(Functional) Characterizations vs (Finite) Tests: Partially Unifying Functional and Inequality-Based Approaches to Testing*

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Abstract

Historically, testing whether decision-makers (DM) obey certain axioms from choice data has taken two rather distinct approaches: the "functional" approach observes the entire "demand function" and puts restrictions on it; while the "revealed preference" approach constructs algebraic inequalities that are used to test finite data. Using restrictions derived in Kubler et al. (2020), we demonstrate a link between revealed preference tests and function-based restrictions. By this, we mean that any functional restriction that *characterizes* the above axioms can be used to construct a well-behaved finite data test. We then develop an algorithm that does this efficiently.

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1 Introduction

Consider an individual decision-maker with complete, transitive, and monotone preferences. While conducting empirical analyses, the analyst often wishes to check whether the decision-maker (DM) satisfies some axiom (e.g., weak separability). A straightforward way to do this is to estimate the DM's demand function (or preference) and then analyze it for said property. However, an analyst will often have access to only limited choice data. This paper asks how and under what conditions such finite data can be used to test a preference for such a property.

When testing preferences, the literature adopts two distinct approaches: the first approach applies conditions involving derivatives of consumer demand and assumes access to the whole demand function; the second is the revealed preference (RP) approach, which derives algebraic conditions (inequalities) on finite choice data. To quote, "The distinction between the two approaches is very important in empirical work. The calculus approach assumes the entire demand function is available for analysis, while the algebraic approach assumes only a finite number of observations on consumer behaviour is available." (Varian, 1983)

In practice, this schism between approaches also extends to testing DMs for more restrictive properties than rationality. However, when confronting finite data, analysts have usually only appealed to the revealed preference/algebraic approach. One reason for this may be that finite sample analogues of demand derivatives or rates of substitution are hard to find. However, the RP approach relies on finite inequalities, which are often hard to enumerate computationally with real data. For example, although testing for weak separability is simple if the demand function is known, testing using revealed preference inequalities is computationally hard. We show that this problem can be solved—under reasonable assumptions, a simple derivative test can be used to construct a computationally efficient test of finite data.

Specifically, consider a calculus-based restriction of a DM's demand function. Assume that demand functions that arise from utilities within our class of interest (e.g., separability) satisfy this restriction. Furthermore, any demand function that arises from a utility that is not within our class fails our test. In other words, this test is an if and only if condition for our DM to lie in the class of interest. We refer to such a

¹To our knowledge, the only other paper which constructs a test of finite data from a procedure relating to derivatives is Aguiar and Serrano (2018) which uses finite data to extend the measure of irrationality for demand functions defined in Aguiar and Serrano (2017).

functional test as a *characterization* and demonstrate that restrictions on demand can be tested but only if they are characterizations, which connects the two strands of the literature. We then construct a general algorithm to convert any such characterization into an efficient test.

The classical RP approach tries to derive *exhaustive* restrictions on the demand functions implied by maximizing behaviour. This means that *any* finite data arising from maximization must satisfy the RP conditions. Every dataset that satisfies these conditions can be seen to arise from utility maximization or, in our case, as arising from the maximization of a weakly separable utility.²

We subtly differ from this—we provide tests of finite classes, and not characterizations. Instead of just testing datasets with choice data for rationalizability, we assume that the data are generated by a DM that is confronted by random choice sets and propose tests with the following properties:

- 1. Any dataset that can be rationalized by a function of our class of interest is accepted, this is akin to standard RP tests.
- 2. Our Proposal: If the DM is not from the class of interest because more choice data are sampled independently, then the probability with which the test rejects the data the DM generates goes to 1.³

The literature on testing refers to the first property as soundness. This means that whenever a dataset is rejected, then this rejection is correct. The second property is completeness, which says that *everything* that can be rejected is rejected. Our proposal is a limiting version of completeness, which is referred to as **asymptotic completeness** (AC henceforth). To state AC in simple words: there may be bad data that passes our test, but the probability of encountering such data goes to zero as the size of the dataset increases.

Under this assumption, we construct a feasible test. This is important for two reasons: first, several properties (e.g., weak separability) have very natural functional tests but no intuitive or simple finite tests. Second, our procedure lets an analyst test large datasets efficiently for properties where exhaustive restrictions are provably hard to check. This is an interesting theoretical contribution because it says that relaxing the if and only if condition on rationalizability can restore computational feasibility.

²Or any other specific subclass we wish to test for.

³The standard RP test would require that if no preference from the class that we are interested in can generate the data, then the data is *always* rejected

Now, let us describe a naive approach to constructing finite tests using functional restrictions and its pitfalls. Mas-Colell (1977) proved that if price demand pairs are generated from an underlying preference, then *any* rationalizing preference will converge to the true underlying preference. Naively, one would think that one could just sample data, pick any preference (demand function) which rationalizes the data, and then test that demand using the derivative test that we already know. However, because of convergence, if the DM is non-separable, then any preference that we pick will fail the test.

And therein lies the rub: if the analyst's preference fails the test, then they do not know whether the true preference fails the test or if they simply picked the wrong one. Moreover, convergence only guarantees that we will eventually be close to the underlying preference. At no point do we know exactly how close we are. To solve this problem, we propose a sophisticated approach that defines bounds on the distance between the preference picked by the analyst and the true underlying preference, which can be constructed after sampling the data.

To do this, we heavily lean on recent work on learning preferences with finite data, particularly Kubler, Malhotra, and Polemarchakis (2020); Beigman and Vohra (2006). This ex-post nature of our bounds lets us define a stronger notion of convergence; that is, certifiable convergence, which can provide explicit bounds to differences between the true preference and the picked preference. We define a **certificate** of radius epsilon as proof that any preference rationalizing the given data is at most epsilon far away from the true underlying preference.

We prove that this kind of convergence holds for data generated from pairwise choices and price-demand pairs. We then leverage mild assumptions on derivative tests (discussed later) to translate this bound into a finite data test and an explicit algorithm to construct it. We now go on to summarise the literature on the topic and how we contribute to it.

1.1 Related Literature

The first approach originated in the work of Slutsky (1915) and Antonelli (1971), who derived necessary and sufficient conditions on derivatives for demand functions to arise from utility maximization.⁴. For a detailed review of tests for classes of preferences,

⁴This the so-called "integrability problem" for demand functions. For a review see Hurwicz and Uzawa (1971)

see Deaton and Muellbauer (1980). However, this literature never attempted to extend these findings to test finite data.

The closest we know of to have come to linking restrictions of demand functions and finite data is Aguiar and Serrano (2018), which extends the measure of irrationality defined in Aguiar and Serrano (2017) to finite data. They demonstrate that their measure of irrationality computed from finite data converges to the true measure for the underlying DM. However, they only concern themselves with the measurement of one specific index. Their arguments rely on the properties of Walrasian demand and thus can only be applied to price data. On the other hand, we concern ourselves with testing for properties and not measuring deviations, demonstrating this can be done for general restrictions. Also, our arguments rely only on preference monotonicity and can thus be applied to more general datasets like pairwise comparisons.

The second approach, which originates in the work of Samuelson (1938, 1939, 1947, 1948), derives algebraic inequalities on the demand data implied by maximizing behaviour. However, the *exhaustive* empirical consequences of utility maximization were provided by Afriat (1967). Afriat used the sufficiency of first-order conditions to construct necessary and sufficient combinatorial conditions for finite data to arise from utility maximization. These conditions are known as the Generalized Axiom of Revealed Preference (GARP). Much of the literature follows a similar tradition, using first-order conditions to construct inequalities that finite data must satisfy.

The last decade has witnessed an explosion of work on RP theory, considering functional form restrictions, inter-temporal models, and generalizing budget sets, among other extensions. To mention a few, Kubler, Selden, and Wei (2014) find the RP characterization for expected utility when there are objective(known) probabilities. Echenique and Saito (2015) solve this problem for the case where probabilities represent subjective beliefs. A similar problem for "translation-invariant" preferences is considered in Chambers, Echinique, and Saito (2016). Polisson, Quah, and Renou (2020) give a general method to construct RP inequalities for budget sets, which apply to several classes of preferences over risk and uncertainty.⁵.

This literature concentrates exclusively on *exhaustive* restrictions that *character-ize*⁶ finite data sets and largely use the sufficiency of first-order conditions that were

⁵For recent extensive reviews, see Crawford and Rock (2014); Echenique (2019); Demuynck and Hjertstrand (2019).

⁶see Chambers and Echenique (2016, p. xv)

pioneered by Afriat.⁷ However, the moment we depart from all but the most straightforward setup, providing an analytic characterization of the inequalities becomes cumbersome and the inequalities are typically non-linear. This non-linearity provides a hurdle to anyone testing choice data because testing if datasets satisfy non-linear inequalities need not be computationally feasible. For example, Cherchye et al. (2015); Echenique (2014) show that testing for weak separability is NP-Hard.

This problem may be solved by finding heuristic algorithms to solve real-world instances of the problem. Cherchye, Demuynck, De Rock, and Hjertstrand (2015) and Hjertstrand, Swofford, and Whitney (2020) take this approach. They use an integer programming approach and demonstrate its attractive behaviour when used on datasets that are typically studied by economists.

We propose a second approach to weaken RP tests' completeness property to another condition that we posit (i.e., Asymptotic Completeness). In our setup, there may be bad data that passes our test but the probability of such data occurring vanishes as the sample size increases. Using this approach, we construct a feasible test of weak separability.

We thus depart from the RP literature in two ways; first, we use local restrictions of the demand function rather than using first-order conditions to construct our test. Then, rather than "RP characterizations," we aim for slightly weaker tests (as described above). Specifically, we demonstrate that any characterization of demand or utility functions can be used to construct a test of finite data. We then demonstrate that the test that we construct is computationally feasible.

On an aside, to the best of our knowledge, this is the first paper to use the certification properties of monotone preferences to construct finite data tests. We believe that explicitly using the certificates that monotonicity provides is an interesting direction for future research in constructing tests of finite observations.

The rest of this paper is structured as follow. We describe our setup and preliminaries in detail in the next section. We then look at a few basic revealed preference and functional tests for preference restrictions, which are previously known. We then discuss the notion of certification and certifiable convergence, showing its applicability in our setup. Using certification, we then illustrate our result using an example (i.e., weak separability) and show that our test satisfies the conditions that we discussed

⁷The only revealed preference test that is not a characterization which we are familiar with is for probabilistic sophistication Epstein (2000).

⁸Polynomial-time in data size.

earlier. We then generalize our results to different families of choice sets and sampling procedures. We finally conclude and discuss some possible directions for future research.

2 Setup

Observations

We first lay out our basic setup and describe the observations that are available to the analyst. The setup is one of individual decision-making, and we assume that the analyst only has access to data on the finite choices that the DM makes.

Items of choice are l-dimensional bundles $x \in \mathbf{X} \subset \mathbb{R}^l$, where **X** is a convex compact set. We refer to the set of complete, transitive, twice differentiable, strongly convex, and monotone preferences as $\mathcal{M} \subset 2^{\mathbf{X} \times \mathbf{X}}$. Every DM has a preference $\succeq \subseteq \mathbf{X} \times \mathbf{X}$. We associate every preference with its graph and write $x \succeq y$ if $(x, y) \in \succeq$, to be read as x is "at least as good as" y.⁹ 10 11

For a collection of choice sets, \mathcal{A} , a choice function, $f: \mathcal{A} \to \mathbf{X}$ associates with every choice set, $\mathbf{A} \subset \mathbf{X}$, $\mathbf{A} \in \mathcal{A}$, an element, $f(\mathbf{A}) \in \mathbf{A}$. Observations are a collection of choice sets and associated choices $(\mathbf{A} \in \mathcal{A}, f(\mathbf{A}))$ or $\{(\mathbf{A}, f)_k\}_{k=1}^n$.

Remark. $\{(\mathbf{A},f)_k\}_{k=1}^n$ is just a dataset and does not assume that the analyst has any knowledge of the underlying choice function f,

A preference relation \succeq rationalizes observations $\{(\mathbf{A}, f)_k\}_{k=1}^n$ if for all $\mathbf{A} \in \mathcal{A}$,

$$f(\mathbf{A})\succeq y$$
 for all $y \in \mathbf{A}$.

Let $\mathcal{C} \subset \mathcal{M}$ be a class of preferences. Call a set of observations $\{(\mathbf{A}, f)_k\}_{k=1}^n$ rationalizable if there exists

$$\succeq \in \mathcal{C}$$
 such that \succeq rationalizes $\{(\mathbf{A}, f)_k\}_{k=1}^n$

We say that an underlying preference $\succeq \in \mathcal{M}$ from generates f if for all choice sets $\mathbf{A} \in \mathcal{A}$,

$$f(\mathbf{A})\succeq y$$
 for all $y \in \mathbf{A}$.

⁹By transitive we mean that $(x,y),(y,z)\in\succeq\Longrightarrow(x,z)\in\succeq$. ¹⁰By convex, we mean $(x,y),(x,z)\in\succeq\Longrightarrow\forall$ $\lambda\in[0,1](x,\lambda y+(1-\lambda z))\in\succeq$

¹¹By monotone we mean $\forall e > 0 \quad (x, x + e) \in \succeq$

refer to the choice function as f_{\succeq} and refer to the corresponding data point generated by choice set **A** as

$$(\mathbf{A}, f_{\succ}(\mathbf{A}))$$

or

$$\{(\mathbf{A}, f_{\succ})_k\}_{k=1}^n$$

for a collection of choice sets and data points.

Finite Data Tests

We now define what we mean by "tests" and how we depart from the revealed preference literature. Let $2^{\{(\mathbf{A},f)_k\}_{k=1}^n}$ be the set of all datasets. We define a test $\mathcal{T}: 2^{\{(\mathbf{A},f)_k\}_{k=1}^n} \to \{0,1\}$ as a function that inputs observations and outputs a 0 or a 1.

We say that a dataset $\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$ passes a test \mathcal{T} if

$$\mathcal{T}\{(\mathbf{A}, f_{\succ})_k\}_{k=1}^n = 1$$

Similarly, we say that a dataset $\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$ fails a test \mathcal{T} if

$$\mathcal{T}\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n = 0$$

Remark. By accepting, we mean what is conventionally termed as failing to reject.

We now define the properties of the tests that we use in this paper.

Definition 1 (Soundness). We call a test \mathcal{T} sound with respect to class \mathcal{C} if,

$$\mathcal{T}\{(\mathbf{A}, f)_k\}_{k=1}^n = 0 \implies \{(\mathbf{A}, f)_k\}_{k=1}^n \text{ is not } \mathcal{C} - rationalizable$$

Soundness means that if the test fails the data, then the answer must be correct; there are no true negatives.

Definition 2 (Completeness). We call a test \mathcal{T} complete with respect to class \mathcal{C} if,

$$\mathcal{T}\{(\mathbf{A}, f)_k\}_{k=1}^n = 1 \implies \{(\mathbf{A}, f)_k\}_{k=1}^n \text{ is } \mathcal{C} - rationalizable$$

Completeness means that only the "correct" preferences pass our test, or there are

no false positives.

We seek to weaken this second property in our approach.

Remark. The literature refers to sound and complete tests as characterizations.

Definition 3 (Asymptotic Completeness). Let (\mathcal{A}) denote a family of choice sets and let ν denote a probability measure over this family. Suppose that $\succeq \in \mathcal{M}$ is some preference and $\{(\mathbf{A}, f_{\succeq})_i\}_{i=1}^n$ is the dataset it generates by i.i.d draws of \mathbf{A} from ν . We say that a $Test \mathcal{T}: 2^{\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n} \to \{0, 1\}$ is Asymptotically Complete w.r.t \mathcal{C} and ν , if,

$$\succeq \notin \mathcal{C} \implies \lim_{n \to \infty} \mu^n \{ \{ (\mathbf{A})^n \} \text{ such that } \mathcal{T}((\mathbf{A}, f_{\succeq})^n) = 0 \} \to 1$$

Where μ^n is the product measure inherited by i.i.d sequences from μ

Demand Functions

In our paper, the analyst never observes whole choice functions. However, we need to define choice functions for specific families of (infinite) choice sets because they are used in revealed preference theory. We define a *demand function*.

$$x: \mathcal{P} \to \mathbf{X}$$

Where $\mathcal{P} \subset \mathbb{R}^l$ represents the set of possible *prices*. For a price $p \in \mathcal{P}$, define a corresponding choice set

$$\mathbf{A}_p := \{ y \in \mathbf{X} : p \cdot y \le 1 \}$$

Definition 4. Call a demand function rational if the dataset

$$D = \bigcup_{p \in \mathcal{P}} (\mathbf{A}_p, x(p))$$

is rationalizable.

Furthermore, refer to the demand function generated by preference \succeq as x_\succeq

Let $\succeq \in \mathcal{M}$ we say that a Utility function $U: \mathbf{X} \to \mathbb{R}$ Represents \succeq if.

$$(x,y)\in\succeq\implies U(x)\geq U(y)$$

The Analysts Problem

The analyst is given a family of choice sets \mathcal{A} , with a measure μ over this family. This generates datasets of the form $\{(\mathbf{A}, f)_k\}_{k=1}^n$. They are then given some other class of preferences $\mathcal{C} \subset \mathcal{M}$. Assuming that there exists a DM with a preference $\succeq \in \mathcal{M}$, that generates the dataset, the analyst must construct a sound and asymptotically complete test of Class \mathcal{C} .

3 Convergence and Bounds to Errors

The Naive Approach

We now highlight the naive approach to the problem and why it does not work. Mas-Colell (1977) shows that given choice data, any preference rationalizing the data converges to the true preference generating the data. Suppose, acting naively, that the analyst samples data, picks a rationalizing preference, and tests it for some property. The problem is as follows

Suppose that the picked preference fails the test for separability. In this case, the analyst does not know if there is no separable preference that generates the data or if they picked the wrong preference.

Adding to this, notice that infinitely many preferences rationalize the data, which renders it impossible for the analyst to test each of them.

We solve this problem by generating bounds on "how close" the picked preference is to the true underlying preference. Once we have bounds on this distance, we can use regularity conditions to generate tests.

The Sophisticated Approach

To illustrate our approach, we first need some formal machinery.

Call a set of observations, $C_{\epsilon} = \{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$ a **certificate** of radius ϵ , if:

$$\succeq^1,\succeq^2$$
 rationalize $\{(\mathbf{A},f_{\succeq})_k\}_{k=1}^n \implies d(\succeq^1,\succeq^2) < \epsilon$

Where the distance between preferences is defined as the measure of the symmetric difference of the graphs.

$$d(\succeq^1,\succeq^2) = \mu^2\{\succeq^1 \cup \succeq^2 - \succeq^1 \cap \succeq^2\}$$

Intuitively, we call something a certificate of radius ϵ if it guarantees that all rationalizing preferences are within epsilon of *each other*. The true preference must rationalize the data, which means that any preference that rationalizes the certificate is within ϵ of the true preference.

Definition 5 (Certifiable Convergence). Let \mathcal{M} be as defined above, and $\nu_{\mathcal{A}}$ be a probability measure over a suitable family \mathcal{A} of choice sets. We say that the pair $(\mathcal{A}, \nu_{\mathcal{A}})$ leads to certifiable convergence if for every $\succeq \in \mathcal{M}, \epsilon > 0$.

$$As \quad n\to\infty \quad \nu^n\left(\{\{(\mathbf{A},f_\succeq)_i\}_{i=1}^n|\{(\mathbf{A},f_\succeq)_i\}_{i=1}^n \text{ is a certificate of radius } \epsilon\}\right)\to 1$$

Where ν^n is the product measure, and datasets are of the form $\{(\mathbf{A}, f_{\succeq})_i\}_{i=1}^n$ This means that as we increase the sample of choice sets, the probability of encountering choice sets such that the data they generate is a certificate of radius epsilon must go to 1 for any epsilon and underlying preference.

By convergence being certifiable, we mean that the *known bound* for the distance between *any preference* that rationalizes the data and the true preference must go to zero for all preferences.

Propostion 3.1 (Kubler, Malhotra, and Polemarchakis (2020)). Let \mathcal{F} be the family of pairs of points or linear budget sets. Furthermore, let $\mu_{\mathcal{F}}$ be the uniform distribution over the respective family. The pair $(\mathcal{F}, \mu_{\mathcal{F}})$ leads to certifiable convergence.

Proof. This theorem follows very simply from Theorem 1 ((Kubler, Malhotra, and Polemarchakis, 2020)).

We now have all of the ingredients that we need to construct our algorithm.

3.1 An Instructive Example-Weak Separability

We now use weak separability as a subclass for which we consider a test. To do this, we appeal to a simple result from demand theory. We now state a straightforward

derivative-based restriction of separability. We think that this is interesting because Echenique (2014) shows that an RP characterization of weak separability is NP-hard.

Definition 6 (Weak Separability). Suppose that the set of I goods can be divided into two subgroups, g_1, g_2 . Furthermore, suppose that x and y are bundles of g_1 and g_2 goods, respectively. We say that the preference \succeq is weakly separable if there exists subutility functions v_1, v_2 for g_1, g_2 , respectively, and a macro utility function u such that $u(v_1(x), v_2(y))$ represents \succeq .

Theorem 3.1. (Deaton and Muellbauer, 1980, p. 128) Suppose that the set of goods can be divided into two subgroups, g_1, g_2 , and set subgroup expenditure $I_k = \sum_{l \in g_k} p_l x_l$. The preference is separable in these two subgroups if and only if

$$\forall (p, I) \in \mathcal{P} \times \mathbb{R} \quad \frac{\partial I_1}{\partial I} \frac{\partial I_2}{\partial I} = \frac{\lambda_{12}}{\mu_{12}}$$

Where λ and μ are independent of price.

Notice that although this is a simple condition to check, the analyst must observe the *entire* demand function to do so.

We now use Theorem 3.1 to construct a testing procedure to illustrate our method. To do that, we first give two definitions and then describe our constructive procedure.

Refer to the class of weakly separable preferences as $S \in \mathcal{M}$, further let the groups of goods be referred to as g_1, g_2 and I_1, I_2 be expenditures as in theorem 3.1. We define the restriction for weak separability $\mathcal{R} : \mathcal{M} \times \mathcal{P} \times \mathbb{R}_+ \to \mathbb{R}$ as:

$$\mathcal{R}(\succeq, p, I) := \frac{\partial}{\partial I} \left(\frac{\partial I_2}{\partial I} \frac{\partial I_1}{\partial I} \right)$$

This is picked as the product within the parenthesis is independent of income. Thus, our restriction must equal zero if the demand function arises from a separate preference.

Claim 3.1.1 (Uniform Continuity). for every $\epsilon > 0$, there exists $\gamma(\epsilon) > 0$ such that $\forall \succeq \in \mathcal{M}, \succeq^* \in \mathcal{S}$

$$d(\succeq^*,\succeq) < \gamma(\epsilon) \implies \left\{ \int_{\mathcal{P}} \left[\mathcal{R}(\succeq^*, p, I) - \mathcal{R}(\succeq, p, I) \right]^2 \right\}^{\frac{1}{2}} < \epsilon$$

$$\implies \left\{ \int [\mathcal{R}(\succeq, p, I)]^2 \right\}^{\frac{1}{2}} < \epsilon$$

In other words, this means that anything close to separable in the space of preferences nearly satisfies our restriction.

Proof. Under our restriction on the space of preferences, convergence is equivalent to the Hausdorff distance between preferences, as defined in Hildenbrand (2015). Under these assumptions, the space of preferences is compact. Therefore, proving continuity suffices to prove what we need. Continuity follows from the continuity of the demand function in the space of preferences, as shown in Mas-Colell (1989b).

Remark. While this function may be hard to compute, it depends only on the underlying space over which the preference is defined and is **independent** of dataset size, which causes it to not affect complexity.

Now, we describe our exact procedure.

Our Procedure

- 1. Pick any sequence $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_n \to 0$
- 2. Sample choice sets(prices) $(\mathbf{A})^n$ from the family of sets \mathcal{F}^n
- 3. Let the underlying preference be $\succeq^* \in \mathcal{M}$, after n observations, the analyst observes $\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$.
- 4. Define radius after n steps as

$$rad(\{(\mathbf{A}, f_{\succ})_k\}_{k=1}^n) = \inf\{\epsilon \in \{\epsilon_n\} | \{(\mathbf{A}, f_{\succ})_k\}_{k=1}^n \text{ is a certificate of radius } \epsilon\}$$

Remark. This infimum exists because \mathcal{M} is bounded, which means that any observation is a certificate of radius 1.

5. Pick any preference \succeq that rationalizes the observations $\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$ and compute the test statistic

$$T_n = \gamma \left(\left\{ \int_{\mathcal{P}} [\mathcal{R}_{\succeq^*}(p, I)]^2 dp \right\}^{\frac{1}{2}} \right)$$

- 6. If $T_n > (rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n))$, the underlying preference \succeq^* can be rejected.
- 7. If $T_n < (rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n))$ draw more samples and repeat

Remark. This procedure will only terminate for non-separable preferences.

Remark. The first thing to remark is that both certification and uniform continuity are needed to construct the test. Observe that to use uniform continuity to construct a statistic, we need a bound for the distance between the true underlying preference and the preference that the analyst picks. Certifiable convergence provides this bound.

Propostion 3.2. Our procedure is **sound**, which means that if the test rejects, then the underlying preference cannot be separable.

Proof. We proceed by contradiction, suppose that the true preference $\succeq^* \in \mathcal{S}$, and

$$(rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n)) < T_n$$

This means that the distance between the chosen preference and the true preference $d(\succeq,\succeq^*) < T_n$, Furthermore, $T_n = \gamma \left(\left\{ \int_{\mathcal{P}} [\mathcal{R}_{\succeq^*}(p,I)]^2 dp \right\}^{\frac{1}{2}} \right)$.

As yet, we have used certification bound the distance between the true preference and the one picked by the analyst. Now by applying uniform continuity, we get:

$$d(\succeq,\succeq^*) < \gamma \left(\left\{ \int_{\mathcal{P}} [\mathcal{R}_{\succeq^*}(p,I)]^2 dp \right\}^{\frac{1}{2}} \right)$$

$$\implies \left\{ \int_{\mathcal{P}} [\mathcal{R}_{\succeq^*}(p,I)]^2 dp \right\} < \left\{ \int_{\mathcal{P}} [\mathcal{R}_{\succeq^*}(p,I)]^2 dp \right\}$$

This is a contradiction, which means that $\succeq^* \notin \mathcal{S}$.

This proposition shows that uniform continuity and a strong notion of convergence allow us to construct a sound test. We do not yet know if the test that we construct has any completeness properties. For asymptotic completeness, we need our restriction to satisfy another condition (which we call regularity). This condition corresponds to the restriction being a characterization of separability.

Claim 3.1.2 (Regularity at 0). for every $\succeq \in \mathcal{M}, \succeq^* \in \mathcal{C}$ and $\epsilon > 0$, there exists $\delta^*(\epsilon) > 0$ such that

$$d^{s}(\succeq^{*},\succeq) > \epsilon \implies \left\{ \int_{\mathcal{P}} [\mathcal{R}(\succeq, p, I)]^{2} \right\}^{\frac{1}{2}} > \delta^{*}(\epsilon)$$

Meaning that if something is far away from our class, then the restriction sends it far away from 0.

Remark. This implies that our restriction is, in fact, a characterization. If there were a non-separable preference that satisfied our restriction, then regularity would be trivially contradicted. We show that the "if and only if" condition on restrictions and uniform continuity is sufficient for regularity in Theorem 5.2.

Using the above claim, we can prove that our procedure terminates if $\succeq^* \notin \mathcal{S}$.

Propostion 3.3. Suppose $\succeq^* \notin \mathcal{S}$ the above procedure must terminate and end in rejection of the generated dataset.

Proof. As $\succeq^* \notin \mathcal{S}$, there must be an n such that

$$\forall \succeq \text{ such that } d(\succeq^*,\succeq) < rad\{(\mathbf{A},f_{\succ})_k\}_{k=1}^n, \succeq \notin \mathcal{S}$$

refer to the minimum such n as n_0 , and $rad\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^{n_0}$ as rad_0 , sample data n_1 , until, $rad\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^{n_1} < \frac{rad_0}{2}$. By the triangle inequality, for any \succeq that rationalizes $\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^{n_1}$

$$\forall \succeq_1 \in \mathcal{S}, \quad \left(d(\succeq_1, \succeq) > \frac{rad_0}{2} \right) \implies \left(\left\{ \int_{\mathcal{P}} [\mathcal{R}(\succeq, p, I)]^2 \right\}^{\frac{1}{2}} > \delta^*(\frac{rad_0}{2}) \right)$$

but this means that $\left\{ \int_{\mathcal{P}} [\mathcal{R}(\succeq, p, I)]^2 \right\}^{\frac{1}{2}}$ is bounded away from 0.

But by Theorem 3.1,

$$\lim_{n\to\infty} rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n) \to 0$$

. This means that eventually

$$\left\{ \int_{\mathcal{P}} [\mathcal{R}(\succeq, p, I)]^2 \right\}^{\frac{1}{2}} > rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n)$$

, which leads to rejection.

Remark. Again notice that rejection requires **both** regularity and certifiable convergence, which is highlighted in the use of theorem 3.1.

Remark. Many of our results depend on the efficient computability of the restrictions and integrals that we use in our analysis. However, this follows from lemma 2 (Kubler and Schmedders, 2010, p. 306), which says that the rationalizing preference can always be picked to be semi-algebraic, which makes symbolic computation feasible.

Remark. Our analysis also requires that the $\epsilon - \delta$ relationship representing uniform continuity is in itself computable. However, this is unproblematic because the computability of the function is independent of the size of the analyst's dataset.

4 General Restrictions and Tests

We now give a general framework for restrictions and our tests and demonstrate their differences from the revealed preference framework.

RP tests do not need to make any assumptions about the underlying data generating preference or sampling method. They would say that any finite data that obeys the restrictions can be treated as if it arises from a (separable) preference. However, our tests flip the approach used by revealed preference theory.

Classic RP Framework

Let the class of interest to the analyst be $\mathcal{C} \subset \mathcal{M}$.

In the classical RP framework, the analyst is playing a game against nature as the adversary. The analyst first generates a set of observations, which pass the RP test if it is possible for nature to pick a preference $\succeq \in \mathcal{C}$, which rationalizes the data.

Formally, we say a set of observations,

$$\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$$
 passes an RP test for $\mathcal{C} \iff \{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$ is $\mathcal{C}-rationalizable$

This is represented by the following flow diagram.



Localized (Our) Setup

The setup that we consider first fixes the preference from which data is being generated and then assumes a measure on the family of choice sets sampled, which allows datasets to inherit the measure on choice sets. Once we have this, we can define AC informally as follows: conditional on nature drawing a preference $\succeq \notin \mathcal{C}$ the **measure of sequences** rejected by our test goes to 1.

The following flowchart shows the setup.



Remark. Soundness is identical in both setups.

This setup shows the difference between the RP and our localized approaches. The revealed preference approach is adversarial, whereas our approach fixes the preference that nature picks and then conducts our analysis.

Restriction and a General Procedure

This section almost exactly follows the above example. We call a function $\mathcal{R}: \mathcal{M} \times \mathbf{X} \to \mathbb{R}$ a Functional Restriction for a class of preferences $\mathcal{C} \subset \mathcal{M}$ if,

$$\succeq \in \mathcal{C} \iff (\forall x \in \mathbf{X} \quad \mathcal{R}(\succeq)(x) = 0)$$

Note that our restrictions must hold at every point, so they are, in a sense, local and not global.

Our Weak Conditions

We need the following two general analogues of conditions 3.1.1 and 3.1.2.

Definition 7 (Uniform Continuity). We call a restriction \mathcal{R} of class \mathcal{C} uniformly continuous at 0 if for every $\succeq \in \mathcal{M}$, $\succeq^* \in \mathcal{C}$ and $\epsilon > 0$, there exists and $\gamma(\epsilon) > 0$ such that:

$$d(\succeq^*,\succeq) < \gamma(\epsilon) \implies \left\{ \int_{\mathbf{X}} [\mathcal{R}(\succeq)(x) - \mathcal{R}(\succeq^*)(x)]^2 \right\}^{\frac{1}{2}} < \epsilon$$

$$= \left\{ \int_{\mathbf{X}} [\mathcal{R}(\succeq)(x)]^2 \right\}^{\frac{1}{2}} < \epsilon$$

We mean that something *close to* a preference that satisfies our restrictions *almost* satisfies our restriction.

Definition 8 (Regularity). Call a restriction \mathcal{R} of class \mathcal{C} regular at 0 if for every $\succeq \in \mathcal{M}, \succeq^* \in \mathcal{C}$ and $\epsilon > 0$, there exists and $\delta(\epsilon) > 0$ such that

$$d(\succeq^*,\succeq) > \epsilon \implies \left\{ \int [\mathcal{R}(\succeq)(x)]^2 \right\}^{\frac{1}{2}} > \delta(\epsilon)$$

We mean that if something is far from our class, the restriction sends it far away from 0.

Remark. Unlike in the previous section, where we showed that weak separability indeed follows these properties, here we assume that these two properties hold.

Remark. In both our definitions, the functions $\delta(\epsilon)$ and $\gamma(\epsilon)$ only depend on the underlying state **X** and the restriction chosen \mathcal{R} . They are independent of the underlying preference and sampling procedure.

4.1 Algorithmic Procedure

This section highlights and describes out how we construct the algorithm to test the smooth restrictions.

Let \mathcal{R} be a smooth restriction for a class $\mathcal{C} \subset \mathcal{M}$. Furthermore, assume that \mathcal{R} is regular at 0. Let \mathcal{F} be a family of choice sets, and $\mu_{\mathcal{F}}$ be a measure over the set of choice sets.

- 1. Pick any sequence $\{\epsilon_n\}_{n=1}^{\infty}$ such that $\epsilon_n \to 0$.
- 2. Sample choice sets \mathcal{A}^n from the family of sets \mathcal{F}^n .
- 3. Let the underlying preference be $\succeq \in \mathcal{M}$, the analyst observes $\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$ after n observations, from an underlying preference $\succeq^* \in \mathcal{M}$.
- 4. Define radius after n steps

$$rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n) = \inf\{\epsilon_n | \{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n \text{ is a certificate of radius } \epsilon\}$$

5. Pick any preference \succeq that rationalizes the observations $\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n$ and compute the test statistic,

$$T_n = \gamma \left(\left\{ \int_{\mathbf{X}} [\mathcal{R}(\succeq)(x) dx]^2 \right\}^{\frac{1}{2}} \right)$$

- 6. If $T_n > (rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n))$, the underlying preference \succeq^* can be rejected.
- 7. If $T_n < (rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n))$ draw more samples and repeat

We now prove the soundness and termination for our algorithm. This proof is almost exactly the proof in 3.1.

Theorem 4.1. If the underlying preference $\succeq^* \in \mathcal{C}$, then our procedure never rejects. If our procedure rejects, then $\succeq^* \notin \mathcal{C}$ (soundness). If $\succeq \notin \mathcal{C}$, then the probability of rejection goes to 1.

Proof of Soundness. Suppose $T_n > ((rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n)))$ and $\succeq^* \in \mathcal{C}$.

By definition, we know,

$$d(\succeq,\succeq^*) < (rad(\{(\mathbf{A}, f_\succeq)_k\}_{k=1}^n) < T_n = \gamma \left(\left\{ \int_{\mathbf{X}} [\mathcal{R}(\succeq)(x) dx]^2 \right\}^{\frac{1}{2}} \right)$$

By uniform continuity, this implies that,

$$\left\{ \int_{\mathbf{X}} [\mathcal{R}(\succeq)(x) dx]^2 \right\}^{\frac{1}{2}} < \left\{ \int_{\mathbf{X}} [\mathcal{R}(\succeq)(x) dx]^2 \right\}^{\frac{1}{2}}$$

which is a contradiction. Therefore, it must be that $\succeq^* \notin \mathcal{C}$

Proof of Termination. Suppose $\succeq^* \notin \mathcal{C}$, by Theorem 3.1

$$\lim_{n\to\infty} (rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n)) \to 0$$

But as $\succeq^* \notin \mathcal{C}$, there must be an n such that

$$d^{s}(\succeq,\succeq^{*}) < (rad(\{(\mathbf{A},f_{\succeq})_{k}\}_{k=1}^{n})) \implies \succeq \notin \mathcal{C}.$$

Refer to the minimum of such n as n_0 , and $rad\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^{n_0}$ as rad_0 . Then, sample until n_1 such that $rad\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^{n_1} < \frac{rad_0}{2}$. By the triangle inequality for any \succeq that

rationalizes $\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^{n_1}$,

$$\forall \succeq_{1} \in \mathcal{C}, d(\succeq_{1}, \succeq) > \frac{rad_{0}}{2} \implies \left\{ \int_{\mathbf{X}} [\mathcal{R}(\succeq)(x)dx]^{2} \right\}^{\frac{1}{2}} > \delta(\frac{rad_{0}}{2})$$

But this means that

$$\left\{ \left[\mathcal{R}(\succeq)(x)dx \right]^2 \right\}^{\frac{1}{2}}$$

is bounded below, and so is the test statistic

$$T_n = \gamma \left\{ \int_{\mathbf{X}} [\mathcal{R}(\succeq)(x) dx]^2 \right\}^{\frac{1}{2}}$$

but we also have

$$\lim_{n\to\infty} (rad(\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n)) \to 0$$

So there must eventually be an n where with high probability,

$$T_n > (rad(\{(\mathbf{A}, f_{\succ})_k\}_{k=1}^n))$$

which proves that $\succeq^* \notin \mathcal{C}$.

Our Test

Definition 9 (A Sound and AC test). Suppose that the analyst has access to a uniformly continuous and regular restriction \mathcal{R} for class \mathcal{C} . Define a Test $\mathcal{T}_{\mathcal{C}}: 2^{\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n} \to \{0, 1\}$ as follows.

P returns 0 on input
$$\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n \implies \mathcal{T}_{\mathcal{C}}\{(\mathbf{A}, f_{\succeq})_k\}_{k=1}^n = 0$$

Otherwise,

$$\mathcal{T}_{\mathcal{C}}\{(\mathbf{A}, f_{\succ})_k\}_{k=1}^n = 1$$

Because this procedure eventually terminates, this test is AC. Because it is correct, the test is sound.

5 Sufficient Conditions on Restrictions

5.1 Uniform Continuity

We have proven that we can construct a sound test with a uniformly continuous restriction. We now try to give simple conditions under which restrictions are uniformly continuous. For this, we need a few definitions.

Definition 10 (Marginal Rate of Substitution). Given a utility function U that represents preference \succeq and some bundle $x \in \mathbf{X}$, define the MRS vector at x as

$$\nabla(x,\succeq) = \frac{\nabla U(x)}{|\nabla U(x)|}$$

which is the gradient normalized to 1.

Remark. This is independent of the choice of the utility function, as shown in Mas-Colell (1989a). In addition, although we only prove this claim for restrictions including first-order derivatives, higher-order ones can be done similarly.

Definition 11 (Inverse Demand). Let $\succeq \in \mathcal{M}$ be a smooth differentiable, strongly convex preference. Define the inverse demand function generated by $\succeq , \pi_{\succeq}(x) : \mathbf{X} \to \mathcal{P}$ as

$$\pi(y) = p$$
 such that $y = x(p, 1)$

This exists and is well-defined by strict convexity.

Consider the space of smooth convex preferences endowed with the topology given by metric d^s .

Lemma (Continuity of Inverse Demand). Hildenbrand (2015) Consider the above defined topology for preferences and the \mathcal{L}^2 norm on the image space. In this setup, inverse demand functions are continuous in preferences. Specifically,

$$\forall \epsilon \exists \delta(\epsilon) > 0$$
, such that $d^s(\succeq^1,\succeq^2) \leq \delta \implies \|\pi_{\succeq^2} - \pi_{\succeq^2}\| < \epsilon$

The dependence of δ only on ϵ is given by the compactness of the space of preferences.¹²

¹²See Grodal (1974)

Now observe the following:

$$\nabla(\succeq, x) = \frac{\pi_{\succeq}(x)}{|\pi_{\succeq}(x)|}$$

If the RHS is continuous in preferences, then so must be the LHS.

Remark. The inverse demand function is never zero because we always consider a bounded space of bundles.

Theorem 5.1. Let $\mathcal{F}: \mathbb{R}^l \to \mathbb{R}$ be a (vector) function that is uniformly continuous in the classical sense. This can be used to generate a uniformly continuous restriction via the marginal rate of substitution.

Proof. By assumption, \mathcal{F} is (uniformly) continuous. Now because the composition of continuous functions is continuous, $\forall \epsilon \exists \delta(\epsilon) > 0$ such that,

$$d^{s}(\succeq^{1},\succeq^{2}) \leq \delta \implies \left\| \mathcal{F}\left(\frac{\pi_{\succeq^{1}}}{|\pi_{\succeq^{1}}|}\right) - \mathcal{F}\left(\frac{\pi_{\succeq^{2}}}{|\pi_{\succeq^{2}}|}\right) \right\| < \epsilon$$
$$\implies \left\| \mathcal{F}\left(\nabla(\succeq^{1},x)\right) - \mathcal{F}\left(\nabla(\succeq^{2},x)\right) \right\| < \epsilon$$

By definition of the \mathcal{L}^2 norm,

$$\left(\int_{\mathbf{X}} \left[\mathcal{F}\left(\nabla(\succeq^{1}, x)\right) - \mathcal{F}\left(\nabla(\succeq^{2}, x)\right) \right]^{2} \right)^{\frac{1}{2}} < \epsilon$$

This is identical to the condition in Definition 7.

5.2 Regularity

Now that we have shown weak conditions under which the restrictions of a class are uniformly continuous, we go on to give conditions under which they are regular.

Take any restriction that is a characterization, meaning that

$$(\mathcal{R}(\succeq, x) = 0 \quad \forall x \in \mathbf{X}) \implies \succeq \in \mathcal{C}$$

We now show that this is sufficient for regularity.

Claim 5.1.1. There is no sequence of preferences $\{\succeq_i\}_{i=1}^n$ such that $\{\int_{\mathbf{X}} \mathcal{R}(\succeq_i, x)\}_{i=1}^n \to 0$ and the sequence $\{\succeq_i\}_{i=1}^n$ does not converge to an element of \mathcal{C} . Contrapositive, we

say that for any sequence of preferences $\{\succeq_i\}_{i=1}^n$

$$\{\int_{\mathbf{X}} \mathcal{R}(\succeq_i, x)\}_{i=1}^n \to 0 \implies \{\succeq_i\}_{i=1}^n \to \succeq^* Where \succeq^* \in \mathcal{C}$$

Proof. Suppose not, by certifiable convergence $\{\succeq_i\}_{i=1}^n$ must have some limit \succeq^* , which is the true underlying preference. By assumption $\succeq^* \notin \mathcal{C}$. However,

$$\lim_{n\to\infty} \{ \int_{\mathbf{X}} \mathcal{R}(\succeq_i, x) \}_{i=1}^n \to 0$$

by continuity, this means that $\{\mathcal{R}(\nabla \cdot \succeq^*) = 0\}$. This implies $\succeq^* \in \mathcal{C}$, which is a contradiction.

We now show that this implies regularity.

Theorem 5.2. Let \mathcal{R} be a restriction that satisfies 5.1.1(Limit Uniqueness). This implies that it satisfies 8(Regularity).

To prove this theorem, we first need one definition,

Definition 12. Define \mathcal{C}^{ϵ} , the set of preferences that are epsilon close to a class \mathcal{C} .

$$\mathcal{C}^{\epsilon} = \{\succeq \in \mathcal{M} \text{ such that } \inf_{\succeq^* \in \mathcal{C}} d^s(\succeq, \succeq^*) \leq \epsilon\}$$

Now we prove our theorem.

Proof. Pick any $\epsilon > 0$, define

$$\delta(\epsilon) = \inf_{\succeq \in (\mathcal{M} - \mathcal{C}^{\epsilon})} \left\{ \int [\mathcal{R}(\nabla \cdot \succeq)]^2 \right\}^{\frac{1}{2}}$$

By limit uniqueness $\delta(\epsilon) > 0$. Furthermore, any preference that is ϵ far from \mathcal{C} must be in $\mathcal{M} - \mathcal{C}^{\epsilon}$),

$$(\forall \succeq^* \in \mathcal{C} \quad d^s(\succeq^*,\succeq) > \epsilon) \implies \succeq \in (\mathcal{M} - \mathcal{C}^{\epsilon})$$

But

$$\succeq \in \mathcal{C}^{\epsilon} \implies \left\{ \int [\mathcal{R}(\succeq, x)]^2 \right\}^{\frac{1}{2}} > \delta(\epsilon)$$

By putting these two together, we get,

$$(\forall \succeq^* \in \mathcal{C} \quad d^s(\succeq^*,\succeq) > \epsilon) \implies \left\{ \int [\mathcal{R}(\succeq,x)]^2 \right\}^{\frac{1}{2}} > \delta(\epsilon)$$

Which implies condition 8.

6 Extensions and Other Properties

6.1 Choice Set Independence

We finally talk about one more attractive property of our approach. Namely, the independence of our procedure from the nature of choice sets that the analyst observes.

Remark. Note that this is starkly different from "revealed preference" tests that do depend on the nature of budget sets¹³ and the sufficiency of first-order conditions. Unlike the classical literature on revealed preference, we do not use the Lagrangian approach, thus avoiding its limitations.

We require one property of choice sets for our tests to work (we refer to this as pairwise discrimination).

Definition 13 (Pairwise Discrimination). Let \mathcal{A} be a family of choice sets. We say that this family is pairwise discriminatory if for every $\succeq \in \mathcal{M}$ and $(x, y) \in \succeq$ there exists finite collection $\{\mathbf{A}_i\}_{i=1}^n \in \mathcal{A}$ such that

$$\succeq^*$$
 rationalizes $\{(\mathbf{A}, f_{\succeq})_i\}_{i=1}^n \implies (x, y) \in \succeq^*$

Remark. As shown in Kubler, Malhotra, and Polemarchakis (2020), pairwise comparisons and linear budget sets are pairwise discriminatory.

We now state our independence result.

Theorem 6.1. Let \mathbf{P} be a procedure that generates a sound and asymptotically complete test for class \mathcal{C} , for a family of choice sets \mathcal{A} , which is pairwise discriminatory. Let \mathcal{A}' also be a family of pairwise discriminatory choice sets. \mathbf{P} must generate a sound and asymptotically complete test for class \mathcal{C} and family \mathcal{A}' .

¹³See Nishimura et al. (2017); Polisson et al. (2020); Forges and Minelli (2009)

Proof. This property follows from certifiable convergence being the sole requirement that we have on the family of choice sets. Kubler, Malhotra, and Polemarchakis (2020) showed that the only condition required for certifiable convergence is pairwise discrimination (13). Our tests inherit this property. □

6.2 The Rationality Assumption

Until now, all of our theorems have assumed "rationality" of preferences, by which we mean completeness, monotonicity, and transitivity. *However, is this assumption needed for our results?*

Looking at separability, we show that this is indeed the case. As long as the analyst is unwilling to assume that the underlying preference is monotonic, there is always an additively separable preference that rationalizes *any* dataset, which means that the only sound test of separability accepts all datasets.

Theorem 6.2. For simplicity, assume that $\mathbf{X} \subset \mathbb{R}^3$. Furthermore, let $\mathcal{A} = \mathbf{X} \times \mathbf{X}$ be the family of all pairwise comparisons. Furthermore, let $\{(\mathbf{A}, f)_k\}_{k=1}^n$ be any finite dataset generated from this family. There exists an additively separable utility function U that rationalizes the data.

Proof. To rationalize any finite dataset $\{(\mathbf{A}, f)_k\}_{k=1}^n$ generated from \mathcal{A} we need to find a utility function U(x, y, z) = U(x) + U(y) + U(z), where for some finite pairs of the form $(x_1, y_1, z_1)^i$, $(x_2, y_2, z_2)^i$,

$$\forall i U((x_1, y_1, z_1)^i) > U((x_2, y_2, z_2)^i)$$

To construct such a function, simply assume integer values of U at the required finite points and assume that

$$U(x, y, z) = \sum_{n=1}^{k} a_i x_i^n + \sum_{n=1}^{k} b_i y_i^n + \sum_{n=1}^{k} c_i z_i^n$$

where k is the size of the set of observations.

Such a polynomial exists by the Lagrange interpolation theorem.

This argument shows that monotonicity is a necessary assumption required for our

results. There may well be no finite analogue of derivative restrictions if monotonicity is relaxed.

7 Conclusion

We give weak sufficient conditions under which functional tests can be used to construct tests of finite data. We then show that these follow from reasonable restrictions on the nature of functional tests. Showing that for our tests to terminate, the functional restrictions must be characterizations. We then construct a test of weak separability.

In the endeavour to unify the two approaches to testing, we define a stronger notion of convergence (i.e., convergence with certification). To our knowledge, we are the first to do this in the field of preference testing. Finally, we show the necessity of monotonicity and hence the certification. We demonstrate that if monotonicity is violated, then our results and tests of finite data disappear. We believe that this will lead to exciting directions for future research in translating commonly used revealed preference axioms into local tests, and exploring the benefits of certification for testing and decision making in general.

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