, e^i, \hat{z}) \in F^i$.

The difference from the definition of a NWELT is that the preference ordering $R^i$ is not being maximized over $B^i_C(p, m^i; z)$ by simultaneous choices of $x^i$ and $e^i$. Instead, $\hat{x}^i$ maximizes $R^i$ w.r.t. $x^i$ over $B^i_P(p, m^i; \hat{e}^i, \hat{z})$ with $\hat{e}^i$ and $\hat{z}$ both fixed, whereas $\hat{e}^i$ is chosen so that the triple $(\hat{x}^i, \hat{e}^i, \hat{z})$ maximizes $R^i$ over $F^i$ by varying $e^i$ with $\hat{x}^i$ and $\hat{z}$ both fixed. Thus, it is as if each individual were represented by two separate and uncoordinated agents who choose the private good net trade and externality vectors independently, but with each of the two agents anticipating the decision of the other. In fact any PNWELT is really a Nash equilibrium in the (generalized) game where each individual $i$ is represented by two such agents, one controlling all the private goods, and the other controlling all the externalities.

## 4. Private Good Constrained Pareto Efficiency

It is rather obvious that widespread externalities typically make NWELT or PNWELT allocations inefficient. A few particular examples of such inefficiency can be found, for example, in Kaneko and Wooders (1994) or Hammond (1993b), but the literature contains many others. Indeed, in the demand revelation game discussed in Section 3.2, a dominant strategy for each individual $i$ is to choose the externality vector $e^i$ without paying any attention to the effect that all individuals choosing this way have on the widespread externality $z$. This gives rise to situations such as those analysed by Schelling (1978). Indeed, as he writes (p. 19): "How well each does for himself in adapting to his social environment is not the same thing as how satisfactory a social environment they collectively create for themselves."

Nevertheless, it is also rather obvious that PNWELT and so NWELT allocations do have at least one constrained efficiency property. For, given any PNWELT $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, \mathbf{m})$, consider the purely private good economy in which the profile $\hat{\mathbf{e}}$ of externality vectors, and so their induced mean $\hat{z}$, are both fixed. In this economy, each individual $i \in I$ has a (conditional) *private good feasible set*

$$X^i_P(\hat{e}^i, \hat{z}) := \{\, x^i \in \Re^G \mid (x^i, \hat{e}^i, \hat{z}) \in F^i \,\} \qquad (4)$$

and a (conditional) *private good preference ordering* $R^i_P(\hat{e}^i, \hat{z})$ on $X^i_P(\hat{e}^i, \hat{z})$ satisfying

$$x^i \; R^i_P(\hat{e}^i, \hat{z}) \; \tilde{x}^i \iff (x^i, \hat{e}^i, \hat{z}) \; R^i \; (\tilde{x}^i, \hat{e}^i, \hat{z}). \qquad (5)$$

This is an ordinary private good economy, in which $(\hat{\mathbf{x}}, p, \mathbf{m})$ is a *Walrasian equilibrium with lump-sum transfers* (or WELT). Each individual's conditional budget set, as defined by (3), can be expressed alternatively as

$$B^i_P(p, m^i; \hat{e}^i, \hat{z}) := \{\, x^i \in X^i_P(\hat{e}^i, \hat{z}) \mid p\, x^i \leq m^i \,\}. \tag{6}$$

Hence, the usual first efficiency theorem of welfare economics (Arrow, 1951) implies that $\hat{\mathbf{x}}$ is weakly Pareto efficient in this constrained economy, and that it will be Pareto efficient if preferences are locally non-satiated in private goods. This property will be called *private good constrained Pareto efficiency*, for obvious reasons.

Conversely, consider any feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ in which the strategic choices of externality vectors $\hat{e}^i$ constitute a conditional Nash equilibrium because, for almost all $i \in I$, one has $(\hat{x}^i, \hat{e}^i, \hat{z})\ R^i\ (\hat{x}^i, e^i, \hat{z})$ whenever $(\hat{x}^i, e^i, \hat{z}) \in F^i$. Impose the general assumptions that each individual $i$'s private good feasible set $X^i_P(\hat{e}^i, \hat{z})$ is convex, and that $i$'s private good preference ordering $R^i_P(\hat{e}^i, \hat{z})$ is both continuous on $X^i_P(\hat{e}^i, \hat{z})$ and locally non-satiated. Also that 0 is an interior point of the aggregate feasible set $\int_I X^i_P(\hat{e}^i, \hat{z})\, d\alpha$. Suppose too that the specific allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ satisfies the non-oligarchy condition that, given any measurable set $A$ of individuals for which $0 < \alpha(A) < 1$, there exists an alternative feasible allocation $\tilde{\mathbf{x}}$ of private goods such that $\tilde{x}^i\ P^i_P(\hat{e}^i, \hat{z})\ \hat{x}^i$ for almost all $i \in A$. Then a necessary condition for $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ to be private good Pareto efficient is that there exist a non-zero price vector $p \in \Re^G$ and transfers $m^i\ (= p\,\hat{x}^i)$ for almost all $i \in I$ such that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, \mathbf{m})$ is a PNWELT. This result is obviously implied by the version of the second efficiency of welfare economics that is presented in Hammond (1993a, especially Sections 3.3.2 and 6.2.2), extending the work of Hildenbrand (1969, 1974).

The significance of this result is that Pareto improvements to any PNWELT allocation are only possible by getting at least a positive proportion of individuals to alter their Nash equilibrium externality vectors. Also, when the general assumptions of the second efficiency theorem are satisfied, then almost any private good constrained Pareto efficient allocation, no matter how severe the Nash constraints may be, can emerge as a PNWELT.

## 5. $f$-Constrained Pareto Efficiency

In this section it will be assumed throughout that each individual $i$'s preference ordering $R^i$ on $F^i$ can be represented by a continuous ordinal utility function $U^i(x^i, e^i, z)$. Define the new *reduced conditional feasible set* for private goods alone by

$$X^i_G(z) := \{ x^i \in \Re^G \mid \exists e^i \in \Re^E : (x^i, e^i, z) \in F^i \}. \tag{7}$$

Thus $X^i_G(z)$ consists of all private good net trade vectors $x^i$ which, given the mean externality $z$, are feasible for $i$ given some suitable choice of $i$'s own externality vector $e^i$. Then define $i$'s *reduced conditional utility function* on $X^i_G(z)$ as

$$U^i_G(x^i; z) := \max_{e^i} \{ U^i(x^i, e^i, z) \mid (x^i, e^i, z) \in F^i \}. \tag{8}$$

This is the maximum utility available to $i$ by choosing the externality vector $e^i$ optimally, given the choice of net trade vector $x^i$ and the mean externality $z$. It is assumed that this maximum always exists. Note that when, for each fixed pair $(e^i, z)$, it is assumed that $U^i(x^i, e^i, z)$ is locally non-satiated in the private good vector $x^i$, then $U^i_G(x^i; z)$ certainly has the same property.

A feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ is said to be *$f$-constrained Pareto efficient* if there is a null set of exceptional agents $N \subset I$ with $\alpha(N) = 0$ such that no finite coalition $C \subset I \setminus N$ outside this exceptional set can find an alternative allocation $(x^i, e^i) \in F^i_C(\hat{z})$ $(i \in C)$ to its members that satisfies both $\sum_{i \in C} x^i = \sum_{i \in C} \hat{x}^i$ and $(x^i, e^i, \hat{z})\ P^i\ (\hat{x}^i, \hat{e}^i, \hat{z})$ for all $i \in C$. This definition is similar to that of the $f$-core; indeed, it is equivalent to having $\hat{\mathbf{x}}$ be in the $f$-core of an economy in which each individual $i \in I$ has initial endowment $\hat{x}^i$ instead of 0, and where individuals and finite coalitions are free to choose their externality vectors as they please, subject only to $(x^i, e^i) \in F^i_C(\hat{z})$.

Because $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ cannot be blocked or improved any by single agent $i \in I \setminus N$, this definition of $f$-constrained Pareto efficiency requires that

$$\hat{e}^i \in \arg\max_{e^i} \{ U^i(\hat{x}^i, e^i, \hat{z}) \mid (\hat{x}^i, e^i, \hat{z}) \in F^i \} \tag{9}$$

for all such $i$. In particular, $U^i(\hat{x}^i, \hat{e}^i, \hat{z}) = U^i_G(\hat{x}^i; \hat{z})$. This obviously implies that no finite coalition $C \subset I \setminus N$ can find alternative net trade vectors $x^i \in X^i_G(\hat{z})$ $(i \in C)$ satisfying both $\sum_{i\in C} x^i = \sum_{i\in C} \hat{x}^i$ and $U_G^i(x^i;\hat{z}) > U_G^i(\hat{x}^i;\hat{z})$ for all $i \in C$. Therefore, it turns out that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ is $f$-constrained Pareto efficient if and only if $\hat{\mathbf{x}}$ is in the $f$-core of the private good economy with, for each $i \in I$, the feasible set $X_G^i(\hat{z})$, utility function $U_G^i(x^i;\hat{z})$ on $X_G^i(\hat{z})$, and initial endowment $\hat{x}^i$.

Now, given any NWELT $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, \mathbf{m})$, the corresponding private good part $(\hat{\mathbf{x}}, p, \mathbf{m})$ must be a WELT in the private good economy with feasible sets $X_G^i(\hat{z})$ and utility functions $U_G^i(x^i;\hat{z})$ $(i \in C)$. By the usual easy argument, it is therefore in the $f$-core of this private good economy when initial endowments are $\hat{\mathbf{x}}$, so it is also $f$-constrained Pareto efficient.

Conversely, let $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ be any $f$-constrained Pareto efficient allocation for which $\hat{\mathbf{x}}$ is non-oligarchic in the private good exchange economy with individual feasible sets $X_G^i(\hat{z})$ and utility functions $U_G^i(x^i;\hat{z})$ defined on $X_G^i(\hat{z})$. Suppose too that, corresponding to the assumptions set out in the penultimate paragraph of Section 4, these feasible sets $X_G^i(\hat{z})$ are convex, that the conditional preference orderings $R_G^i(\hat{z})$ represented by $U_G^i(x^i;\hat{z})$ are continuous and locally non-satiated w.r.t. rational vectors, and that 0 is in the interior of $\int_I X_G^i(\hat{z})d\alpha$. Suppose too that the correspondence from $i$ to the graph of the preference relation $R_G^i(\hat{z})$ has a measurable graph. Then it will be proved in Section A.1 of the appendix that there must exist prices $p \neq 0$ and transfers $\mathbf{m}$ for which $(\hat{\mathbf{x}}, p, \mathbf{m})$ is a WELT, given $\hat{z}$. Because (9) is true for almost every agent $i \in I$, it follows that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, \mathbf{m})$ must be a NWELT.

The last two paragraphs have shown that NWELT allocations are characterized by the property of $f$-constrained Pareto efficiency in exactly the same way as WELT allocations are characterized by orthodox Pareto efficiency.

## 6. Lindahl–Pigou Equilibria

The characterization results of Sections 4 and 5 extend fairly readily to distributional and even to general widespread externalities, as defined in Section 3.1. In this section and the next, however, it is crucial that only the mean externality vector $z$ is relevant.

Now, *Lindahl prices* take the form of individualized price vectors $q^i \in \Re^E$ $(i \in I)$ for the mean externality vector $z$ with the property that the profile $\mathbf{q} : I \to \Re^E$ is a measurable function. Corresponding *Pigou taxes* (or subsidies if the appropriate component is negative) take the form of a collective price vector $t \in \Re^E$ with $t = \bar{q} := \int_I q^i d\alpha$. In other words, the Pigou tax on each externality should be set equal to the mean of the individualized Lindahl prices for that externality. Given the private good price vector $p$, the Pigou tax vector $t$, the profile of Lindahl price vectors $\mathbf{q} : I \to \Re^E$, and the lump-sum transfers $\mathbf{m}$, each individual $i \in I$ will be confronted with the budget constraint $p\,x^i + t\,e^i - q^i\,z \leq m^i$. The corresponding *Lindahl–Pigou budget set* is

$$B^i_{LP}(p, t, q^i, m^i) := \{\, (x^i, e^i, z) \in F^i \mid p\,x^i + t\,e^i - q^i\,z \leq m^i \,\}. \tag{10}$$

Then $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, t, \mathbf{q}, \mathbf{m})$ is a *Lindahl–Pigou equilibrium with lump-sum transfers* (or LPELT) if it consists of a feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ together with prices $(p, t, \mathbf{q})$ and lump-sum transfers $\mathbf{m}$ such that, for almost all $i \in I$ one has both $(\hat{x}^i, \hat{e}^i, \hat{z}) \in B^i_{LP}(p, t, q^i, m^i)$ and also $p\,x^i + t\,e^i - q^i\,z > m^i$ whenever $(x^i, e^i, z) \in F^i$ with $(x^i, e^i, z)\ P^i\ (\hat{x}^i, \hat{e}^i, \hat{z})$.

As explained by Foley (1970) and Milleron (1972), such an equilibrium is equivalent to one in a purely private good economy where each individual is able to buy a personalized copy $z^i$ of the mean externality vector $z$. In a continuum economy, the commodity space then becomes an infinite-dimensional function space. Of course, market clearing requires that $z^i = z$ for almost all $i \in I$. Also, each individual $i \in I$ is prevented from trading in the other individuals' personalized copies of $z$, so is restricted to a finite-dimensional subspace. Because of this natural equivalence, it is no surprise that any LPELT is weakly Pareto efficient, without any constraints apart from physical feasibility. A very short direct proof is given below, however.

Suppose that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, t, \mathbf{q}, \mathbf{m})$ is any LPELT. Let $(\mathbf{x}, \mathbf{e})$ define any measurable function from $I$ to $\Re^G \times \Re^E$ with the property that, for $z = \int_I e^i$, one has both $(x^i, e^i, z) \in F^i$ and $(x^i, e^i, z)\ P^i\ (\hat{x}^i, \hat{e}^i, \hat{z})$ for almost all $i \in I$. Then $p\,x^i + t\,e^i - q^i\,z > m^i$ for almost all $i \in I$, and so

$$0 = \int_I m^i d\alpha < \int_I (p\,x^i + t\,e^i - q^i\,z)\,d\alpha = p\int_I x^i d\alpha + t\,z - \left(\int_I q^i d\alpha\right)\,z = p\int_I x^i d\alpha, \tag{11}$$

where the last equality holds because $t = \int_I q^i d\alpha$. This is incompatible with $\int_I x^i d\alpha = 0$, so $(\mathbf{x}, \mathbf{e}, z)$ cannot be a feasible allocation. Therefore $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ must be weakly Pareto efficient. The corresponding proof of ordinary Pareto efficiency when preferences for private goods are locally non-satiated is very similar; the key step is to realize that, for almost all $i \in I$, one has $p\,x^i + t\,e^i - q^i\,z \geq m^i$ whenever $(x^i, e^i, z) \in F^i$ with $(x^i, e^i, z)\ R^i\ (\hat{x}^i, \hat{e}^i, \hat{z})$.

In general the second efficiency theorem is considerably harder to prove in this setting, because of the infinite-dimensional commodity space (cf. Roberts' (1973) proof of existence of Lindahl equilibrium in a continuum economy). For the case of symmetric allocations in an economy with a finite number of different types of agent, finite-dimensional arguments such as those in Hammond (1994) are applicable. Nevertheless, as is well known, Lindahl equilibria generally give rise to free rider problems, even in continuum economies, as discussed in Hammond (1979). And there are also the fundamental non-convexities noticed by Starrett (1972), which create further obstacles to Lindahl pricing. For these reasons, the second efficiency theorem will not be discussed here any further.

Thus, the widespread externality problem can only be solved completely, and the economy made to reach a fully Pareto efficient allocation, if the price mechanism is extended to include Lindahl prices for each component of the mean externality vector, as well as Pigou taxes or subsidies. This is because each component of the mean externality vector is like a public good (or bad). Further confirmation of this comes in the next Section from the characterization results for externality constrained Pareto efficiency.

## 7. Pigou–Walrasian Equilibrium

Without Lindahl pricing, or some other more robust device for solving the public good problem, full Pareto efficiency will generally be lost. This is true even if the rights and duties to create externalities are distributed efficiently as a result of Pigou taxes or subsidies.

Formally, consider price systems of the form $(p, t) \in \Re^G \times \Re^E$ and lump-sum transfers $m^i$ $(i \in I)$, where $\mathbf{m} : I \to \Re$ is a measurable function. The corresponding budget constraint of each individual $i \in I$ is then $p\, x^i + t\, e^i \leq m^i$. Given the mean externality vector $z$, each individual $i \in I$ will be confronted with the (conditional) *Pigou–Walrasian budget set*

$$B^i_{PW}(p, t, m^i; z) := \{\, (x^i, e^i) \in F^i_C(z) \mid p\, x^i + t\, e^i \leq m^i \,\}. \tag{12}$$

Then the combination $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, t, \mathbf{m})$ is a *Pigou–Walrasian equilibrium with lump-sum transfers* (or PWELT) if $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ is a feasible allocation satisfying, for almost all $i \in I$, both $(\hat{x}^i, \hat{e}^i, \hat{z}) \in B^i_{PW}(p, t, m^i; z)$ and also $p\, x^i + t\, e^i > m^i$ whenever $(x^i, e^i) \in F^i_C(\hat{z})$ with $(x^i, e^i, \hat{z})\ P^i\ (\hat{x}^i, \hat{e}^i, \hat{z})$. Thus, $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, p, t, \mathbf{m})$ is effectively a WELT in the economy with complete competitive markets in private goods and externalities, but with the mean externality vector $\hat{z}$ fixed.

Not surprisingly, therefore, any PWELT allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ must be *externality constrained weakly Pareto efficient*, in the sense that there is no strictly Pareto superior feasible allocation $(\mathbf{x}, \mathbf{e}, z)$ with $z = \hat{z}$. And, if preferences for private goods are locally non-satiated, then $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ must be *externality constrained Pareto efficient* in the sense that there is no Pareto superior feasible allocation $(\mathbf{x}, \mathbf{e}, z)$ with $z = \hat{z}$. That is, a (strict) Pareto improvement is impossible without altering the mean externality vector. In particular, this mean externality vector must be distributed efficiently between different individuals.

The arguments of Section 5 apply equally well to PWELT allocations. Any PWELT allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ must therefore be *$f$-externality constrained Pareto efficient*. That is, except for a null set of agents $N \subset I$, there can be no finite coalition $C \subset I \setminus N$ with net trade and externality vectors $(x^i, e^i)$ $(i \in C)$ satisfying both $(x^i, e^i) \in F_C^i(\hat{z})$ and $(x^i, e^i, \hat{z})\ P^i\ (\hat{x}^i, \hat{e}^i, \hat{z})$ for all $i \in C$, as well as $\sum_{i \in C} x^i = \sum_{i \in C} \hat{x}^i$ and $\sum_{i \in C} e^i = \sum_{i \in C} \hat{e}^i$. So, since finite coalitions lack the power to alter the mean externality vector, almost none of them are able to find improvements by reallocating private goods and/or externality vectors between their members.

Of course, the difference between ordinary $f$-constrained and $f$-externality constrained Pareto efficient allocations is that the former still allow there to be $f$-Pareto improvements involving finite coalitions who do not change their total externality vector, whereas the latter excludes them. In fact, therefore, $f$-externality constrained Pareto efficiency turns out to be less demanding, even though it requires aggregate externality vectors to be distributed efficiently within each finite coalition. The point is that, given the level of widespread externalities, an $f$-constrained Pareto efficient allocation effectively allows both individuals and finite coalitions to choose their externality vectors optimally without any further constraint, whereas an $f$-externality constrained Pareto efficient allocation can fix individuals' externality vectors arbitrarily, and allows any finite coalition to change its members' externality vectors only by redistributing the aggregate externality vector of that coalition. In fact, if the widespread externalities have their levels fixed, then the rights to create them are distributed efficiently even if their price is zero. Thus, the efficiency gains from using non-zero Pigou prices must come entirely from altering the levels of the widespread externalities; efficient distribution occurs even at a zero price.

These two versions of the first efficiency theorem apply to any PWELT, including one for which the equilibrium Pigou tax/subsidy vector $t = 0$. But in this case a PWELT is actually a NWELT, as considered in Sections 4 and 5. So, unless the public good problem is given due attention, there is no guarantee that introducing markets for externalities will do anything to improve efficiency; in fact, such markets could even produce a Pareto inferior allocation. Note in particular that the standard gains from trade arguments do not apply because opening up externality markets in order to reach a PWELT affects the widespread externalities, and could make them worse.

Second efficiency theorems can be proved in the usual way for externality constrained Pareto efficient allocations, and also for $f$-externality constrained Pareto efficient allocations. Indeed, let $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ be any externality constrained Pareto efficient allocation. Then $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ is certainly Pareto efficient in the economy with $\hat{z}$ fixed. To apply previous proofs that there exist price vectors $(p, t)$ and lump-sum transfers $\mathbf{m}$ such that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, t, \mathbf{m})$ is a PWELT given $\hat{z}$, some slight variations in previous assumptions are required. Specifically, one should make the general assumptions that the conditional feasible sets $F^i_C(\hat{z})$ $(i \in I)$ are convex subsets of $\Re^G \times \Re^I$, and that $(0, \hat{z})$ is an interior point of $\int_I F^i_C(\hat{z}) d\alpha$. Then too, the usual proof requires the specific assumption that $(\hat{\mathbf{x}}, \hat{\mathbf{e}})$ is non-oligarchic in the economy where $\hat{z}$ is fixed, but where both private goods and externality vectors can be redistributed. These conditions are enough to ensure that the externality constrained Pareto efficient allocation can be decentralized as a PWELT, given $\hat{z}$.

Also, under the same assumptions as in the previous paragraph, for any non-oligarchic $f$-externality constrained Pareto efficient allocation $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$, there will exist price vectors $(p, t)$ and lump-sum transfers $\mathbf{m}$ such that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, t, \mathbf{m})$ is a PWELT given $\hat{z}$.

These second efficiency theorems are especially important because they show how *any* non-oligarchic externality constrained or $f$-externality constrained Pareto efficient allocation, even one in which the aggregate externality vector is actually very inefficient, can emerge from perfect competitive markets for private goods and for the rights and duties to create externalities. Actually, individuals' preferences for mean externality vectors play no role in establishing whether $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, t, \mathbf{m})$ is a PWELT; it is only conditional preferences given $\hat{z}$ that count. Thus externality markets, or the equivalent Pigou taxes and subsidies, are certainly no panacea.

To conclude, note how Starrett's (1972) fundamental non-convexities actually create no difficulties at all for these standard results concerning Pigou–Walrasian equilibria. This is because of the way in which such non-convexities enter the feasible sets $F^i$, and make the utility functions $U^i(x^i, e^i, z)$ violate concavity. In fact, fundamental non-convexities are quite compatible with each conditional feasible set $F^i_C(z)$ being convex and each utility function $U^i(x^i, e^i, z)$ concave as a conditional function of $(x^i, e^i)$ alone. Indeed, in the case of an external diseconomy, the appropriate component of the partial gradient vector $U^i_z(x^i, e^i, z)$ is negative, yet $U^i(x^i, e^i, z)$ may well be bounded below even as the corresponding component of $z$ grows infinitely large; this is incompatible with having the negative marginal utility non-increasing as $z$ increases, which is what concavity of $U^i(x^i, e^i, z)$ would require.

## Appendix

*A.1. The Second $f$-Constrained Efficiency Theorem*

The following is the result promised in Section 5.

THEOREM. *Let $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ be any $f$-constrained Pareto efficient allocation which is non-oligarchic in the private good exchange economy with individual feasible sets $X^i_G(\hat{z})$ defined by (7), and utility functions $U^i_G(x^i; \hat{z})$ defined on $X^i_G(\hat{z})$ by (8). Suppose too that the sets $X^i_G(\hat{z})$ are convex, that the conditional preference orderings $R^i_G(\hat{z})$ represented by $U^i_G(x^i; \hat{z})$ are continuous and locally non-satiated w.r.t. rational vectors, and that 0 is in the interior of $\int_I X^i_G(\hat{z})\, d\alpha$. Suppose too that the correspondence from $i$ to the graph of the preference relation $R^i_G(\hat{z})$ has a measurable graph. Then there exist prices $p \neq 0$ and transfers $\mathbf{m}$ such that $(\hat{\mathbf{x}}, p, \mathbf{m})$ is a WELT, given $\hat{z}$, and so $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, \mathbf{m})$ is a NWELT.*

PROOF: Let $\hat{\mathbf{x}}$ be any feasible allocation in the private good exchange economy with individual feasible sets $X^i_G(\hat{z})$. For each $i \in I$, define the strict preference set

$$T^i := \{\, t \in \Re^G \mid t + \hat{x}^i \; P^i_G(\hat{z}) \; \hat{x}^i \,\}. \tag{13}$$

Now, following the beginning of Aumann's (1964, p. 45) and Hildenbrand's (1982, pp. 843–4) proof of core equivalence, first define $Q^G$ as the set of vectors in $\Re^G$ with all coordinates rational. Second, for every $t \in Q^G$, define the (measurable) set $I(t) := \{\, i \in I \mid t \in T^i \,\}$. Third, let

$$I' := I \setminus \bigcup \{\, I(t) \mid t \in Q^G,\ \alpha(I(t)) = 0 \,\}. \tag{14}$$

Because $Q^G$ is countable, the set $I'$ is measurable and satisfies $\alpha(I') = 1$. Finally, let $K := \text{co}\left[\bigcup_{i\in I'}(Q^G \cap T^i)\right]$.

Suppose that $0 \in K$. Then there exist individuals $i_k \in I'$, vectors $t_k \in Q^G \cap T^{i_k}$, and *rational* convex weights $r_k$ $(k = 1, \ldots, m)$ such that $t_k$ and $\sum_{k=1}^m r_k\, t_k = 0$. By definition of $I'$, it follows that $\alpha(I(t_k)) > 0$ $(k = 1, \ldots, m)$. Because the convex weights are rational, there exist natural numbers $n_k$ $(k = 1, \ldots, m)$ such that $\sum_{k=1}^m n_k\, t_k = 0$. Now define $C$ as any finite coalition consisting of $n_k$ members of each set $I(t_k)$, for $k = 1, \ldots, m$. Also, define $x^i := t_k + \hat{x}^i$ for each $i \in I(t_k) \cap C$ $(k = 1, \ldots, m)$. Then $x^i\ P^i_G(\hat{z})\ \hat{x}^i$ for all $i \in C$, while

$$\sum\nolimits_{i\in C} x^i = \sum\nolimits_{k=1}^m n_k\, t_k + \sum\nolimits_{i\in C} \hat{x}^i = \sum\nolimits_{i\in C} \hat{x}^i. \tag{15}$$

Note that the coalition $C$ can be selected to include any individual in the set $\cup_{k=1}^m I(t_k)$ whose measure is positive. This implies that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ cannot be $f$-constrained Pareto efficient.

The previous paragraph shows that, if $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z})$ is $f$-constrained Pareto efficient, then $0 \notin K$. So, by the separating hyperplane theorem, there exists $p \neq 0$ such that $p\,t \geq 0$ for all $t \in K$. Now, given any $i \in I'$ and any $x^i \in X^i_G(\hat{z})$ satisfying $x^i\ P^i_G(\hat{z})\ \hat{x}^i$, local non-satiation w.r.t. rational vectors implies that there is an infinite sequence $x^i_m \to x^i$ with $x^i_m\ P^i_G(\hat{z})\ x^i$ and $x^i_m - \hat{x}^i \in Q^G$ $(m = 1, 2, \ldots)$. Thus, the vector $x^i - \hat{x}^i$ is the limit of points in $Q^G \cap T^i$. From this it follows that, for all $i \in I'$ and so for almost all $i \in I$, one has $p\,x^i \geq p\,\hat{x}^i$ whenever $x^i\ P^i_G(\hat{z})\ \hat{x}^i$. Because of ordinary local non-satiation, a standard argument then establishes that for almost all $i \in I$, one has $p\,x^i \geq p\,\hat{x}^i$ whenever $x^i\ R^i_G(\hat{z})\ \hat{x}^i$. In other words, $(\hat{\mathbf{x}}, p, \mathbf{m})$ must be a "compensated" Walrasian equilibrium given the lump-sum transfers $m^i := p\,\hat{x}^i$ for all $i \in I$. But then, under the other hypotheses of the theorem, the proof in Hammond (1993a) of the corresponding version of the second efficiency theorem (Prop. 6.2.2, p. 105) establishes that $(\hat{\mathbf{x}}, p, \mathbf{m})$ is a WELT in the conditional private good economy with $\hat{z}$ given. As argued in Section 5, this implies that $(\hat{\mathbf{x}}, \hat{\mathbf{e}}, \hat{z}, p, \mathbf{m})$ is a NWELT. ∎

The second $f$-externality constrained efficiency theorem discussed in Section 7 is proved in exactly the same way.

*A.2. Existence of Nash–Walrasian Equilibrium*

The following existence proof will assume that individuals' feasible sets $F^i$ are closed, the conditional feasible sets $X^i_G(z)$ are convex, and that both preference orderings $R^i$ and $R^i_G(z)$ are continuous and locally non-satiated. But it will also require the additional assumption that the conditional feasible set correspondences $F^i_C(z)$ and $E^i(x^i, z)$ are continuous (i.e., both upper and lower hemi-continuous — u.h.c. and l.h.c.), where the latter correspondence is defined by

$$E^i(x^i, z) := \{ e^i \in \Re^E \mid (x^i, e^i, z) \in F^i \} \tag{16}$$

for all $(x^i, z) \in \Re^G \times \Re^E$. Note that, because $z \mapsto\!\!\!\to F^i_C(z)$ is, by assumption, a l.h.c. correspondence, a routine proof shows that $z \mapsto\!\!\!\to X^i_G(z)$ is also l.h.c.

Furthermore, the following *boundedness* assumption will be invoked. Suppose that almost every $i \in I$ has bounds $\underline{x}^i, \underline{e}^i, \bar{e}^i \in \Re^G$ which are the values of measurable functions on $I$ whose integrals $\underline{x} := \int_I \underline{x}^i \, d\alpha$, $\underline{e} := \int_I \underline{e}^i \, d\alpha$, and $\bar{e} := \int_I \bar{e}^i \, d\alpha$ are all finite vectors. Let

$$E^i := \{ e^i \in \Re^E \mid \underline{e}^i \leqq e^i \leqq \bar{e}^i \} \quad \text{and} \quad Z := \{ z \in \Re^E \mid \underline{e} \leqq z \leqq \bar{e} \}. \tag{17}$$

Also, define the individual externality correspondence

$$\eta^i(x^i, z) := \arg\max_{e^i} \{ U^i(x^i, e^i, z) \mid e^i \in E^i(x^i, z) \}. \tag{18}$$

Then the bounds $\underline{x}^i, \underline{e}^i, \bar{e}^i$ must be such that, for almost all $i \in I$, one has $x^i \geqq \underline{x}^i$ whenever $(x^i, e^i, z) \in F^i$; also whenever $z \in Z$, there must exist an optimal externality vector $\hat{e}^i \in E^i \cap \eta^i(x^i, z)$. By a direct application of Berge's maximum theorem, continuity of the correspondence $E^i(x^i, z)$ evidently implies that the restricted correspondence defined by $\hat{\eta}^i(x^i, z) := \eta^i(x^i, z) \cap E^i$ is non-empty valued and u.h.c. In addition, because $E^i$ is compact and $z \mapsto\!\!\!\to F^i_C(z)$ is a u.h.c. correspondence, a routine proof shows that $z \mapsto\!\!\!\to X^i_G(z)$ is also u.h.c. So the latter is actually a continuous correspondence.

Next, define $\Delta := \{ p \in \Re^G_+ \mid \sum_{g \in G} p_g = 1 \}$. Suppose that there are income distribution rules $m^i(p, z)$ defined for all $i \in I$, all $p \in \Delta$, and all $z \in Z$, and that these rules are homogeneous of degree one in $p$, continuous in $(p, z)$ and measurable in $(i, p, z)$. Suppose these rules have the properties that, for all $(p, z) \in \Delta \times Z$, both $\int_I m^i(p, z) \, d\alpha = 0$ and, for almost all $i \in I$, there exists $x^i \in X^i_G(z)$ satisfying $p \, x^i < m^i(p, z)$. Suppose too that, for almost all $i \in I$, there exists an upper bound $\bar{x}^i \in \Re^G$ such that $m^i(p, z) \leq p \, \bar{x}^i$ for all $(p, z) \in \Delta \times Z$, where $i \mapsto \bar{x}^i$ defines a measurable function $\bar{\mathbf{x}} : I \to \Re^G$ for which $\int_I \max_g \{ \bar{x}^i_g - \underline{x}^i_g \mid g \in G \} \, d\alpha$ is finite.

Finally, for each good $g \in G$, let $v_g$ denote the corresponding unit vector in $\Re^G$. Then suppose that there is a non-null set $I_g \subset I$ of individuals such that, whenever $x^i \in X^i_G(z)$

with $p\,x^i \leq m^i(p,z)$, there exists $\lambda > 0$ such that $x^i + \lambda v_g \in X^i_G(z)$ with $x^i + \lambda v_g \; P^i_G(z) \; x^i$. This assumption ensures that any equilibrium price vector must be strictly positive, because otherwise the set $I_g$ of individuals would have unlimited demands for good $g$. Of course, it is implied by the usual free disposal and monotonicity assumptions, but is much weaker.

THEOREM. *Under all the assumptions stated above, a NWELT exists.*

PROOF: For each closed and convex feasible set $X_G \subset \Re^G$, continuous preference ordering $R_G$ on $X_G$, price vector $p \in \Delta$, and income level $m \in \Re$, define the budget set

$$\beta(X_G, R_G, p, m) := \{\, x \in X_G \mid p\,x \leq m \,\}. \tag{19}$$

Give the space of pairs $(X_G, R_G)$ the topology of closed convergence. Then, arguing as in Hildenbrand (1974, pp. 99–100), the correspondence $\beta(X_G, R_G, p, m)$ has a closed graph, and is l.h.c. wherever there exists $x \in X_G$ such that $p\,x < m$.

Now define the demand set

$$\phi(X_G, R_G, p, m) := \{\, x \in \beta(X_G, R_G, p, m) \mid x' \; P_G \; x \Longrightarrow p\,x' > m \,\}. \tag{20}$$

Because the preference ordering $R_G$ is continuous, this demand correspondence has non-empty compact values and is also u.h.c. wherever $\beta(X_G, R_G, p, m)$ is compact and contains an $x$ with $p\,x < m$.

Next, for each integer $k = 1, 2, \ldots$, define

$$\Delta_k := \{\, p \in \Delta \mid \forall g \in G : p_g \geq 1/(\#G + k) \,\}. \tag{21}$$

Now, if $(p,z) \in \Delta_k \times Z$ and $x^i \in X^i_G(z)$ with $p\,x^i \leq m^i(p,z)$, then boundedness implies that $x^i \geqq \underline{x}^i$ and also, because of the assumptions regarding the income distribution rule, that

$$\begin{aligned} x^i_g &\leq (1/p_g) \left[ m^i(p,z) - \sum\nolimits_{h \neq g} p_h \, \underline{x}^i_h \right] \leq (1/p_g) \left[ p\,\bar{x}^i - \sum\nolimits_{h \neq g} p_h \, \underline{x}^i_h \right] \\ &= \bar{x}^i_g + (1/p_g) \sum\nolimits_{h \neq g} p_h \, (\bar{x}^i_g - \underline{x}^i_h) \leq \bar{x}^i_g + k \max_g \{\, \bar{x}^i_g - \underline{x}^i_g \mid g \in G \,\}. \end{aligned} \tag{22}$$

It follows that $x^i \leqq \bar{x}^i_k$ where the hypotheses of the existence theorem imply that $\bar{x}^i_k$ is integrable and that $\bar{x}_k := \int_I \bar{x}^i_k \, d\alpha$ is finite.

For all $i \in I$ and all $(p,z) \in \Delta_k \times Z$, define

$$\begin{aligned} \beta^i_k(p,z) &:= \beta(X^i_G(z), R^i_G(z), p, m^i(p,z)) \\ \xi^i_k(p,z) &:= \{\, (x^i, e^i) \in \phi(X^i_G(z), R^i_G(z), p, m^i(p,z)) \times E^i \mid e^i \in \hat{\eta}^i(x^i, z) \,\}. \end{aligned} \tag{23}$$

Then the assumptions regarding the income distribution rule imply that the budget set $\beta^i_k(p,z)$ is everywhere compact and contains an $x$ satisfying $p\,x < m^i(p,z)$. From this it follows that the demand correspondence $\phi(X_G, R_G, p, m)$ has non-empty compact values and is u.h.c. in $(X_G, R_G, p, m)$ whenever $(X_G, R_G, p, m) = (X^i_G(z), R^i_G(z), p, m^i(p,z))$ for some $(p,z) \in \Delta_k \times Z$.

To see that the correspondence $\xi^i_k : \Delta_k \times Z \to \Re^G \times \Re^E$ has a closed graph, consider any infinite sequence $(x^i_m, e^i_m, p_m, z_m)$ satisfying $(x^i_m, e^i_m) \in \xi^i(p_m, z_m)$ $(m = 1, 2, \ldots)$ that converges to $(\hat{x}^i, \hat{e}^i, \hat{p}, \hat{z})$ as $m \to \infty$. Then $(X^i_G(z_m), R^i_G(z_m))$ converges to $(X^i_G(\hat{z}), R^i_G(\hat{z}))$ in the topology of closed convergence. Also, $x^i_m \in \phi(X^i_G(z_m), R^i_G(z_m), p_m, m^i(p_m, z_m))$ and $e^i_m \in \hat{\eta}^i(x^i_m, z_m)$ $(m = 1, 2, \ldots)$. So, because the correspondence $\phi(X_G, R_G, p, m)$ is u.h.c. at $(X^i_G(\hat{z}), R^i_G(\hat{z}), \hat{p}, m^i(\hat{p}, \hat{z}))$, it follows that $\hat{x}^i \in \phi(X^i_G(\hat{z}), R^i_G(\hat{z}), \hat{p}, m^i(\hat{p}, \hat{z}))$. In addition, because the correspondence $(x^i, z) \mapsto\!\!\!\rightarrow \hat{\eta}^i(x^i, z)$ is u.h.c., it follows that $\hat{e}^i \in \hat{\eta}^i(\hat{x}^i, \hat{z})$, and so $(\hat{x}^i, \hat{e}^i) \in \xi^i_k(\hat{p}, \hat{z})$. Therefore $\xi^i_k(\cdot)$ does have a closed graph.

Because the function $i \mapsto m^i(p,z)$ and the correspondence $i \mapsto\!\!\!\rightarrow (X^i_G(z), R^i_G(z))$ are both measurable on $I$, so is the correspondence $i \mapsto\!\!\!\rightarrow \xi^i_k(p,z)$. By the boundedness assumption and the definition of $E^i$, the correspondence is also non-empty valued. Also, because of local non-satiation, $(x^i, e^i) \in \xi^i_k(p,z)$ implies $p\,x^i = m^i(p,z)$. In addition, it must be true that $\underline{x}^i \leqq x^i \leqq \bar{x}^i_k$ and $\underline{e}^i \leqq e^i \leqq \bar{e}^i$ whenever $(x^i, e^i) \in \xi^i_k(p,z)$ for any $(p,z) \in \Delta_k \times Z$, and so $\xi^i_k(p,z)$ is integrably bounded on $\Delta_k \times Z$. By Hildenbrand (1974, p. 73, Prop. 8), it follows that the aggregate net demand correspondence $\xi_k : \Delta_k \times Z \twoheadrightarrow \Re^G \times \Re^E$ with $\xi_k(p,z) := \int_I \xi^i_k(p,z)\, d\alpha$ is well defined and has a closed graph. Its values are non-empty and, because the measure $\alpha$ is non-atomic, convex. Let $X_k := \{\, x \in \Re^G \mid \underline{x} \leqq x \leqq \bar{x}_k \,\}$. Then $\xi_k(p,z) \in X_k \times Z$ for all $(p,z) \in \Delta_k \times Z$.

Define the price adjustment correspondence $\psi^P_k : X_k \twoheadrightarrow \Delta_k$ by

$$\psi^P_k(x) := \arg\max_{\tilde{p}} \{\, \tilde{p}\, x \mid \tilde{p} \in \Delta_k \,\}. \tag{24}$$

Evidently $\psi^P_k$ has a closed graph and non-empty convex values. Then consider the correspondence $\psi_k : \Delta_k \times X_k \times Z \twoheadrightarrow \Delta_k \times X_k \times Z$ defined by $\psi_k(p, x, z) := \psi^P_k(x) \times \xi_k(p,z)$. As a product of closed graph correspondences with non-empty convex values, it too has a closed graph and non-empty convex values. Moreover, the domain $\Delta_k \times X_k \times Z$ is the product of non-empty compact convex sets. So Kakutani's fixed point theorem can be applied. Each correspondence $\psi_k$ therefore has a fixed point satisfying $(p_k, x_k, z_k) \in \psi_k(p_k, x_k, z_k)$ $(k = 1, 2, \ldots)$. So there exist infinite sequences of price vectors $p_k \in \Delta_k$, quantity vectors $x_k \in \Re^G$, aggregate externality vectors $z_k \in \Re^E$, together with integrably bounded and measurable functions $\mathbf{x_k} : I \to \Re^G$ and $\mathbf{e_k} : I \to \Re^E$, such that: (i) $(x^i_k, e^i_k) \in \xi^i(p_k, z_k)$ a.e. in $I$; (ii) $x_k = \int x^i_k$ and $z_k = \int e^i_k$; (iii) $p\, x_k \leq p_k\, x_k = 0$ for all $p \in \Delta_k$. But then, because

$(1/\#G)\,(1,1,\ldots,1) \in \Delta_k$, (iii) above implies that $\sum_{g\in G} x_{kg} \leq 0$ for all $k$. So the sequence of fixed points $(p_k, x_k, z_k)$ always lies in the compact set $\Delta \times X \times Z$, where

$$X := \{\, x \in \Re^G \mid x \geqq \underline{x} \quad \text{and} \quad \sum\nolimits_{g\in G} x_g \leq 0 \,\}. \tag{25}$$

Hence there must exist some subsequence of $(p_k, x_k, z_k)$ $(k = 1, 2, \ldots)$ which converges to a limit point $(p^*, x^*, z^*) \in \Delta \times X \times Z$. Moreover, Fatou's Lemma in many dimensions (see, for instance, Hildenbrand, 1974, p. 69) can now be applied to show that there exists a subsequence $k(m)$ $(m = 1, 2, \ldots)$, together with some $p \in \Delta$ and some measurable functions $\mathbf{x} : I \to \Re^G$ and $\mathbf{e} : I \to \Re^E$ such that: (iv) $\int x^i \leqq x^*$ and $\int e^i = z^*$ (because each $e^i_k$ lies in the integrably bounded set $E^i$); and also, in the limit as $m \to \infty$, so: (v) $p_{k(m)} \to p^*$; (vi) $\int x^i_{k(m)} = x_{k(m)} \to x^*$ and $\int e^i_{k(m)} = z_{k(m)} \to z^*$; (vii) $x^i_{k(m)} \to x^i$ and $e^i_{k(m)} \to e^i$ a.e. in $I$.

Now, for any positive integers $m$ and $r$ such that $k(m) \geq r$, (iii) implies that $p \int x^i_{k(m)} \leq 0$ for all $p \in \Delta_r \subset \Delta_{k(m)}$. Because of (vi), taking the limit as $m \to \infty$ gives $p\,x^* \leq 0$ for all $p \in \Delta_r$. Since this is true for any positive integer $r$, one has $p\,x^* \leq 0$ for all $p \in \Delta^0 = \cup_{r=1}^{\infty} \Delta_r$. Hence $x^* \leqq 0$. Also, since (i) implies that $(x^i_{k(m)}, e^i_{k(m)}) \in \xi^i(p_{k(m)}, z_{k(m)})$ a.e. in $I$, and since (v) and (vii) above are both true, the closed graph property of the compensated demand correspondence implies that $(x^i, e^i) \in \xi^i(p^*, z^*)$ a.e. in $I$.

Finally, if $p^*_g = 0$ for some $g \in G$, then $\xi^i(p^*, z^*)$ would be empty for all $i$ in the non-null set $I_g$, a contradiction. Therefore $p^* \gg 0$. But $\int x^i \leqq x^* \leqq 0$. Also $p^*\,x^i = m^i(p^*, z^*)$ a.e. in $I$, so $p^* \int x^i = \int m^i(p^*, z^*) = 0$. It follows that $\int x^i = x^* = 0$. Because $\int e^i = z^*$, this proves that $(\mathbf{x}, \mathbf{e}, z^*, p^*, \mathbf{m}(p^*, z^*))$ is a NWELT. ∎

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