Pricing and Hedging Catastrophic Derivatives using Robust Optimal Control Techniques

Stephen Weston, Deutsche Bank AG, London

Mark Salmon, Warwick Business School

Climate Change and Financial Markets Workshop

Warwick University

11 January 2008

Abstract

Feedback control techniques developed in engineering are applied to solve the problem of finding a pricing model for catastrophic derivatives that is both robust and stable in operation. Specific application of non-linear robust control Lyapunov techniques are found to provide a pricing paradigm for catastrophe based derivatives, the relative performance of which, is measurably superior in terms of stability and robustness to the popular equilibrium and actuarial models that dominate the approach to pricing of these emerging instruments.

1 Contents

- Introduction and problem statement
- Robust pricing and hedging models
  1. Brief review of existing robust approaches
  2. Non-linear robust approach
- Results and comparisons
- Conclusions

2 Introduction and problem statement

- Rising losses from catastrophic events over the past 10 years culminating in the largest single loss event – hurricane Katrina in 2005, the cost of which currently stands at \$81bn and will probably top \$100bn by the time all claims are settled according to Swiss Re and PCS..

- Katrina was the costliest natural disaster in U.S. history.

- Up until end 2005, aggregate insured losses had increased every year since 1970.

- Rising catastrophic losses have adversely impacted the solvency of the insurance industry.

- So. . . ..

Insured Losses, U.S. Catastrophes, 1997 - 2006[1]

Year	Number of Catastrophes	Number of claims (mns)	Original $ bns	In 2006 $ bns
1997	25	1.6	2.6	3.3
1998	37	3.6	10.1	12.5
1999	27	3.2	8.3	10
2000	24	1.5	4.6	5.4
2001	20	1.5	26.5	30.2
2002	25	1.8	5.9	6.6
2003	21	2.7	12.9	14.1
2004	22	3.4	27.5	29.3
2005	24	4.4	62.3	64.3
2006	33	2.3	9.2	9.2

(1) Includes catastrophes causing insured losses to the industry of at least $25 million and affecting a significant number of policy holders and insurers.
(2) Adjusted to 2006 dollars by the Insurance Information Institute.

Source: ISO's Property Claims Services unit; Insurance Information Institute.

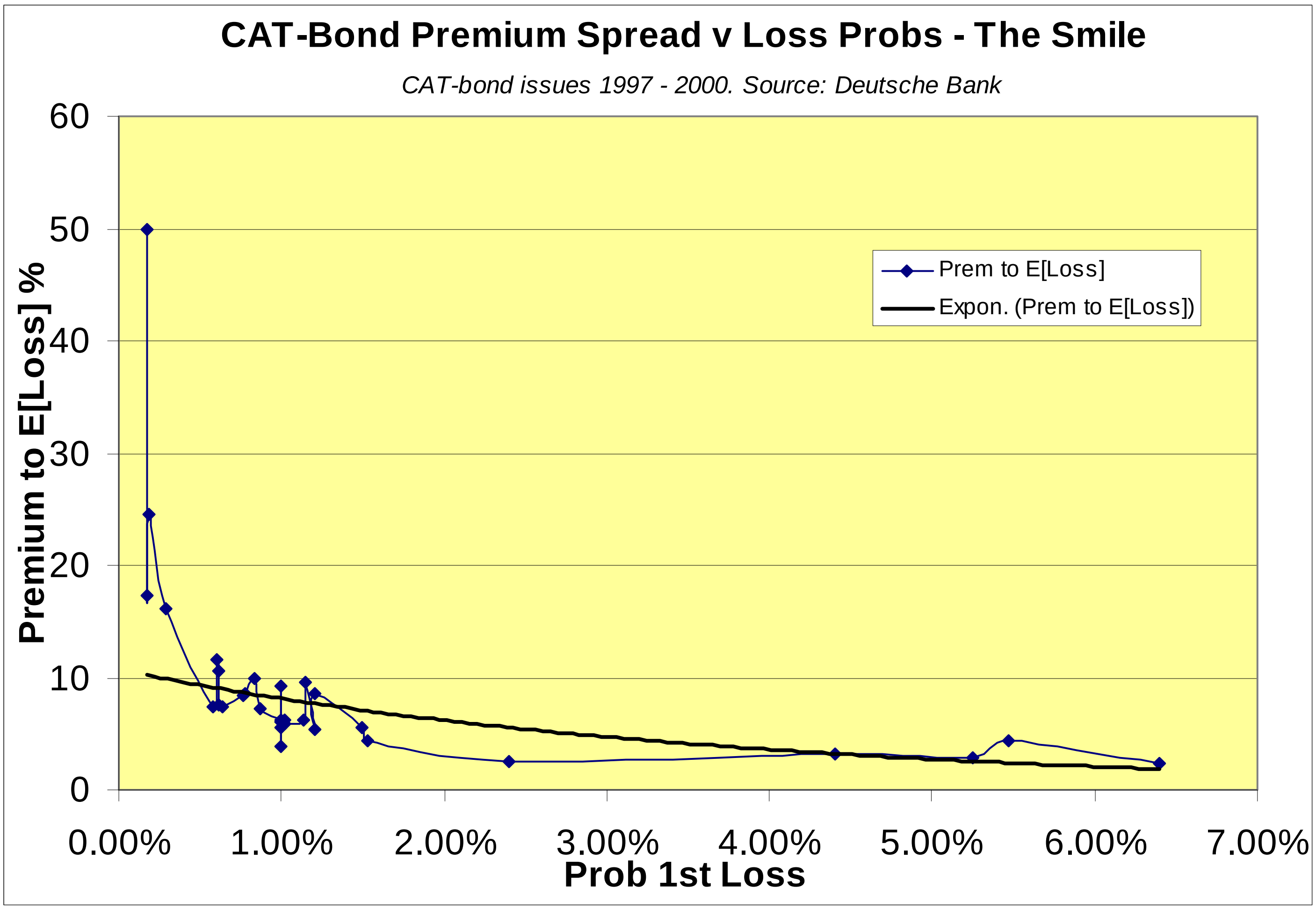
- There has been a search for new products to provide coverage against catastrophe risks.

- Possible solutions tried so far have been:

  1. CAT-linked bonds.
  2. CAT-linked swaps.
  3. Exchange traded CAT-linked futures and options.

- Features of these solutions:

  1. Spread premium is very high relative to the expected loss of bond principal.
  2. Presence of a smile in relationship between premium spreads and loss probabilities.



CAT-Bond Premium Spread v Loss Probs - The Smile
CAT-bond issues 1997 - 2000. Source: Deutsche Bank
Premium to E[Loss] %
60
50
40
30
20
10
0
Prem to E[Loss]
Expon. (Prem to E[Loss])
0.00%
1.00%
2.00%
3.00%
4.00%
5.00%
6.00%
7.00%
Prob 1st Loss

- The main problem with CAT derivative valuation is that insurance markets are incomplete, so the use of no-arbitrage arguments is not guaranteed to generate a unique price process.

- Two main approaches to the pricing problem:

  1. Partial/general equilibrium models – derive risk premium from equilibrium model; e.g. Aase (1993), Cummins & Geman (1995), Geman & Yor (1997 and Muermann (2003).

  2. Actuarial models – e.g. Wang (2002) and Young (2005).

- Problems with all of the models used so far:

  1. Do not explain the presence and persistence of the smile.

  2. Do not explain why premium spreads for CAT risks are so high.

3. Do not explain why premium spreads fall when moving from low probability of occurrence to high probability of occurrence CAT securities.

- It is worth noting that Wang's (2002) transform with two factors (to capture greed and fear as well as risk aversion) does do a better job of explaining these problems, but the linkage between the two factors and the underlying economic model is not grounded in a relevant economic model.

- The major problem with the existing approaches is that they are principally concerned with pricing risk and not with capturing the uncertainty surrounding the catastrophic event(s).

- Clearly, the presence of unmodeled uncertainty is leading to misspecification in existing models, so we need to look for an alternative approach that deals explicitly with uncertainty and its consequent model misspecification.

3 Robustness

- What is needed are models that are robust in the face of such misspecification and uncertainty.

- What exactly do we mean by robust?

- Webster's dictionary defines robust as: "..performing without failure under a wide range of conditions...".

- A good place to look for an approach that performs without failure under a wide range of conditions by explicitly incorporating uncertainty, is in engineering where there are known techniques for dealing with both model and parameter uncertainty.

- Robust control theory is an engineering technique that originated in the need to deal with systems that have modeling uncertainty.

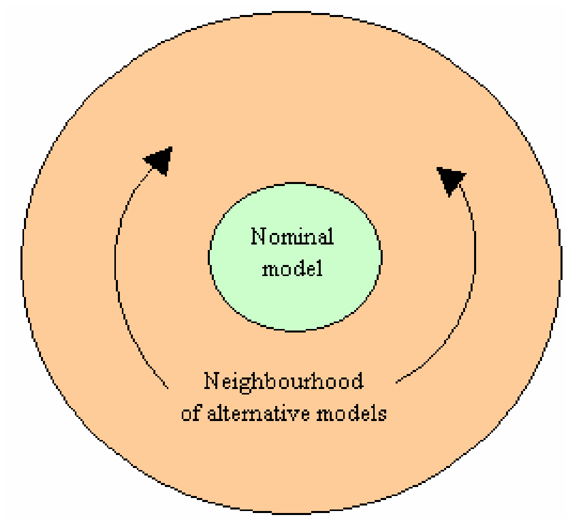
- Modeling uncertainty arises due to:

  1. Uncertain parameters.
  2. Unmodeled dynamics.
  3. Intentional model simplification – linearisation and/or model reduction.
  4. Inexact data – insufficient or inconsistent experimental results.

- The traditional models referred to above assume that decision making is "open-loop" - i.e. once decisions are made they are not altered by feeding back the effects of subsequent events and actions, so uncertainty remains regarding 1-4 above.

- In contrast, robust control theory deals with uncertainty by explicitly incorporating the effects of feedback into the decision making process through the use of a "closed-loop" decision making model.

- A nominal model is constructed to describe a financial system and then used to generate optimal decision rules: e.g. the optimal hedge.



- Uncertainty means that the optimal decision rules can frequently be overly sensitive to small changes in the model specification, measurement, data and disturbances.

- The objective is therefore to identify a type of decision rule that works well for a neighbourhood of alternative models (including the nominal model) and so is unaffected by modeling inadequacies, data errors, mis-measurement and uncertainty; such decision rules are said to be robust.

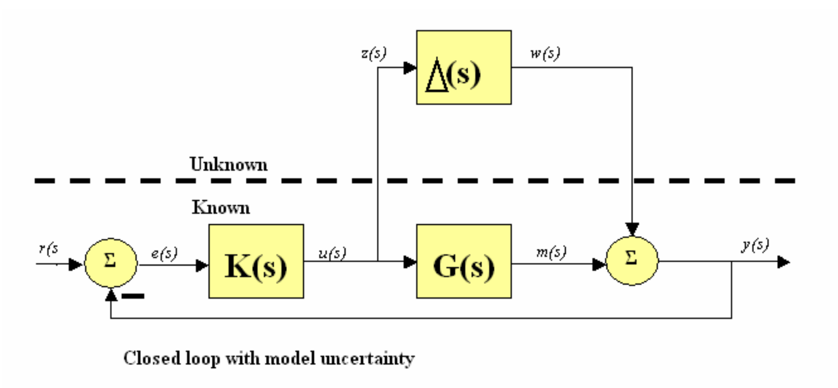
- The key problem is to jointly achieve robustness and stability with respect to disturbances and model uncertainty.

- Closed loop feedback is used to capture and reduce the impact of uncertainty.

- Using a state space approach within the robust optimal control framework allows the valuation problem to be represented by classes of tractable differential equations.

- Conceptually, the nominal model, G, and the controller process, K, are connected together by a closed feedback loop that allows uncertainty to enter the system.

Figure 2.1: Incorporating Feedback To Reduce Uncertainty



- Given G, the robust control problem involves constructing a controller, K(in this case the hedging rule), that guarantees closed-loop stability and performance in the presence of any uncertainty belonging to a given family or neighbourhood of admissible models in the neighbourhood of the nominal model.

3.1 Review of existing robust approaches

- The Black & Scholes (1973) model is generally formulated in terms of a stochastic framework.

- Many stochastic control problems are set up in terms of a cost criterion which is some sort of expectation of a functional of the process.

- But for option pricing, the seller would like to guarantee a certain bound on the cost - i.e. for almost every path of the noise process, the option writer would like to ensure that no loss will be incurred.

- So, even in the simple stochastic framework the option pricing problem is very nearly formulated as a robust control problem.

- There are two main ways of incorporating disturbances or noise processes into systems:

1. As stochastic processes with associated filtrations and probability measures.

2. Treat the disturbances as unknown, but deterministic processes of finite energy - this is the approach taken in the $H\infty$ and (more generally) the robust optimal control methodologies.

- Four derivations of the robust optimal control problem have been tried:

  1. Deterministic, differential games - McEneaney (1995), replaces Ito integral with Stratonovich integral to price option using stop-loss hedging assumption.

  2. Game theoretic - Bernhard (2001, 2005) uses game theory to solve linear HJB equation indirectly using $H\infty$ techniques.

  3. Ambiguity aversion - Zhu (2007) uses a modified equilibrium model, solving the resulting linear HJB equation directly.

4. Game theoretic - Weston & Salmon (2008) solve non-linear Lyapunov equations directly.

- The first one does not explain the smile.

- The second one explains the smile, but does not produce robust hedges.

- The third one explains the smile and produces robust hedges.

- So, what are the steps to applying robust optimal control theory to pricing derivative securities?

- The first three approaches also result in placing an inflated value on the catastrophic risk.

- McEneaney (1995) uses a combination of the relationship between the Ito and Stratonovich integrals and robust control techniques to obtain a simple option price independent of volatility - the price derived corresponds to a stop-loss hedging strategy (see Hull, 1993 for example).

- Zhu (2007) in contrast, uses a generalised equilibrium framework by allowing a representative agent to act in a robust control framework against model misspecification with respect to rare events, then derives the corresponding equivalent martingale measure.

- The discussion will now be around the use of a nonlinear robust control framework as a means of dealing with the uncertainty around model specification with respect to rare events in the sense of Anderson, Hansen and Sargent (2000).

3.2 Non-linear robust approach

- So if the application of the basic robust optimal control model produces high prices what are the benefits of incorporating the inherent non-linearities into the option pricing model?

  1. Solving the HJI equation is frequently impractical for all but a small and relatively simple set of linear models.

  2. Robust control Lyapunov equations can be used to construct an optimal control rule directly without having to solve the HJI equation.

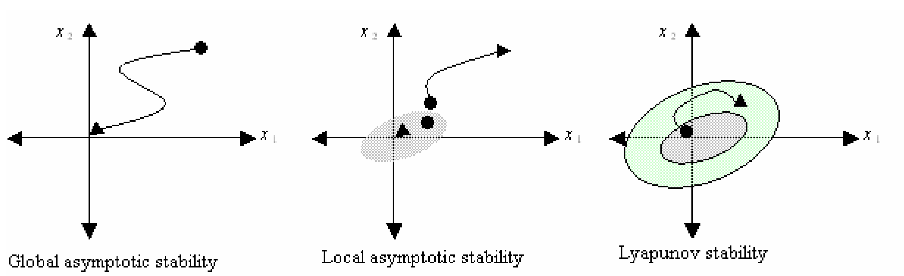
  3. The existence of a Lyapunov function is sufficient to prove stability and performance in the required region.

  4. If the value function, $V(x)$, is negative definite, the equilibrium is asymptotically stable.

5. Can be easily cast in a state space setting making it flexible for dealing with securities that have prices which depend on different states.

- Use non-linearities in the control function to improve stabilisability and robustness - don't fight reality - conceptually it works as follows:

Figure 2.2: Stability Compared



- How then does this basic idea translate into to a robust optimal control Lyapunov approach ?

- To begin with, consider a simple representation of a system of catastrophic and financial variables, (complete with a number of simplifying assumptions, most of which will be successively relaxed) and assume there is some interest rate generating process, $F(x)$, where x is a vector of state variables describing the interest rate.

- Assume also that there exists a hedging strategy, $G(x)u$ that uses a non-catastrophe related zero coupon bond (whose value is determined by x) to hedge the CAT bond.

- The final element is a disturbance input, $H(x)w$, capable of capturing catastrophic shifts.

- It is assumed that u, the control or hedging strategy, is used to balance the hedging portfolio. These elements are linked together to form the following system

$$\dot{x} = F(x) + G(x)u + H(x)w \qquad (1)$$

F, G and H are all assumed to be continuous functions.

- It is also assumed that the system is stabilisable and that the state is available for feedback - not an unreasonable assumption given that G is the hedging policy which will be dynamic and feedback into the model.

- The key assumption, however, is that a control Lyapunov function is known for this system. In other words, assume that a $\mathbb{C}^1$ (i.e. continuous in the first derivative), positive definite function of the form

$$V : \mathcal{X} \longrightarrow \mathbb{R}_+ \tag{2}$$

  is known, such that

$$\inf_{u \in U} \nabla V(x) \cdot [F(x) + G(x) u] < -\alpha_V(x) \tag{3}$$

  for all $x \neq 0$ and for some function α_V.

- The critical concept is to use the control Lyapunov function V as a robust control Lyapunov function for the uncertain system of equation 1.

- As Freeman &Kokotovic (1995) point out, this robust control Lyapunov function can be chosen independently of the uncertainty so that there is no knowledge of the structure of the disturbances, H.

- This means that for a CAT bond, it is possible to derive a robust control Lyapunov function without any knowledge of the structure of the catastrophic disturbance.

- This is a very strong feature of the model compared with the actuarial and equilibrium approaches where the assumption is made that there is a probability distribution for the catastrophic events and is one that ensures its robustness in the presence of uncertainty surrounding the likely arrival of catastrophic events.

- From control theory it is known that to be a robust control Lyapunov function V must satisfy

$$\inf_{u \in U} \sup_{w \in B} \nabla V(x) \cdot [F(x) + G(x)u + H(x)w] < -\alpha_V(x) \tag{4}$$

for all $x \neq 0$.

- Having provided an overview of the model, how does the robust optimal control compare with the actuarial and equilibrium models?

- In order to be able to answer this question, it is first necessary to provide a precise definition of the variables in the model.

- As far as the financial market variables are concerned, Cox and Pedersen assume these to be modelled on the filtered probability space $\Omega^{(1)}, \wp^{(1)}$,Þ$_1$, where $\Omega^{(1)}$ is taken to be finite such that it represents all paths that the financial variable can take over the time $k = 0, 1, ..., T$.

- However, the point of the robust approach is to move away from using a specific form of probability distribution to characterise the state space for the variables in the model.

- As Cox and Pedersen point out, their results also hold for infinite sample spaces, so the extension to a more general notion of a state-space seems intuitively acceptable.

- The key concept in making this transition for the purposes of robustness is the need to deal with the initial information state.

- It is known (e.g. Helton and James 1999) that careful choice of the initial state makes an enormous difference in the implementability of the controller or hedging process $G(x)$ and strongly affects the dynamic behaviour of the system.

- Therefore, within the robust control Lyapunov approach, we will consider four finite dimensional Euclidean spaces: the state space (interest rate or financial variable such as the price of a discount bond) χ, the control or hedging space $\mathcal{U}$, the disturbance or catastrophe generating space $\mathcal{W}$ and the measurement space $\mathcal{Y}$. Given a continuous function $f : \chi \times \mathcal{U} \times \mathcal{W} \times \mathbb{R} \rightarrow \chi$, a differential equation can be formed

$$\dot{x} = f\left(x, u, w, t\right) \tag{5}$$

  where $x \in \chi$ is the state variable, $u \in \mathcal{U}$ is the control or hedging input, $w \in \mathcal{W}$ is the catastrophic disturbance input and $t \in \mathbb{R}$ is the time variable.

- Associated with the differential equation 5 are admissible measurements, admissible disturbances and admissible controls - with each being characterised by a set-valued constraint.

- Taking the admissible measurements first, a measurement for equation 5 is a function $y : \chi \times \mathbb{R}$ such that $y(\cdot, t)$ is continuous for each fixed $t \in \mathbb{R}$ and $y(x, \cdot)$ is locally L_∞ for each fixed $x \in \chi$ (i.e. bounded on a neighbourhood of every point). Assuming a measurement constraint of the form $Y : \chi \times \mathbb{R} \rightsquigarrow \mathcal{Y}$, then a measurement $y(x, t)$ is deemed admissible when $y(x, t) \in Y(x, t)$ for all $(x, t) \in \chi \times \mathbb{R}$.

- The importance of this definition is that it allows for measurement uncertainty due to imperfections in the measurement process, perhaps because there may be several different measurement trajectories associated with a single state trajectory.

- In equation 5, a disturbance is a function $w : \chi \times \mathcal{U} \times \mathbb{R} \rightarrow \mathcal{W}$, such that $w(\cdot, \cdot, t)$ is continuous for each fixed $t \in \mathbb{R}$ and $w(x, u, \cdot)$ is locally L_∞ for each fixed $(x, u) \in \chi \times \mathcal{U}$.

- Therefore, given a disturbance constraint $W : \chi \times \mathcal{U} \times \mathbb{R} \rightsquigarrow \mathcal{W}$, it is possible to state that a disturbance $w(x, u, t)$ is admissible when $w(x, u, t) \in W(x, u, t)$ for all $(x, u, t) \in \chi \times \mathcal{U} \times \mathbb{R}$.

- This is central to the modelling of the catastrophe space because admissible disturbances can include both exogenous disturbances such as catastrophes and feedback disturbances, such that they encompass a large class of memoryless model and input uncertainties and form part of the basis of the approach in yielding guaranteed stability framework for robust non-linear control.

- In equation 5, a control is a function $u : \mathcal{Y} \times \mathbb{R} \rightarrow \mathcal{U}$ such that $u(\cdot, t)$ exhibits continuity for each fixed $t \in \mathbb{R}$ and $u(y, \cdot)$ is locally L_∞ for each fixed $y \in \mathcal{Y}$.

- Following the same approach, given a control constraint $U : \mathcal{Y} \times \mathbb{R} \rightsquigarrow \mathcal{U}$, it is possible to say that a control is admissible when $u(y, t) \in U(y, t)$ for $(y, t) \in \mathcal{Y} \times \mathbb{R}$ and that $u(y, t)$ is jointly continuous in (y, t).

- As F&K point out, it might be expected that a constant control constraint $U(y, t) \equiv U_0$ should be enough but for the purposes of our model there are valid and desirable reasons for allowing the constraint to depend on the measurement y.

- The most glaringly obvious example is that it might be desired not to hedge the CAT bond using some possibly expensive strategy when the value of the CAT bond remains within an acceptably "normal" region.

- The function f, taken with the set valued constraints U, W and Y, comprises a system

$$\sum = f\left(x, u\left(y\left(x, t\right)\right)\right) \tag{6}$$

and a solution, $x\left(t\right)$, to this system solves the initial value problem

$$\dot{x} = f\left(x, u\left(y(x,t)\,, t\right), w(x, u(y(x,t), t), t), t\right) \qquad x\left(t_0\right) \tag{7}$$

given a measurement $y(x,t)$, a disturbance $w\left(x, u, t\right)$, a control $u\left(y, t\right)$ and an initial condition $\left(x_0, t_0\right) \in \chi \times \mathbb{R}$.

- Classical existence theorems from control theory guarantee that the right hand side of equation 7 is continuous in x and locally L_∞ in t, which means that solutions to $\sum$ always exist (locally in t) but need not necessarily be unique.

- It is also important to note that the above formulation can also include fixed order dynamics by redefining the system $\sum$.

- For example, fixed order dynamics can be imposed by adding auxiliary variables to the state, control and measurement variables but the essential point that emerges from this problem statement is that solutions to $\sum$ are robustly, globally and asymptotically stable.

- In order to derive a catastrophe derivative valuation framework based on this approach, it is necessary to take into account three particular issues:

1. First, it must be remembered that for non-linear systems the feedback gain between inputs and outputs at each state depends on initial conditions.

2. Second, the non-linearities inherent in the model, such as convexity of the payoff function, must be modelled as part of the initial conditions.

3. Third, there must be an existing methodology for calculating the required quantities.

- Fortunately, robust control Lyapunov analysis satisfies all three demands and is the approach upon which the following analysis is constructed.

- At its simplest, a control Lyapunov function for a system of the form $\dot{x} = f(x, u)$ is a $\mathbb{C}^1$ positive definite, radially bounded function $V(x)$ such that

$$\begin{aligned} x &\neq 0 && (8) \\ \inf_{u \in U} \nabla V(x) \cdot f(x, u) &< 0 && (9) \end{aligned}$$

  where U is a convex set of admissible values of the control variable, such that the derivative of the function can be made negative pointwise by the choice of control values.

- A function $V \in \mathcal{V}(\mathcal{X})$ is a robust control Lyapunov for a system $\mathbf{\Sigma}$ when there exist $c_v \in \mathbb{R}_+$ and $\alpha_v \in \mathcal{P}(\mathcal{X})$ such that

$$\inf_{u \in U(y,t)} \sup_{x \in Q(y,c,t)} \sup_{w \in W(u,t)} \left[L_f V(x, u, w, t) + \alpha_v(x, t) \right] \quad (10)$$

  for all $y \in \mathcal{Y}$, all $t \in \mathbb{R}$ and all $c > c_v$; and where $L_f V$ is a Lyapunov derivative.

- This formulation of the robust control Lyapunov function is important as it is generalisable in a number of directions such that it provides a significant degree of flexibility.

- Note that both control and disturbance inputs enter the equation and that the definition copes with both measurement feedback and state feedback.

- This capability to deal with feedback is particularly valuable when devising valuation models as it means that more realistic hedging strategies can be represented.

- Notice also that the term c_v enables the modeler to cope with the three issues in stabilisability (in addition to asymptotic stabilisability) identified above.

- Finding a function V that is a solution to $\boldsymbol{\Sigma}$ and is also robustly globally uniformly and asymptotically stable and which also converges to a residual and compact set $\Omega \in \mathcal{X}$ necessitates finding admissible controls known as pointwise min-norm control laws, which are so called because at each point x, their value is the unique element of $\mathcal{U}$ of a minimum norm that satisfies the control constraint $U(x)$ whilst also making the worst-case Lyapunov derivative at least as negative as $-\alpha_v(x)$.

- The good news from a computational perspective is that it is possible to compute the value of a pointwise min-norm control law at any point x by solving a convex, static minimisation programming problem that is completely determined by the data $\mathbf{\Sigma}$, V and α_V.

- The further good news is that this static problem has a simple explicit solution in a wide variety of circumstances, a number of which are directly applicable to the CAT bond valuation problem.

- The only restriction is that the system must be jointly affine in u and w.

- To see how this works in practice, take an example of a system $\dot{x} = f(x, u, w)$ for continuous functions f_0, f_1 and f_2

$$\dot{x} = f_0(x) + f_1(x)\,u + f_2(x)\,w \tag{11}$$

and suppose that V is a robust control Lyapunov function for this system such that $D : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, then

$$D(x, u) := \max_{w \in W(x)} \left[L_f V(x, u, w) + \alpha_v(x) \right] \tag{12}$$

which, upon substituting, gives

$$\begin{aligned} D(x, u) &= \nabla V(x) \cdot f_0(x) + \nabla V(x) \cdot f_1(x)\,u && (13) \\ &+ \parallel \nabla V(x) \cdot f_2(x) \parallel + \alpha_v(x) && (14) \end{aligned}$$

Using the simplifications

$$\begin{aligned} \psi_0(x) &:= \nabla V(x) \cdot f_0(x) + \parallel \nabla V(x) \cdot f_2(x) \parallel + \alpha_v(x) \\ \psi_1(x) &= [\nabla V(x) \cdot f_1(x)]^T \end{aligned}$$

and defining $K : \mathcal{X} \rightsquigarrow \mathcal{U}$, gives

$$K(x) = \left\{ u \in U : \psi_0(x) + \psi_1^T(x)\,u < 0 \right\} \tag{17}$$

which finally gives the simplified expression for the pointwise min-norm control law

$$m(x) = \left\{ \begin{array}{ll} -\frac{\psi_0(x)\psi_1(x)}{\psi_1^T(x)\psi_1(x)} & \text{when } \psi_0(x) > 0 \\ 0 & \text{when } \psi_0(x) \leq 0 \end{array} \right\} \tag{18}$$

for all $x \in V^{-1}(c_v, \infty)$.

- This then is the expression that will enable the calculation of the robust control law that will provide both a hedging strategy in the face of both a catastrophe and disturbances to the underlying interest rate environment.

- The initial rigidity of joint affineness of the control and the disturbance can be relaxed through an integral back stepping procedure described by Freeman &Kokotovic (1995, 1997), but it was found that such relaxation added relatively little to the robustness or stability of the results described in the next section.

4 Empirical results

- The basic zero coupon CAT bond presented in these results has the following structure. It is assumed to pay an amount, Z, at maturity, T, contingent upon a threshold time $\tau > T$.

- The no arbitrage, present value (discounted at a continuously compounded rate of R) of the zero coupon CAT bond associated with a threshold loss level, D, catastrophic flow, M, an aggregate loss process, L and a distribution of incurred losses, F, that pays Z at maturity, is given by

$$V_t^1 = E\left[Z \exp\left\{-R\left(t, T\right)\right\}\left(1 - N_t\right)|\mathcal{F}_t\right]$$

where $\tau = \inf\left(t : L_t \geqslant D\right)$ and $N_t = I(L_t \geqslant D)^*$.

*Baryishnikov et al (1998) also show that this is a doubly stochastic Poisson process with intensity

$$\lambda_s = m_s\left\{1 - F\left(D - L_s\right)\right\} I\left(L_s < D\right)$$

- It is assumed that the threshold event is the time at which the accumulated losses exceed the threshold level, D, i.e. $\tau = \inf\left(t : L_t \geqslant D\right)$. To simplify the computations, the zero coupon CAT bond valuations reported are assumed to redeem at par of \$100 at maturity if aggregate losses do not exceed the threshold level, D. However, unlike Burnecki, Kukla and Taylor (2001) it is assumed that if accumulated losses exceed the threshold, then the bond holder receives a recovery amount, B_t, calculated as

$$B = Z - N_t$$

where N_t is stated as percentage of Z.

- The following results are therefore reported for varying levels of accumulated catastrophic losses incurred prior to maturity, expressed as a percentage. To achieve this scaling, the PCS loss data was simply re-based to 100 at the beginning of the calculation period.

- Figure 2.3 therefore illustrates the behaviour of the price of a series of such zero coupon CAT bonds (assumed to have been issued at a discount with accretion to par and with price expressed as a percentage of par) for given combinations of time to maturity and percentage loss.

- The valuations were produced using the Cox-PIDE model. The CAT bond valuations are at increasing monthly maturities from 1 month out to 12 months (e.g. 1m maturity, 2m maturity, 3m maturity etc), all with identical issue date of 01 August 1992. The results in figure 2.3 are for CAT bonds based on PCS loss data for the 12 months beginning 01 August 1992 for the National index, which was chosen specifically because it contained the largest (at that time) and most costly world insurance loss in the form of hurricane Andrew which occurred on 23 August 1992 and produced total insured losses of $15.5bn in 1992 dollar terms (or $20.8bn in 2004 dollar terms).

- Figure 2.3 presents a number of interesting features. First, it is clear that increasing the threshold loss level increases the value of the CAT bond. This behaviour makes intuitive sense, since raising the threshold loss level means less likelihood of losses consuming the entire value of the bond. For comparative and sanity check purposes, it is encouraging to note that the profile shown in figure 2.3 is consistent with results produced by both Burnecki, Kukla and Taylor (2003) and Baryshnikov, Mayo and Taylor (2001) from similar studies of CAT bond pricing using PCS data.

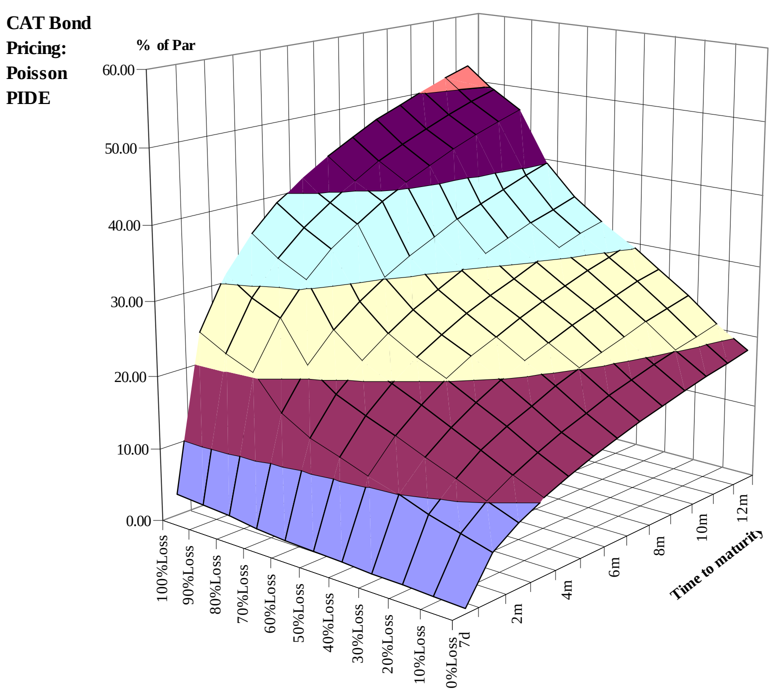


Figure 2.3: Short-term CAT bond valuation using Cox-PIDE

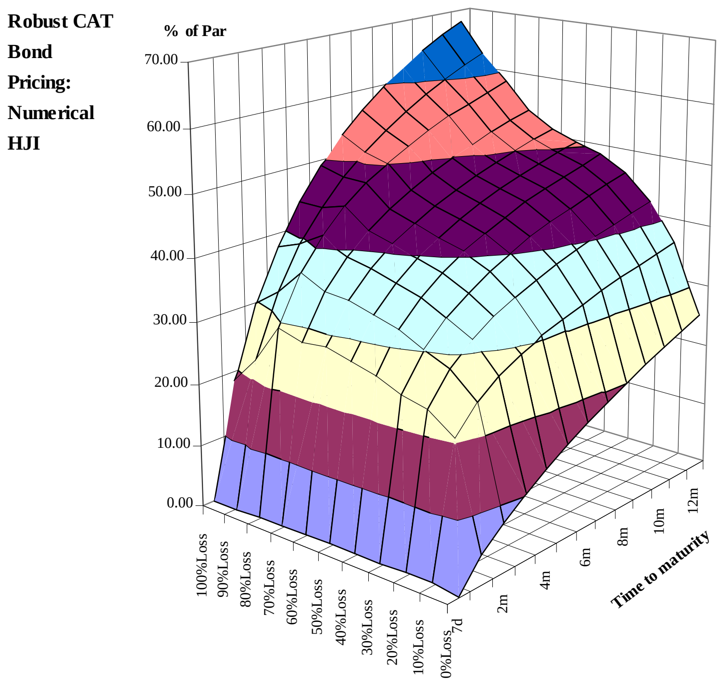


Figure 2.4: Short-term CAT bond valuation using numerical HJI

- Now compare the results in figure 2.3 with the results of performing the same valuation exercise using the same period PCS loss index data for the same overall set of maturities and loss levels, but using instead the robust numerical HJI model to value the CAT bond. The results of these robust model computations are shown in figure 2.4. What is immediately, though not unsurprisingly apparent, is that the robust numerical HJI model results in much higher prices for the CAT bond across almost all maturities and levels of threshold loss.

- There are a number of possible explanations for this result:

  1. First and trivially, could be that the Cox-PIDE model fundamentally undervalues the value of catastrophic events, that by their very definition are much less likely to occur and would therefore be in the tail of a distribution. This is clearly one

possible explanation. However, following Burnecki, Kukla and Taylor (2003), the results reported in figure 2.3 were in fact generated using the heavy-tailed Burr distribution to fit the PCS loss distribution, so this explanation is only partially acceptable[†]. Therefore, it would appear that despite using a heavy tailed loss distribution, the Cox-PIDE model appears to undervalue the CAT bond; the undervaluation is dramatically higher at lower loss levels and shorter times to maturity and then declines with both loss level and maturity to around 30% for the 12 month and 100% loss level combination.What is also interesting is the pattern of differences between the two approaches. Figure 2.5 shows the percentage undervaluation between Cox-PIDE and numerical HJI. The pattern of differences are worthy of note. First, it should be born in mind that the results in figures 2.3 and 2.4 are limited to bonds

[†]See Appendix 2 for details of the fitting procedure and some of the results of the calibration.

with maturities ranging from 1 month to 1 year. The undervaluation appears much greater for the shorter maturities and lower loss levels (rising to a peak for the 2 month and 30% threshold loss level), before tailing-off to an over-valuation of around 30% for the longer maturities and higher loss levels[‡]. An immediately obvious question is why does the pattern of differences change so significantly? One possible explanation is that the Cox-PIDE fundamentally undervalues the impact of smaller, individual catastrophes that occur more frequently, but that the undervaluation effect is eroded with the effects of time and as loss levels rise.

2. A second possible explanation is that the pattern of undervaluation could also be a function of the volatility of claims estimates due to the way in which the catastrophes are reported and

[‡]Note that all of the results used to plot the figures contained in this chapter are provided in numerical form in Appendix 3).

the index adjusted. This would appear to fit with the fact that there is a distinct pattern to the way in which catastrophes are reported. Typically, an initial estimate of the number of claims and their total value gets published and then refined during the development period. The initial uncertainty around the losses associated with a catastrophe therefore declines as estimates become firmer. Detailed examination of data for individual catastrophes appears to bear this out, as catastrophes exhibit a higher level of volatility in initial claims estimates (due to the initial lack of hard facts as losses take time to assess and claims then take further time to complete and submit), which then declines as facts emerge, loss estimates crystalise and claim numbers and values stabilise.

3. A third possible explanation is that the mis-valuation could be attributable to seasonal effects. There

is a well known and heavily documented[§] pattern of hurricane and tropical storm occurrence in the southern USA during the late summer and autumn. This weather pattern accounts for a significant proportion of the insured losses that form the PCS index. Even though the heavy-tailed Burr distribution was used to fit the PCS data, it may be the case the Burr distribution is simply not capable of adequately capturing the effects of the well known pattern to the occurrence of hurricanes and storms in the south eastern USA during this period. The spike in catastrophes is clearly evident in the steep slope in both figures 2.3 and 2.4, which both clearly show the dramatic impact of hurricane Andrew in August and September 1992. Figure 2.5 shows that the Cox-PIDE model significantly undervalues the impact on the CAT bond price of this catastrophe for short dated maturities when the

[§]See for example the Insurance Information Institute website at: www2.iii.org/facts.

number and size of claims is still at its most volatile, indicating a lack of robustness with respect to the occurrence of large catastrophic events.

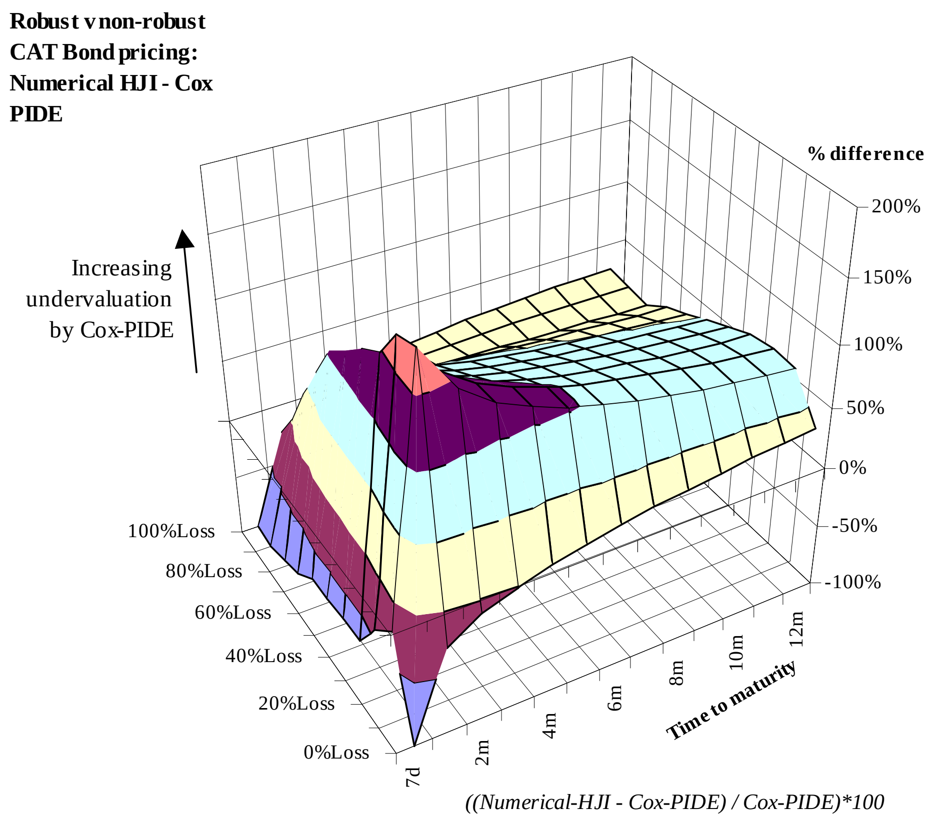


Figure 2.5: Short-term CAT bond model valuation: Numerical HJI - Cox-PIDE

- An immediately interesting question is whether or not the cost of robustness varies according to the robust model approach employed. Therefore, to investigate whether the pattern exhibited in the results of the numerical HJI model was mainly a function of a particular facet of the HJI model, a linear robust control Lyapunov model was used to value CAT bonds using the same range of maturities, the same underlying PCS loss data and the same range of loss levels as used for the Cox-PIDE and numerical HJI models. Figure 2.6 provides the results of these CAT bond valuations for the linear robust control Lyapunov valuation model developed earlier in this paper.

- What is immediately clear is that all three models display relatively smooth monotonic valuation profiles throughout the ranges of loss levels and time to maturity. This is to some extent to be expected as all three models contain significant linearisations. The numerical HJI model used in this analysis is, in particular, an explicitly linear model - which, as already explained earlier in this paper means that significant control effort (and therefore associated higher control cost, which is in turn reflected in a higher valuation) can be wasted attempting to combat inherent non-linearities.

- The final logical step is therefore to extend the analysis by including the non-linear version of the robust control Lyapunov model in its piece-wise min-norm form. Once again, the same maturities, loss levels and PCS data were used to produce comparable CAT bond valuations, the results of which are reported in figure 2.8. Examination of figure 2.8 immediately shows the benefit of explicitly incorporating the non-linearities, as the behaviour of CAT bond value around low loss levels and short time to maturity is now much smoother than in the simple linear Lyapunov case shown in figure 2.6.

- Worthy of note is the interesting behaviour arises when the relative performance of the non-linear Lyapunov model is compared with the non-robust Cox-PIDE model - as shown in figure 2.9.

- Two features are worthy of comment:

  1. First, is that there is now a much lower range of variability in valuation around the Cox-PIDE model, suggesting that robustness appears to have been achieved at a much lower cost by explicitly incorporating non-linearities into the robust control Lyapunov model.

  2. The second interesting feature is the pronounced double peakiness in valuation differences. This is most pronounced around the 50% and 80% loss levels. For the very short dated CAT bonds this would appear to coincide with the re-estimation volatility associated with the uncertainty surrounding the losses from hurricane Andrew. The fact

that this behaviour is far less pronounced in the case of the longer dated CAT bonds seems to lend support to such conjecture.

- The contrast in performance between the three robust models is clearly seen when comparing the results in figure 2.5 (numerical HJI v Cox-PIDE), figure 2.7 (linear Lyapunov v Cox-PIDE) and figure 2.9 (non-linear Lyapunov v Cox-PIDE).

- The first point to note is that the numerical HJI model exhibits a much smoother difference profile with respect to the Cox-PIDE model compared with the linear Lyapunov model, but at a higher cost than the non-linear Lyapunov model. Figure 2.7 shows an interesting pattern of differences which is most marked around the lower loss levels and shorter times to maturity.

- The numerical HJI model imposes a much greater cost penalty to robustness than the linear Lyapunov model, which is reflected in much higher valuations. This pattern is particularly pronounced in the short-term and low loss cases which in the context of the current analysis are precisely those CAT bonds most subject to the impact of hurricane Andrew. The pattern of differences then falls away, becoming far less significant in the case of increased time to maturity and higher loss levels.

- The final step in this CAT bond research was to analyse the importance of time to maturity in determining the cost of robustness. Focus has so far been limited to short dated CAT bonds, but the critical question is whether examining bonds with a maximum maturity of only 12 months is likely to exacerbate or hide any valuation patterns that may be associated with achieving robustness.

- On the one hand, identifying the cost of robustness may be argued to be a simpler task by concentrating on short-dated CAT bonds. Unfortunately, on the other hand, little can be inferred about the dynamics of robustness over time if attention is restricted to such a short space of time.

- The next logical step is therefore to extend the maturity of the CAT bonds for all four models. Accordingly, all four CAT bond models were therefore re-run using the same loss levels, but using instead 10 years worth of PCS data beginning 01 January 1990 and ending 31 December 1999. The time to maturity of the longest CAT bond was extended to 10 years at 6 monthly intervals, i.e. 6m, 12,m, 18m,...,108m, 114m, 120m. In other words, CAT bonds with 20 different maturities ranging from 6 months to 10 years (but all with an identical start date of 01 January 1990) were valued using each of the four models. The results of each set of valuations is reported in figures 2.10 (Cox-PIDE), 2.11 (numerical HJI), 2.12 (linear Lyapunov) and 2.13 (non-linear Lyapunov), with comparisons to Cox-PIDE being presented in figures 2.14 (numerical HJI v Cox-PIDE), 2.15 (linear Lyapunov v Cox-PIDE) and 2.16 (non-linear Lyapunov v Cox-PIDE).

- The results for these longer maturity CAT bonds provide a number of further insights into the robustness and stability of the three robust valuation models. The first and most obvious feature to emerge from the long-dated CAT bond valuations is the fundamentally different shapes of the valuation surfaces when compared with those generated for the short dated CAT bonds. The most interesting set of results is for the numerical HJI valued bonds shown in figure 2.11, which exhibit an extremely high implied cost of robustness as can be vividly seen in figure 2.14. Detailed examination of the results revealed that the principal reason for this behaviour was that for the longer dated bonds severe cost penalties were being incurred by the numerical HJI algorithm in order to ensure stable solutions. These cost penalties translated directly into higher valuations as the HJI model consumed increasing numbers of processing cycles searching for a stable solution to satisfy the robustness and stability constraints.

- The second feature of interest is the presence of a much clearer valuation differential between the numerical HJI model on the one hand and the two robust control Lyapunov models on the other hand.

The numerical HJI algorithm used in the computations follows the standard power series approach of Al'brecht (1961) for solving infinite time optimal control problems¶. Why does this differential occur and how should it be interpreted? To answer these questions, consider that in contrast to the numerical HI algorithm, the two Lyapunov based models both use the Freeman and Kokotovic approach of finding a meaningful cost function such that the given robust control Lyapunov function is the corresponding value function. This implicitly provides a solution to the equivalent linear HJI equation, thereby enabling the direct computation of the robust optimal control law. Therefore, providing that the cost function belongs

¶Al'brecht's (slightly modified) method solves the HJI partial differential equation in the neighbourhood of the origin using a power series method. This ultimately reduces the quadratic terms of the HJI pde to an easily solvable Riccati equation and a linear optimal feedback rule. This is then solved using function SB02PD ported from the Slicot library as explained in Appendix 1. This function solves the continuous algebraic Riccati equations using the matrix sign function method with condition and forward error bound estimates.

to a meaningful class of cost functions, the resulting control law is robust and guaranteed to inherit all the required optimality properties. The robust control Lyapunov approach uses an inverse optimal robust stabilisation problem of finding a meaningful cost function, such that a given robust control Lyapunov function is the corresponding value function. This results in a solution to the equivalent linear HJI problem that is both stable *and* robust. In the case of the non-linear robust optimal control Lyapunov model, this further translates into solutions that take advantage of the non-linearities in the valuation problem that exist explicitly because the catastrophic events are driven by highly complex non-linear relationships. The outcome in the case of the non-linear robust control Lyapunov model is smoother and less expensive robustness - in other words, the cost of robustness is lower in the non-linear case as the solution takes advantage of the non-linearities rather than fighting against them, which translates directly into lower robustness costs.

- The final issue to consider is whether the relative performance of the three robust models presented so far provide sufficient information to draw definitive conclusions about the cost of robust valuation in the face of massive catastrophic events such as hurricane Andrew? Looking first at the short-term valuation results, all three robust models can be seen to exhibit substantial differences compared to the Cox-PIDE model for the 1-6 month securities and up to around the 50% loss level. The pattern of differences then appears to be less pronounced for the 6-12 month securities and higher loss levels. This may at least in part be due to the volatility of the claims estimates referred to above being handled differently in the cost function within the models. the liner and non-linear Lyapunov approaches

- As far as the long-term valuation results are concerned, the impact and tail effects of hurricane Andrew can be seen quite clearly between the four models. Once again, the numerical HJI model exhibits extreme cost penalties right across the loss level spectrum for the shorter maturities and in the middle of the loss threshold range, but these penalties fall off very rapidly as the initial impact effect of Andrew decays. The cost of robustness for the numerical HJI is therefore far higher even for the very longest bonds at all but the very lowest loss levels. Detailed examination of the diagnostics for the numerical HJI once again reveals the same explanation as in the shorter dated case. What is also worthy of note is that the pattern of extreme cost penalties appears to have quite a lengthy tail to its decay structure. The tail has three discernible phases, which can best be observed by looking at the longest dated bonds. The first phase covers the initial impact of Andrew and appears to last out to around 3 years. During this phase the cost penalties begin extremely high, but fall off very rapidly. The second phase is from around 3 to 5 years, during which time the cost penalties continue to fall but at a much slower pace. The final phase, from 5 to 10 years sees the cost penalties flattening out, but still remaining high.

- In contrast, the two Lyapunov models no longer continue to attract extremely high cost penalties compared with the Cox-PIDE model as can be clearly seen when comparing figures 2.14, 2.15 and 2.16. What is even more interesting is that the two Lyapunov models actually exhibit valuations below the Cox-PIDE for some combinations of shorter maturities and higher loss levels. Closer examination of the diagnostics for the linear Lyapunov model revealed that incorporation and consequent influence of feedback yielded smoother solutions so that the cost penalties associated with achieving robustness were substantially reduced. In the case of the non-linear Lyapunov model the extra influence of the lower cost penalties associated with incorporating the non-linear dynamics further reduced the costs of robustness. This finding for the non-linear robust control Lyapunov function is a significant finding as it underlines that the costs of robustness to uncertainty may not be so high as to make robust strategies unaffordable. The point is that in times when catastrophic events do not occur - which by their very definition tends to be most of the time - the costs of robustness make it totally uneconomic as a valuation methodology. However, when catastrophic events are brought into the picture, the costs of robustness become far more acceptable compared the possible levels of loss, which may include bankruptcy or ruin at the limit.

- Notwithstanding the above possible explanations for the valuation differences between the Cox-PIDE and the numerical HJI models, arguably the more interesting question is whether the robust model actually *overvalues* the benefits of robustness. One way of answering this question is by resorting to a comparison of the CAT bond valuations with the out-turn in the PCS index[‖]. The answer to this question can

[‖] An equally valuable cross-check would be to compare the results of the CAT bond models with the traded prices of puts and calls in the CBOT options prices. However, the option contracts did not trade for the entire period of interest of the short-dated CAT bonds. See Appendix 2 for details of the periods covered by the PCS options data available from CBOT.

be only partially inferred from the results presented in figures 2.3, 2.4 and 2.5. Hurricane Andrew occurred on 23rd August 1992, so the first month in which estimates of likely loss were fully available was September 1992. The relative differences in value before and after the impact of hurricane Andrew can be seen in figure 2.5.

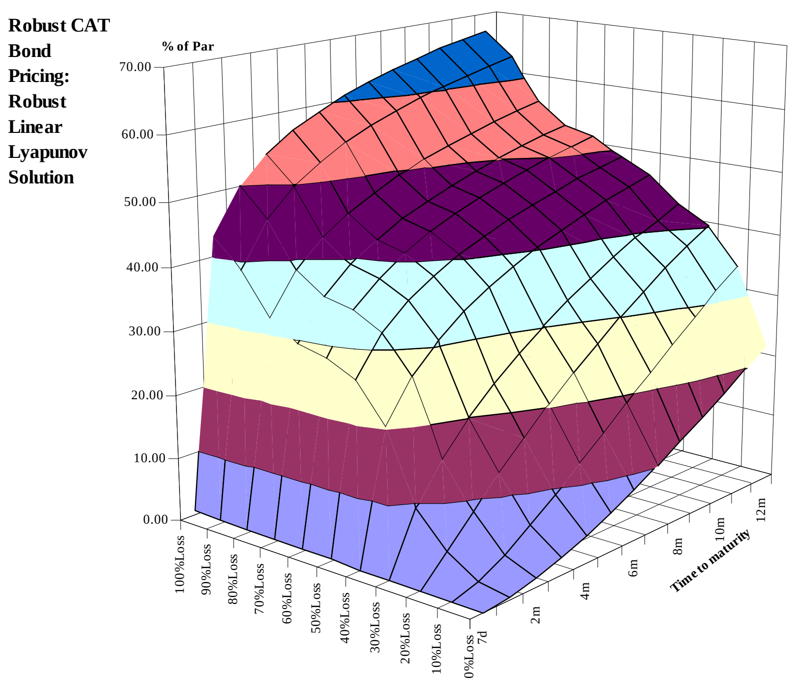


Figure 2.6: Short-term CAT bond valuation using Linear Lyapunov

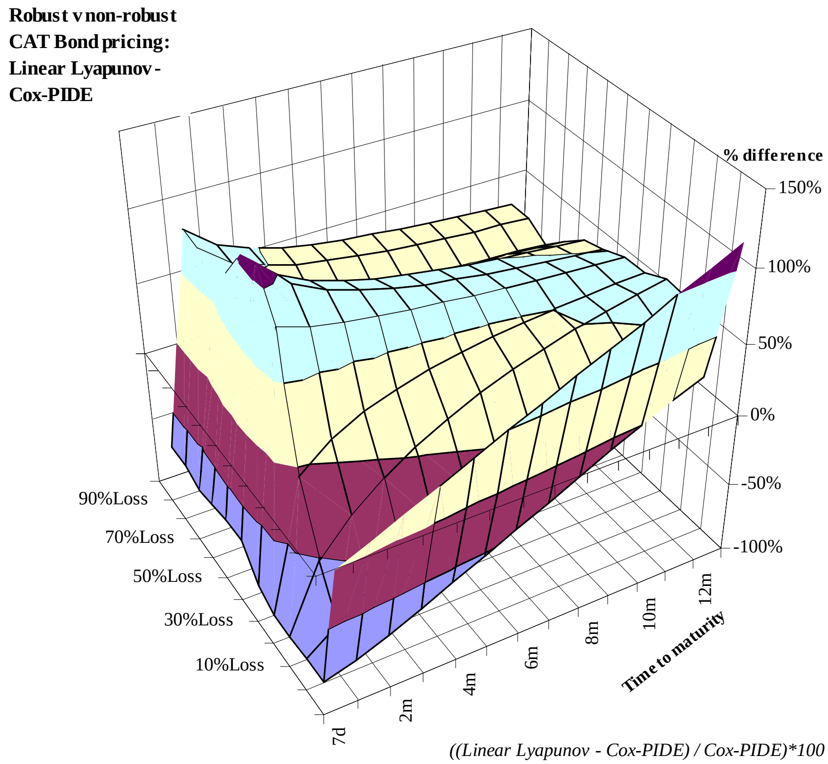


Figure 2.7: Short-term CAT bond model valuation: Linear Lyapunov - Cox-PIDE

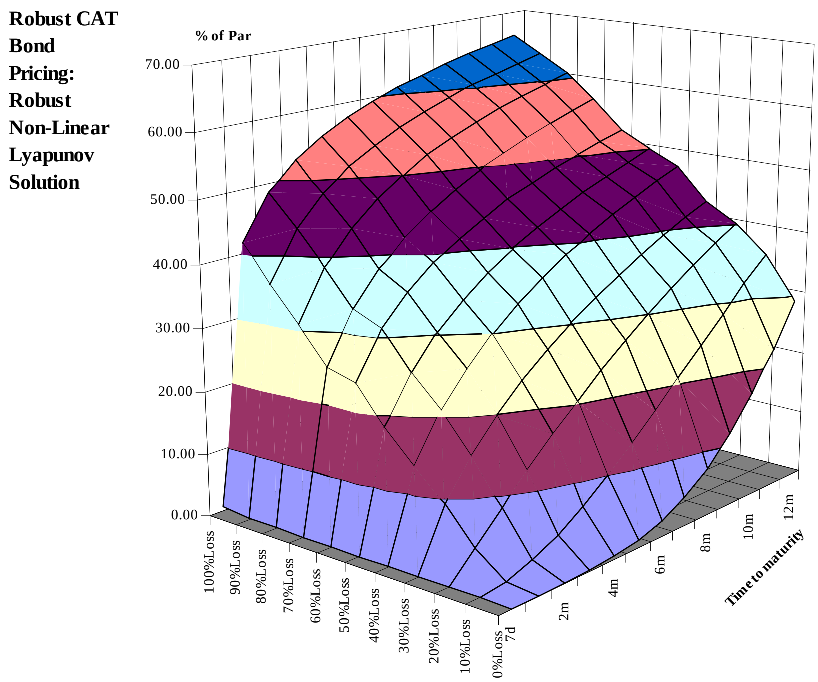


Figure 2.8: Short-term CAT bond valuation: Non-Linear Lyapunov

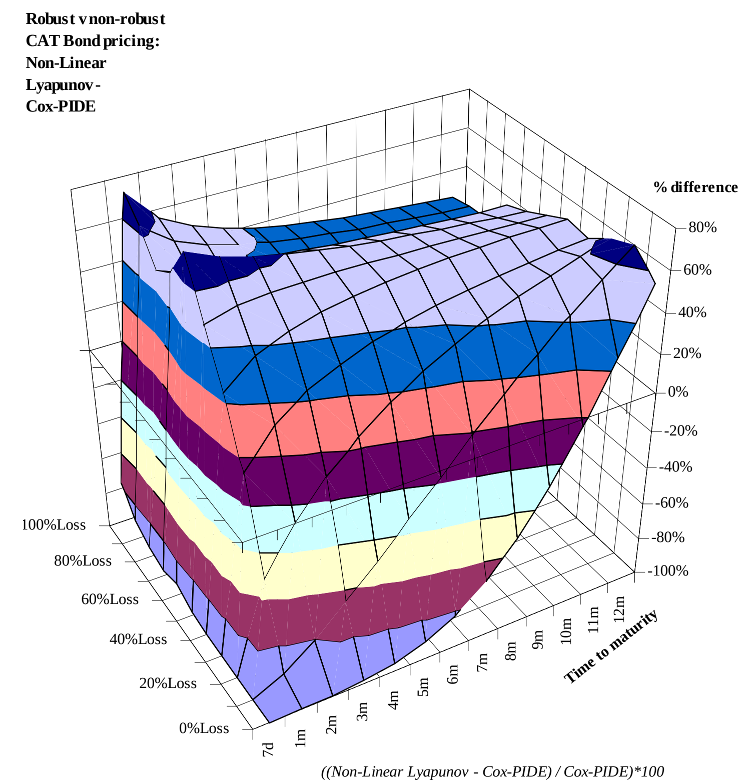


Figure 2.9: Short-term CAT bond model valuation: Non-Linear Lyapunov v Cox-PIDE

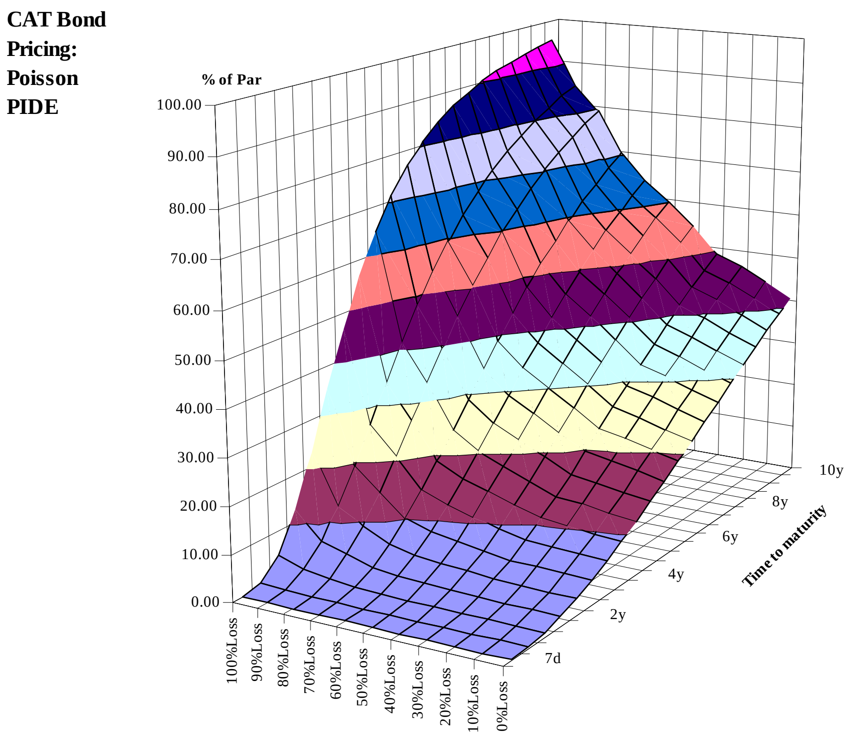


Figure 2.10: Long-term CAT bond valuation: Cox-PIDE

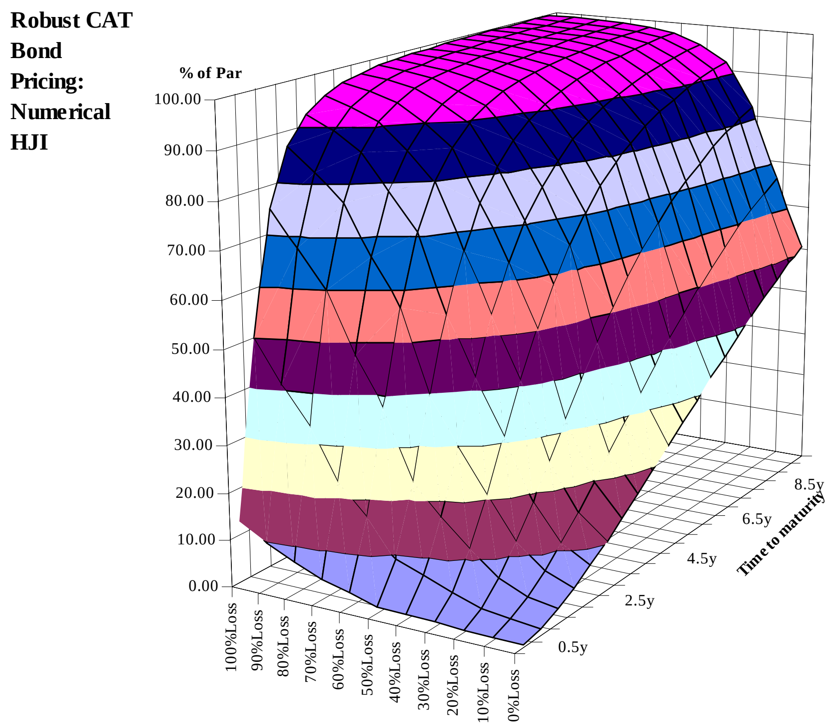


Figure 2.11: Long-term CAT bond valuation: Numerical HJI

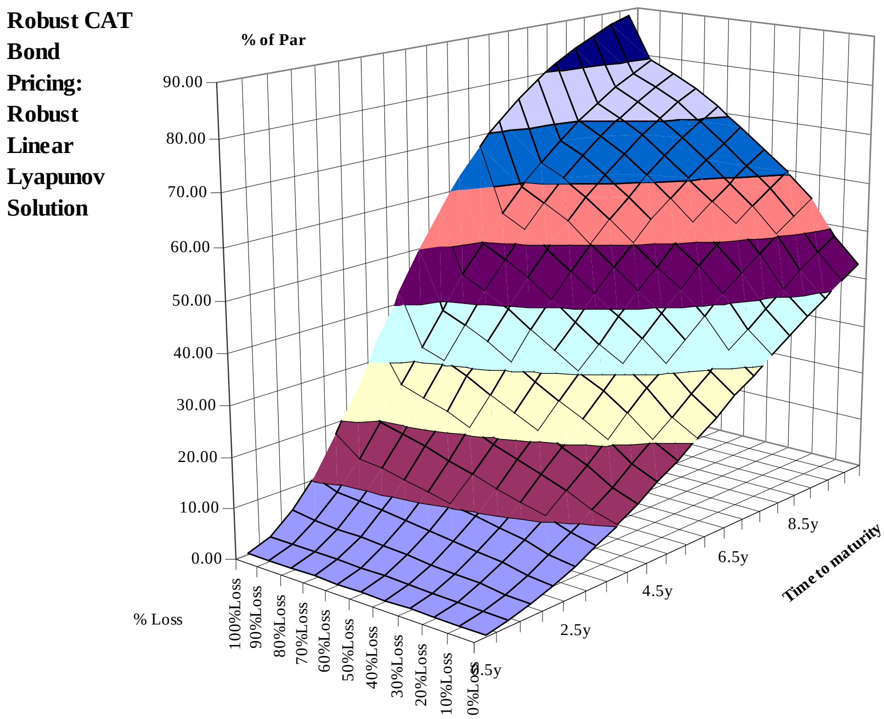


Figure 2.12: Long-term CAT bond valuation: Linear Lyapunov

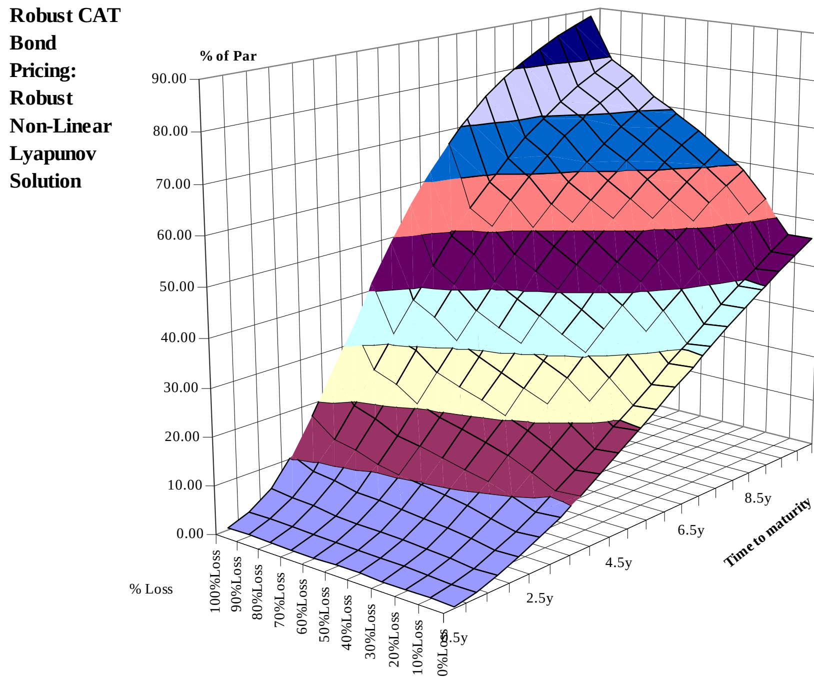


Figure 2.13: Long-term CAT bond valuation: Non-Linear Lyapunov

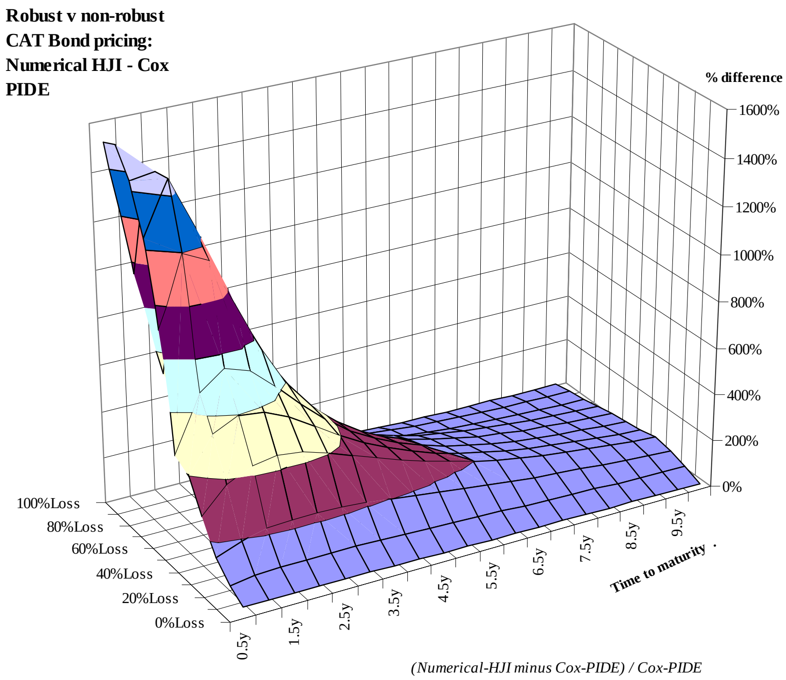
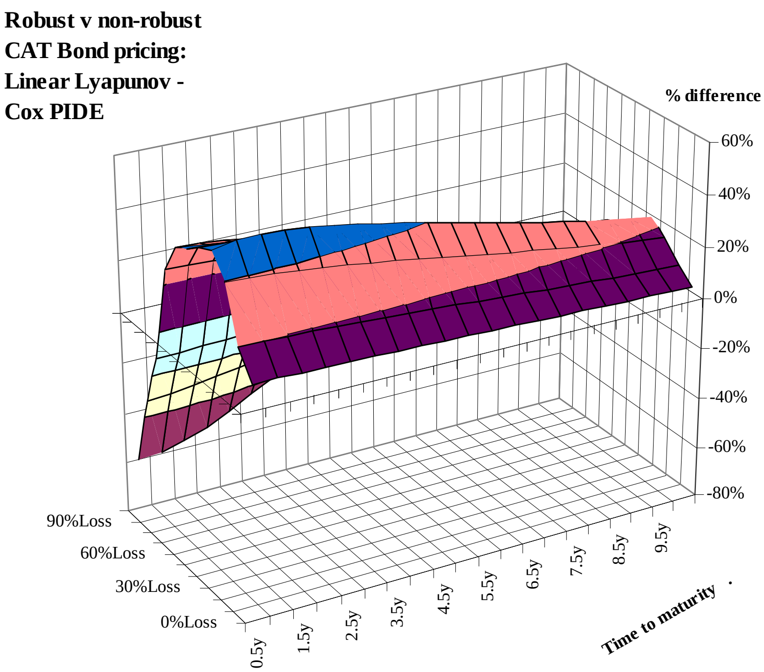


Figure 2.14: Long-term CAT bond valuation: Numerical HJI v Cox-PIDE



(Linear Lyapunov minus Cox-PIDE) / Cox-PIDE

Figure 2.15: Long-term CAT bond valuation: Linear Lyapunov v Cox-PIDE

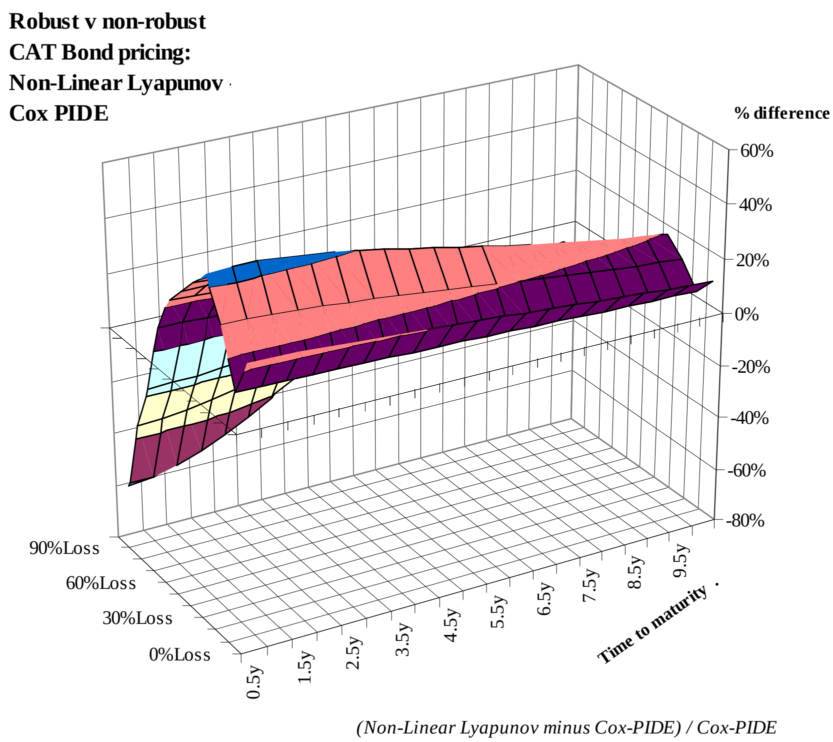


Figure 2.16: Long-term CAT bond valuation: Non-Linear Lyapunov v Cox-PIDE

- Having analysed the theoretical costs of robustness, it is relevant to consider how such valuation techniques would actually have fared in practice. There are two points here:

  1. First, actual property insurance claims of approximately USD 60 billion were made between 1990

and 1996 (Canter, Cole, and Sandor; 1996) resulting in the insolvency of a number of insurance firms. These bankruptcies were brought on in the wake of hurricanes Andrew (Florida and Louisiana affected, 1992), Opal (Florida and Alabama, 1995) and Fran (North Carolina, 1996), which caused combined damage totalling USD 19.7 billion (Canter, Cole, and Sandor; 1996). These, along with the Northridge earthquake (1994) and similar disasters, led to an interest in alternative means for underwriting insurance. In 1995, when the CAT bond market was born, the primary and secondary (or reinsurance) industries had access to approximately USD 240 billion in capital (Canter, Cole, and Sandor; 1996; Cummins and Danzon; 1997). Given the capital level constraints necessary for the reinsuring of property losses and the potential for single-event losses in excess of USD 100 billion, this was clearly insufficient. A loss of $100 billion would consume approximately 30 - 40% of the equity capital of the US insurance industry but would be less than 0.5% of the value of the US stock and bond markets. Whether such problems could have been avoided by insurers attempting more sophisticated risk management through the issuance of CAT bonds that had been valued using robust valuation methods is difficult to assess directly. However, what can be concluded is that in terms of total revenue, the prevailing Cox-PIDE model produced fundamental mis-valuations that would have resulted in a significant lack of risk mitigation and cashflow - particularly if the CAT bonds had been short-term securities.

2. Could CAT bonds have been issued in amounts that would have been sufficient to enable the required risk mitigation? It is undoubtedly the case that the international capital markets provided a potential source of risk appetite for the reinsurance market. An estimated capitalisation of the international financial markets around the time of

hurricane Andrew, of about USD 19 trillion underwent an average daily fluctuation of approximately 70 basis points or USD 133 billion (Sigma; 1996). So, clearly the capacity to bear such large amounts of catastrophic risk was (and remains) much greater in the capital markets. However, the under-capitalisation of the reinsurance industry (and the consequent potential default risk) meant that there was a tendency for CAT reinsurance prices to be highly volatile which discouraged many potential issuers from using CAT bonds. This was reflected in the traditional insurance market, with rates on line being significantly higher in the years following catastrophes and dropping off in the intervening years (Sigma; 1997; Froot and O'Connell; 1997). This heterogeneity in pricing had a very strong damping effect, forcing many re-insurers to leave the market, which in turn has adverse consequences for the primary insurers. A number of reasons for this volatility have been advanced (see for example Cummins and Danzon; 1997 and Winter; 1994).

- Some of the traditional assumptions of derivative security pricing are not correct when applied to these instruments due to the properties of the underlying contingent stochastic processes. There is evidence that certain catastrophic natural events have (partial) power-law distributions associated with their loss statistics (Barton and Nishenko; 1994), which if true, would overturns the traditional log-normal assumption of derivative pricing models and makes robustness hard if not impossible to achieve without using non-linear models.

- There are also well-known statistical difficulties associated with the moments of power-law distributions**, thus rendering it impossible to employ traditional pooling methods and consequently the central limit theorem. Given that heavy-tailed or large

**This has become a significant research topic in its own right - see for example, "Multifractal Power Law Distributions: Negative and Critical Dimensions and Other “Anomalies, Explained by a Simple Example", by Benoit B. Mandelbrot, in the Journal of Statistical Physics, Vol. 110, Nos. 3–6, March 2003.

deviation results assume, in general, that at least the first moment of the distribution exists, there will be difficulties with applying extreme value theory to this problem (Embrechts, Resnick, and Samorodnitsky; 1999). It would seem that these characteristics may render traditional actuarial or derivatives pricing approaches ineffective.

- Although there is some similarity with the valuation of defaultable bonds, there are additional features to modelling the CAT bond price which are not to be found in models of ordinary corporate or government securities. The main feature is that the trigger event that underlies CAT bond pricing is dependent on both the frequency and severity of natural disasters. The Cox-PIE model in this section is used to reduce to a minimum any assumptions about the underlying distribution functions in the interests of generality of application. However, given the daily availability of PCS loss data, it is also appears to be reasonable to assume that loss levels are instantaneously measurable and updatable, which makes it straightforward to adjust the underlying process to accommodate a development period and it is this feature that is explicitly included in the next section where results of CAT option valuation are reported that once again use PCS loss data.

- There is a natural similarity between the pricing of catastrophe bonds and the pricing of defaultable bonds. Defaultable bonds, by definition, must contain within their pricing model a mechanism that accounts for the potential (partial or complete) loss of their principal value. Defaultable bonds yield higher returns, in part, because of this potential defaultability. Similarly, CAT bonds are offered at high yields because of the unpredictable nature of the catastrophe process. With this characteristic in mind, a number of pricing models for defaultable bonds have been advanced (e.g. Jarrow and Turnbull, 1995, Duffie and Singleton, 1999, Zhou and 1997). The trigger event for the default process has similar statistical characteristics to that of the equivalent catastrophic event pertaining to CAT bonds.[††]

[††]In an allied application to mortgage insurance, the similarity between catastrophe and default in the log-normal context has been commented on (Kau and Keenan; 1996).

5 Conclusions

- This research has developed and applied modern robust control techniques to produce practical solutions to real-world financial problems in derivative pricing, hedging and risk management.

- Robustness in decision making has been shown to be important, achievable in a cost effective manner and computationally tractable in these 3 distinct cases.

- Specifically incorporating non-linearities is shown to be important because doing so:

  1. Ameliorates the impact of model reduction particularly as there are a broader range of controls from which to choose.

  2. Increases the range and efficiency of control laws available making robustness less costly and less wasteful.

3. Produces a robust and stable solution methodology that is flexible, tractable and computationally efficient.

References

[1] Aase, K., (1999), "An equilibrium model of catastrophe insurance futures and spreads", Geneva Papers on Risk and Insurance Theory 24, 69-96.

[2] Berkowitz, J., (2001), “Getting the right option price with the wrong model”, Graduate School of Management, University of California at Irvine.

[3] Bernhard, P., (2005), “The Robust Control Approach To Option Pricing and Interval Models: An Overview”, in

[4] Black, F. and Scholes, M., (1973), “The Pricing of Options and Corporate Liabilities”, Journal of Political Economy, 81(3).

[5] Cummins, J.D. & Geman, H., (1995), "Pricing catastrophe insurance futures and call spreads: an arbitrage approach", Journal of Fixed Income 4, 46-57.

[6] Geman, H. & Yor, M., (1997), "Stochastic time changes in catastrophe option pricing", Insurance: Mathematics and Economics 21, 185-193.

[7] Heston, S., (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options", Review of Financial Studies.

[8] Hofmann, N., Platen, E. and Schweizer, (1992), "Option Pricing under incompleteness and stochastic volatility", Mathematical Finance,

[9] Hull, J. and White, A., (1987), “The pricing of options on assets with stochastic volatilities”, Journal of Finance.

[10] Insurance Standards Organisation, Annual Reports, 1995 and 1996.

[11] MacEneaney, W.M., (1996), “A Robust Control Framework for Option Pricing”, Department of Mathematics, Carnegie Mellon University.

[12] Muermann, A., (2003), "Market price of insurance risk implied by catastrophe derivatives", working paper, The Wharton School of the University of Pennsylvania.Annual Report, International Institute for Applied Systems Analysis, 2003.

[13] Sargent, T.J. (2001), “Robust control and model uncertainty”, paper presented to American Economic Association, New Orleans, January.

[14] Rubenstein,

[15] Stein, E.M. and Stein, J.C., (1991), “Stock price distributions with stochastic volatility: an analytic approach”, Review of Financial Studies.

[16] Veronesi, P., (2004), “Belief-dependent Utilities, Aversion to State-Uncertainty and Asset Prices”, University of Chicago, NBER and CEPR, January 8, 2004.

[17] Wang, S., (2002), “A Universal Framework for Pricing Financial and Insurance Risks”, ASTIN Bulletin, 32(2).

[18] Wiggins, J., (1987), “Options Values Under Stochastic Volatility: Theory and Empirical Evidence”, Journal of Financial Economics.

[19] Young, V.R., (2005), "Premium Principles", in Encyclopedia of Actuarial Science, edited by Teugels, J.L. and Sundt, B., published by Wiley.