Asymptotic Theory for Range-Based Estimation of Quadratic Variation of Discontinuous Semimartingales*

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Abstract

We propose using realized range-based estimation to draw inference about the quadratic variation of jump-diffusion processes. Moreover, we construct a new test of the hypothesis that an asset price has a continuous sample path. Simulated data shows that our approach is efficient, the test is well-sized and more powerful than a return-based t-statistic for sampling frequencies normally used in empirical work. Applied to equity data, we find that the jump process is not as active as reported in previous work.

JEL Classification: C10; C22; C80.

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1. Introduction

Continuous time finance has to a large extent been developed by assuming that the return-generating process moves along a continuous sample path. To this date, the workhorse of both applied and theoretical econometrics is by far the stochastic volatility model (e.g., Hull & White (1987) or Heston (1993)). Modeling asset prices as a continuous function of time is appealing, but it also heavily contrasted by the many abrupt changes found in the data. A series of papers has progressed by using low-frequency data to estimate jump-diffusion processes and/or test for the presence of a jump component (see, e.g., Aït-Sahalia (2002), Andersen, Benzoni & Lund (2002), Pan (2002), Chernov, Gallant, Ghysels & Tauchen (2003), Eraker, Johannes & Polson (2003), Johannes (2004), and the references therein). The empirical evidence from these studies is at best mixed, however. Indeed, diffusive models can generate sample paths that resemble those of discontinuous processes, except at sufficiently high time resolutions.

To this end, inference about the jump component has progressed rapidly, following the harnessing of high-frequency data. Recent papers, under the heading "realized multi-power variation," has build a non-parametric framework for backing out several variational measures of the price process (see, e.g., BN-S (2007)). For example, realized variance is a sum of squared intraday returns and converges in probability to the quadratic variation of all semimartingales, as the sampling frequency tends to infinity (e.g., Protter (2004)). In jump-diffusion models, the quadratic variation consists of the integrated variance and squared jumps. Bi-power variation, a related statistic proposed by BN-S (2004), has an intriguing robustness property, however, and can consistently estimate the integrated variance. Thus, we can draw inference about the jump component, if any, by studying realized variance less bi-power variation. BN-S (2006) derived the joint asymptotic distribution of these estimators - computed under the null of a continuous sample path - and developed a non-parametric test on this basis. The evidence from high-frequency data is more convincing and strongly suggests that jumps are frequent and account for significant proportion of quadratic variation.

In this paper, we propose using realized range-based multi-power variation to estimate the quadratic variation. Moreover, we construct a new non-parametric test for jump detection. Our motivation is that high-frequency range-based estimation of integrated variance is very efficient (see, e.g., Parkinson (1980), Christensen & Podolskij (2005), or Dijk & Martens (2006)). Hence, we expect that range-based inference about the jump component is powerful. In addition, due to market microstructure noise, realized variance and bi-power variation are often sparsely sampled (e.g., at the 5-minute frequency), which entails a loss of information that is partly recovered by sampling a range. The properties of the high-low has, however, been neglected in the context of jump-diffusion processes.

Our paper contributes in several directions. First, we extend the asymptotic results on the realized range-based variance - introduced by Christensen & Podolskij (2005) for Brownian semimartingales - to cover the jump-diffusion setting. Although this appears to be a minor nuisance, we are going to show that for such processes realized range-based variance is an inconsistent estimator of the quadratic variation. Second, we suggest a modified range-based estimator that - with a simple correction - restores the consistency. Third, we introduce range-based bi-power variation, derive its probability limit and asymptotic distribution under the null of a continuous sample path. Fourth, we develop a procedure based on the range for testing the hypothesis of no jump component.

The paper proceeds as follows. In section 2, we set notation and invoke a standard arbitrage-free continuous time semimartingale framework. We briefly review the theory of realized variance and then switch attention to realized range-based variance within a jump-diffusion model. A Monte Carlo simulation is conducted in section 3 to illustrate the disentangling of the components of quadratic variation using range-based bi-power variation and uncover the finite sample properties of the jump detection t-statistic. In section 4, we progress with an empirical application by looking at Merck high-frequency data for a 5-year period. In section 5, we conclude. An appendix contains the derivations of our results.

2. A Jump-Diffusion Semimartingale

In this section, we propose a non-parametric method based on the price range for consistently estimating the components of quadratic variation. Moreover, we introduce a new test for drawing inference about the jump part. The theory is developed for a univariate log-price, say $p = (p_t)_{t \geq 0}$, defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$. p evolves in continuous time and is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e. a collection of σ-fields with $\mathcal{F}_u \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $u \leq t < \infty$.

Throughout the paper, we restrict p to be member of the class of jump-diffusion semimartingales that satisfies the generic representation:[1]

$$p_t = p_0 + \int_0^t \mu_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u + \sum_{i=1}^{N_t} J_i, \tag{2.1}$$

where $\mu = (\mu_t)_{t\geq 0}$ is locally bounded and predictable, $\sigma = (\sigma_t)_{t\geq 0}$ is càdlàg and $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. Also, $N = (N_t)_{t\geq 0}$ is a finite-activity simple counting process that has an associated sequence of non-zero random variables, $J = \{J_i\}_{i=1,\dots,N_t}$.[2]

Equation (**2.1**) with $N = 0$ is called a Brownian semimartingale and we write $p \in BSM$ to reflect this in the following. Note that, without loss of generality, we can restrict the functions μ and σ to be bounded. Moreover, as $t \mapsto \sigma_t$ is càdlàg, all powers of σ are locally integrable with respect to Lebesgue measure, so that for any t and $s > 0$, $\int_0^t \sigma_u^s \mathrm{d}u < \infty$ (e.g., Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006)).

We assume that high-frequency data are available through $[0, t]$, which is the sampling period and will be thought of as representing a trading day. At any two sampling times t_{i-1} and t_i, such that $0 \leq t_{i-1} \leq t_i \leq t$, we define the intraday return of p over $[t_{i-1}, t_i]$ by:

$$r_{t_i,\Delta_i} = p_{t_i} - p_{t_{i-1}}, \tag{2.2}$$

where $\Delta_i = t_i - t_{i-1}$.

With this notation at hand, we introduce the object of interest; the quadratic variation process. The theory of stochastic integration states that this quantity exists for all semimartingales. Its relevance to financial economics is stressed in several papers (e.g., Andersen, Bollerslev & Diebold (2002)). The definition of quadratic variation is given by:

$$\langle p \rangle_t = \operatorname*{p-lim}_{n\to\infty} \sum_{i=1}^{n} r_{t_i,\Delta_i}^2, \tag{2.3}$$

[1] Asset prices must be semimartingales under rather weak conditions (e.g., Back (1991)).

[2] A simple counting process, N, is of finite-activity provided $N_t < \infty$ for $t \geq 0$, almost surely. In this paper, we do not explore infinite-activity processes, although these models have been studied in the context of realized multi-power variation (e.g., Barndorff-Nielsen, Shephard & Winkel (2006) or Woerner (2004*a*, 2004*b*)).

for any sequence of partitions $0 = t_0 < t_1 < \ldots < t_n = t$ such that $\max_{1 \leq i \leq n} \{\Delta_i\} \to 0$ as $n \to \infty$ (e.g., Protter (2004)). In our setting, $\langle p \rangle_t$ reduces to:

$$\langle p \rangle_t = \int_0^t \sigma_u^2 \mathrm{d}u + \sum_{i=1}^{N_t} J_i^2, \tag{2.4}$$

i.e. the integrated variance plus the sum of squared jumps. We are interested in analyzing $\langle p \rangle_t$, its distinct components, and testing the hypothesis $H_0 : p \in BSM$ against $H_a : p \notin BSM$. The latter will be done by examining whether $\langle p \rangle_t = \int_0^t \sigma_u^2 \mathrm{d}u$.

2.1. Return- and Range-Based Notation

$\langle p \rangle_t$ is latent and must be backed out from discrete high-frequency data. To do this, we form statistics of the observed part of the sample path of p. The basis is an equidistant grid $t_i = i/n$, $i = 0, \ldots, [nt]$, where n is the sampling frequency and $[x]$ denotes the integer part of x. We then construct equidistant returns:

$$r_{i\Delta,\Delta} = p_{i/n} - p_{(i-1)/n}, \tag{2.5}$$

for $i = 1, \ldots, [nt]$.[3] We also assume that each interval $[(i-1)/n, i/n]$ contains $m+1$ recordings of p at time points $t_{(i-1)/n + j/mn}$, $j = 0, 1, \ldots, m$ and define intraday ranges:

$$s_{p_{i\Delta,\Delta,m}} = \max_{(i-1)/n \leq s,t \leq i/n} \{p_t - p_s\}, \tag{2.6}$$

for $i = 1, \ldots, [nt]$. Below, we also need the range of a standard Brownian motion, which is denoted by $s_{W_{i\Delta,\Delta,m}}$, simply replacing p with W in Equation (**2.6**).

Note that m can be any natural number, or possibly infinity. In the latter setting, we suppress the dependence on m and write

$$s_{p_{i\Delta,\Delta}} = \sup_{(i-1)/n \leq s,t \leq i/n} \{p_t - p_s\}, \tag{2.7}$$

[3]In practice, high-frequency data are irregularly spaced and equidistant prices are imputed from the observed ones. Two approaches are linear interpolation (e.g., Andersen & Bollerslev (1997)) or the previous-tick method suggested by Wasserfallen & Zimmermann (1985). The former has an unfortunate property in connection with quadratic variation, see Hansen & Lunde (2006, Lemma 1).

with the same convention for $s_{W_{i\Delta,\Delta}}$.[4] Finally, m is allowed to be a function of n, but this dependence is also dropped for notational ease.

2.2. Realized Variance and Bi-Power Variation

The availability of high-frequency data in financial economics has inspired the development of a powerful toolkit for measuring the variation of asset price processes. Under the heading realized multi-power variation, this framework builds on powers of absolute returns over non-overlapping intervals (e.g., BN-S (2007)).

More formally, given $\{r_{i\Delta,\Delta}\}_{i=1,\ldots,[nt]}$, we define realized multi-power variation by setting:

$$MPV^n_{(r_1,\ldots,r_k),t} = n^{r_+/2-1} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} \frac{1}{\mu_{r_j}} |r_{(i+j-1)\Delta,\Delta}|^{r_j}, \tag{2.8}$$

with $k \in \mathbb{N}$, $r_j \geq 0$ for all j, $r_+ = \sum_{j=1}^{k} r_j$, $\mu_{r_j} = \mathbb{E}\left(|\,\phi\,|^{r_j}\right)$, and $\phi \sim N(0,1)$.[5]

Equation **(2.8)** boils down to many econometric estimators for suitable choices of k and the r_k's. The most popular is realized variance ($k = 1$ and $r_1 = 2$):

$$RV^n_t = \sum_{i=1}^{[nt]} r^2_{i\Delta,\Delta}, \tag{2.9}$$

RV^n_t is the sum of squared returns and, by definition, consistent for $\langle\, p\,\rangle_t$ of all semimartingales as $n \to \infty$. Hence, it follows from Equation **(2.4)** that:

$$RV^n_t \xrightarrow{p} \int_0^t \sigma^2_u \mathrm{d}u + \sum_{i=1}^{N_t} J^2_i. \tag{2.10}$$

RV^n_t measures the total variation induced by the diffusive and jump component. BN-S (2004) introduced (realized) bi-power variation that can be used to separate these parts. The estimator was extended in Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) to weaker

[4] We assume that $[0, t]$ is divided into $[nt]$ equidistant subintervals $[(i-1)/n, i/n]$, $i = 1, \ldots, [nt]$, for simplicity. The asymptotic results for irregular intervals $[t_{i-1}, t_i]$ is derived in a similar manner (e.g., Christensen & Podolskij (2005)). Moreover, under some balance conditions, we can allow for varying number of points and positions in the irregular intervals.

[5] In the empirical application, a small sample correction $n/(n-k+1)$ is applied to the realized return- and range-based estimators. We omit it in this section to ease notation.

conditions. The (first-order) bi-power variation is defined as ($k = 2$, $r_k = 1$):

$$BV_t^n = \frac{1}{\mu_1^2} \sum_{i=1}^{[nt]-1} |r_{i\Delta,\Delta}||r_{(i+1)\Delta,\Delta}|. \tag{2.11}$$

Then, it holds that:

$$BV_t^n \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u. \tag{2.12}$$

Intuitively, as N is bounded, the probability of having jumps in consecutive returns goes to zero as $n \to \infty$. Thus, for n sufficiently large, all returns with a jump are paired with continuous returns. The latter converges in probability to zero, so the limit is unaffected by the product.

2.2.1. A Return-Based Theory for Jump Detection

BN-S (2004) coupled the stochastic convergence in (**2.12**) with a central limit theorem (CLT) for (RV_t^n, BV_t^n), computed under the null of a continuous sample path:

$$\sqrt{n} \begin{pmatrix} RV_t^n - \int_0^t \sigma_u^2 \mathrm{d}u \\ BV_t^n - \int_0^t \sigma_u^2 \mathrm{d}u \end{pmatrix} \xrightarrow{d} MN\left(\mathbf{0}, \int_0^t \sigma_u^4 \mathrm{d}u \begin{bmatrix} 2 & 2 \\ 2 & 2+\nu_1 \end{bmatrix}\right), \tag{2.13}$$

where $\nu_1 = \left(\pi^2/4\right) + \pi - 5 \simeq 0.6091$ and $\int_0^t \sigma_u^4 \mathrm{d}u$ is the *integrated quarticity.* Note that RV_t^n is more efficient than BV_t^n. Applying the delta-method to the joint asymptotic distribution of (RV_t^n, BV_t^n), we can construct a non-parametric test of H_0 as:

$$\frac{\sqrt{n}\left(RV_t^n - BV_t^n\right)}{\sqrt{\nu_1 \int_0^t \sigma_u^4 \mathrm{d}u}} \xrightarrow{d} N(0,1). \tag{2.14}$$

The CLT in (**2.14**) is infeasible, because it depends on $\int_0^t \sigma_u^4 \mathrm{d}u$. To implement a feasible test, we must replace $\int_0^t \sigma_u^4 \mathrm{d}u$ with a consistent estimator that is computed directly from the data. To avoid eroding the power of the test, it is preferable to use an estimator that is robust under H_a. A natural statistic is *quad-power quarticity* ($k = 4$; $r_k = 1$):

$$QQ_t^n = \frac{1}{\mu_1^4} \sum_{i=1}^{[nt]-3} |r_{i\Delta,\Delta}||r_{(i+1)\Delta,\Delta}||r_{(i+2)\Delta,\Delta}||r_{(i+3)\Delta,\Delta}|. \tag{2.15}$$

Now, it holds both under H_0 and H_a that $QQ_t^n \xrightarrow{p} \int_0^t \sigma_u^4 \mathrm{d}u$ as $n \to \infty$. Hence, this allows us to construct a feasible t-statistic:

$$z_{RV_t^n, BV_t^n, QQ_t^n} = \frac{\sqrt{n}\left(RV_t^n - BV_t^n\right)}{\sqrt{\nu_1 QQ_t^n}} \xrightarrow{d} N(0,1). \tag{2.16}$$

The linear t-statistic in Equation (**2.16**) can be interpreted as a Hausman (1978) test. Under H_a, $RV_t^n - BV_t^n \xrightarrow{p} \sum_{i=1,\ldots,N_t} J_i^2 \geq 0$, so the test is one-sided and positive outcomes go against H_0.[6] Thus, we reject H_0 if $z_{RV_t^n, BV_t^n, QQ_t^n}$ exceeds some prescribed quantile, z_α, in the right-hand tail of the $N(0,1)$ (α is the significance level). Simulation studies in Huang & Tauchen (2005) and BN-S (2006), however, show that (**2.16**) is a poor description of the actual coverage probabilities for sampling frequencies relevant to empirical work. BN-S (2006) suggested a ratio-statistic to improve the asymptotic approximation:

$$z^r_{RV_t^n, BV_t^n, QQ_t^n} = \frac{\sqrt{n}\left(1 - BV_t^n / RV_t^n\right)}{\sqrt{\nu_1 QQ_t^n / \left(BV_t^n\right)^2}} \xrightarrow{d} N(0,1). \qquad \textbf{(2.17)}$$

BN-S (2004) noted that by Jensen's inequality:

$$\frac{QQ_t^n}{\left(BV_t^n\right)^2} \xrightarrow{p} \frac{\displaystyle\int_0^t \sigma_u^4 \mathrm{d}u}{\left(\displaystyle\int_0^t \sigma_u^2 \mathrm{d}u\right)^2} \geq 1/t, \qquad \textbf{(2.18)}$$

with equality in the homoscedastic setting, $\sigma_t = \sigma$. As nothing prevents $QQ_t^n / \left(BV_t^n\right)^2 < 1/t$ for finite n, it turns out to be better to construct a modified ratio-statistic:

$$z^{ar}_{RV_t^n, BV_t^n, QQ_t^n} = \frac{\sqrt{n}\left(1 - BV_t^n / RV_t^n\right)}{\sqrt{\nu_1 \max\left\{QQ_t^n / \left(BV_t^n\right)^2, 1/t\right\}}} \xrightarrow{d} N(0,1). \qquad \textbf{(2.19)}$$

2.3. Realized Range-Based Variance and Bi-Power Variation

The starting point of range-based estimation of volatility was the seminal paper by Feller (1951), where the distribution of the range was derived. Parkinson (1980) transformed Feller's result into an estimator of a constant diffusion coefficient, $\sigma_t = \sigma$, and the theoretical foundations have then been lifted in a series of papers (e.g., Rogers & Satchell (1991), Alizadeh, Brandt & Diebold (2002), or Christensen & Podolskij (2005)). In particular, Christensen & Podolskij (2005) generalized the theory to work for basically all Brownian semimartingales and proposed *realized range-based variance* as a non-parametric equivalent to RV_t^n:

$$RRV_{b,t}^{n,m} = \frac{1}{\lambda_{2,m}} \sum_{i=1}^{[nt]} s^2_{p_{i\Delta},\Delta,m}, \qquad \textbf{(2.20)}$$

[6] Recently, Jiang & Oomen (2005) proposed a two-sided swap-variance test that exploits information in the higher-order moments of asset returns.

where

$$\lambda_{r,m} = \mathbb{E}\left(s_{W,m}^{r}\right), \tag{2.21}$$

and

$$s_{W,m} = \max_{0 \leq s,t \leq m} \left\{W_{t/m} - W_{s/m}\right\}, \tag{2.22}$$

is the range of a standard Brownian motion measured from m increments over a unit interval. The $\lambda_{r,m}$ scalars remove the downward bias reported in the range-based literature (e.g, Garman & Klass (1980) or Rogers & Satchell (1991)). To our knowledge, there is no closed-form solution for $\lambda_{r,m}$, so we must resort to numerical estimates.

Moreover, $\lambda_{r,m}$ is not necessarily finite for arbitrary choices of $r \in \mathbb{R}$ and $m \in \mathbb{N} \cup \{\infty\}$. The next lemma presents a sufficient condition to ensure this.

Lemma 1 *With $r > -m$, it holds that*

$$\lambda_{r,m} < \infty. \tag{2.23}$$

This result has some interesting implications that we discuss further below. Note, in particular, that $\lambda_r \equiv \lambda_{r,\infty}$ is finite for all $r \in \mathbb{R}$.

First, we review the asymptotic results developed for $RRV_{b,t}^{n,m}$ and extend these in a number of ways. To prove a CLT, Christensen & Podolskij (2005) imposed some regularity conditions on the process σ:

(V) σ is everywhere invertible (V_1) and satisfies:

$$\sigma_t = \sigma_0 + \int_0^t \mu'_u \mathrm{d}u + \int_0^t \sigma'_{u-} \mathrm{d}W_u + \int_0^t v'_{u-} \mathrm{d}B'_u, \tag{$\mathbf{V_2}$}$$

where $\mu' = (\mu'_t)_{t \geq 0}$, $\sigma' = (\sigma'_t)_{t \geq 0}$, $v' = (v'_t)_{t \geq 0}$ are adapted càdlàg processes with μ' also predictable and locally bounded, and $B' = (B'_t)_{t \geq 0}$ is a Brownian motion independent of W.

Assumption V_1 was a technical condition required in the proofs, but it is satisfied for almost all Brownian semimartingales. V_2 is sufficient, but probably not necessary, and could be weakened to include a jump process. As above, we can also restrict the functions μ', σ', v' and σ^{-1} to be bounded by a constant (see, e.g., Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006)).

The next proposition summarizes the properties of $RRV_{b,t}^{n,m}$ and is reproduced from Christensen & Podolskij (2005).

Proposition 1 *Assume that $p \in BSM$. Then, as $n \to \infty$*

$$RRV_{b,t}^{n,m} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u, \tag{2.24}$$

*where the convergence holds locally uniform in t and uniformly in m. Moreover, if condition **(V)** holds and $m \to c \in \mathbb{N} \cup \{\infty\}$:*

$$\sqrt{n}\left(RRV_{b,t}^{n,m} - \int_0^t \sigma_u^2 \mathrm{d}u\right) \xrightarrow{d} MN\left(0, \Lambda_c^R \int_0^t \sigma_u^4 \mathrm{d}u\right), \tag{2.25}$$

where $\Lambda_c^R = \left(\lambda_{4,c} - \lambda_{2,c}^2\right)/\lambda_{2,c}^2$.

The scalar c in the asymptotic variance of $RRV_{b,t}^{n,m}$ determines the efficiency compared to RV_t^n. If $m \to 1$ as $n \to \infty$, $\Lambda_m^R \to 2$ because $RRV_{b,t}^{n,1} = RV_t^n$. If $m \to \infty$ as $n \to \infty$, $\Lambda_m^R \to 0.4073$ (approximately), so $RRV_{b,t}^{n,m}$ is up to five times more accurate than RV_t^n, which is a direct extension of Parkinson (1980).

Maintaining $p \in BSM$, a consistent estimator of $\int_0^t \sigma_u^4 \mathrm{d}u$ is given by the *realized range-based quarticity*:

$$RRQ_t^{n,m} = \frac{n}{\lambda_{4,m}} \sum_{i=1}^{[nt]} s_{p_{i\Delta},\Delta,m}^4 \xrightarrow{p} \int_0^t \sigma_u^4 \mathrm{d}u, \tag{2.26}$$

so,

$$\frac{\sqrt{n}\left(RRV_{b,t}^{n,m} - \int_0^t \sigma_u^2 \mathrm{d}u\right)}{\sqrt{\Lambda_m^R RRQ_t^{n,m}}} \xrightarrow{d} N(0,1). \tag{2.27}$$

With this result, we can construct confidence intervals for $\int_0^t \sigma_u^2 \mathrm{d}u$. It will be clear, however, that neither $RRV_{t,b}^{n,m}$ nor $RRQ_t^{n,m}$ are appropriate choices, if p exhibits discontinuities.

2.3.1. How About a Jump-Diffusion Model

A drawback of the above analysis is that the jump component of Equation (**2.1**) is excluded. To the best of our knowledge, there is no theory for estimating quadratic variation of jump-diffusion processes with the high-low. This raises the question of whether the convergence in probability extends to that situation. The answer, unfortunately, is negative. In fact, $RRV_{b,t}^{n,m}$ is downward biased if $N \neq 0$ (and $m \neq 1$), as the subscript b indicates.

Theorem 1 *If p satisfies* **(2.1)**, *then as* $n \to \infty$*:*

$$RRV_{b,t}^{n,m} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u + \frac{1}{\lambda_{2,m}} \sum_{i=1}^{N_t} J_i^2, \tag{2.28}$$

where the convergence holds locally uniform in t and uniformly in m.

Theorem 1 shows that $RRV_{b,t}^{n,m}$ is inconsistent, apart from stochastic volatility models or $m = 1$. Nonetheless, the structure of the problem opens the route for a modified intraday high-low statistic that is also consistent for the quadratic variation of the jump component.

Inspired by bi-power variation, we might exploit the corollary:

$$BV_t^n + \lambda_{2,m} \left(RRV_{b,t}^{n,m} - BV_t^n \right) \xrightarrow{p} \langle p \rangle_t . \tag{2.29}$$

This defies the nature of our approach, however, so we opt for other ways of correcting $RRV_{b,t}^{n,m}$. In particular, we introduce the idea of (realized) range-based bi-power variation.

Definition 1 *Range-based bi-power variation with parameter* $(r, s) \in \mathbb{R}_+^2$ *is defined as:*

$$RBV_{(r,s),t}^{n,m} = n^{(r+s)/2-1} \frac{1}{\lambda_{r,m}} \frac{1}{\lambda_{s,m}} \sum_{i=1}^{[nt]-1} s_{p_{i\Delta,\Delta,m}}^{r} s_{p_{(i+1)\Delta,\Delta,m}}^{s}, \tag{2.30}$$

Remark 1 *In the definition,* $(i+1)$ *may be replaced with* $(i+q)$*, for any finite positive integer q. Such "staggering" has been suggested for* BV_t^n *in Andersen, Bollerslev & Diebold (2006) and BN-S (2006). Moreover, Huang & Tauchen (2005) show that extra lagging can alleviate the impact of microstructure noise by breaking the serial correlation in returns.*

$RBV_{(r,s),t}^{n,m}$ is composed of range-based cross-terms raised to the powers (r, s) and constitutes a direct analogue BV_t^n. The parameter determines $n^{(r+s)/2-1}$, which is required to balance the order of the estimator and produce non-trivial limits, as confirmed in Theorem 2.

Theorem 2 *If* $p \in BSM$*, then as* $n \to \infty$

$$RBV_{(r,s),t}^{n,m} \xrightarrow{p} \int_0^t |\sigma_u|^{r+s} \mathrm{d}u, \tag{2.31}$$

where the convergence holds locally uniform in t and uniformly in m.

Corollary 1 *Set $r = 0$:*

$$RPV_{(s),t}^{n,m} \xrightarrow{p} \int_0^t |\sigma_u|^s \mathrm{d}u, \tag{2.32}$$

with the convention $RPV_{(s),t}^{n,m} \equiv RBV_{(s,0),t}^{n,m}$. This estimator is called realized range-based power variation with parameter $s \in \mathbb{R}_+$.

The theory implies that for $r \in (0, 2)$:

$$RBV_{(r,\,2-r),t}^{n,m} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u, \tag{2.33}$$

i.e. $RBV_{(r,\,2-r),t}^{n,m}$ provides an alternative way of drawing inference about $\int_0^t \sigma_u^2 \mathrm{d}u$. Moreover, $RBV_{(r,\,2-r),t}^{n,m}$ will also estimate $\int_0^t \sigma_u^2 \mathrm{d}u$ under H_a, as we show below.

In this paper, we will mainly focus on the first-order range-based bi-power variation, defined as $RBV_{(1,1),t}^{n,m} \equiv RBV_t^{n,m}$. Obviously:

$$RBV_t^{n,m} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u. \tag{2.34}$$

This subsection will be closed by introducing a new range-based estimator that is consistent for $\langle p \rangle_t$ of the jump-diffusion semimartingale in (**2.1**):

$$RRV_t^{n,m} \equiv \lambda_{2,m} RRV_{b,t}^{n,m} + (1 - \lambda_{2,m})\, RBV_t^{n,m} \xrightarrow{p} \int_0^t \sigma_u^2 \mathrm{d}u + \sum_{i=1}^{N_t} J_i^2, \tag{2.35}$$

i.e., we form a linear combination of the downward biased $RRV_{b,t}^{n,m}$ and $RBV_t^{n,m}$ using the weights $\lambda_{2,m}$ and $1 - \lambda_{2,m}$.

2.3.2. Asymptotic Distribution Theory

The consistency of $RBV_{(r,s),t}^{n,m}$ does not offer any information about the rate of convergence. In practice, market microstructure noise effectively puts a bound on n (e.g., at the 5-minute frequency) and it is therefore of interest to know more about the sampling errors. Therefore, we will now extend the convergence in probability of $RBV_{(r,s),t}^{n,m}$ to a CLT.[7]

[7] To prove the CLT, we exploit stable convergence. A sequence of random variables, $(X_n)_{n\in\mathbb{N}}$, *converges stably in law with limit* X, defined on an extension of $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, if and only if for every $\mathcal{F}$-measurable, bounded random variable Y and any bounded, continuous function g, the convergence $\lim_{n\to\infty} \mathbb{E}\left[Y g(X_n)\right] = \mathbb{E}\left[Y g(X)\right]$ holds. Throughout the paper, $X_n \xrightarrow{d_s} X$ denotes stable convergence. Note that it implies weak convergence by setting $Y = 1$ (see, e.g., Rényi (1963) or Aldous & Eagleson (1978) for more details).

Theorem 3 *Given $p \in BSM$ and **(V)** are satisfied, then as $n \to \infty$ and $m \to c \in \mathbb{N} \cup \{\infty\}$*

$$\sqrt{n}\left(RBV^{n,m}_{(r,s),t} - \int_0^t |\sigma_u|^{r+s} \mathrm{d}u \right) \xrightarrow{d_s} \sqrt{\Lambda_c^{B_{r,s}}} \int_0^t |\sigma_u|^{r+s} \mathrm{d}B_u, \tag{2.36}$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion defined on an extension of $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ and independent from the σ-field $\mathcal{F}$, and

$$\Lambda_c^{B_{r,s}} = \frac{\lambda_{2r,c}\lambda_{2s,c} + 2\lambda_{r,c}\lambda_{s,c}\lambda_{r+s,c} - 3\lambda^2_{r,c}\lambda^2_{s,c}}{\lambda^2_{r,c}\lambda^2_{s,c}}. \tag{2.37}$$

Remark 2 *Note that the rate of convergence is not influenced by m and no assumptions on the ratio n/m are required.*

Remark 3 *Suppose that $p_t = \int_0^t \sigma_u \mathrm{d}W_u$, where σ is independent of W and bounded away from zero. If we have slightly more data than the moment condition in Lemma 1 requires (e.g., $r, s > -m+1$), then Theorem 2 and 3 allows for negative values of (r, s). In principle, this means $RBV^{n,m}_{(r,s),t}$ can estimate integrals with negative powers of σ, e.g., $\int_0^t \sigma_u^{-2} \mathrm{d}u$. Unfortunately, it does not seem possible for general processes without extra assumptions. Nevertheless, it is an intriguing feature of the range, as BV^n_t cannot estimate such quantities.*

The critical feature of Theorem 3 is that B is independent of σ. This implies that the limit process in Equation **(2.36)** has a mixed normal distribution:

$$\sqrt{n}\left(RBV^{n,m}_{(r,s),t} - \int_0^t |\sigma_u|^{r+s} \mathrm{d}u \right) \xrightarrow{d} MN\left(0, \Lambda_c^{B_{r,s}} \int_0^t |\sigma_u|^{2(r+s)} \mathrm{d}u \right). \tag{2.38}$$

Again, the distribution theory in **(2.38)** is infeasible. This problem is resolved below, when we construct a non-parametric range-based test for jump detection.

[INSERT FIGURE 1 ABOUT HERE]

The variance factor of $RBV^{n,m}_t$, $\Lambda_m^{B_{1,1}} \equiv \Lambda_m^B$, is plotted in Figure 1 for all values of m that integer divide 23,400. As m increases, there is less sampling variation, because each element of $\left\{ s_{p_{i\Delta},\Delta,m} \right\}_{i=1,\ldots,[nt]}$ is based on more increments. An appealing feature of this graph is that Λ_m^B decreases fast initially, so a large part of the variance reduction is caused by a few increments. A striking result is that $\Lambda_m^B \to \left(\lambda_2^2 + 2\lambda_1^2\lambda_2 - 3\lambda_2^4\right)/\lambda_1^4 \simeq 0.3631$ as $m \to \infty$, which is lower than the asymptote of Λ_m^R of about 0.4073. The break-even point, defined such that $\Lambda_m^R = \Lambda_m^B$

(approximately), is a stunning low $m = 3$. Thus, $RBV_t^{n,m}$ is more efficient than $RRV_{b,t}^{n,m}$ for almost every m under H_0. This contradicts both the comparison of (RV_t^n, BV_t^n) and our intuition. Note that $\Lambda_1^B = 2.6098$ is (roughly) the constant appearing in the CLT of BV_t^n. Hence, $RBV_t^{n,m}$ is up to 7.2 times more efficient than BV_t^n (as $m \to \infty$).

2.3.3. A Range-Based Theory for Jump Detection

Now, we will extend the univariate convergence in Proposition 1 and Theorem 3 to cover the joint bivariate asymptotic distribution of $\left(RRV_{b,t}^{n,m}, RBV_t^{n,m}\right)$. This result is then exploited to propose a new non-parametric test of H_0.

Theorem 4 *If $p \in BSM$ and* **(V)** *holds, then as $n \to \infty$ and $m \to c \in \mathbb{N} \cup \{\infty\}$*

$$\sqrt{n} \begin{pmatrix} RRV_{b,t}^{n,m} - \int_0^t \sigma_u^2 \mathrm{d}u \\ RBV_t^{n,m} - \int_0^t \sigma_u^2 \mathrm{d}u \end{pmatrix} \xrightarrow{d} MN\left(\mathbf{0}, \int_0^t \sigma_u^4 \mathrm{d}u \begin{bmatrix} \Lambda_c^R & \Lambda_c^{RB} \\ \Lambda_c^{RB} & \Lambda_c^B \end{bmatrix}\right), \qquad \textbf{(2.39)}$$

with

$$\Lambda_c^{RB} = \frac{2\lambda_{3,c}\lambda_{1,c} - 2\lambda_{2,c}\lambda_{1,c}^2}{\lambda_{2,c}\lambda_{1,c}^2}. \qquad \textbf{(2.40)}$$

The proof of Theorem 4 is a simple extension of Equation (**2.25**) and (**2.38**), so we omit it. By the delta-method, it follows that under H_0 (note the subscripting):

$$\frac{\sqrt{n}\left(RRV_t^{n,m} - RBV_t^{n,m}\right)}{\sqrt{\nu_m \int_0^t \sigma_u^4 \mathrm{d}u}} \xrightarrow{d} N(0,1), \qquad \textbf{(2.41)}$$

where $\nu_m = \lambda_{2,m}^2 \left(\Lambda_m^R + \Lambda_m^B - 2\Lambda_m^{RB}\right)$.

How do these results change, as we move from H_0 towards the jump-diffusion semimartingale of Equation (**2.1**)? Of course, $RRV_t^{n,m}$ picks up both the diffusive and jump component, but the numerator of (**2.41**) could also change with $RBV_t^{n,m}$. We also mentioned above that $RRQ_t^{n,m}$ is not a suitable estimator of $\int_0^t \sigma_u^4 \mathrm{d}u$. More precisely, it is not consistent under H_a. The next result clarifies these statements and is similar to Theorem 5 in BN-S (2004).

Theorem 5 *If p satisfies* **(2.1)**, *then:*

$$RBV^{n,m}_{(r,s),t} \xrightarrow{p} \begin{cases} \int_0^t |\sigma_u|^{r+s} \mathrm{d}u, & \max(r,s) < 2, \\ X_t^*, & \max(r,s) = 2, \\ \infty, & \max(r,s) > 2, \end{cases} \tag{2.42}$$

where X_t^* is some stochastic process.

The proof of Theorem 5 follows the logic of BN-S (2004) and is omitted. Thus, $RRQ^{n,m}_t \xrightarrow{p} \infty$ as $n \to \infty$ under H_a and $RBV^{n,m}_{(r,s),t}$ cannot estimate $\int_0^t \sigma_u^4 \mathrm{d}u$, as $\max(r,s) < 2$ restricts $r+s < 4$. It is straightforward, however, to define range-based multi-power variation analogous to Equation **(2.8)**. Provided $\max(r_1, \ldots, r_k) < 2$, such estimators are robust to the jump component and capable of estimating higher-order integrated power variation. These generalizations will be discussed elsewhere. For now, we introduce *range-based quad-power quarticity*:

$$RQQ^{n,m}_t = \frac{n}{\lambda^4_{1,m}} \sum_{i=1}^{[nt]-3} s_{p_{i\Delta,\Delta,m}} s_{p_{(i+1)\Delta,\Delta,m}} s_{p_{(i+2)\Delta,\Delta,m}} s_{p_{(i+3)\Delta,\Delta,m}}. \tag{2.43}$$

Now, both under H_0 and H_a: $RQQ^{n,m}_t \xrightarrow{p} \int_0^t \sigma_u^4 \mathrm{d}u$, so

$$z_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t} = \frac{\sqrt{n}\left(RRV^{n,m}_t - RBV^{n,m}_t\right)}{\sqrt{\nu_m RQQ^{n,m}_t}} \xrightarrow{d} N(0,1). \tag{2.44}$$

We expect that a transformation of this convergence also improves the coverage probabilities of the range-based t-statistic. As above, we will work with an adjusted ratio-statistic:

$$z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t} = \frac{\sqrt{n}\left(1 - RBV^{n,m}_t / RRV^{n,m}_t\right)}{\sqrt{\nu_m \max\left\{RQQ^{n,m}_t / \left(RBV^{n,m}_t\right)^2, 1/t\right\}}} \xrightarrow{d} N(0,1). \tag{2.45}$$

3. Monte Carlo Simulation

In this section, a Monte Carlo experiment is conducted to inspect the small sample properties of the asymptotic results. We untangle the two parts of $\langle p \rangle_t$ with $RBV^{n,m}_t$ and evaluate $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ for jump detection. At the end, we also compare our test with $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$. The simulated Brownian semimartingale is:

$$\begin{aligned} \mathrm{d}p_u &= \sigma_u \mathrm{d}W_u \\ \mathrm{d}\ln\sigma_u^2 &= \theta(\omega - \ln\sigma_u^2)\mathrm{d}u + \eta \mathrm{d}B_u, \end{aligned} \tag{3.1}$$

where W and B are independent standard Brownian motions. According to this model, instantaneous log-variance evolves as a mean reverting Orstein-Uhlenbeck process with parameter vector (θ, ω, η). The values $(\theta, \omega, \eta) = (0.032, -0.631, 0.374)$ are collected from an empirical study by Andersen, Benzoni & Lund (2002).

To produce a discontinuous sample path for p, we follow BN-S (2006) and allocate j jumps uniformly in each unit of time, $j = 1, 2$. Hence, the reported power is the conditional probability of rejecting the null, given j jumps. We generate jump sizes by drawing independent $N(0, \sigma_J^2)$ variates and consider the values $\sigma_J^2 = 0.05, 0.10, \dots, 0.25$ to uncover the impact on power.

The remaining settings are: $T = 100,000$ replications of (**3.1**) are made for all σ_J^2, with the proportion of trading each day amounting to 6.5 hours, or 23,400 seconds. This choice reflects the number of opening hours at NYSE, from which our empirical data are collected. We set $p_0 = 0$, $\ln \sigma_0^2 = \omega$ and generate a realization of (**3.1**) such that a new observation of p is recorded every 20th second ($mn = 1170$). Again, this is calibrated to match our real data. $RRV_t^{n,m}$, $RBV_t^{n,m}$, and $RQQ_t^{n,m}$ are then computed for $n = 39$, 78, and 390 ($m = 30$, 15, and 3), corresponding to 10-, 5-, and 1-minute sampling.

3.1. Simulation Results

In the upper row of Figure 2, we print $RBV_t^{n,m}$ and the integrated variance for 200 iterations of the model with $j = 1$ and $\sigma_J^2 = 0.10$. The second row shows $(RRV_t^{n,m} - RBV_t^{n,m})^+$ against the squared jump, where $(x)^+ = \max(0, x)$. The maximum correction applied to $RRV_t^{n,m} - RBV_t^{n,m}$ was suggested by BN-S (2004) in the context of realized variance, and as:

$$RRV_t^{n,m} - RBV_t^{n,m} \xrightarrow{p} \sum_{i=1}^{N_t} J_i^2 \geq 0, \qquad \textbf{(3.2)}$$

we also expect a better finite sample behavior here, although the modified estimator has the disadvantage of being biased.

[INSERT FIGURE 2 ABOUT HERE]

As n increases, both statistics converge to their population counterparts. At $n = 78$, they are usually quite accurate, although $RBV_t^{n,m}$ has a larger RMSE relative to $(RRV_t^{n,m} - RBV_t^{n,m})^+$.

According to the CLT, the conditional variance of $RBV_t^{n,m}$ is $\Lambda_m^B \int_0^t \sigma_u^4 \mathrm{d}u$ for Brownian semi-martingales. There is some indication that the errors bounds of $RBV_t^{n,m}$ increase with σ - most pronounced at $n = 390$ - but, of course, in our setting jumps are interacting.

[INSERT FIGURE 3 ABOUT HERE]

The distribution of $z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}}$ is plotted in Figure 3 with $\sigma_J^2 = 0.00$. From Equation (**2.44**), the t-statistic then converges to the $N(0,1)$ as $n \to \infty$, which the kernel-based densities confirm. The approximation is not impressive for moderate n, but the focal point is the right-hand tail, where the rejection region is located. Testing at a nominal level of $\alpha = 0.01$ with critical value $z_\alpha = 2.326$, for example, yields actual sizes of 2.236, 1.856 and 1.292 percent, respectively. At $\alpha = 0.05$ - or $z_\alpha = 1.645$ - the type I errors are 6.497, 6.071 and 5.504 percent, in both situations leading to a modest over-rejection. This finding is consistent with the Monte Carlo studies on $z^{ar}_{RV_t^n,BV_t^n,QQ_t^n}$ in Huang & Tauchen (2005) and BN-S (2006).

[INSERT TABLE 1 ABOUT HERE]

The bottom part of Table 1 examines the power of $z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}}$ across the jump size domain $\sigma_J^2 = 0.00, 0.05, \ldots, 0.25$ with $j = 1$ and $j = 2$. The numbers reflect the proportion of t-statistics that exceeded $z_\alpha = 2.326$ (i.e. no size-correction).

There is a substantial type II error for $j = 1$ and small σ_J^2, but it diminishes as we depart from the null distribution. At $\sigma_J^2 = 0.10$, the rejection rates are 0.234, 0.321 and 0.490 for $n = 39$, 78, and 390. The power improves more quickly for $j = 2$, reflecting the increase in the jump variation. Consistent with BN-S (2006), we also find that power is roughly equal for $\sigma_J^2 = x$ and $j = 1$ compared to $\sigma_J^2 = x/2$ and $j = 2$, showing that the main constituent affecting the properties of $z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}}$ under the alternative is the variance of the jump process: $j\sigma_J^2$. As an example, consider $j = 2$ and $\sigma_J^2 = 0.05$; here the fraction of t-statistics above $z_\alpha = 2.326$ is 0.218, 0.340 and 0.585.

Note, however, that the relationship is much weaker at $n = 390$. Across simulations, there is a pronounced pattern that - keeping $j\sigma_J^2$ fixed - the t-statistic tends to prefer a higher value of σ_J^2 at the expense of j for low n, while the opposite holds for large n. Intuitively, at higher sampling frequencies two small breaks in p appear more abrupt, while they are drowned by the variation of the continuous part for infrequent sampling.

As the simulation is designed, m is greater than 1. Hence, the range-based t-statistic ought to be more powerful than the return-based version. We construct RV_t^n, BV_t^n, and QQ_t^n and report $z^{ar}_{RV_t^n,BV_t^n,QQ_t^n}$ in the right-hand side of Table 1. In general, the range-based t-statistic is superior and, in particular, has a much higher probability of detecting small jumps at lower sampling frequencies. Interestingly, though, $z^{ar}_{RV_t^n,BV_t^n,QQ_t^n}$ has slightly better size properties than $z^{ar}_{RRV_t^{n,m},RBV_t^{n,m},RQQ_t^{n,m}}$.

4. Empirical Application

We illustrate some features of the theory for a component of the Dow Jones Industrial Average as of April 8, 2004. Our exposition is based on Merck (MRK).

High-frequency data for Merck was extracted from the Trade and Quote (TAQ) database for the sample period January 3, 2000 to December 31, 2004, a total of 1,253 trading days. We restrict attention to midquote data from NYSE.[8] The raw data were filtered for outliers and we discarded updates outside the trading session from 9:30AM EST to 4:00PM EST.

[INSERT TABLE 2 ABOUT HERE]

Table 2 reports the amount of tick data. We exclude zero returns - $r_{\tau_i} = 0$ - and non-zero returns that result from reversals - $r_{\tau_i} \neq 0$ but $\Delta r_{\tau_i} = 0$ - where $r_{\tau_i} = p_{\tau_i} - p_{\tau_{i-1}}$ and τ_i is the arrival time of the ith tick, from consideration to compute mn. There is a lot of empirical support for adopting this convention, because counting such returns induce an upward bias in mn - due to price repetitions and bid-ask bounce - thus a downward bias in the range-based estimates. Hence, for a price change to affect mn, we require both that $r_{\tau_i} \neq 0$ and $\Delta r_{\tau_i} \neq 0$. On average, this reduces the mn numbers by one-third (one-half) for the quote (trade) data relative to using $r_{\tau_i} \neq 0$.

To account for the irregular spacing of high-frequency data, we exploit *tick-time sampling* (e.g., Hansen & Lunde (2006)). This procedure sets the sampling times t_i, $i = 1, \ldots, [nt]$, at

[8] The analysis based on transaction data is available at request.

every mth new quotation and, apart from end effects, has the advantage of fixing the number of returns in each interval $[t_{i-1}, t_i]$. Of course, tick-time sampled grids are irregular in calendar-time, but this is not a problem provided that we also use a tick-time estimator of the conditional variance. We set $m = 15$, such that on average for our sample period $n = 78$ is used, which corresponds to 5-minute sampling.

[INSERT TABLE 3 ABOUT HERE]

We progress by constructing a time series of $RRV_t^{n,m}$, $RBV_t^{n,m}$, $RQQ_t^{n,m}$, RV_t^n, BV_t^n and QQ_t^n (indexed by $t = 1, \ldots, T = 1,253$ in this section). Table 3 reports some descriptive statistics of the series. The variances of $RRV_t^{n,m}$ and $RBV_t^{n,m}$ are smaller than those of RV_t^n and BV_t^n. The reduction is most pronounced for $RBV_t^{n,m}$, although its mean is also somewhat smaller than that of BV_t^n. There is a high positive correlation between the pairs $(RRV_t^{n,m}, RV_t^n)$, $(RBV_t^{n,m}, BV_t^n)$ and $(RQQ_t^{n,m}, QQ_t^n)$, reflecting that they are estimating the same part of p. Note also the large differences in the mean and variance of $RQQ_t^{n,m}$ and QQ_t^n. As seen from the table, the maximum of QQ_t^n is more than twice that of $RQQ_t^{n,m}$.

[INSERT FIGURE 4 ABOUT HERE]

Figure 4 plots $RRV_t^{n,m}$ ($RBV_t^{n,m}$) measured against the left (right) y-axis. Both series are reported in annualized standard deviation form. The correlation coefficient of the two series is a high 0.901. Moreover, they exhibit a strong serial dependence, reflecting the volatility clustering in the data. The first five autocorrelations of $RRV_t^{n,m}$ are 0.523, 0.461, 0.383, 0.361, 0.383, and 0.722, 0.644, 0.564, 0.539, 0.563 for $RBV_t^{n,m}$. Intuitively, $RBV_t^{n,m}$ is more persistent than $RRV_t^{n,m}$, because it is robust against the (less persistent) jump component. The most important feature of this graph is that some of the spikes appearing in $RRV_t^{n,m}$ are not matched by $RBV_t^{n,m}$. On these days, the estimators associate a large proportion of $\langle p \rangle_t$ to the jump part, which we now review in more detail.

[INSERT FIGURE 5 ABOUT HERE]

$z^{ar}_{RRV_t^{n,m}, RBV_t^{n,m}, RQQ_t^{n,m}}$ and $z^{ar}_{RV_t^n, BV_t^n, QQ_t^n}$ are plotted in Figure 5. We truncate the y-axis at 0, as negative outcomes of the t-statistics are never against the null. The horizontal line represents a cut-off value of $z_\alpha = 2.326$, i.e. the 0.99 quantile from a standard normal distribution. There is a marked difference between these series; $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ identifies few discontinuities and $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$ extremely many.

[INSERT TABLE 4 ABOUT HERE]

Table 4 underscores this by reporting the number of rejections at a 5% and 1% significance level. It also shows three measures of the fraction of $\langle p \rangle_t$ explained by jumps. At the 5% level, $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ rejects H_0 141 times, while $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$ does so 289 times out of 1,253; a two-fold increase. If there were no jumps at all in the data, we would expect the two t-statistics to reject 76 and 68 times, respectively, based on the actual size of the tests reported in the simulation section. At the 1% level, the numbers are down to 48 and 151 rejections, as opposed to the expected 23 and 16 under H_0.

Nonetheless, $RRV^{n,m}_t - RBV^{n,m}_t$ induces a higher proportion of $\langle p \rangle_t$ - 16.9% - when all positive, also insignificant, jump terms are counted. This is because the difference in means is somewhat higher than for $RV^n_t - BV^n_t$, which explains 10.9% of $\langle p \rangle_t$. Accounting for sampling uncertainty, however, the numbers are aligned, as the range-based t-statistic regards many of the small jump contributions as insignificant. At the 1%-level, $RRV^{n,m}_t - RBV^{n,m}_t$ ($RV^n_t - BV^n_t$) accounts for 5.6% (6.1%) of $\langle p \rangle_t$. The difference is supposed to be caused by many very small jumps that $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$ picks up, but $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ fails to detect. This explanation, however, is contradicted by the simulations that revealed $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ has higher power to unearth these.

4.1. A Robustness Property

It is an open question, how much of $\langle p \rangle_t$ that is induced by the jump process. To conclude our empirical application, we therefore offer an explanation of the differences noted above by drawing the data for Thursday, August 24, 2000 in Figure 6.

This trading day exhibits some sudden shifts in p, but there is no conspicuous jump. In the morning there are several downticks, but the price then trends upwards for the rest of the day. It appears quiet and still $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t} = 3.688$, which is a huge rejection. In contrast,

$z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$ equals 0.560 and the t-statistics thus lead to opposite conclusions about the sample path.

We studied Merck's price fluctuations on many of the days in our sample, where $z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$ exceeds the 1% size-corrected significance level. Out of those, several contains "run-ups" in the price, a feature that is quite relevant in empirical analysis. Almost none of these days were rejected by $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$, pointing towards a higher degree of robustness in these cases.

There are (at least) two explanations for this. First, $s_{p_{i\Delta,\Delta},m}$ and $\lambda_{r,m}$ are functions of m - the number of returns. A streak of negative or positive returns inflates m and $s^r_{p_{i\Delta,\Delta},m}$ is scaled harder by $\lambda_{r,m}$, for all r; implicitly recognizing that a big one-sided move was composed of many small returns. A low m causes a softer $\lambda_{r,m}$ scaling so that large ranges are more likely to be associated with a jump. Of course, the $\lambda_{r,m}$ scalars interact in complicated ways to compute the t-statistic, but the end product is indeed decreasing, such that m and $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ are negatively related (keeping $\{s_{p_{i\Delta,\Delta},m}\}_{i=1,\ldots,[nt]}$ fixed).

Second, the substitution of $\int_0^t \sigma^2_u \mathrm{d}u$ and $\int_0^t \sigma^4_u \mathrm{d}u$ relies on consistent estimators. Due to sampling variation, $BV^{n,m}_t$ and $QQ^{n,m}_t$ are likely - at the 5-minute frequency - to deviate somewhat from their population values, relative to $RBV^{n,m}_t$ and $RQQ^{n,m}_t$. For instance, notice from Table 3 that the variance of QQ^n_t is five times larger than $RQQ^{n,m}_t$. Indeed, the theoretical efficiency of the range is higher for estimating $\int_0^t \sigma^4_u \mathrm{d}u$; under the null, the asymptotic variance of QQ^n_t is more than 9.7 times bigger than $RQQ^{n,m}_t$ (as $m \to \infty$). Hence, QQ^n_t under- or overshoots by relatively much, and a too low QQ^n_t estimate can move the t-statistic into the rejection region, even on days where $1 - BV^n_t / RV^n_t$ is borderline.

To sum up, realized range-based estimation offers a powerful frame for drawing inference about the jump part, if any, of asset prices processes. The range-based t-statistic seems more robust against falsely signaling jumps from rapid, continuous shifts in the price. We also believe that empirical estimation of integrated quarticity plays a key role.

5. Conclusions and Directions for Future Research

This paper develops some tools for using high-frequency data to conduct range-based inference about the jump component of asset prices.

A Monte Carlo simulation indicates that separating the components of the quadratic variation with realized range-based statistics is quite efficient for sampling frequencies relevant to empirical work. Moreover, testing the null of a continuous sample path leads to t-statistics with significantly higher power under the alternative compared to realized variance, although the size is slightly worse off.

Our empirical application documents the potential efficiency gains that can be achieved by using range-based estimators to back out the quadratic variation. We also uncovered some interesting findings, which suggest that jumps are not as frequent as reported earlier in the high-frequency literature. In particular, the range-based t-statistic signals far fewer days with jumps than the return-based t-statistic. On this account, we offered two explanations, but further research is required to make a stronger conclusion.

The theory developed here casts new light on the properties of the price range, but there are still several problems left hanging for ongoing and future research. First, realized range-based variance was somewhat informally motivated by appealing to the sparse sampling of realized variance caused by market microstructure noise. It is not clear how such contaminations affect the range, and we are currently pursuing a paper on this topic. Second, there is nothing sacred about the high-low. Garman & Klass (1980), among others, construct estimators of a constant diffusion coefficient by combining the daily range and return. Their procedure extends to general semimartingales and intraday data, which suggests that further efficiency gains are waiting. Indeed, other (non-standard) functionals of the sample path could be more informative about the quadratic variation. Third, it will be interesting to connect non-parametric historical volatility measurements using the intraday high-low statistics with model-based forecasting. Fourth, it is also worth considering bootstrap methods to refine the asymptotic approximation, as suggested by Gonçalves & Meddahi (2005) in the context of realized variance.

A. **Appendix of Proofs**

A.1. **Proof of Lemma 1**

The assertion is trivial for $r \geq 0$ and is a consequence of Burkholder's inequality (e.g., Revuz & Yor (1998, pp. 160)). Now, assume $-m < r < 0$ and note that:

$$s_{W,m} = \max_{0 \leq s,t \leq m} \left\{ W_{t/m} - W_{s/m} \right\} \geq \max_{1 \leq i \leq m} \left\{ |W_{i/m} - W_{(i-1)/m}| \right\}$$
$$\stackrel{d}{=} \frac{1}{\sqrt{m}} \max_{1 \leq i \leq m} \{|\phi_i|\} \equiv M_\phi,$$

where ϕ_i, $i = 1, \ldots, m$, are IID standard normal random variables. Then, we have the inequality $\lambda_{r,m} \leq \mathbb{E}\big(M_\phi^r\big) < \infty$ (for $-m < r < 0$). ■

A.2. **Proof of Theorem 1**

This theorem is proved by decomposing $RRV_{b,t}^{n,m}$ into a continuous, jump and mixed part. We adopt the additional notation:

$$p_t^b = p_0 + \int_0^t \mu_u \mathrm{d}u + \int_0^t \sigma_u \mathrm{d}W_u, \qquad p_t^j = p_0 + \sum_{i=1}^{N_t} J_i^2.$$

Then, using the finite-activity property of N_t, it follows that:

$$\sum_{i=1}^{[nt]} s^2_{p^j_{i\Delta,\Delta},m} \xrightarrow{p} \sum_{i=1}^{N_t} J_i^2, \qquad \sum_{i=1}^{[nt]} s^2_{p^b_{i\Delta,\Delta},m} \xrightarrow{p} \lambda_{2,m} \int_0^t \sigma_u^2 \mathrm{d}u, \qquad 2 \sum_{i=1}^{[nt]} s_{p^b_{i\Delta,\Delta},m} s_{p^j_{i\Delta,\Delta},m} \xrightarrow{p} 0,$$

uniformly in m, where the second convergence is from Christensen & Podolskij (2005). ■

In the upcoming theorems, we shall first prove the case with $m = \infty$, and then extend these results to $m < \infty$. In the rest of the appendix, we also make the replacements:

$$g(x) = \frac{1}{\lambda_{r,m}} x^r, \qquad h(x) = \frac{1}{\lambda_{s,m}} x^s,$$

for $x \in \mathbb{R}_+$.

We also fix some notation before proceeding. For the processes X^n and X, we denote by $X^n \xrightarrow{p} X$, the uniform convergence:

$$\sup_{s \leq t} |X_s^n - X_s| \xrightarrow{p} 0,$$

for all $t > 0$. If X^n has the form:

$$X_t^n = \sum_{i=1}^{[nt]} \zeta_i^n,$$

for an array (ζ_i^n) and $X^n \xrightarrow{p} 0$, (ζ_i^n) is said to be *Asymptotically Negligible* (AN). The constants appearing below are denoted by C, or C_p if they depend on an external parameter p. To prove our asymptotic results, we require some technical preliminaries.

A.3. **Preliminaries I**

First, we define:

$$\beta_i^n = \sqrt{n} |\sigma_{\frac{i-1}{n}}| s_{W_{i\Delta,\Delta}}, \qquad \beta_i'^n = \sqrt{n} |\sigma_{\frac{i-1}{n}}| s_{W_{(i+1)\Delta,\Delta}}, \tag{A.1}$$

and

$$\rho_x(f) = \mathbb{E}\left[f\left(|x| s_W\right)\right],$$

where $s_W = \sup\limits_{0 \leq s,t \leq 1} \{W_t - W_s\}$ and f is a real-valued function. Note that

$$\rho_x(g) = |x|^r.$$

We consider an adapted càdlàg and bounded process ν, and the function $f(x) = x^p$, for $p > 0$. Then, we introduce

$$U_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \nu_{\frac{i-1}{n}} \Big(f\left(\beta_i^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(f) \Big), \tag{A.2}$$

$$U_t'^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \Big(g\left(\beta_i^n\right) h\left(\beta_i'^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(g) \, \rho_{\sigma_{\frac{i-1}{n}}}(h) \Big). \tag{A.3}$$

Next, we prove a central limit theorem.

Lemma 2 *If $p \in BSM$:*

$$U_t^n \xrightarrow{d_s} U_t = \sqrt{\lambda_{2p} - \lambda_p^2} \int_0^t \nu_u \sigma_u^p \mathrm{d}B_u, \tag{A.4}$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, defined on an extension of the filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$ and independent from the σ-field $\mathcal{F}$.

Lemma 3 *If $p \in BSM$:*

$$U_t'^n \xrightarrow{d_s} \sqrt{\frac{\lambda_{2r}\lambda_{2s} + 2\lambda_r\lambda_s\lambda_{r+s} - 3\lambda_r^2\lambda_s^2}{\lambda_r^2\lambda_s^2}} \int_0^t |\sigma_u|^{r+s} \mathrm{d}B_u.$$

We only prove Lemma 3, leaving Lemma 2 that is shown with similar techniques. But before doing so, we will mention an estimate that is obvious under $p \in BSM$:

$$\mathbb{E}\left[|\beta_i^n|^q\right] + \mathbb{E}\left[|\beta_i'^n|^q\right] \leq C_q, \tag{A.5}$$

for all $q > 0$. Lemma 2 and 3 also imply

$$\frac{1}{n}\sum_{i=1}^{[nt]} \nu_{\frac{i-1}{n}} f\left(\beta_i^n\right) \xrightarrow{p} \int_0^t \nu_u \rho_{\sigma_u}(f)\,\mathrm{d}u, \tag{A.6}$$

$$\frac{1}{n}\sum_{i=1}^{[nt]} g\left(\beta_i^n\right) h\left(\beta_i'^n\right) \xrightarrow{p} \int_0^t \rho_{\sigma_u}(g)\,\rho_{\sigma_u}(h)\,\mathrm{d}u. \tag{A.7}$$

Proof of Lemma 3

By simple computations, we obtain the decomposition

$$U_t'^n = \sum_{i=2}^{[nt]+1} \zeta_i^n + \gamma_1^n - \gamma_{[nt]+1}^n,$$

with

$$\zeta_i^n = \frac{1}{\sqrt{n}}\left(g\left(\beta_{i-1}^n\right)\left(h\left(\beta_{i-1}'^n\right) - \rho_{\sigma_{\frac{i-2}{n}}}(h)\right) + \left(g\left(\beta_i^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\right)\rho_{\sigma_{\frac{i-1}{n}}}(h)\right),$$

$$\gamma_i^n = \frac{1}{\sqrt{n}}\left(g\left(\beta_i^n\right) - \rho_{\sigma_{\frac{i-1}{n}}}(g)\right)\rho_{\sigma_{\frac{i-1}{n}}}(h).$$

Set:

$$\rho_{i-2,i-1}^n(g,h) = \int g\left(\sigma_{\frac{i-1}{n}}x\right) h\left(\sigma_{\frac{i-2}{n}}x\right)\delta(\mathrm{d}x),$$

where

$$\delta(x) = 8\sum_{j=1}^{\infty}(-1)^{j+1} j^2 \phi(jx),$$

is the density function of s_W (e.g., Feller (1951)). Note the identity

$$\mathbb{E}\left[|\zeta_i^n|^2 \mid \mathcal{F}_{\frac{i-1}{n}}\right] = \frac{1}{n}\Bigg(g\left(\beta_{i-1}^n\right)^2\left(\rho_{\sigma_{\frac{i-2}{n}}}\left(h^2\right) - \rho^2_{\sigma_{\frac{i-2}{n}}}(h)\right) + \rho^2_{\sigma_{\frac{i-1}{n}}}(h)\left(\rho_{\sigma_{\frac{i-1}{n}}}\left(g^2\right) - \rho^2_{\sigma_{\frac{i-1}{n}}}(g)\right)$$

$$+ 2g\left(\beta_{i-1}^n\right)\rho_{\sigma_{\frac{i-1}{n}}}(h)\left(\rho_{i-2,i-1}^n(g,h) - \rho_{\sigma_{\frac{i-2}{n}}}(h)\,\rho_{\sigma_{\frac{i-1}{n}}}(g)\right)\Bigg).$$

As

$$\sup_{i\leq[nt]+1}\left|\rho^n_{i-2,i-1}(gh)-\rho_{\sigma_{\frac{i-2}{n}}}(gh)\right|\xrightarrow{p}0,$$

it holds by (A.6) that

$$\sum_{i=2}^{[nt]+1}\mathbb{E}\left[|\zeta^n_i|^2\mid\mathcal{F}_{\frac{i-1}{n}}\right]\xrightarrow{p}\frac{\lambda_{2r}\lambda_{2s}+2\lambda_r\lambda_s\lambda_{r+s}-3\lambda_r^2\lambda_s^2}{\lambda_r^2\lambda_s^2}\int_0^t|\sigma_u|^{2(r+s)}\mathrm{d}u,$$

and

$$\sup_{i\leq[nt]}|\gamma^n_i|\xrightarrow{p}0,$$

for any t. Trivially,

$$\mathbb{E}\left[\zeta^n_i\mid\mathcal{F}_{\frac{i-1}{n}}\right]=0.$$

As $W\overset{d}{=}-W$, we also get

$$\mathbb{E}\left[\zeta^n_i\left(W_{\frac{i}{n}}-W_{\frac{i-1}{n}}\right)\mid\mathcal{F}_{\frac{i-1}{n}}\right]=0.$$

Next, assume N is a bounded martingale on $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq0},\mathbb{P}\right)$ and orthogonal to W (i.e. with quadratic covariation $\langle W,N\rangle_t=0$, almost surely). As $g(\beta^n_i)$ is a functional of W times $|\sigma_{\frac{i-1}{n}}|^r$, it follows from Clark's representation theorem (e.g., Karatzas & Shreve (1998, Appendix E)):

$$g(\beta^n_i)-\rho_{\sigma_{\frac{i-1}{n}}}(g)=\frac{1}{\lambda_r}|\sigma_{\frac{i-1}{n}}|^r\int_{\frac{i-1}{n}}^{\frac{i}{n}}H^n_u\mathrm{d}W_u,$$

for some predictable function H^n_u. Similarly for $h\left(\beta'^n_{i-1}\right)-\rho_{\sigma_{\frac{i-2}{n}}}(h)$. Hence,

$$\mathbb{E}\left[\left(g(\beta^n_i)-\rho_{\sigma_{\frac{i-1}{n}}}(g)\right)\left(N_{\frac{i}{n}}-N_{\frac{i-1}{n}}\right)\mid\mathcal{F}_{\frac{i-1}{n}}\right]=0,$$

$$\mathbb{E}\left[\left(h\left(\beta'^n_{i-1}\right)-\rho_{\sigma_{\frac{i-2}{n}}}(h)\right)\left(N_{\frac{i}{n}}-N_{\frac{i-1}{n}}\right)\mid\mathcal{F}_{\frac{i-1}{n}}\right]=0,$$

as N is orthogonal to W. Finally

$$\mathbb{E}\left[\zeta^n_i\left(N_{\frac{i}{n}}-N_{\frac{i-1}{n}}\right)\mid\mathcal{F}_{\frac{i-1}{n}}\right]=0.\tag{A.8}$$

Now, Lemma 3 follows from Theorem IX 7.28 in Jacod & Shiryaev (2002). ■

A.4. **Preliminaries II**

We define the process

$$U\left(g,h\right)_t^n = \frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\Big\{g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right)$$

$$- \mathbb{E}\left[g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right]\Big\}, \tag{A.9}$$

In this subsection, we show that

$$U\left(g,h\right)_t^n - U_t^{\prime n} \xrightarrow{p} 0, \tag{A.10}$$

and, therefore,

$$U\left(g,h\right)_t^n \xrightarrow{d_s} \sqrt{\frac{\lambda_{2r}\lambda_{2s}+2\lambda_r\lambda_s\lambda_{r+s}-3\lambda_r^2\lambda_s^2}{\lambda_r^2\lambda_s^2}}\int_0^t |\sigma_u|^{r+s}\mathrm{d}B_u.$$

Start with:

$$\xi_i^n = \sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n, \qquad \xi_i^{\prime n} = \sqrt{n}s_{p_{(i+1)\Delta,\Delta}} - \beta_i^{\prime n}, \tag{A.11}$$

and observe that

$$\xi_i^n \leq \sqrt{n}\left(\sup_{(i-1)/n\leq s,t\leq i/n}\left|\int_s^t \mu_u \mathrm{d}u\right| + \sup_{(i-1)/n\leq s,t\leq i/n}\left|\int_s^t \left(\sigma_{u-}-\sigma_{\frac{i-1}{n}}\right)\mathrm{d}W_u\right|\right),$$

with a similar inequality for $\xi_i^{\prime n}$. Now we present a simple Lemma.

Lemma 4 *With $p \in BSM$, it holds that*

$$\frac{1}{n}\sum_{i=1}^{[nt]}\mathbb{E}\left[|\xi_i^n|^2 + |\beta_{i+1}^n - \beta_i^{\prime n}|^2\right] \to 0, \tag{A.12}$$

for all $t > 0$.

Proof

The boundedness of μ and Burkholder's inequality yield

$$\mathbb{E}\left[|\xi_i^n|^2\right] \leq C\left(\frac{1}{n} + n\mathbb{E}\left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} |\sigma_{u-}-\sigma_{\frac{i-1}{n}}|^2\mathrm{d}u\right]\right).$$

Moreover,

$$\mathbb{E}\left[|\beta_{i+1}^n - \beta_i^{\prime n}|^2\right] \leq C\mathbb{E}\left[|\sigma_{\frac{i}{n}} - \sigma_{\frac{i-1}{n}}|^2\right]$$

$$\leq Cn\mathbb{E}\left[\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(|\sigma_{u-}-\sigma_{\frac{i-1}{n}}|^2 + |\sigma_{u-}-\sigma_{\frac{i}{n}}|^2\right)\mathrm{d}u\right].$$

Hence, the left-hand side of (A.12) is smaller than

$$C\left(\frac{t}{n}+\int_0^t \mathbb{E}\left[|\sigma_{u-}-\sigma_{\frac{[nu]}{n}}|^2+|\sigma_{u-}-\sigma_{\frac{[nu]+1}{n}}|^2\right]\mathrm{d}u\right).$$

As σ is càdlàg, the last expectation converges to 0 for almost all u and is bounded by a constant. Thus, the statement follows from Lebesgue's theorem. ■

To prove the convergence in Equation (A.10), we reproduce the univariate version of Lemma 6.2 and 4.7 from Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006).

Lemma 5 *Let (ζ_i^n) be an array of random variables satisfying*

$$\sum_{i=1}^{[nt]}\mathbb{E}\left[|\zeta_i^n|^2 \mid \mathcal{F}_{\frac{i-1}{n}}\right]\xrightarrow{p}0, \tag{A.13}$$

for all t. If further each ζ_i^n is $\mathcal{F}_{\frac{i+1}{n}}$-measurable:

$$\sum_{i=1}^{[nt]}\left(\zeta_i^n-\mathbb{E}\left[\zeta_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]\right)\xrightarrow{p}0.$$

Lemma 6 *Assume that for all $q>0$*

1. *f and k are real functions on $\mathbb{R}$ of at most polynomial growth.*

2. *γ_i^n, $\gamma_i'^n$, $\gamma_i''^n$ are $\mathbb{R}$-valued random variables.*

3. *The process*

$$Z_i^n=1+\left|\gamma_i^n\right|+\left|\gamma_i'^n\right|+\left|\gamma_i''^n\right|,$$

satisfies

$$\mathbb{E}\left[(Z_i^n)^q\right]\le C_q.$$

If k is continuous and

$$\frac{1}{n}\sum_{i=1}^{[nt]}\mathbb{E}\left[\left|\gamma_i'^n-\gamma_i''^n\right|^2\right]\to 0, \tag{A.14}$$

then for all $t>0$

$$\frac{1}{n}\sum_{i=1}^{[nt]}\mathbb{E}\left[f^2\left(\gamma_i^n\right)\left(k\left(\gamma_i'^n\right)-k\left(\gamma_i''^n\right)\right)^2\right]\to 0. \tag{A.15}$$

Now, we prove (A.10). Define:

$$\zeta_i^n = \frac{1}{\sqrt{n}}\Big(g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right) h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) - g\left(\beta_i^n\right) h\left(\beta_i'^n\right)\Big), \tag{A.16}$$

and note that ζ_i^n is $\mathcal{F}_{\frac{i+1}{n}}$-measurable and

$$U\left(g,h\right)_t^n - U_t'^n = \sum_{i=1}^{[nt]} \left(\zeta_i^n - \mathbb{E}\left[\zeta_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]\right).$$

By Lemma 5, it is sufficient to show that:

$$\sum_{i=1}^{[nt]} \mathbb{E}\left[\left|\zeta_i^n\right|^2\right] \to 0. \tag{A.17}$$

Recall the identity:

$$\sqrt{n}s_{p_{i\Delta,\Delta}} = \beta_i^n + \xi_i^n,$$

and, therefore,

$$\begin{aligned}\left|\zeta_i^n\right|^2 \leq \frac{C}{n}\Big(& h^2\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right)\left(g\left(\beta_i^n + \xi_i^n\right) - g\left(\beta_i^n\right)\right)^2 \\ & + g^2\left(\beta_i^n\right)\left(h\left(\beta_{i+1}^n + \xi_{i+1}^n\right) - h\left(\beta_{i+1}^n\right)\right)^2 + g^2\left(\beta_i^n\right)\left(h\left(\beta_{i+1}^n\right) - h\left(\beta_i'^n\right)\right)^2\Big).\end{aligned}$$

Now, (A.17) is a result of (A.5), Lemma 4 and 6. ■

A.5. **Proof of Theorem 2**

$m = \infty$: First, we define

$$V_t^n = \frac{1}{n}\sum_{i=1}^{[nt]} \eta_i^n,$$

$$V_t'^n = \frac{1}{n}\sum_{i=1}^{[nt]} \eta_i'^n,$$

with

$$\eta_i^n = \mathbb{E}\left[g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right) h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

$$\eta_i'^n = \rho_{\sigma_{\frac{i-1}{n}}}\left(g\right)\rho_{\sigma_{\frac{i-1}{n}}}\left(h\right).$$

The convergence in (A.10) means that:

$$RBV^{n}_{(r,s),t} - V^{n}_{t} \xrightarrow{p} 0,$$

and by Riemann integrability:

$$V'^{n}_{t} \xrightarrow{p} \int_{0}^{t} |\sigma_u|^{r+s} \mathrm{d}u.$$

Thus, it remains to show:

$$V^{n}_{t} - V'^{n}_{t} \xrightarrow{p} 0.$$

As,

$$\eta^{n}_{i} - \eta'^{n}_{i} = \sqrt{n} \mathbb{E} \left[\zeta^{n}_{i} \mid \mathcal{F}_{\frac{i-1}{n}} \right],$$

a sufficient condition is that:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \mathbb{E} \left[|\zeta^{n}_{i}| \right] \to 0. \tag{A.18}$$

From the Cauchy-Schwarz inequality:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \mathbb{E} \left[|\zeta^{n}_{i}| \right] \leq \left(t \sum_{i=1}^{[nt]} \mathbb{E} \left[|\zeta^{n}_{i}|^{2} \right] \right)^{\frac{1}{2}},$$

Now, (A.18) follows from (A.17).

$m < \infty$: We introduce

$$\beta^{n,m}_{i} = \sqrt{n} |\sigma_{\frac{i-1}{n}}| s_{W_{i\Delta,\Delta},m}, \qquad \beta'^{n,m}_{i} = \sqrt{n} |\sigma_{\frac{i-1}{n}}| s_{W_{(i+1)\Delta,\Delta},m}, \tag{A.19}$$

which are discrete analogues of β^{n}_{i} and β'^{n}_{i} from (A.1). Also, put

$$\rho^{m}_{x} (f) = \mathbb{E} \left[f \left(|x| s_{W,m} \right) \right],$$

where $s_{W,m}$ was defined in (**2.22**). Note that:

$$\rho^{m}_{x} (g) = |x|^{r},$$

We proceed with the decomposition

$$RBV^{n,m}_{(r,s),t} - \int_{0}^{t} |\sigma_u|^{r+s} \mathrm{d}u = U^{n,m}_{t}(1) + U^{n,m}_{t}(2) + U^{n,m}_{t}(3),$$

with $U_t^{n,m}(k)$ given by:

$$U_t^{n,m}(1) = \frac{1}{n}\sum_{i=1}^{[nt]}\left(g\left(\sqrt{n}s_{p_{i\Delta,\Delta,m}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta,m}}\right) - g\left(\beta_i^{n,m}\right)h\left(\beta_i'^{n,m}\right)\right),$$

$$U_t^{n,m}(2) = \frac{1}{n}\sum_{i=1}^{[nt]}\left(g\left(\beta_i^{n,m}\right)h\left(\beta_i'^{n,m}\right) - \rho^m_{\sigma_{\frac{i-1}{n}}}(g)\,\rho^m_{\sigma_{\frac{i-1}{n}}}(h)\right),$$

$$U_t^{n,m}(3) = \frac{1}{n}\sum_{i=1}^{[nt]}\rho^m_{\sigma_{\frac{i-1}{n}}}(g)\,\rho^m_{\sigma_{\frac{i-1}{n}}}(h) - \int_0^t|\sigma_u|^{r+s}\mathrm{d}u.$$

Then,

$$U_t^{n,m}(3) = \lambda_{r,m}\lambda_{s,m}\left(\frac{1}{n}\sum_{i=1}^{[nt]}|\sigma_{\frac{i-1}{n}}|^{r+s} - \int_0^t|\sigma_u|^{r+s}\mathrm{d}u\right).$$

The boundedness of $\lambda_{r,m}$ (for fixed r) yields the convergence:

$$U_t^{n,m}(3) \xrightarrow{p} 0, \quad \text{as } n\to\infty,$$

uniformly in m. From the calculation of the conditional variance in the proof of Lemma 3, we also get

$$U_t^{n,m}(2) \xrightarrow{p} 0, \quad \text{as } n\to\infty,$$

uniformly in m. We decompose $U_t^{n,m}(1)$ further into:

$$U_t^{n,m}(1) = U_t^{n,m}(1.1) + U_t^{n,m}(1.2),$$

where

$$U_t^{n,m}(1.1) = \frac{1}{n}\sum_{i=1}^{[nt]}h\Big(\beta_i'^{n,m}\Big)\Big(g\left(\sqrt{n}s_{p_{i\Delta,\Delta,m}}\right) - g\left(\beta_i^{n,m}\right)\Big),$$

$$U_t^{n,m}(1.2) = \frac{1}{n}\sum_{i=1}^{[nt]}g\Big(\beta_i^{n,m}\Big)\Big(h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta,m}}\right) - h\left(\beta_i'^{n,m}\right)\Big).$$

We only show that:

$$U_t^{n,m}(1.1) \xrightarrow{p} 0,$$

uniformly in m. The corresponding result for $U_t^{n,m}(1.2)$ is proved equivalently. First, assume that $r\geq 1$. Then,

$$|h\left(\beta_i'^{n,m}\right)\left(g\left(\sqrt{n}s_{p_{i\Delta,\Delta,m}}\right) - g\left(\beta_i^{n,m}\right)\right)| \leq \frac{r}{\lambda_{r,m}}h\left(\beta_i'^{n,m}\right)\left(\sqrt{n}s_{p_{i\Delta,\Delta,m}} + \beta_i^{n,m}\right)^{r-1}|\sqrt{n}s_{p_{i\Delta,\Delta,m}} - \beta_i^{n,m}|.$$

The estimate:

$$\mathbb{E}\left[|\sqrt{n}s_{p_{i\Delta,\Delta},m}|^q + |\beta_i^{n,m}|^q + |\beta_i'^{n,m}|^q\right] \leq C_q,$$

holds for all $q > 0$. Thus:

$$\mathbb{E}\Big[|U_t^{n,m}(1.1)|\Big] \leq \frac{C}{n}\sum_{i=1}^{[nt]}\Big(\mathbb{E}\Big[|\sqrt{n}s_{p_{i\Delta,\Delta},m} - \beta_i^{n,m}|^2\Big]\Big)^{\frac{1}{2}}.$$

By Hölder's inequality:

$$\mathbb{E}\Big[|U_t^{n,m}(1.1)|\Big] \leq \left(\frac{Ct}{n}\sum_{i=1}^{[nt]}\mathbb{E}\Big[|\sqrt{n}s_{p_{i\Delta,\Delta},m} - \beta_i^{n,m}|^2\Big]\right)^{\frac{1}{2}}. \tag{A.20}$$

Note:

$$|\sqrt{n}s_{p_{i\Delta,\Delta},m} - \beta_i^{n,m}| \leq \sqrt{n}\left(\sup_{(i-1)/n\leq s,t\leq i/n}|\int_s^t \mu_u \mathrm{d}u| + \sup_{(i-1)/n\leq s,t\leq i/n}|\int_s^t\Big(\sigma_{u-} - \sigma_{\frac{i-1}{n}}\Big)\mathrm{d}W_u|\right),$$

with the right-hand side independent of m. Now, from (A.20), (A.5) and Lemma 4:

$$U_t^{n,m}(1.1) \xrightarrow{p} 0,$$

uniformly in m. Second, assume that $r < 1$. Then,

$$|g\left(\sqrt{n}s_{p_{i\Delta,\Delta},m}\right) - g\left(\beta_i^{n,m}\right)| \leq \frac{1}{\lambda_{r,m}}|\sqrt{n}s_{p_{i\Delta,\Delta},m} - \beta_i^{n,m}|^r.$$

We obtain the inequality:

$$\mathbb{E}\left[|U_t^{n,m}(1.1)|\right] \leq \left(\frac{Ct^{\frac{2}{r}-1}}{n}\sum_{i=1}^{[nt]}\mathbb{E}\left[|\sqrt{n}s_{p_{i\Delta,\Delta},m} - \beta_i^{n,m}|^2\right]\right)^{\frac{r}{2}}.$$

Therefore,

$$U_t^{n,m}(1.1) \xrightarrow{p} 0,$$

uniformly in m. Hence, the proof is complete. ■

A.6. **Proof of Theorem 3**

In light of the previous results, Theorem 3 follows from the convergence

$$\sqrt{n}\left(RBV_{(r,s),t}^n - \int_0^t |\sigma_u|^{r+s}\mathrm{d}u\right) - U(g,h)_t^n \xrightarrow{p} 0,$$

which can be shown by proving that the array

$$\zeta_i^n = \frac{1}{\sqrt{n}} \mathbb{E}\left[g\left(\sqrt{n} s_{p_{i\Delta,\Delta}}\right) h\left(\sqrt{n} s_{p_{(i+1)\Delta,\Delta}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] - \sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \rho_{\sigma_u}(g) \rho_{\sigma_u}(h) \, \mathrm{d}u,$$

is AN. To accomplish this, we split ζ_i^n into:

$$\zeta_i^n = \zeta_i'^n + \zeta_i''^n,$$

where

$$\zeta_i'^n = \frac{1}{\sqrt{n}} \left(\mathbb{E}\left[g\left(\sqrt{n} s_{p_{i\Delta,\Delta}}\right) h\left(\sqrt{n} s_{p_{(i+1)\Delta,\Delta}}\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] - \mathbb{E}\left[g\left(\beta_i^n\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] \mathbb{E}\left[h\left(\beta_i'^n\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right]\right), \tag{A.21}$$

$$\zeta_i''^n = \sqrt{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\rho_{\sigma_u}(g) \rho_{\sigma_u}(h) - \rho_{\sigma_{\frac{i-1}{n}}}(g) \rho_{\sigma_{\frac{i-1}{n}}}(h) \right) \mathrm{d}u. \tag{A.22}$$

It follows from Barndorff-Nielsen, Graversen, Jacod, Podolskij & Shephard (2006) that $\zeta_i''^n$ is AN. Next, we prove that the sequence $\zeta_i'^n$ is AN. Under V_2, we introduce the random variables:

$$\begin{aligned}
\zeta(1)_i^n = \sqrt{n} \sup_{(i-1)/n \le s,t \le i/n} \Bigg(& \int_s^t \sigma_{\frac{i-1}{n}} \mathrm{d}W_u + \int_s^t \mu_{\frac{i-1}{n}} \mathrm{d}u \\
& + \int_s^t \left\{ \sigma'_{\frac{i-1}{n}} \left(W_u - W_{\frac{i-1}{n}}\right) + v'_{\frac{i-1}{n}} \left(B'_u - B'_{\frac{i-1}{n}}\right) \right\} \mathrm{d}W_u \Bigg) - \beta_i^n,
\end{aligned} \tag{A.23}$$

$$\begin{aligned}
\zeta(2)_i^n = \sqrt{n} \Bigg\{ & \sup_{(i-1)/n \le s,t \le i/n} \Big(\int_s^t \mu_u \mathrm{d}u + \int_s^t \sigma_u \mathrm{d}W_u \Big) - \sup_{(i-1)/n \le s,t \le i/n} \Bigg(\int_s^t \sigma_{\frac{i-1}{n}} \mathrm{d}W_u + \int_s^t \mu_{\frac{i-1}{n}} \mathrm{d}u \\
& + \int_s^t \left\{ \sigma'_{\frac{i-1}{n}} \left(W_u - W_{\frac{i-1}{n}}\right) + v'_{\frac{i-1}{n}} \left(B'_u - B'_{\frac{i-1}{n}}\right) \right\} \mathrm{d}W_u \Bigg) \Bigg\}.
\end{aligned} \tag{A.24}$$

We have

$$\xi_i^n = \zeta(1)_i^n + \zeta(2)_i^n,$$

and a similar decomposition holds for $\xi_i'^n$. Now we present a simple Lemma, which is shown at the end of this subsection.

Lemma 7 *If $p \in BSM$ and assumption V_2 holds, then for any $q > 0$*

$$\mathbb{E}\left[|\xi_i^n|^q\right] \le C n^{-\frac{q}{2}}, \tag{A.25}$$

uniformly in i.

We have

$$\zeta_i'^n = \mathbb{E}\left[\delta_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

with δ_i^n defined by:

$$\delta_i^n = \frac{1}{\sqrt{n}}\left(g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) - g\left(\beta_i^n\right)h\left(\beta_i'^n\right)\right).$$

Observe the decomposition

$$\delta_i^n = \frac{1}{\sqrt{n}}g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right)\left(h\left(\sqrt{n}s_{p_{(i+1)\Delta,\Delta}}\right) - h\left(\beta_i'^n\right)\right) + \frac{1}{\sqrt{n}}\left(g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right) - g\left(\beta_i^n\right)\right)h\left(\beta_i'^n\right)$$

$$\equiv \delta_i'^n + \delta_i''^n.$$

We now show that the sequence

$$\mathbb{E}\left[\delta_i''^n \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

is AN. The AN property of the sequence $\mathbb{E}\left[\delta_i'^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]$ is shown in a similar way. For a proof of this property, set

$$A_i^n = \left\{\left|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n\right| > \frac{\beta_i^n}{2}\right\}.$$

We obtain the decomposition

$$\begin{aligned}
g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right) - g\left(\beta_i^n\right) &= \left(g\left(\sqrt{n}s_{p_{i\Delta,\Delta}}\right) - g\left(\beta_i^n\right)\right)I_{A_i^n} - \nabla g\left(\beta_i^n\right)\left(\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n\right)I_{A_i^n} \\
&\quad + \left(\nabla g\left(\bar{\gamma}_i^n\right) - \nabla g\left(\beta_i^n\right)\right)\left(\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n\right)I_{\left(A_i^n\right)^c} + \nabla g\left(\beta_i^n\right)\left(\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n\right) \\
&\equiv \vartheta_i^n(1) + \vartheta_i^n(2) + \vartheta_i^n(3) + \vartheta_i^n(4),
\end{aligned} \tag{A.26}$$

where $\bar{\gamma}_i^n$ is some random variable located between $\sqrt{n}s_{p_{i\Delta,\Delta}}$ and β_i^n. Now:

$$\delta_i''^n = \delta_i''^n(1) + \delta_i''^n(2) + \delta_i''^n(3) + \delta_i''^n(4),$$

with

$$\delta_i''^n(k) = \frac{1}{\sqrt{n}}h\left(\beta_i^n\right)\vartheta_i^n(k).$$

To complete the proof, it suffices that

$$\mathbb{E}\left[\delta_i''^n(k) \mid \mathcal{F}_{\frac{i-1}{n}}\right],$$

are AN for $k = 1$, 2, 3, and 4.

The term $\mathbb{E}\left[\delta_i''^n(1) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$:

With $r \geq 1$:

$$|\vartheta_i^n(1)| \leq C|\sqrt{n}s_{p_{i\Delta,\Delta}} + \beta_i^n|^{r-1}\frac{|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n|^{1+s}}{(\beta_i^n)^s},$$

for some $s \in (0,1)$. As μ and σ are bounded:

$$\mathbb{E}\left[|\sqrt{n}s_{p_{i\Delta,\Delta}}|^p\right] \leq C_p, \tag{A.27}$$

for all $p > 0$. With $r < 1$:

$$|\vartheta_i^n(1)| \leq C\frac{|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n|^{1+r/2}}{(\beta_i^n)^{1-r/2}}. \tag{A.28}$$

Now,

$$\mathbb{E}\left[\delta_i''^n(1) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = \frac{1}{\sqrt{n}}\rho_{\sigma_{\frac{i-1}{n}}}(h)\,\mathbb{E}\left[\vartheta_i^n(1) \mid \mathcal{F}_{\frac{i-1}{n}}\right].$$

For all $r > 0$, we now obtain by (A.5), (A.27), (**2.23**), Lemma 7 and Hölder's inequality:

$$\mathbb{E}\left[|\vartheta_i^n(1)|\right] \leq Cn^{-\frac{q}{2}},$$

for some $q > 1$, and so $\mathbb{E}\left[\delta_i''^n(1) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$ is AN.

The term $\mathbb{E}\left[\delta_i''^n(2) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$:

As in the previous step, for some $s \in (0,1)$, we obtain:

$$|\vartheta_i^n(2)| \leq C(\beta_i^n)^{r-1-s}|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n|^{1+s}, \quad \text{for } r \geq 1,$$

$$|\vartheta_i^n(2)| \leq C(\beta_i^n)^{r/2-1}|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n|^{1+r/2}, \quad \text{for } r < 1. \tag{A.29}$$

The AN property of $\mathbb{E}\left[\delta_i''^n(2) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$ is now a consequence of Equation (A.5), (**2.23**), Lemma 7 and Hölder's inequality.

The term $\mathbb{E}\left[\delta_i''^n(3) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$:

For $r \geq 2$:

$$|\vartheta_i^n(3)| \leq C|\sqrt{n}s_{p_{i\Delta,\Delta}} + \beta_i^n|^{r-2}|\sqrt{n}s_{p_{i\Delta,\Delta}} - \beta_i^n|^2.$$

For $r < 2$:

$$|\vartheta_i^n(3)| \leq C(\beta_i^n)^{r-2} |\sqrt{n} s_{p_{i\Delta,\Delta}} - \beta_i^n|^2 I_{(A_i^n)^c}.$$

By definition of A_i^n:

$$|\vartheta_i^n(3)| \leq C(\beta_i^n)^{r/2-1} |\sqrt{n} s_{p_{i\Delta,\Delta}} - \beta_i^n|^{1+r/2}, \tag{A.30}$$

for $r < 2$. That $\mathbb{E}\left[\delta_i''^n(3) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$ is AN is a consequence of the above.

The term $\mathbb{E}\left[\delta_i''^n(4) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$:

First, we find a stochastic expansion for

$$\xi_i^n = \sqrt{n} s_{p_{i\Delta,\Delta}} - \beta_i^n,$$

defined in (A.11). Recall:

$$\xi_i^n = \zeta(1)_i^n + \zeta(2)_i^n,$$

with $\zeta(1)_i^n$ and $\zeta(2)_i^n$ defined by (A.23) and (A.24), respectively. Set

$$\begin{aligned} f_{in}(s,t) &= \sqrt{n}\sigma_{\frac{i-1}{n}}(W_t - W_s), \\ g_{in}(s,t) &= n\int_s^t \mu_{\frac{i-1}{n}} \mathrm{d}u + n\int_s^t \left\{\sigma'_{\frac{i-1}{n}}\left(W_u - W_{\frac{i-1}{n}}\right) + v'_{\frac{i-1}{n}}\left(B'_u - B'_{\frac{i-1}{n}}\right)\right\} \mathrm{d}W_u \\ &= \mu_{\frac{i-1}{n}} g_{in}^1(s,t) + \sigma'_{\frac{i-1}{n}} g_{in}^2(s,t) + v'_{\frac{i-1}{n}} g_{in}^3(s,t), \end{aligned}$$

to achieve the identity:

$$\zeta(1)_i^n = \sup_{(i-1)/n \leq s,t \leq i/n} \left(f_{in}(t,s) + \frac{1}{\sqrt{n}} g_{in}(t,s)\right) - \sup_{(i-1)/n \leq s,t \leq i/n} f_{in}(t,s).$$

Imposing assumption V_1:

$$\begin{aligned} (t_{in}^*(W), s_{in}^*(W)) &= \arg\sup_{(i-1)/n \leq s,t \leq i/n} f_{in}(s,t) \\ &= \arg\sup_{(i-1)/n \leq s,t \leq i/n} \sqrt{n}(W_t - W_s) \\ &\overset{d}{=} \arg\sup_{0 \leq s,t \leq 1} (W_t - W_s). \end{aligned}$$

A standard result then states that the pair $(t_{in}^*(W), s_{in}^*(W))$ is unique, almost surely (e.g., Revuz & Yor (1998)). In the next Lemma - which is proved at the end of this subsection - a stochastic expansion of the quantity $\zeta(1)_i^n$ is given.

Lemma 8 *Given assumption V_1:*

$$\zeta(1)_i^n = \frac{1}{\sqrt{n}}\Big\{g_{in}\left(t_{in}^*(W), s_{in}^*(W)\right) + \tilde{g}_{in}\Big\},$$

where

$$\mathbb{E}\big[|\tilde{g}_{in}|^p\big] = \mathrm{o}(1), \tag{A.31}$$

for all $p > 0$ and uniformly in i.

Note also that:

$$\left(t_{in}^*(W), s_{in}^*(W)\right) = \left(s_{in}^*(-W), t_{in}^*(-W)\right).$$

As $(W, B') \overset{d}{=} -(W, B')$ and $\nabla g(\beta_i^n)$ is an even functional of W:

$$\mathbb{E}\left[\nabla g(\beta_i^n)\, g_{in}^k\left(t_{in}^*(W), s_{in}^*(W)\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0,$$

for $k = 1$, 2, and 3. Hence,

$$\mathbb{E}\left[\nabla g(\beta_i^n)\, g_{in}\left(t_{in}^*(W), s_{in}^*(W)\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right] = 0. \tag{A.32}$$

For the quantity $\zeta(2)_i^n$, it holds that

$$\zeta(2)_i^n \leq \sqrt{n}\bigg(\int_{\frac{i-1}{n}}^{\frac{i}{n}} |\mu_u - \mu_{\frac{i-1}{n}}|\mathrm{d}u \tag{A.33}$$

$$+ \sup_{(i-1)/n \leq s,t \leq i/n} \int_s^t \bigg\{\int_{\frac{i-1}{n}}^u \mu_r' \mathrm{d}r + \int_{\frac{i-1}{n}}^u \Big(\sigma_{r-}' - \sigma_{\frac{i-1}{n}}'\Big)\mathrm{d}W_r + \int_{\frac{i-1}{n}}^u \Big(v_{r-}' - v_{\frac{i-1}{n}}'\Big)\mathrm{d}B_r'\bigg\}\mathrm{d}W_u\bigg).$$

The next lemma is also proved later on.

Lemma 9 *For $q \geq 2$, it holds that*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\Big(\mathbb{E}\left[|\zeta(2)_i^n|^q\right]\Big)^{\frac{1}{q}} \to 0,$$

for all $t > 0$.

Using Hölder's inequality, it follows that

$$|\,\mathbb{E}\left[\delta_i''^n(4) \mid \mathcal{F}_{\frac{i-1}{n}}\right]| = \frac{1}{\sqrt{n}}\rho_{\sigma_{\frac{i-1}{n}}}(h)\,|\,\mathbb{E}\left[\nabla g(\beta_i^n)\left(\zeta(1)_i^n + \zeta(2)_i^n\right) \mid \mathcal{F}_{\frac{i-1}{n}}\right]|$$

$$\leq \frac{1}{\sqrt{n}}\rho_{\sigma_{\frac{i-1}{n}}}(h)\bigg(|\,\mathbb{E}\left[\nabla g(\beta_i^n)\,\zeta(1)_i^n \mid \mathcal{F}_{\frac{i-1}{n}}\right]|$$

$$+ \left(\mathbb{E}\left[(\nabla g(\beta_i^n))^p\right]\right)^{\frac{1}{p}}\left(\mathbb{E}\left[|\zeta(2)_i^n|^q\right]\right)^{\frac{1}{q}}\bigg), \tag{A.34}$$

for some $p > 1, q \geq 2$ with $(r-1)p > -1$ and $1/p + 1/q = 1$. Finally, by combining (**2.23**), (A.31), (A.32) and Lemma 9, we get the AN property of the sequence $\mathbb{E}\left[\delta_i''^n(4) \mid \mathcal{F}_{\frac{i-1}{n}}\right]$. Hence, Theorem 3 with $m = \infty$ is proved.

$m < \infty$: To show the theorem with $m < \infty$, we want to point out that the main structure of the previous proof can be adapted directly. The difference is the moment condition:

$$\lambda_{r,m} < \infty,$$

for $r > -m$. The estimates (A.28), (A.29), (A.30) and (A.34), however, were formulated such that we can use this condition without changing the proof (for all $m \in \mathbb{N}$). ■

Proof of Lemma 7

Note that:

$$\mathbb{E}\left[|\zeta(1)_i^n|^q\right] \leq Cn^{\frac{q}{2}}\Bigg(\sup_{(i-1)/n \leq s,t \leq i/n} \left|\int_s^t \mu_{\frac{i-1}{n}}\,\mathrm{d}u\right|^q$$

$$+\sup_{(i-1)/n \leq s,t \leq i/n} \left|\int_s^t \left\{\sigma'_{\frac{i-1}{n}}\left(W_u - W_{\frac{i-1}{n}}\right) + v'_{\frac{i-1}{n}}\left(B'_u - B'_{\frac{i-1}{n}}\right)\right\}\mathrm{d}W_u\right|^q\Bigg).$$

The boundedness of μ, σ', v' and Burkholder's inequality give

$$\mathbb{E}\left[|\zeta(1)_i^n|^q\right] \leq Cn^{-\frac{q}{2}},$$

and the term $\zeta(2)_i^n$ is handled equivalently. ■

Proof of Lemma 8

We need a deterministic version of Lemma 8:

Lemma 10 *Given two continuous functions $f, g : I \to \mathbb{R}$ on a compact set $I \subseteq \mathbb{R}^n$, assume t^* is the only point where the maximum of the function f on I is achieved. Then, it holds:*

$$M_\epsilon(g) \equiv \frac{1}{\epsilon}\left[\sup_{t\in I}\{f(t) + \epsilon g(t)\} - \sup_{t\in I}\{f(t)\}\right] \to g(t^*) \quad \text{as} \quad \epsilon \downarrow 0.$$

Proof of Lemma 10

Construct the set

$$\bar{G} = \Big\{ h \in C(I) \mid h \text{ is constant on } B_\delta(t^*) \cap I \text{ for some } \delta > 0 \Big\}.$$

As usual, $C(I)$ is the set of continuous functions on I and $B_\delta(t^*)$ is an open ball of radius δ centered at t^*. Take $\bar{g} \in \bar{G}$ and recall $\bar{g}$ is bounded on I. Thus, for ϵ small enough:

$$\begin{aligned}
\sup_{t \in I} \{ f(t) + \epsilon \bar{g}(t) \} &= \max \left\{ \sup_{t \in I \cap B_\delta(t^*)} \{ f(t) + \epsilon \bar{g}(t) \}, \sup_{t \in I \cap B_\delta^c(t^*)} \{ f(t) + \epsilon \bar{g}(t) \} \right\} \\
&= \sup_{t \in I \cap B_\delta(t^*)} \{ f(t) + \epsilon \bar{g}(t) \} \\
&= f(t^*) + \epsilon \bar{g}(t^*).
\end{aligned}$$

So,

$$M_\epsilon(\bar{g}) \to \bar{g}(t^*),$$

$\forall\, \bar{g} \in \bar{G}$. Now, let $g \in C(I)$. As $\bar{G}$ is dense in $C(I)$, $\exists\, \bar{g} \in \bar{G} : \bar{g}(t^*) = g(t^*)$ and $|\bar{g} - g|_\infty < \epsilon'$ ($|\cdot|_\infty$ is the sup-norm). We see that $|M_\epsilon(\bar{g}) - M_\epsilon(g)| < \epsilon'$, and

$$|M_\epsilon(g) - g(t^*)| \leq |M_\epsilon(\bar{g}) - \bar{g}(t^*)| + |M_\epsilon(g) - M_\epsilon(\bar{g})| \to 0.$$

Thus, the assertion is established. ■

Now, $(t_{in}^*(W), s_{in}^*(W))$ is unique and the functions g_{in}, f_{in} are continuous, both almost surely. Thus, Lemma 8 is shown by replicating the proof of Lemma 10 for g_{in} and f_{in}. More precisely, the random function $\bar{g}_{in} \in \bar{G}$ that is constant in a neighbourhood of $(t_{in}^*(W), s_{in}^*(W))$ must be constructed. The rest goes along the lines of Lemma 10. ■

Proof of Lemma 9

From (A.33) and repeated use of the Hölder and Burkholder inequalities plus the boundedness of μ', we get:

$$\begin{aligned}
\mathbb{E}\left[|\zeta(2)_i^n|^q \right] \leq C_q n^{\frac{q}{2}} \Bigg(& n^{-q+1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |\mu_u - \mu_{\frac{i-1}{n}}|^q \mathrm{d}u + n^{-\frac{3q}{2}} + n^{-q+1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |\sigma'_{u-} - \sigma'_{\frac{i-1}{n}}|^q \mathrm{d}u \\
& + n^{-q+1} \int_{\frac{i-1}{n}}^{\frac{i}{n}} |v'_{u-} - v'_{\frac{i-1}{n}}|^q \mathrm{d}u \Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]}\left(\mathbb{E}\left[|\zeta\left(2\right)_i^n|^q\right]\right)^{\frac{1}{q}} &\leq C_q t^{\frac{q-1}{q}}\Bigg(\sum_{i=1}^{[nt]}\mathbb{E}\Big[\int_{\frac{i-1}{n}}^{\frac{i}{n}}|\mu_u-\mu_{\frac{i-1}{n}}|^q\mathrm{d}u+\int_{\frac{i-1}{n}}^{\frac{i}{n}}|\sigma'_{u-}-\sigma'_{\frac{i-1}{n}}|^q\mathrm{d}u \\ &\quad+\int_{\frac{i-1}{n}}^{\frac{i}{n}}|v'_{u-}-v'_{\frac{i-1}{n}}|^q\mathrm{d}u\Big]\Bigg)^{\frac{1}{q}}+\mathrm{o}\left(1\right) \\ &= C_q t^{\frac{q-1}{q}}\Bigg(\mathbb{E}\Big[\int_0^t|\mu_u-\mu_{\frac{[nu]}{n}}|^q+|\sigma'_{u-}-\sigma'_{\frac{[nu]}{n}}|^q \\ &\quad+|v'_{u-}-v'_{\frac{[nu]}{n}}|^q\mathrm{d}u\Big]\Bigg)^{\frac{1}{q}}+\mathrm{o}\left(1\right).\end{aligned}$$

As σ' and v' are càdlàg, the proof is complete. ■

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Table 1: Finite sample properties of t-statistics for jump detection.

	$z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$				$z^{ar}_{RV^n_t, BV^n_t, QQ^n_t}$		
Size:	$n = 39$	$n = 78$	$n = 390$		$n = 39$	$n = 78$	$n = 390$
$\alpha = 0.01$	2.236	1.856	1.292		1.417	1.291	1.164
0.05	6.497	6.071	5.504		5.607	5.447	5.114
0.10	10.769	10.673	10.282		10.379	10.186	10.008
Power:				$j = 1$			
$\sigma^2_J = 0.05$	11.982	18.385	33.970		3.805	6.777	23.314
0.10	23.440	32.096	48.995		8.215	15.317	38.592
0.15	31.882	41.576	57.492		12.895	22.694	47.714
0.20	37.890	47.217	62.343		17.144	28.172	53.138
0.25	42.582	51.886	65.832		20.873	32.916	57.572
				$j = 2$			
$\sigma^2_J = 0.05$	21.767	33.971	58.519		6.003	12.673	43.301
0.10	40.940	55.274	75.695		14.091	27.806	63.913
0.15	53.817	66.833	83.029		22.654	39.789	73.940
0.20	61.641	73.426	86.930		29.419	48.280	79.307
0.25	67.074	77.683	89.234		35.410	54.375	82.817

This table reports the finite sample properties of the range- and return-based t-statistic for testing the null of a continuous sample path at the sampling frequencies $n = 39$, 78, and 390 ($m = 30$, 15, and 3). We show the actual size of the tests at an $\alpha = 0.01$, 0.05, and 0.10 nominal level of significance. In the bottom, the power of the test is computed with $j = 1$ or $j = 2$ independent $N\left(0, \sigma^2_J\right)$ jumps, where $\sigma^2_J = 0.05, 0.10, \ldots, 0.25$ and the test is conducted at an $\alpha = 0.01$ nominal level with critical point $z_\alpha = 2.326$.

Table 2: Number of tick data pr. trading day.

Ticker	Trades			Quotes		
	All	$\#r_{\tau_i} \neq 0$	$\#\Delta r_{\tau_i} \neq 0$	All	$\#r_{\tau_i} \neq 0$	$\#\Delta r_{\tau_i} \neq 0$
MRK	2891	1314	706	5537	1750	1246

The table contains information about the filtering of the Merck high-frequency data. All numbers are averages across the 1,253 trading days in our sample from January 3, 2000 through December 31, 2004. $\#r_{\tau_i} \neq 0$ is the daily amount of tick data left after counting out price repetitions in consecutive ticks. $\#\Delta r_{\tau_i} \neq 0$ also removes instantaneous price reversals.

Table 3: Sample statistics for estimators of $\langle p \rangle_t$, $\int_0^t \sigma_u^2 \mathrm{d}u$ and $\int_0^t \sigma_u^4 \mathrm{d}u$.

	Mean	Var.	Skew.	Kurt.	Min.	Max.	Correlation					
$RRV_t^{n,m}$	7.266	64.143	5.554	52.411	0.597	117.679	1.000	0.901	*0.986*	0.888	0.798	0.727
$RBV_t^{n,m}$	6.077	30.446	3.681	24.774	0.495	61.589		1.000	0.916	*0.976*	0.843	0.759
RV_t^n	7.063	67.200	5.271	47.118	0.372	116.096			1.000	0.927	0.810	0.747
BV_t^n	6.459	48.860	4.353	31.893	0.284	82.273				1.000	0.855	0.796
$RQQ_t^{n,m}$	0.286	0.843	10.751	154.839	0.001	16.114					1.000	*0.972*
QQ_t^n	0.381	2.598	13.280	231.051	0.000	33.576						1.000

Sample statistics of the annualized percentage $RRV_t^{n,m}$, $RBV_t^{n,m}$, $RQQ_t^{n,m}$, RV_t^n, BV_t^n and QQ_t^n for Merck from January 3, 2000 up to December 31, 2004. The table prints the mean, variance, skewness, kurtosis, minimum and maximum of the various time series, plus the correlation matrix. $RQQ_t^{n,m}$ and QQ_t^n are multiplied further by 100 to improve the scale.

Table 4: Proportion of $\langle p \rangle_t$ due to the jump component.

		$\alpha = 0.05$		$\alpha = 0.01$	
	$\%JV$	#rej	$\%JV_s$	#rej	$\%JV_s$
$RRV_t^{n,m}$	16.966	141 (76)	7.984	48 (23)	5.559
RV_t^n	10.938	289 (68)	8.012	151 (16)	6.091

The proportion of $\langle p \rangle_t$ of Merck due to the jump component is reported using three criteria. $\%JV = \sum_{t=1}^{T} (RRV_t^{n,m} - RBV_t^{n,m})^+ / \sum_{t=1}^{T} RRV_t^{n,m}$, is an overall measure of average jump contribution for the $T = 1,253$ trading days in our sample period running from January 3, 2000 up to December 31, 2004 (the definition for RV_t^n is identical). A subscript s in the next two indicates that only significant terms in the numerator of $\%JV$ are counted, either at the $\alpha = 0.05$ or $\alpha = 0.01$ nominal level. #rej is the amount of times $z^{ar}_{RRV_t^{n,m}, RBV_t^{n,m}, RQQ_t^{n,m}}$ and $z^{ar}_{RV_t^n, BV_t^n, QQ_t^n}$, rejected the null of a continuous sample path, respectively. The parenthesis contain the number of rejections (rounded to the nearest integer) that would be expected, in absence of jumps, based on the actual size of the tests from Table 1.

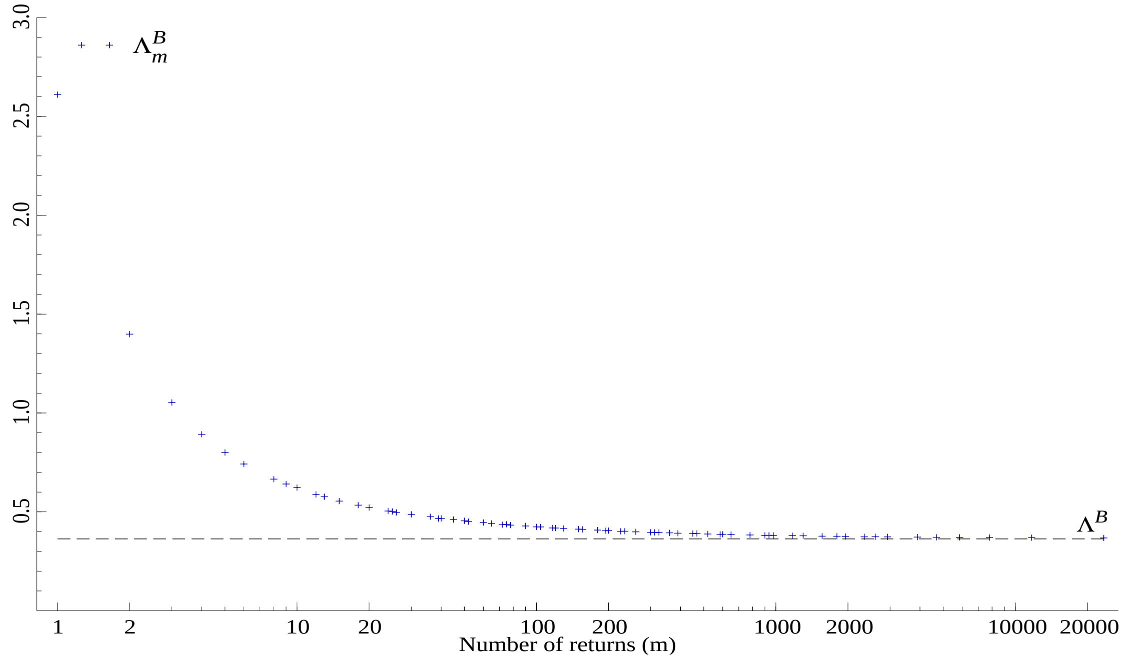

Figure 1: Λ_m^B against m on a log-scale. All estimates are based on a simulation with 1,000,000 repetitions, and the dashed line represents the asymptotic value.

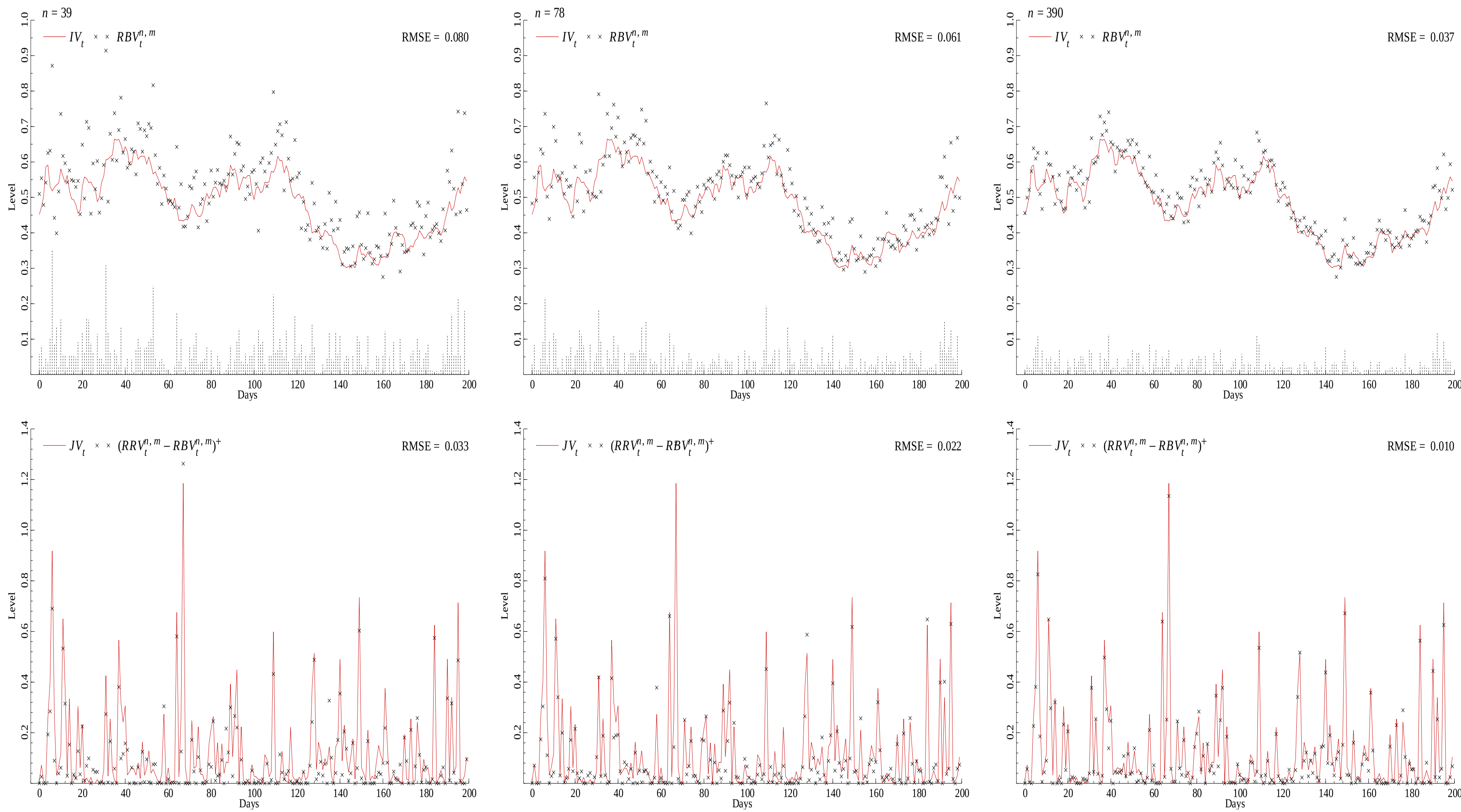

Figure 2: $RBV_t^{n,m}$ and $(RRV_t^{n,m} - RBV_t^{n,m})^+$ are plotted against the population values of the integrated variance (IV_t) and jump variation (JV_t) for 200 iterations with $j = 1$ and $\sigma_J^2 = 0.10$, using $n = 39$, 78, and 390 (column-wise). The vertical dashed lines in the first row are the absolute sampling errors of $RBV_t^{n,m}$. We also report the root mean squared error (RMSE) across all 100,000 simulations.

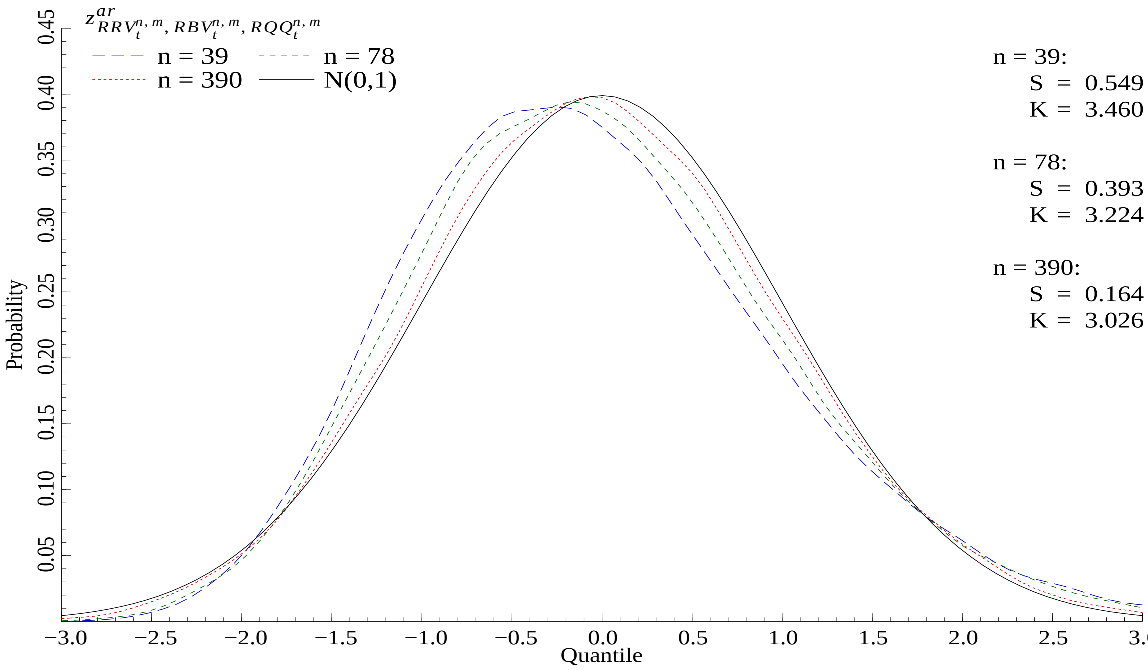

Figure 3: The null distribution of $z^{ar}_{RRV^{n,m}_t,RBV^{n,m}_t,RQQ^{n,m}_t}$ is drawn against the sampling frequencies $n = 39$, 78, and 390 ($m = 30$, 15, and 3). We compute the coefficient of skewness and kurtosis for each n and superimpose a standard normal for visual reference (the solid line). The figure is based on a simulation with 100,000 repetitions, as detailed in the main text.

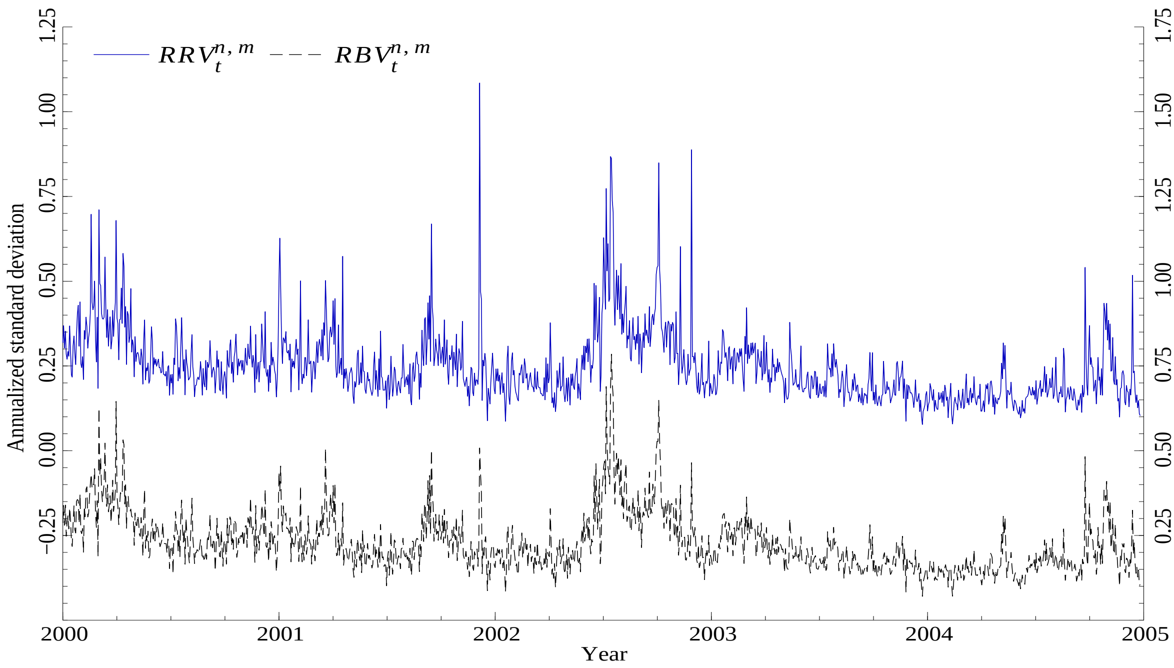

Figure 4: $RRV_t^{n,m}$ and $RBV_t^{n,m}$ are drawn over the sample period January 3, 2000 through December 31, 2004. The time series are constructed from tick-time sampled ranges of Merck, setting $m = 15$, and are reported in annualized standard deviation form. $RRV_t^{n,m}$ is measured against the left y-axis and $RBV_t^{n,m}$ is read off from the right.

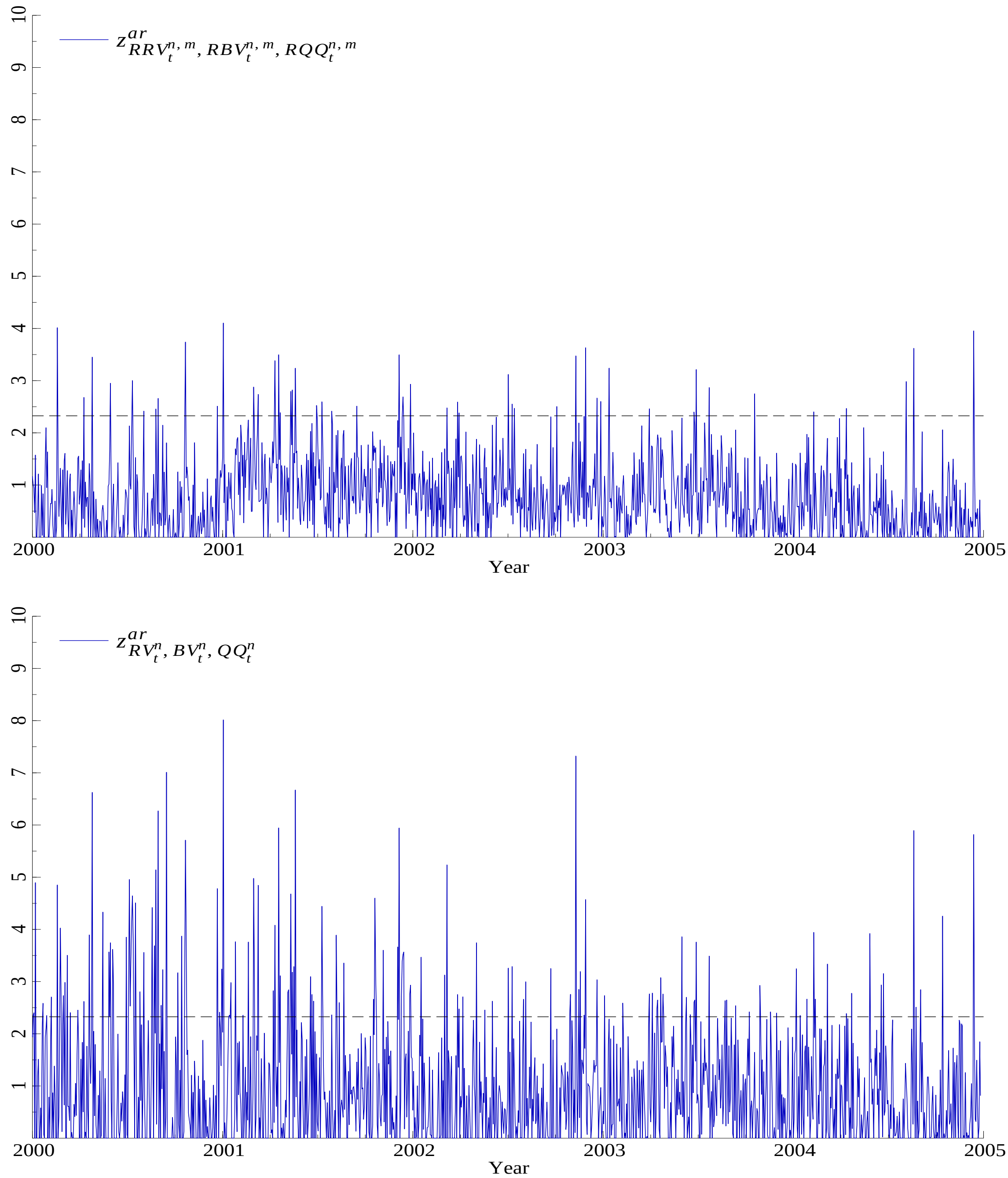

Figure 5: We plot $z^{ar}_{RRV^{n,m}_t, RBV^{n,m}_t, RQQ^{n,m}_t}$ and $z^{ar}_{RV^{n}_t, BV^{n}_t, QQ^{n}_t}$ for the 1,253 trading days running from January 3, 2000 through December 31, 2004. The horizontal dashed line corresponds to the 0.99 quantile of a standard normal distribution.

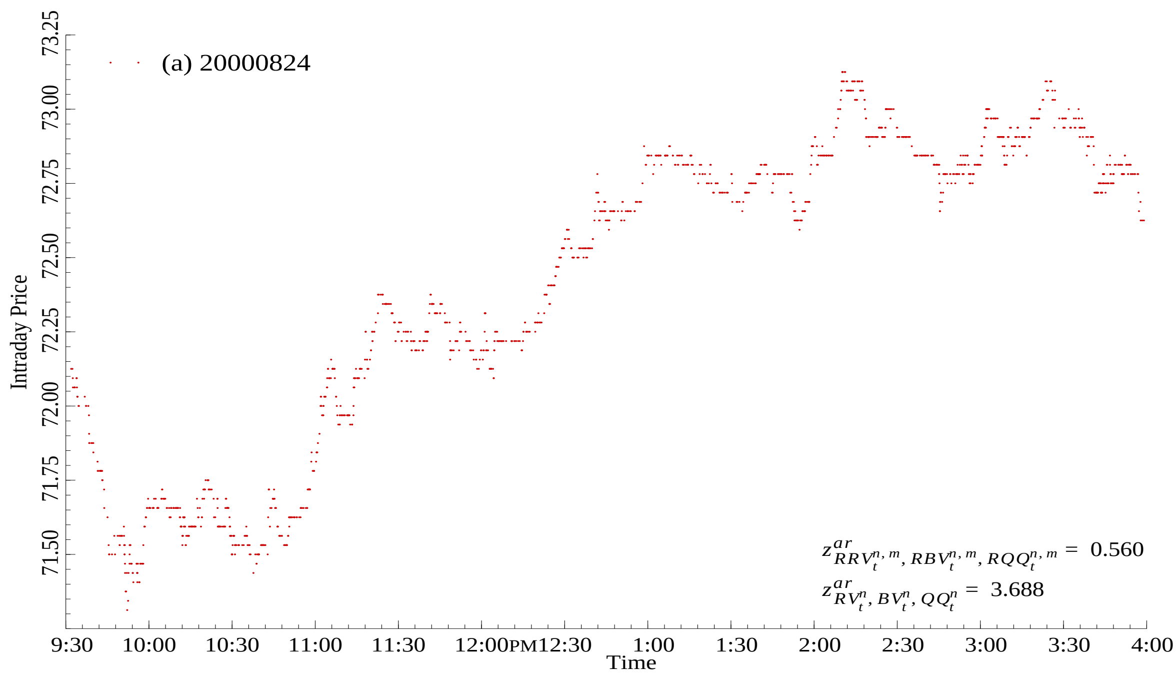

Figure 6: The intraday price variation of Merck on August 24, 2000 is shown, and we report the outcome of both $z^{ar}_{RRV^{n,m}_t,RBV^{n,m}_t,RQQ^{n,m}_t}$ and $z^{ar}_{RV^{n}_t,BV^{n}_t,QQ^{n}_t}$.