Informational Black Holes in Financial Markets

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ABSTRACT

We study how well primary financial markets allocate capital when information about investment opportunities is dispersed across market participants. We set up a canonical capital-raising in which an entrepreneur attempts to raise financing for a new project from informed investors in a common-value auction. Information generated in the auction affects the decision of whether to start the project or not. We show that the bidding behavior of investors fails to convey their information, and that larger markets with more aggregate information may reduce efficiency. Contrary to the predictions of standard theory, the revenues of an entrepreneur can be decreasing in the size of the market, early releases of information may be suboptimal, and collusion among investors may benefit the entrepreneur.

JEL Codes: D44, D82, G10, G20.

The main role of primary financial markets is to channel resources to firms with worthwhile projects, a process that requires information. The unprecedent growth of the financial sector over the last three decades has led to a proliferation of informed intermediaries to fulfill this role. Today, a large and growing number of business angels, venture capitalists, and private equity firms alongside traditional commercial banks use their expertise to identify whether firms seeking capital have sufficiently good investment opportunities. In light of these developments, it is natural to ask whether a larger market populated with more experts leads to better investment decisions and a lower cost of capital for entrepreneurs seeking financing.

There are two compelling reasons for the answer to be "Yes." First, increased competition between investors should reduce their informational rents and drive down the cost of capital for entrepreneurs. Second, when the market as an aggregate possesses more information about the viability of a project, investment decisions should become more efficient—which should further drive down the cost of capital.

The message in our paper is a more pessimistic one. We set up a canonical capital-raising model where informed investors compete for the right to finance an entrepreneur in a common-value auction. We show that larger markets often lead to less efficient investment decisions, and lower revenues to the entrepreneur. Our results hold regardless of which of the standard auction formats is used (first price, second price, or ascending price), and regardless of how much of the information generated in the auction is made available to the person making the investment decision.

The important departure from the standard auction-theory setting of Milgrom and Weber (1982), where a pre-existing asset is sold, is that in our setting the auction affects whether the project gets started or not. If no bids are received, or if the winning investor together with the entrepreneur decide that the project is not worthwhile after observing the outcome of the auction, the project is cancelled.

The fact that information generated in the auction has real productive value changes some of the main insights of standard auction theory. In a standard auction of an existing asset, sellers gain from having more bidders because competition reduces the informational rent of bidders (see Kremer (2002), Bulow and Klemperer (1996) and Bali and Jackson (2002)). Furthermore, anyone who observes the bids in a standard auction learns all the information the market possesses, since bids are strictly increasing in the signals of bidders.[1] Hence, in the standard setting, larger auctions typically generate

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[1] In our setting, it is natural to assume that bids are revealed to the agent making the investment decision, although our main results do not depend on this assumption. The literature on information aggregation in auctions has mainly studied whether the *price* set in the auction converges to the true value of the asset as the market grows large. In an ascending-price auction, the price aggregates market information because bidders can condition their strategies on the dropout behavior of competitors (see

more information.

In our setting, just as in the standard setting, increased competition increases the fraction of surplus captured by the entrepreneur. However, we show that increasing the size of the market may lead to a deterioration of the information produced in the auction. The decreased informativeness of the auction in turn leads to worse investment decisions, lower total surplus, and lower rents to the entrepreneur.

These results stem from the interaction of the investment decision and the bidding behavior of investors. If an investor with a sufficiently pessimistic signal wins the right to finance the project, he assumes that the project is negative NPV and not worth investing in. Relatively pessimistic investors therefore either abstain from bidding or bid zero.[2] As a result, all their information is pooled together and lost—they fall into an "informational black hole". This loss of information is costly, and leads to investment mistakes of two types—some projects that would have been worth pursuing had all market information been utilized do not get financed, while some that are not worth pursuing get financed.

The problem is exacerbated as the market grows larger, because of the winners curse. In a larger market, even an investor with somewhat favorable information will conclude that the project is not worth investing in if he wins, since winning implies that all other investors are more pessimistic. Hence, the informational black hole grows with the size of the market, and we show that for some reasonable distributional assumptions the social surplus as well as the expected revenues to the entrepreneur can fall with the size of the market.

Our results have normative implications for how entrepreneurs should maximize revenues that drastically contrast with the prescriptions of standard auction theory. In particular, our findings might explain why we often see entrepreneurs engage in so-called proprietary transactions, where they negotiate a financing deal with a single venture capitalist rather than engaging in a more competitive search. Similarly, in acquisition procedures investment banks working on behalf of a selling firm often restrict the set of invited bidders, and there is no evidence that this practice reduces seller revenues (see Boone and Mulherin (2007)).

When firms cannot commit to restrict the number of bidders[3], we show that the equilibrium size of the financial sector may be inefficiently large. This happens because the marginal investor does not internalize the negative externality he imposes

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Kremer (2002) and Han and Shum (2004)). In sealed-bid auctions, such as first-price and second-price auctions, the price itself does not aggregate all information as the market grows large except in the very special circumstances outlined in Milgrom (1979), but the bids do.

[2]Investors are free to submit negative bids, but never do so in equilibrium.

[3]To commit to restrict the number of bidders, a firm needs to commit not to consider unsolicited offers, because ex post it is always optimal to consider all offers.

on allocational efficiency when he enters the market. We show that social welfare can decrease with a decrease in the cost of setting up an informed intermediary, and that policies aimed at restricting the size of the market can lead to Pareto improvements.

We further show that in our setting, efficiency can be improved by allowing a sufficiently large number of investors to receive a stake in the project if this is practically feasible. This is in contrast to the standard setting, where revenues are maximized by concentrating the allocation to the highest bidder. In a multi-unit auction where the number of units grows with the number of bidders, a loser's curse balances out the winner's curse (as shown in Pesendorfer and Swinkels (1997) for standard multi-unit auctions) which in our setting leads to higher participation and a recovery of information aggregation, and hence a higher surplus. This may be one rationale for crowd-funding, in which start-ups seek financing on a platform that looks very much like a multi-unit auction. The finding may also explain why IPO allocations are rationed to increase the number of winning participants.

A related solution is to allow syndicates or consortia consisting of multiple investors to submit joint "club bids" in the auction. Club bids and syndicates are common practice among both angel investors, venture capitalists, and private equity firms, and have been the subject of investigation by competition authorities for creating anti-competitive collusion. Indeed, in a standard auction setting, club bids reduce the expected revenues of the seller. In our setting, the opposite may hold—because club bids reduce the winner's curse problem, it encourages participation, which increases the efficiency of the market.

We also show that the famous "linkage principle" of Milgrom and Weber (1982) may fail in our setting. The linkage principle holds that any value-relevant information that can be revealed before an auction should be revealed in order to lower the informational rent of bidders. For example, if an entrepreneur can postpone seeking financing until some public information about market conditions is revealed, he should do so. In our setting, to the contrary, it is often better to attempt financing of the project *before* some value-relevant information is revealed. The reason is that residual uncertainty creates an option value to the project which makes less optimistic bidders participate, which in turn increases the information aggregation properties of the market.

Our analysis is complicated by the fact that in our setting, unlike in the standard auction setting, there are multiple equilibria even if strategies are restricted to be symmetric. Because the size of the informational black hole affects the efficiency of investment decisions, there are strategic complementarities among investors. When an investor expects others to bid zero over a larger signal interval he expects surplus from the auction to be lower because of the lost information. As a result, he also bids zero

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over a larger signal interval, making the expectation of a larger informational black hole self-fulfilling. We show that there can be a continuum of equilibria, which are ranked in their efficiency by the size of the informational black hole. While the most efficient of these equilibria lead to perfect information aggregation as the market grows large, we show that efficient equilibria are very fragile and do not exist if submitting a bid has an arbitrarily small cost, or if bids have to be made in discrete increments, no matter how fine the bidding grid is.

In the main part of our analysis, we model competition as happening through one of the standard auction formats. These market structures approximate most real-world financing procedures, including informal settings where investors approach the entrepreneur with unsolicited offers. As an extension we discuss sufficient conditions under which informational black holes and the resulting inefficiencies can appear in general market mechanisms. First, we have to assume that the mechanism cannot split the allocation of the project rights over several investors, perhaps because cash flows are non-contractible or because coordination among several creditors is costly ex post. This rules out the use of multi-unit auctions and club bidding. Second, we assume a mechanism has to be regret free in that investors can default on the mechanism ex post if they are not happy with the outcome, and that it should not be profitable for unserious investors without information to enter the mechanism. These two restrictions make it impossible to reward or punish investors who do not receive an allocation in the mechanism, which limits the scope of eliciting information from them. Third, we assume that the mechanism should be ex-post efficient, in that the project is started if and only if it is positive NPV given the information revealed in the mechanism. These restrictions turn out to be sufficient for the existence of informational black holes even in optimal mechanisms. Finally, if we impose that the mechanism also has to be robust to the introduction of arbitrarily small costs of submitting a bid, we show that even an optimal mechanism cannot achieve higher efficiency than a standard auction.

Our paper is related to several different strands of literature. At a general level, our paper is related to the literature on the social value and optimal size of financial markets. Several papers have argued that gains associated with purely speculative trading or rent-seeking activities can attract too many entrants into financial markets (see, e.g., Murphy, Shleifer and Vishny (1991) and Bolton, Santos and Scheinkman (2016)). We provide an alternative mechanism in which each market participant possesses valuable information for guiding real production, but competition inhibits the effective use of information. Also, complementary to our results, a few papers in auction theory show that restricting the number of bidders can be optimal if agents have heterogeneous preferences or valuations. Bulow and Klemperer (2002) show that this

can happen in an auction in which bidder valuations depend on a common value component that is the sum of the independently drawn bidder signals and a (very small) private value component. Samuelson (1985) and Levin and Smith (1994) show that it may be optimal to restrict entry in auctions where bidders incur a participation cost.

Our paper is also related to the literature on information aggregation in auctions. With a few exceptions discussed below, this literature has focussed on the question of whether the price in an auction of an existing asset converges to the true value of the asset as the market grows large. Kremer (2002) and Han and Shum (2004) show that the price in ascending-price auctions always aggregates information in this sense. Wilson (1977), Milgrom (1979), and Milgrom (1981) show that in first-price and second-price auctions the price aggregates information only under special assumptions about the signal distribution, but do not study how the informativeness of the auction varies with market size. For multi-unit auctions, Pesendorfer and Swinkels (1997) show that the price converges to the true value of the asset in uniform-price auctions if the number of units sold also grows sufficiently large. In all these settings, information is irrelevant for surplus and a seller would always prefer the market to be as large as possible and would always prefer to sell all units to the highest bidder, regardless of whether prices aggregate information or not. Our paper complements these results in three ways. First, we show that once information is valuable for production, the ascending-price auction no longer aggregates information as the market grows large, and observing bids does not improve information aggregation in first- and second-price auctions. Second, we show that not only do the auctions not aggregate information as the market grows large, but the informational content can decrease with market size. Third, by making the link between surplus creation and information explicit, we show that surplus and entrepreneurial profits can be improved by limiting the size of the market in the single-unit case, or by running a multi-unit auction if this is feasible.

Related to our results, Rostek and Weretka (2012) show that smaller markets can be better at aggregating information than larger markets. They consider a double-auction where traders have imperfectly correlated values rather than common values. Rostek and Weretka (2012) show that prices may become less informative with market size if correlation of values is heterogenous across traders.

Similar to our paper, Atakan and Ekmekci (2014) also study a setting where information learnt in an auction is valuable for production decisions, but in the context of a uniform-price auction of many objects where the buyer of each object can make a separate decision about how to use it. In their setting, the bidder's value function is non-monotonic in his signal which they show can result in failure of information aggregation. Neither the assumption of non-monotonicity nor the assumption that multiple

winners take different actions, which are essential for their results, are natural in the project financing setting we are focussing on.

Like us, Broecker (1990) studies a project financing setting. He considers a special case of our model when first-price auctions are used, signals are binary, and investors who provide financing do not have the option to cancel a project after an offer is accepted. Broecker (1990) does not study information aggregation and surplus specifically and does not consider the effect of reducing the number of bidders, releasing information, revealing bids, or allowing bidders to endogenously decide on the investment after the auction is over.

More generally, the link between the informativeness of financial markets (such as stock markets) and real decisions by firms or governments is studied in the relatively recent "feed back" literature (for a summary of this literature, see Bond, Edmans and Goldstein (2012)). The closest to our work in this literature are the papers by Bond and Goldstein (2014) and Goldstein, Ozdenoren, Yuan (2011) who show that when an economic actor takes real decisions based on the information in asset prices, they affect the incentives to trade on this information in an endogenous way that may destroy the informational efficiency of the market. None of these papers analyze the effect of market size on efficiency, which is one of our main objectives. Furthermore, our paper shows that informational and allocational efficiency can fail even in the primary market for capital, where investors directly bear the consequences of their actions.

1.   Model setup

We consider a penniless entrepreneur seeking outside financing for a new project from a set of N potential investors indexed by $i \in \{1, ..., N\}$.[4] All agents are risk neutral. The project requires an investment of I and yields a random cash flow V if started. The project can be of two types: good (G) and bad (B), where a good project is positive net present value and a bad project is negative net present value:

$$E(V - I|G) > 0 > E(V - I|B). \tag{1}$$

The assumption of two types of projects is for convenience only—all of our results generalize to cases with more types or a continuum of types. The investment amount I can also be interpreted more broadly as an opportunity cost foregone if the project is started. For example, it can represent the outside option of the entrepreneur in another

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[4]Although we assume the entrepreneur has zero wealth to invest in the project, this is not essential for our results. Our results generalize to situations where the entrepreneur has either wealth or other assets to pledge against the project.

employment. Alternatively, V can represent the cash flows of an existing asset in a particular use, while I is the value in an alternative use. What is important is that V is the uncertain variable about which the market has dispersed information, while I is either a known quantity or a random variable about which all available information is public.

No one knows the type of the project, but investors each get a noisy private signal $S_i \in [0, 1]$ about project type. Signals are drawn independently from a distribution with cumulative distribution function $F_G(s)$ and density $f_G(s)$ if the type is good, and from a distribution with cdf $F_B(s)$ and density $f_B(s)$ if the type is bad. We make the following assumption about the signal distribution:

ASSUMPTION 1: *Signals satisfy the monotone likelihood ratio property (MLRP):*

$$\forall s > s', \quad \frac{f_G(s)}{f_B(s)} \geq \frac{f_G(s')}{f_B(s')}.$$

Both $f_G(s)$ and $f_B(s)$ are continuously differentiable at $s = 1$, $f_B(1) > 0$, and $\lambda \equiv f_G(1)/f_B(1) > 1$.

Without loss of generality, we will also assume that $f_G(s)$ and $f_B(s)$ are left-continuous and have right limits everywhere. Assumption 1 ensures that higher signals are at least weakly better news than lower signals. Assuming that densities are continuously differentiable at the top of the signal distribution simplifies our proofs, but is not essential for our results.

We denote the likelihood ratio at the top of the distribution by λ, a quantity that will be important in our asymptotic analysis. Assuming $\lambda > 1$ ensures that MLRP is strict over a set of non-zero measure, which in turn implies that as $N \to \infty$, an observer of all signals would learn the true type with probability one. Therefore, for large enough N, the aggregate market information is valuable for making the right investment decision.

To focus our analysis on the most interesting case, we make the stronger assumption that the signal of a single investor can take on values such that the project can be either negative or positive NPV:

ASSUMPTION 2: $E(V - I|S_i = 0) < 0 < E(V - I|S_i = 1).$

Assumption 2 is not essential for our results, what matters is that the investment decision is non-trivial conditional on observing a sufficient number of signals, which is already guaranteed by Assumption 1.

Although the signal space is continuous with no probability mass points, it can be used to represent discrete signals by letting the likelihood ratio $f_G(s)/f_B(s)$ follow a step-function which jumps up at a finite set of points. All signals within an interval over which the likelihood ratio is constant are informationally equivalent and represent the same underlying discrete signal. Following Pesendorfer and Swinkels (1997), we call such intervals "equivalence intervals." Representing discrete signals as equivalence intervals is a convenient way of making strategies pure when they are mixed in the discrete space: one can think of a continuous signal s as a combination of a discrete signal and a random draw from the equivalence interval, where a different draw can result in a different strategy even when the underlying discrete signal is the same.

Investors compete with each other to finance the project by submitting offers to the entrepreneur. We assume that the entrepreneur can only accept financing from a single investor. The assumption that outside ownership has to be concentrated is realistic in many corporate finance contexts, where a dispersed ownership structure can lead to free-riding and coordination problems that impede the running of the firm (see, for example, Myers (1977), Grossman and Hart (1980), Shleifer and Vishny (1986), and Gertner and Scharfstein (1991)). In Section 4.2, we show that if the assumption of concentrated ownership is relaxed, the efficiency of the market can be improved.

We model competition as happening through one of the standard single-unit auction formats (first-price, second-price, and ascending-price auctions). These market structures approximate most real-world selling procedures, including informal settings where investors approach the entrepreneur with unsolicited offers.[5]

Our results hold both for *cash* auctions, in which investors submit cash bids for full ownership of the right to start the project, and *security* auctions, in which investors finance the project in exchange for a security backed by the cash flow V of the project. One example of a security auction is a setting where banks offer loans at interest rate R_i and the bank which submits the lowest interest rate gets to finance the project, while another is a setting where venture capitalists offer to finance the project in exchange for an equity stake. Although the real-world applications we have in mind are usually security auctions, we choose to focus on cash auctions to make the exposition as transparent as possible and to simplify comparison with the standard auction literature. We show that all results hold for security auctions in Section 6.1.

In a first-price cash auction, investors submit sealed cash bids for ownership of the project rights. The highest bidder wins the auction, pays his bid to the seller, and gets the right to decide whether to start the project or not. A second-price auc-

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[5]In a companion paper (Axelson and Makarov (2016)), we show that most of our results are robust to modelling competition as a sequential search market in which an entrepreneur visits investors in sequence.

tion is the same except that the winning investor pays the bid of the runner-up. An ascending-price auction proceeds as follows. Bidding starts at 0 and the price is gradually increased until all but one investor remains. All investors can see at which price other investors drop out, and an investor who has dropped out cannot reenter the auction. The last remaining investor wins the auction and pays the price at which the runner-up dropped out, and then decides whether to invest or not.

The ascending-price auction is of special importance for two reasons. First, it is probably the best approximation to most real-world settings, be it formal auction procedures or informal rounds of bidding where bidders have the chance to react to competitors. Second, because bidders can make their strategies contingent on the drop-out behavior of competitors, the ascending-price auction has the best information aggregation properties of all standard auctions (including multi-unit auctions; see Kremer (2002) and Han and Shum (2004)), as well as generating the highest revenues to the seller (see Milgrom and Weber (1982) for revenue comparisons between standard auction formats and Lopomo (2000) for a mechanism-design approach.) Thus, our results about the failure of information aggregation are the starkest for the ascending-price auction.

For first- and second-price auctions, where bids are sealed, we need to specify what post-auction information is made available to the decision maker. Our benchmark assumption is that the winner gets to observe all the submitted bids before making the investment decision. This is a natural assumption in our setting since the entrepreneur is always at least weakly better off revealing this information to whoever makes the investment decision, and since in the context of security auctions, the entrepreneur herself will be one of the decision makers. Under this assumption, our results on the failure of the auction to generate valuable information are the most striking. In fact, we will show that whether the investment decision can be made contingent on the bid history or not is irrelevant for efficiency once we require equilibria to satisfy natural robustness criteria.

2. Equilibrium bidding

In this section, we show that no standard auction can produce valuable information beyond what is contained in the top two signals. In Section 3, we use this fact to analyze how market size affects investment efficiency and entrepreneurial reveneus.

We begin this section by focussing on a specific equilibrium of the second-price auction to build intuition. We then show that the introduction of a real investment decision in an otherwise standard auction setting leads to the existence of multiple

equilibria, and introduce two natural robustness criteria to eliminate fragile equilibria. In Section 2.3, we show that our results extend to first-price and ascending-price auctions as well.

As a benchmark, we review the standard auction theory setting where there is no investment decision to be made. For this purpose, assume that the investment into the project has already been made by the entrepreneur, whereafter the project is sold in an auction. Thus, the auction is of an asset that pays a random amount V.

We denote the order statistics of the N signals received by investors by $Y_{1,N}, ..., Y_{N,N}$ so that $Y_{1,N}$ represents the highest signal, $Y_{2,N}$ represents the second-highest signal, et cetera. As shown in Milgrom (1981), in the second price auction it is an equilibrium for a bidder with signal s to bid $b(s)$ given by:

$$b(s) = E(V|Y_{1,N} = Y_{2,N} = s).$$

A bidder bids his value of the asset conditional on just marginally winning the auction, which happens when he has the highest signal ($Y_{1,N} = s$) and the second highest signal is the same ($Y_{2,N} = s$). Deviating by bidding higher would make a bidder win in situations where the price is higher than his valuation conditional on winning; while deviating by bidding lower would make a bidder lose in situations where the price would have been lower than his valuation.[6]

Figure 1 shows the equilibrium bidding function in the standard setting. Bids are strictly increasing in the signal of a bidder, which implies that anyone who observes the history of bids ex post can recover all the information available in the market. This is also true in first-price and ascending-price auctions. Thus, the auction generates all relevant information possessed by the bidders about project cash flows.

We now turn to our setting in which after winning, an investor has to decide whether to invest I and start the project or not. Thus, unlike in the standard setting, information learnt in the auction has real value. Proposition 1 describes a particularly transparent equilibrium, in which investors simply lower their bids by I relative to the standard setting to reflect the investment amount, and cap their bids at zero to reflect the option of not investing:

PROPOSITION 1: *In the second-price auction, there is an equilibrium where investors bid according to*

$$b(s) = \max(E(V - I|Y_{1,N} = Y_{2,N} = s), 0).$$

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[6] The fact that bids are revealed to the winner after the auction has no impact on bidding strategies, since this information cannot be used for anything ex post in the standard setting.

Investors with $S_i \leq s_N$ bid zero, where s_N is defined as

$$s_N = \sup s : E(V - I|Y_{1,N} = Y_{2,N} = s)) \leq 0. \tag{2}$$

The winner invests in the project if $Y_{2,N} > s_N$ or if his own signal is sufficiently high, and otherwise does not invest.

We postpone the proof until Proposition 2, which considers a more general case. Figure 2 shows the equilibrium bidding function relative to the standard setting. Investors with signals below the threshold s_N bid zero and do not invest if they win the auction. To see why, suppose an investor with a signal equal to the threshold s_N expects all other investors to follow the strategy in Proposition 1. If he wins with a bid of zero, all he learns from observing other bids (which are all zero) is that all other investors have signals below s_N. By the definition of s_N, his updated NPV of the project is then negative. Therefore, he never invests after winning, which justifies his bid of zero.[7]

Investors with signals above s_N submit strictly positive bids which are strictly increasing in their signal. If such an investor wins and the second highest bid is also strictly positive, the updated NPV of the project is positive by the definition of s_N, so the winner will invest. If all other bids are zero, the winner may or may not invest depending on how high his own signal is. Hence, investors with signals above s_N expect to sometimes win when the project is positive NPV, which justifies their positive bids. Bids are strictly increasing since investors with higher signals attach strictly higher NPV to the project. Relative to the standard setting, only signals above s_N can be recovered from observing bids. Signals below the threshold s_N cannot be recovered, since all bids are zero. We therefore call the signal range $[0, s_N]$ the *informational black hole*, and the threshold s_N the *black-out level*.

The existence of the informational black hole leads to inefficient investment behavior relative to the situation where all signals are observed because a winner will assume that all investors who bid zero had "average" signals. In particular, when signals in the black hole are close to the black-out level, the project will often not be undertaken even though it is positive NPV, while if signals in the black hole are very pessimistic the project will often be undertaken even though it is negative NPV. This loss of efficiency leads to a reduced surplus, and hence lower expected revenues to the entrepreneur relative to the first best. The magnitude of investment inefficiencies is determined purely by the size of the informational black hole and does not depend on the particular

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[7]We allow for the possibility of negative bids, but because investors always have the option to abandon the project they never submit negative bids in equilibrium.

shape of the bidding function, as long as bids outside of the informational black hole are strictly increasing. We use this fact below to extent our results to other auction formats.

2.1. Strategic complementarities and multiple equilibria

Because the size of the informational black hole affects the efficiency of investment decisions, there are strategic complementarities among investors. When an investor expects others to bid zero over a large signal interval so that the informational black hole is larger, he expects surplus from the auction to be lower because of the lost information, which justifies bidding lower and in particular bidding zero for higher signal realizations. Hence, the expectation of a larger informational black hole can be self-fulfilling. We next show that this feedback loop can lead to a continuum of equilibria characterized by different sizes of the informational black hole.

Proposition 2 establishes an upper and a lower bound on the equilibrium black-out level and shows that any black-out level in between can be supported in equilibrium:

PROPOSITION 2: *Define the threshold $\underline{s}_N$ as the highest signal such that*

$$E[V - I|Y_{1,N} =, \ldots, = Y_{N,N} = \underline{s}_N] \leq 0, \tag{3}$$

and the threshold $\overline{s}_N$ as the highest signal such that

$$E(V - I|Y_{1,N} = \overline{s}_N) \leq 0. \tag{4}$$

For any $\hat{s} \in [\underline{s}_N, \overline{s}_N]$, there is a symmetric monotone equilibrium in the second-price auction with black-out level $\hat{s}$, in which an investor with a signal s bids

$$b(s; \hat{s}) = E\left[\max\left(E[V - I|\boldsymbol{S}_{>\hat{s}}], 0\right)|Y_{1,N} = Y_{2,N} = s\right], \tag{5}$$

where $\boldsymbol{S}_{>\hat{s}}$ is the signal vector of investors censored below $\hat{s}$:

$$\boldsymbol{S}_{>\hat{s}} \equiv \{\max(S_i, \hat{s})\}_{i=1}^{N}.$$

There is no symmetric monotone equilibrium with a black-out level outside this range.

Proof. The upper bound $\overline{s}_N$ on feasible black-out levels is defined such that an investor who learns only that he has the top signal will invest if and only if his signal is above $\overline{s}_N$. To see why this is an upper bound, suppose to the contrary that there is an equilibrium in which an investor with a signal slightly above $\overline{s}_N$ is supposed to bid zero. If such an

investor wins the auction, his updated NPV of the project is strictly positive from the definition of $\overline{s}_N$, so he makes strictly positive profits when winning. By an arbitrarily small increase of his bid, he is guaranteed to receive this profit without affecting the price he pays, a profitable deviation.

The lower bound $\underline{s}_N$ on feasible black-out levels is defined such that the project just breaks even conditional on *all* investors having this signal. Suppose to the contrary that there is an equilibrium where an investor with a signal $s < \underline{s}_N$ bids a strictly positive amount. When such an investor wins the auction in a monotone equilibrium, other investors have signals weakly below his. By the definition of $\underline{s}_N$, the project is therefore always negative NPV when such an investor wins, which is inconsistent with a strictly positive bid.

We next show that we can support any black-out level $\hat{s} \in [\underline{s}_N, \overline{s}_N]$ in equilibrium. An investor who expects the black-out level to be $\hat{s}$ will assume that if he wins, he will be able to recover all signals above the black-out level when making his investment decision, which is equivalent to observing the censored vector of signals $\mathbf{S}_{>\hat{s}}$ defined in the proposition. Since a winner will invest only if the NPV is positive conditional on observing $\mathbf{S}_{>\hat{s}}$, this is an auction of an option to invest which has random value $\max\left(E[V - I|\mathbf{S}_{>\hat{s}}], 0\right)$. The equilibrium bidding function $b(s; \hat{s})$ then takes the standard form derived in Milgrom (1981): Investors bid their value of the project rights conditional on just marginally winning. The bidding function (5) will indeed constitute an equilibrium in our setting if it is consistent with the belief that the black-out level is $\hat{s}$, that is, if $b(s; \hat{s})$ is zero for $s \leq \hat{s}$ and is strictly positive and increasing for $s > \hat{s}$. Notice that investors with signals below the black-out level $\hat{s}$ learn only that all signals are in the informational black hole when they win, which results in zero option value of the project for any $\hat{s} \leq \overline{s}_N$. Therefore, it is optimal for them to bid zero. To prove that $b(s; \hat{s})$ is strictly positive for $s > \hat{s}$ notice that if an investor with a signal above the the black-out level $\hat{s} \geq \underline{s}_N$ wins the auction then there is a positive probability that all other investors have their signals in the interval $[\underline{s}_N, \hat{s}]$, which results in positive option value, and therefore, a positive bid. The proof that $b(s; \hat{s})$ is strictly increasing for $s > \hat{s}$ is the same as in Milgrom (1981). *Q.E.D.*

The feedback effect from the destruction of information to the value of the option to invest allows the black-out level to take any value in the range $[\underline{s}_N, \overline{s}_N]$. The least efficient equilibrium is the one with the highest black-out level $\overline{s}_N$. In this equilibrium, only information in the highest signal affects the investment decision and no other information can be used. The equilibrium in Proposition 1 with black-out level s_N is more efficient because the top two signals can affect the investment decision. Finally, the equilibrium with the lowest black-out level $\underline{s}_N$ is the most efficient because

investment can be conditioned on the largest set of information.[8] We next show that equilibria with black-out levels below s_N are very fragile, so that the equilibrium in Proposition 1 is in fact the most efficient robust equilibrium.

2.2. Robust equilibria

In this section, we introduce two robustness criteria. The first one requires that an equilibrium is a limit of equilibria in auctions where bids have to be made in increments of some $\delta > 0$ as we let δ go to zero. Since all real-world markets have discrete price grids we view this as a natural requirement. We call such an equilibrium δ-*bid robust*.

Our second robustness criterion requires that an equilibrium is a limit of equilibria in auctions where investors have to incur some cost $\varepsilon > 0$ for submitting a bid as we let ε go to zero. We allow investors to not submit a bid to avoid this cost. We call such an equilibrium ε-*cost robust*.

The equilibria in the standard setting in all auction formats are both δ-bid and ε-cost robust. In our setting, Proposition 3 shows that the more efficient equilibria with black-out levels below s_N are not δ-bid robust, and that equilibria with black-out levels below $\overline{s}_N$ are not ε-cost robust.

PROPOSITION 3: *There is no δ-bid robust symmetric monotone equilibrium in the second-price auction with black-out level below s_N. There is no ε-cost robust symmetric monotone equilibrium in the second-price auction with black-out level below $\overline{s}_N$.*

Proof. See the Online Appendix.

The formal proof is in the appendix. Here we provide a sketch of the proof. Consider an equilibrium with a black-out level $\hat{s} < s_N$, and an investor with a signal s very slightly above $\hat{s}$ who submits the minimal bid δ. If he wins the auction at price zero, so that all other bids are in the informational black hole, he concludes that the project is negative NPV and he does not invest. If he wins when only one other investor bids δ, the updated NPV is also negative by the definition of s_N. Hence, he loses the price δ. The only circumstance in which the investor can make profits from investing is when there are at least two other investors who bid δ. But for small δ, as we show in the formal proof, the probability of tying at δ with more than one investor becomes negligible relative to the loss event of tying with just one investor. Hence the investor cannot break even with a non-zero bid, which contradicts that the black-out level is $\hat{s} < s_N$.

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[8]Note that even this equilibrium has inefficiencies relative to the first best where all signals are observed because the black-out level $\underline{s}_N$ is strictly positive.

Note that this argument does not extend to equilibria with black-out levels $\hat{s} \geq s_N$ because by the definition of s_N the project is positive NPV when a winner outside of the black hole ties with one other investor. However, if investors have to incur some arbitrarily small cost for submitting a bid (but can stay out of the auction for free), a parallel argument shows that the only viable equilibrium black-out level is the upper bound $\overline{s}_N$ even when bids do not need to be in discrete increments. To see this, consider again an investor very slightly above a candidate black-out level $\hat{s} < \overline{s}_N$. From the definition of $\overline{s}_N$, such an investor can only make profits if at least one other investor submits a lower but strictly positive bid. But the probability of this event becomes arbitrarily small for investors arbitrarily close to the black-out level, so that they cannot recoup the cost of submitting a bid. Therefore, the equilibrium unravels so that the only viable threshold is $\overline{s}_N$.

2.3.   Ascending and First-Price Auctions

We now extend our results to the ascending-price and first-price auction formats. The logic for the first-price auction is the same as for the second-price auction. Given a candidate black-out level $\hat{s}$ we can view our setting as an auction of an object with value $\max\left(E[V - I|\mathbf{S}_{>\hat{s}}], 0\right)$. This is the value of the option to do the project for someone who expects to observe all signals above the black-out level. In the first-price auction, the winner can infer all signals above the black-out level by observing bids ex post.

Constructing an equilibrium then follows the same steps as in the standard setting of Milgrom and Weber (1982), with the extra condition that the candidate black-out level has to be consistent with the equilibrium bidding function. As in Milgrom and Weber (1982), an equilibrium bid in the first price auction is an average of the bids $b(s; \hat{s})$ investors with lower signals would have submitted in the second-price auction:

$$b^I(s; \hat{s}) = \int_0^s b(s'; \hat{s}) dL(s'|s), \tag{6}$$

where $b(s'; \hat{s})$ is the bidding function (5) from the second-price auction specified in Proposition 2, and

$$L(s'|s) = \exp\left(\int_s^{s'} \frac{h(s'|s)}{H(s'|s)} dt\right).$$

The function $H(\cdot|s)$ is the distribution of $Y_{2,N}$ conditional on $Y_{1,N} = s$ and $h(\cdot|s)$ is the associated conditional density function.

Note that since $b(s'; \hat{s})$ is strictly positive if and only if $s' > \hat{s}$, the same is true for $b^I(s)$, so that the bidding function is consistent with the black-out level $\hat{s}$. Following the same steps as the ones in the proof of Theorem 14 of Milgrom and Weber (1982)

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one can then show that bidding strategies $b^I(s; \hat{s})$ form an equilibrium in the first-price auction for any black-out level $\hat{s} \in [\underline{s}_N, \overline{s}_N]$.

The equilibria of the first-price auction turn out to be even more fragile than for the second-price auction. This is because a winner has to pay his own bid, so that he incurs a loss whenever he does not invest. Following similar steps as for the second-price auction, one can show that only the equilibrium with the highest black-out level $\overline{s}_N$ is δ-bid and ε-cost robust.[9]

Next consider the ascending-price auction. For black-out levels in the interval $\hat{s} \in [s_N, \overline{s}_N]$, exactly the same arguments as for the first-price and second-price auctions can be used to construct equilibria as in Milgrom and Weber (1982), where the object for sale has value $\max(E[V - I|\mathbf{S}_{>\hat{s}}], 0)$. If the price goes above zero, which happens only if at least two investors stay in, the project is always positive NPV from the definition of s_N, so that the auction is completely standard.

However, for black-out levels below s_N we have to take special care in defining how investors can react when other investors drop out as the price increases above zero. When multiple investors drop out at price zero, investors who otherwise would stay in the auction may want to drop out immediately as well. Modelling this requires either that we allow players to condition their actions on the simultaneous actions of other players, or that investors can drop out just as the price goes above zero. The first alternative is logically inconsistent, while the second is not well defined when price is increased continuously. For this reason we model price as increasing in discrete increments, and study equilibria in the limit as the size of the increments go to zero. Proposition 4 shows that the feasible equilibrium black-out levels are then exactly the same as the ones we derived for the robust second-price auctions in Proposition 3.

PROPOSITION 4: *There is no δ-bid robust symmetric monotone equilibrium in the ascending-price auction with black-out level below s_N. There is no ε-cost robust symmetric monotone equilibrium in the ascending-price auction with black-out level below $\overline{s}_N$.*

Proof. See the Online Appendix.

The argument that the black-out level cannot be below s_N with discrete bids follows a similar logic as for the second-price auction. Suppose to the contrary that there is an equilibrium in which the black-out level is some signal $\hat{s} < s_N$, so that an investor with a signal just slightly above $\hat{s}$ stays in the auction until the price is slightly positive. This investor can win under three circumstances. First, he can win if all other bidders

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[9]The formal proof is in the Online Appendix.

drop out at zero, in which case it is optimal not to start the project, which involves zero profits because the price is also zero. Second, he can win if only one other bidder stays in the auction and this bidder has a signal below s_N, in which case it is also optimal not to start the project. Since the price is positive, this involves some losses. Third, he can win if more than two other bidders stays at positive prices, which could imply that the project is positive NPV. But in this scenario he only wins if other bidders have lower signals than him, a very small probability event. The expected profits will therefore be negative.

The argument for why an arbitrarily small cost ε of submitting a bid leads to the maximum black-out level is the same as for the second-price auction. For any candidate lower black-out level, an investor just above the threshold would be unable to recoup his cost because the probability of winning when the project is positive NPV is too small.

2.4. Post-auction information

In the previous analysis, we have assumed that the investment decision can be made contingent on the whole history of bids. For the first- and second-price auctions, this requires that all the losing bids in the auction are disclosed to the winner before he makes the investment decision.[10] We next show that that any $\delta-$ or $\varepsilon-$robust equilibrium is independent of the amount of post-auction information disclosed. In particular, any $\delta-$ or $\varepsilon-$robust equilibrium exists even if there is no disclosure of bids after the auction.

PROPOSITION 5: *Any $\delta-$ or $\varepsilon-$robust equilibrium is independent of the amount of post-auction information disclosed. Under no disclosure, the set of equilibria coincides with the set of $\delta-$robust equilibria.*

Proof. To prove the first part of the proposition, note that the only situation where disclosing bids potentially reveals more information than just observing the price is when the price is strictly positive—if the price is zero, all losing bids must be zero as well. We have shown above that any robust equilibrium has a black-out level of at least s_N. But from the definition of s_N, in such equilibria the winner always invests when the price is strictly positive. Hence, investment behavior is independent of the information contained in losing bids (other than the price), and so post-auction disclosure has no effect on pay offs and equilibrium strategies. To prove the second part of the proposition, first consider a second-price auction with no post-auction disclosure of

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[10]In ascending-price auction, no such disclosure is necessary since each bidder observes drop-out behavior during the auction.

bids. Suppose contrary to the claim in the proposition that there exists an equilibrium with a black-out level $\hat{s}$ strictly below s_N. From the definition of s_N, an investor with signal $s \in [\hat{s}, s_N]$ who wins the auction will conclude that the project is negative NPV. Hence, he cannot break even with a strictly positive bid, so no such equilibrium exists. Next, consider the first-price auction with no post-auction disclosure of bids. Suppose contrary to the claim in the proposition that there exists an equilibrium with a black-out level $\hat{s}$ strictly below $\overline{s}_N$. From the definition of $\overline{s}_N$, an investor with signal $s \in [\hat{s}, \overline{s}_N]$ who wins the auction will conclude that the project is negative NPV. Hence, he cannot break even with a strictly positive bid, so no such equilibrium exists. *Q.E.D.*

Proposition 5 shows that irrespective of whether the full bid history is observed or not, robust equilibria can generate no more useful information than what is contained in the top two order statistics of the signal distribution. Next, we use this fact to study the effect of market size on investment efficiency and entrepreneurial revenues.

3.   Market size and informational efficiency

In this section, we first show that even as the market grows infinitely large so that aggregate information is perfect, substantial investment mistakes occur. We then show that small markets can create both higher social surplus and higher entrepreneurial revenues than large markets. Finally, we endogenize the size of the market by assuming that investors have some cost of acquiring information and show that inefficiently large financial markets can occur in equilibrium.

3.1.   Surplus in large markets

In our setting, as the market grows larger, the stronger winner's curse leads to a larger informational black hole. Proposition 6 below shows that the informational black hole approaches the whole range of signals as N goes to infinity, and characterizes limiting investment behavior:

PROPOSITION 6: *The black-out levels s_N and $\overline{s}_N$ go to 1 with N:*

$$\overline{s}_N = 1 - \frac{a_1}{f_B(1)} \frac{1}{N} + o\left(\frac{1}{N}\right), \quad s_N = 1 - \frac{a_2}{f_B(1)} \frac{1}{N} + o\left(\frac{1}{N}\right),$$

where a_1 and a_2 are strictly positive constants with $a_2 > a_1$, and

$$\lim_{\lambda \to \infty} a_i = 0, \quad \lim_{\lambda \to \infty} \lambda a_i = \infty, \quad i = 1, 2.$$

Both over- and under-investment happens with positive probability as N goes to infinity. For equilibria with black-out level s_N:

$$\lim_{N \to \infty} \Pr(\textit{Investment } |B) = 1 - e^{-a_2}(1 + a_2),$$
$$\lim_{N \to \infty} \Pr(\textit{No Investment } |G) = e^{-\lambda a_2}(1 + \lambda a_2).$$

For equilibria with black-out level $\overline{s}_N$:

$$\lim_{N \to \infty} \Pr(\textit{Investment}|B) = 1 - e^{-a_1},$$
$$\lim_{N \to \infty} \Pr(\textit{No Investment } |G) = e^{-\lambda a_1}.$$

Proof: See the Online Appendix.

Proposition 6 shows that the first-best is achieved if and only if the likelihood ratio increases without bound at the top signals. Our result is similar to that in Milgrom (1979), Pesendorfer and Swinkels (1997), and Kremer (2002) who show that under the same condition, the price in a standard first and second-price auctions converges to the true value of the asset. This is no coincidence, since the price summarizes the information contained in one of the top order statistics, which is also the information used for investment decisions in our setting. In contrast to the standard setting, in our setting nothing extra can be learnt from observing the full bid history—in particular, the ascending-price auctions does no better than the second-price auction.

As Pesendorfer and Swinkels (1997) note, the condition of infinite λ is very strong—it requires that there is a signal s such that an observer of that signal can rule out that the project is bad. Therefore, in what follows, we focus on the case of finite λ.

3.2. Smaller versus larger markets

We next show that not only is the first best not achieved in the limit, but surplus can actually go down as the market grows. Since investment depends entirely on the realization of either the highest or the second highest signal among bidders, increasing the market size is beneficial only if top signals become more informative as the "sample size" of signals grows.

PROPOSITION 7: If $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ is a decreasing (increasing) function at $s = 1$ then there is an $\underline{N}$ such that surplus decreases (increases) with $N > \underline{N}$ for equilibrium black-out levels s_N and $\bar{s}_N$ in all auction formats.

Proof: See the Online Appendix.

The ratio $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ is a conditional likelihood ratio, which measures the informativeness of the top signal s if signals are restricted to be drawn from the interval $[0, s]$. If this ratio decreases with s, it means that not much of the information in the signal distribution is concentrated at the top end. Adding bidders then reduces efficiency, since it shifts the distribution of the pivotal order statistics $Y_{1,N}$ and $Y_{2,N}$ towards the less informative part of the distribution.

One example where the market becomes less efficient as the size increases is when information is coarse such that signals can take on only a finite number of discrete values. In our continuous representation, a discrete signal corresponds to an interval $(a, b]$ such that the likelihood ratio $f_G(s)/f_B(s))$ is flat for $s \in (a, b]$. At the top of the signal distribution, the likelihood ratio is then a constant λ over some interval $(a, 1]$, so that

$$\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)} = \lambda \frac{F_B(s)}{F_G(s)},$$

which decreases in s. Intuitively, the highest signal in a very large market will almost surely be in the highest interval regardless of the quality of the project. Hence, the realization of the top signal is not particularly informative. In a smaller market, on the other hand, observing that the top signal is in the highest interval makes it more likely that the project is good rather than bad.

Figure 3 plots surplus as a function of the market size for binary signals. We assume that if the project is good, investors get only high signals, while if the project is bad, they are equally likely to get high and low signals. This binary signal structure can be represented by setting $f_B(s) = 1$ for all $s \in [0, 1]$, and setting $f_G(s) = 0$ for $s \in [0, 1/2]$ and $f_G(s) = 2$ for $s > 1/2$. We provide the full calculations for this example in the appendix.

In line with the results of Proposition 7 we can see in Figure 3 that in the least efficient equilibrium social surplus declines with the market size for all N—surplus is maximized with a single investor. In the most efficient robust equilibrium surplus is maximized with two investors and then declines with market size.

We next consider entrepreneurial revenues as a function of market size. If the entrepreneur has the power to pick the number of bidders, he will do so in order to maximize revenues rather than surplus. The private optimum may differ from the

social optimum if the entrepreneur captures only part of the surplus. In our setting, the *split* of the surplus between the entrepreneur and investors has similar comparative statics with respect to the number of bidders as in the standard auction theory setting of Milgrom and Weber (1982), where surplus itself is fixed. In particular, the fraction of surplus captured by the entrepreneur goes to one with N in all auction formats. Hence, if surplus increases with N, there is no conflict between the private and social optimum—the entrepreneur will prefer the maximal number of bidders.

The non-trivial case is when surplus decreases with N. Will the entrepreneur find it optimal to restrict the number of bidders even though this may entail surrendering a higher fraction of the surplus to investors? Our answer is a qualified "Yes". The next proposition gives a sufficient condition for when this is the case.

PROPOSITION 8: *Suppose that there exists an $\varepsilon > 0$ such that $f_G(s)/f_B(s) = \lambda$ for $s \in [1 - \varepsilon, 1]$. Then, there exists some $\underline{N}$ such that revenue is strictly decreasing in N for $N \geq \underline{N}$.*

Proof: We know that there exists some $\underline{N}$ such that $\overline{s}_N \geq 1 - \varepsilon$ for all $N \geq \underline{N}$. Over this interval, $\frac{f_G(s)}{F_G(s)} / \frac{f_B(s)}{F_B(s)}$ is strictly decreasing, and so from Proposition 7, surplus is decreasing in N for $N > \underline{N}$. All bidders must make the same expected profits since they are in the same equivalence interval. Since some bidders do not participate, the expected bidder profits are zero, and hence revenues coincide with surplus. *Q.E.D.*

To understand this result, note that surplus decreases with N when the top of the signal distribution is relatively flat, so that investors who draw high signals are informationally close to each other. But when this is the case, investors also capture little informational rent even for moderate levels of N. In other words, increasing N beyond a certain level has little effect on the split of revenues but a large negative effect on surplus. As an illustration, in the example of Figure 3 where investors get binary signals, bidders earn exactly zero surplus whenever $N > 1$ because of competition between informationally identical bidders from the top equivalence interval. Hence, whenever surplus is maximized at some market size $N > 1$, the social optimum coincides with the entrepreneur's private optimum. For the least efficient equilibrium, the social optimum is to have one investor. For this case the entrepreneur may prefer inviting an extra bidder despite the loss of surplus in order to increase competition.[11]

The conditions in Proposition 8 are sufficient but not necessary for the entrepreneur to prefer a smaller market. As Example 1 in the next section shows, the entrepreneur

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[11]Even for $N = 1$, the entrepreneur can capture the full surplus if he has enough commitment power to set an appropriate reserve price.

will prefer a smaller market whenever the likelihood ratio does not increase too steeply at the top of the signal distribution.

Our results provide one explanation for why so many capital raising situations involve negotiations with a restricted set of investors rather than an auction open to everyone.

3.3. Can financial markets be too big?

In the previous section we established that small markets may be preferable both from the entrepreneur's and from a social surplus perspective. In this section we show that the equilibrium size of the market can be too large relative to both the social and the entrepreneurial optimum, and can be Pareto inferior relative to a market with one less investor.

If the entrepreneur can commit to seek financing from a restricted set of investors, the market can obviously never be larger than what is optimal for the entrepreneur. However, restricting the set of potential investors may be difficult in practice because it is ex post optimal for the entrepreneur to consider any offer he receives, even if the offer is unsolicited. In this section we therefore assume no commitment so that investors can enter any auction.

So far, we have assumed that investors observe signals for free to make our results on the failure of information aggregation in large markets as striking as possible. In order to have a non-trivial equilibrium market size, we now assume that investors face some costs of gathering information.

Assume that each potential investors i has a cost c_i of gathering information about the project, and that c_i is strictly increasing. We focus on the case where $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ is a decreasing function around $s = 1$ so that social surplus (gross of investor costs) is maximized at a finite market size. The socially optimal market size net of costs is then even smaller.

We also assume that MLRP holds strictly, which ensures that investors have strictly positive expected profits from participating in the market gross of their information gathering cost. We then have the following result:

PROPOSITION 9: *Suppose that $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ is a decreasing function around $s = 1$ and that MLRP holds strictly. Then, there is a $c > 0$ such that if sufficiently many investors have costs of gathering information below c, the equilibrium size of the market is larger than the socially optimal size. Lowering information gathering costs can lead to a decrease in social surplus.*

Proof: See the Online Appendix.

The proposition shows that there is no reason to believe that markets will become more efficient as information technology improves. This is in contrast to the predictions of Samuelson (1985) and Levin and Smith (1994) who study information costs in an otherwise standard auction theory setting. In both papers, the optimal size of the market goes to infinity as costs go to zero.

Proposition 9 shows that there can be too much entry in equilibrium relative to the social optimum. The next example shows that both investors and the entrepreneur can be better off if entry is restricted.

Example 1: Suppose that $f_B(s) \equiv 1$ and $f_G(s)$ is a truncation to the interval $[0, 1]$ of a normal distribution with mean 1 and standard deviation 0.75. The likelihood ratio $f_G(s)/f_B(s)$ is strictly increasing over $[0, 1]$, so MLRP holds strictly. Also, because the derivative of the likelihood ratio is zero at $s = 1$, the ratio $\frac{f_G(s)}{F_G(s)}/\frac{f_B(s)}{F_B(s)}$ is a decreasing function around $s = 1$. We assume that the net present value for a good project is 0.75, while a bad project has an NPV of minus one.

Panel A of Figure 4 shows social surplus gross of investor costs and the expected revenues to the entrepreneur as a function of the size of the market. The figure is drawn for the most efficient robust equilibrium where the black-out level is s_N. Social surplus is maximized at a market size of three, while the entrepreneur's revenues are maximized at a market size of four. The entrepreneur prefers a somewhat larger market size than what maximizes social surplus because increased competition between investors reduces their share of the surplus.

Panel B shows expected gross profits to investors from participating in the auction as a function of market size, as well as a particular specification for the cost c_i of information gathering for each investor. In equilibrium, investors will enter as long as expected profits cover their cost, so that for the specific costs drawn in the figure the first ten investors will enter in equilibrium with investor ten indifferent between entering and staying out. Hence, the equilibrium market size is larger than both the social optimum and the entrepreneur's optimum.

Now suppose that every investor's cost was just slightly larger. This would be the case if, for example, tax rates on venture capitalist profits are increased slightly. The equilibrium market size would drop to nine, which would constitute a Pareto improvement. Participating investors would make higher profits because of both reduced competition and more efficient investment decisions. The entrepreneur's revenues would increase because the increased surplus from more efficient investment outweighs the loss from reduced competition. Finally, the investor who drops out of the market is no worse off since he was just breaking even before.

4. Strategies for reducing the winner's curse

The source of inefficiency in our model is the effect the winner's curse has on the participation of pessimistic investors, an effect that becomes stronger as the market grows larger. In this section we discuss a number of strategies that can help to alleviate the winner's curse. First, we show that it may be beneficial to raise capital before important information is learnt in order to increase the option value embedded in the project. Second, we show that allowing a larger set of investors to co-finance the project helps reduce the winner's curse. Third, in contrast to results for standard auctions, we show that allowing investors to collude ex ante via bidding clubs can also improve efficiency and revenues. Finally, we discuss how adding an appropriately designed derivative market where investors can bet on project failures might eliminate the informational black hole. All these "fixes" rely on alternative trading mechanisms that may not always be implementable in practice. In Section 5 we provide a systematic treatment of the conditions that lead to informational black holes in general mechanisms.

4.1. Choosing when to finance and the linkage principle

Suppose that there is some exogenous signal affiliated with the value of the project that gets realized either before or after the auction. For example, this could be a signal about demand conditions for the products the project is meant to create, or any information the entrepreneur might have about the project that can be credibly communicated to the investors. The question we ask is whether it is better to run the auction before or after this information is released.

For standard auctions, where no action is taken, the linkage principle of Milgrom and Weber (1982) suggests that it is better to run the auction after all value-relevant information is realized in order to lower the informational asymmetry between bidders. However, in our setting we have an extra effect: If the signal is revealed after the auction but before the investment decision is made, the project has some real option value when bids are submitted, and so even investors with low signals might want to participate. This could break the destruction of information.

We now give an example where the linkage principle fails in our setting. Suppose that a public signal $S_P \in \{s_G, s_B\}$ will be released at date t, where $\Pr(S_P = s_G | B) = 0$ and $\Pr(S_P = s_G | G) = q$, $q \in (0, 1)$. Hence, when the public signal is s_G, the project NPV is positive regardless of the investors' signals.

Suppose first that the entrepreneur runs the auction after the public information

is released, as the linkage principle prescribes. We now calculate the expected surplus generated by the auction. With probability $q \Pr(G)$ the public signal reveals that the project is good, so surplus is $E(V - I|G)$. With probability $(1-q)\Pr(G) + 1 - \Pr(G)$, the public signal is s_B and the updated prior on the project being good is $\Pr(G|s_B) = \frac{\Pr(G)(1-q)}{\Pr(G)(1-q)+(1-\Pr(G))} < \Pr(G)$, in which case the auction generates some surplus W, which from Proposition 7 is strictly below the first-best surplus. The expected surplus is then

$$q\Pr(G)E(V - I|G) + ((1-q)\Pr(G) + 1 - \Pr(G))W < \Pr(G)E(V - I|G).$$

Suppose to the contrary that the entrepreneur runs the auction before the public signal is released, and that winners can wait to observe the public signal before they make the decision to start the project. In this case, everyone participates in the auction and there is no informational black hole. To see this, notice that even for the most pessimistic investors, the option to do the project has some strictly positive value since there is always some strictly positive probability that the public signal will reveal the project to be good. It is then easy to verify that bids will be strictly positive and strictly increasing in signals for all N. As a result, all informational properties of the auction are the same as in the standard setting. In particular, ascending-price auctions aggregate all information and leads to first-best investment decisions when the market grows large, and the same holds for first-price and second-price auction if bids are revealed ex post. Furthermore, the expected revenue converges to the expected surplus as N goes to infinity. Hence, the seller is better off running the auction before the public signal is revealed.

Remark 1: Our exercise in this section compares the effect of running the auction before or after some public release of information, rather than asking whether releasing information is better than never releasing it at all. In the standard model of Milgrom and Weber (1982) this distinction is irrelevant, since ex post releases of information have no impact on the expected value of the asset up for sale. If the choice is whether to release information before the auction or never, Theorem 18 of Milgrom and Weber (1982) can be applied to show that the linkage principle holds for the least efficient equilibria. Whether this version of the linkage principle holds for our wider set of equilibria is an open question.

Remark 2: The results in this section show that if the decision to start the project can be postponed indefinitely and costlessly, and if there is any possibility that the project can become positive net present value sometime in the future even for the most pessimistic investors, then the informational black hole will be eliminated and the auction will properly aggregate information. Hence an important underlying assumption

for our results is that the option to start the project has some natural expiration date, or that there are sufficient costs associated with keeping the option alive. We believe this to be a natural assumption for most real options.

4.2. Dispersed ownership

In the previous sections we assumed that only one investor ends up with a stake in the project. In this section we allow for the possibility that $K > 1$ investors can co-finance the project. Allowing for more investors to receive an allocation weakens the winner's curse and hence encourages more investors to submit non-zero bids, which has a positive effect on efficiency. Pesendorfer and Swinkels (1997) show that the K-unit auction has a unique symmetric monotone equilibrium in the standard setting and that the auction fully aggregates information as $N \to \infty$ if and only if K satisfies the "double largeness" condition: $K \to \infty$ and $N - K \to \infty$.

While there are multiple equilibria in our setting, we show that the aggregation properties of K-unit auction mirror those of Pesendorfer and Swinkels (1997). In particular, inefficiencies persist as long as K is finite, even if the bids are made known after the auction and are incorporated in the investment decision. The case of finite K seems reasonable in most corporate finance situations. If K is allowed to grow proportionately with N, we show that inefficiencies disappear in the limit.

Specifically, we assume that the K highest bidders who submit nonzero bids share the investment costs and the project's payoff. Each winner pays the bid submitted by the $K + 1^{st}$ highest bidder. If there are less than K investors who submit nonzero bids the project is cancelled. Otherwise the K highest bidders get the right to finance the project. In principle, winning investors may disagree about the decision to start the project. When K grows with N we show that for large N all winning investors agree on the investment decision. When K is finite we consider the optimistic scenario in which all winning investors share their information with each other and jointly decide whether to start the project.

PROPOSITION 10: *In the K-unit auction, for any finite K, the limiting surplus is strictly lower than the first-best expected surplus. If K/N goes to some constant larger than zero and smaller than one, then the expected surplus converges to the first-best expected surplus.*

Proof: See the Online Appendix.

Our results in this section can be used to explain why firms explicitly ration the allocation of shares in initial public offerings so that a larger number of investors receive

an allocation. It can also explain why entrepreneurs often allow a number of venture capitalists to co-invest, and the increasing popularity of crowd-funding platforms.

In related work, Atakan and Ekmekci (2014) study a large multi-unit auction in which each unit can be put to a different use, and show that the price does not fully aggregate information. Their equilibria are specific to the multi-unit setting and fail to exist in a single-unit setting or when K is finite. Our results are the reverse— information is aggregated when double-largeness holds but not when K is finite. The non-revealing equilibria in Atakan and Ekmekci (2014) require that winners of different units take different actions. In contrast, we assume that winners have to take a joint action (start the project or not), which is the appropriate assumption in a project financing context.

4.3. Syndicates and club bids

We now study a setting in which investors can form consortia and submit a joint bid. We provide an example in which allowing such "club bids" has a positive effect on surplus and revenues. This is in contrast to the intuition from the standard setting, where collusion among bidders tends to lower seller revenues.

A full analysis of club bidding is challenging for several reasons. First, club formation is an endogenous process which may lead to clubs of different size, which would require analysis of auctions with asymmetric bidders. Second, there may be incentive problems within the club that prevent full sharing of information among club members. Third, even if information is freely shared within the club, the resulting information is multidimensional, which makes analysis of the resulting auction technically challenging.

Dealing with these issues is beyond the scope of our paper and we therefore consider a simplified setting where we assume clubs are of equal and exogenously given size, and that information is freely shared within the club. We also assume that individual signals are distributed as in Proposition 8 and that the market is sufficiently large, which as we explain below makes it possible to handle multidimensional signals in a straightforward way.

We assume that there are $N \times M$ investors in the market. We will contrast two market settings. In the first, there is no collusion among investors and everyone submits bids independently. In the second, investors are randomly allocated to N symmetric clubs each consisting of M investors, whereupon each club submits a joint bid in the auction. Our question is whether an auction with club bids generates more revenue than a non-collusive auction.

As a benchmark, we first consider the standard auction setting where the asset

for sale is already in place. In this setting, surplus is always the same. Under the assumptions of Proposition 8, the results in Axelson (2008) imply that in the first-price and second-price auctions, larger clubs lead to lower revenues when the number of participants is large.

In our investment setting, suppose we hold the number of club members M fixed and let the number of clubs N grow large. Recall that Proposition 8 assumes that individual signals have a constant likelihood ratio $\lambda = f_G(s)/f_B(s)$ over some interval at the top of the signal distribution, which is a sufficient condition for the entrepreneur to prefer smaller markets. If the number of clubs N is large enough, only clubs where all members have signals in the top interval will participate because of the winner's curse. The likelihood ratio corresponding to a situation where M members have signals in the top interval is then λ^M. Since $\lambda > 1$, this likelihood ratio increases in the size of the club—in other words, the fact that all members in a club are optimistic is a stronger signal the more members there are.

We show in the proof of Proposition 7 that the asymptotic surplus is an increasing function of the likelihood ratio at the top of the signal distribution, which is a natural consequence of the fact that a signal with a higher likelihood ratio is more informative and leads to smaller investment mistakes. It then follows immediately that for a large enough market larger clubs lead to higher social surplus. Furthermore, we show in the proof of Proposition 8 that all this surplus goes to the entrepreneur, and hence the entrepreneur is better off with club bidding.

There are two forces favoring club bidding in our setting. First, club bidding reduces the effective number of bidders, which is beneficial when markets are inefficiently large, even if the club would submit a bid based on the signal of only one member. Second, signals become more informative whenever there is some information sharing within the club. When these effects outweigh the reduced competition, the entrepreneur gains. Our theory provides a benign rationale for the prevalent use of club bids in private equity and the use of syndicates in venture capital that has come under scrutiny by competition authorities.[12]

4.4. Shorting markets

The informational black hole appears because pessimistic investors have no incentive to bid in the auction. It could therefore be in the interest of the entrepreneur to create a market which rewards pessimistic investors for expressing their views, in a similar way that short sellers in equity markets can profit on their information when they think

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[12]See Bailey (2007) for further discussion.

a stock is overvalued. We now discuss how creation of such a market can remove the informational black hole.

There are at least three problems in constructing such a market. First, a derivatives market in which investors can take zero-sum bets would not be possible because there are no gains from trade due to the pure common value nature of the project, and so the no-trade theorem applies. As a result, any side market would have to be subsidized and would not appear spontaneously.

Second, one has to be careful in the design of the contract to avoid further informational black holes to appear. For example, a contract which is short the cash flows of the project relies on the project actually being started, and so would not be attractive to the most pessimistic investors. Similarly, a bet on whether the project is started or not would have a black hole where only the most pessimistic investors participate. Finally, a side market can lead to negative externalities on the original financing market due to strategic interactions.

Addressing all these issues rigorously goes beyond the scope of the current paper. Here we just conjecture a design that may reduce or eliminate the informational black hole. For example, suppose the entrepreneur subsidizes a side market and sells a contract which promises to pay \$1 if the project is not started, or if the project is started but fails, and pays \$0 if the project is started and succeeds. The entrepreneur then sells the project rights and the shorting contract in two independent, simultaneous auctions, whereafter all bids are revealed so that information from the shorting market can be used when making the investment decision. We conjecture that in a sufficiently large market, bids in the shorting market will be strictly decreasing in investor signals, and hence observing the bids in the shorting market is equivalent to observing all signals. This would eliminate the informational black hole in the original market and lead to a first-best solution.

5. When do informational black holes exist in general mechanisms?

The previous section illustrates a number of special examples of augmented selling procedures that eliminate the informational black hole. In fact, it is well-know that in a pure common value setting such as ours, there are mechanisms that can fully extract the information of bidders at virtually no cost for the entrepreneur if no restrictions are put on allowable mechanisms (see for example McAfee, McMillan and Reny (1989)). These mechanisms have been criticized for their sometimes esoteric structure and for their lack

of "robustness" to small changes in the environment, which is one of the reasons that our main focus in this paper is on the tried and tested standard auction procedures. Nonetheless, it is natural to ask what type of robustness criteria are needed for our results to go through in a mechanism design setting where general selling mechanisms are allowed.

We show two results. First, we develop a set of robustness criteria under which any mechanism in which investors either report their true signal or nothing has equilibria with informational black holes. Second, we show that if we also require mechanisms to be ε-cost robust, then an optimal mechanism cannot improve on the efficiency of standard auctions $\bar{s}_N$.

Consider a mechanism in which investors either report their true signal or nothing (which we denote by a report of $\emptyset$). We denote a set of reports by $R = \{r_1, ..., r_N\}$. A mechanism is a function $Q(R) = \{q_1(R), ..., q_N(R)\}$, which for each set of reports R assigns probability $q_i(R)$ that investor i gets allocated the project rights, an outcome $A(R) \in \{0, 1, ..., N\}$ of the lottery $Q(R)$, where $A(R)$ is the winning investor ($A(R) = 0$ is the situation where the seller keeps the project rights), and a set of transfers $t(R, A(R)) = \{t_1(R, A(R)), ..., t_N(R, A(R))\}$ from investors to the seller (which could be negative, if bidders are paid by the seller). A bidder who gets allocated the project rights and does not walk away from the mechanism gets the net project payoff $E(V - I|R)$ if the project is started.

The first robustness condition we impose rules out mechanisms that split the allocation over several winners, such as a K-unit auction or collusion among investors.

Condition 1: *(Winner-take-all) The project is indivisible, with non-contractible cash flows, and the mechanism must allocate the project to the highest-signal investor or no one at all if no signals are reported.*

Notice that it is not enough to require that the project can only be allocated to one winner, because the equilibrium of the K-unit auction can be implemented by allocating the *entire* project to one of the K highest bidders through a lottery, rather than splitting the allocation over many investors. Hence, we require that the mechanism is such that it allocates the project to the highest signal investor.

There are two possible ways to justify this condition: First, if the highest signal investor also has some small private value component which is higher than other investors (such as lower costs or better skills in running the project), it is ex post efficient to allocate the project to him, and the highest-signal investor would be allocated the project in a renegotiation proof mechanism. Second, the highest signal investor will also have the highest ex post willingness to pay, so a seller without sufficient commit-

ment power may be tempted to allocate the full project rights to him. The next two conditions put restrictions on the type of admissible transfers.

Condition 2: *(Fly-by-night free) No investor without private information can strictly profit from entering the mechanism.*

Condition 3: *(Regret free) No investor would prefer ex post to exit the mechanism.*

Condition 2 ensures that the mechanism is not swamped by unserious "fly-by-night" operators masquerading as serious bidders but without private information. If there is an infinite supply of such fly-by-night operators, a mechanism that rewards them for revealing their "signal" would quickly run out of money.[13] Imposing this condition ensures that losers in the auction never get any positive transfers. Condition 3 ensures that losers never pay.[14] The combination of conditions 2 and 3 makes it impossible to give investors a strict incentive to reveal their information if they expect to never implement the project when they win the auction. Finally, we require that the mechanism is renegotiation proof in the following sense:

Condition 4: *(Renegotiation proof) The project is implemented if and only if it is positive NPV conditional on the information revealed in the mechanism.*

If the mechanism is not required to be renegotiation proof, an entrepreneur with personal wealth could eliminate the informational black hole by promising to fund the project with some small probability independent of bids. This would give all investors an incentive to bid something strictly positive, and bids would be strictly increasing in signals.

We show in Proposition 11 that conditions 1-4 are sufficient for informational black holes to exist as the outcome in any mechanism. If we also assume that equilibria have to be ε-cost robust, that is, robust to introducing an arbitrarily small cost for investors to reveal their signal, we show that any equilibrium must contain a black hole of the maximal size.

PROPOSITION 11: *Under conditions 1-4, it is incentive compatible in any direct mechanism for investors not to reveal their signal below the black-out level $\overline{s}_N$. Under conditions 1-4, an optimal ε-cost robust equilibrium has black-out level $\overline{s}_N$.*

Proof: See the Online Appendix.

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[13]See Rajan (1992) and Axelson, Stromberg, and Weibach (2009) for related robustness criteria.
[14]See Lopomo (2000) and Bergemann and Morris (2005) for related robustness criteria.

6. Other robustness issues

6.1. Security auctions

We first show that all our results remain true in the case of security auctions, in which investors finance the project in exchange for part of the profits. Suppose the project's payoff in our setting is either 0 or $1 + X$. Then, a security auction takes a particularly simple form: investors submit interest rates $R_i \in [0, X]$ at which they are willing to finance the project. The auction proceeds in the same way as for cash auctions. In a first-price and second-price auctions, the winner is the investor who submits the lowest interest rate. In an ascending-price auction, bidding starts at interest rate X and the interest rate is lowered until only one investor remains.

We assume that the decision to start the project rests with the entrepreneur unless the interest rate set in the auction is X, in which case the entrepreneur gives up all the cash flow rights, and therefore control rights are transferred to the winning investor.

Consider the first-price auction. Notice that whenever the winning bid is below X the entrepreneur always starts the project. Hence, an investor who submits a bid below X should be prepared to finance and start the project if he wins the auction. Hence, the black-out region in the first-price security auction is exactly the same as the one in the first-price cash auction.

In the second-price and ascending-price security auctions, a winner who gets to finance the project at the interest rate X has an option not to start it. This is the same option that a winner in the cash auction has when he wins and pays 0. Thus, there is one-to-one mapping between the size of the informational black hole in the second-price and ascending-price security and cash auctions.

Because social surplus depends solely on the size of the informational black hole, social surplus is the same in the security auction as in the cash auction.

6.2. Assets in place and entrepreneurial wealth

We have assumed that the entrepreneur has no wealth of his own to finance the project, and no other assets that can be pledged to investors in exchange for financing. The model easily extends to the case of an existing firm raising financing for a new project, where the firm could either use some of its cash to co-finance the project or issue securities that are backed not only by the cash flows of the new project but also by the existing assets of the firm.

First, imagine that the entrepreneur has some wealth w, and issues an equity stake backed by a fraction $1 - w$ of the cash-flows of the project, where the winner invests

$1 - w$ and the entrepreneur invests w to start the project if they find it optimal to do so. It is easy to see that this leads to the exact same equilibria as when there is no wealth, except that all prices and bids are scaled down by a factor $1 - w$. Hence, surplus is exactly the same independent of the wealth of the entrepreneur. The only change is that revenues of the entrepreneur go up with wealth, since the fraction of surplus captured by investors goes down by a factor $1 - w$. This effect reinforces our result in Proposition 8 that revenues can go down with the size of the market: as w goes to one, revenues will behave in exactly the same way as surplus.

One can also show that the entrepreneur would never want to subsidize investors by giving up a larger share of the project than $1 - w$. Doing so would lower equilibrium black-out levels, but only because investors sometimes would find it optimal to pursue negative NPV projects, which would lead to a destruction of surplus.

Now suppose that the entrepreneur does not have liquid wealth, but has an existing firm with assets that can be pledged to back the security issue. For example, suppose the firm has assets in place with random but positive cash flows Z uncorrelated with the project's cash flows and that the firm issues new shares backed by both the assets in place and the new project. Suppose the firm runs a security auction in which investors bid the fraction of shares α they are willing to accept in exchange for the capital needed to finance the project. The most pessimistic investors would then submit a bid of $1/E(Z + 1)$; this is the fraction of shares needed to break even on an investment of 1 if the project is not pursued and the money raised is kept within the firm. The equilibrium black-out level below which investors submit this bid would be exactly the same as in our original model, so surplus would also remain the same. Again, as in the case of wealth, the entrepreneur would capture a larger share of the surplus the larger the value of the existing assets are, but investment efficiency would not be improved.

7. Conclusion

Our paper studies how well primary financial markets allocate capital when informtion is dispersed among market participants, and how the efficiency of the market is affected by market size. We show that financing offers made by investors fail to convey their private information once information has real value for guiding investment decisions, and that the resulting investment inefficiencies can grow larger with the size of the market. Our analysis shows that several intuitive prescriptions from standard theory need to be reexamined when information has a real allocational role: a more competitive, larger financial market may reduce welfare and entrepreneurial revenues, early releases of information may be suboptimal, and collusion among investors may

be beneficial for an entrepreneur seeking financing.

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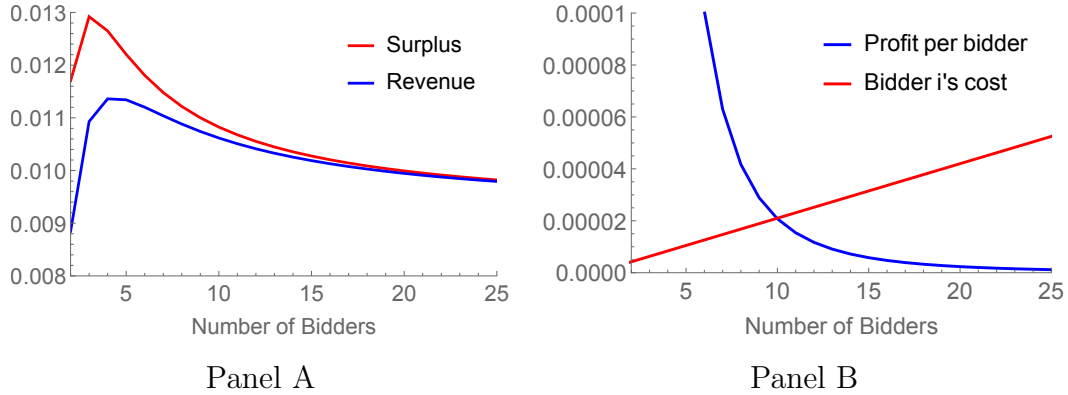
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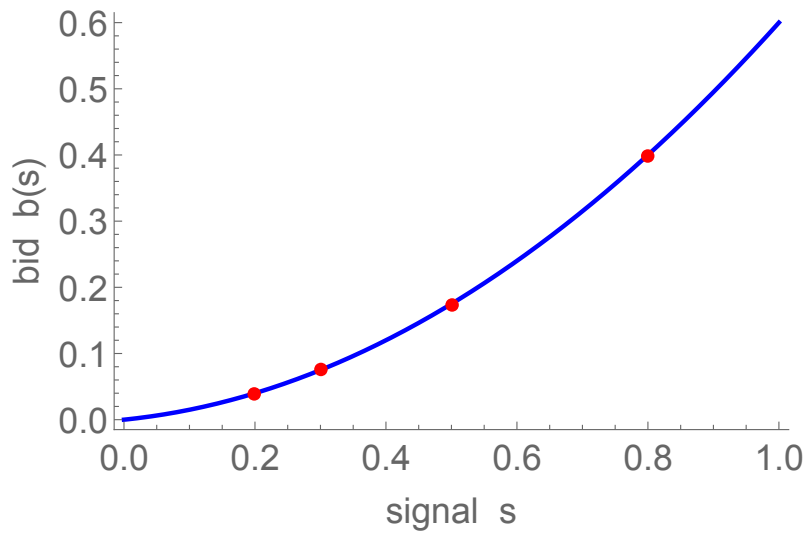
Figure 1. Bids in the standard setting. Figure 1 shows the equilibrium bidding function in the standard setting for four bidders.
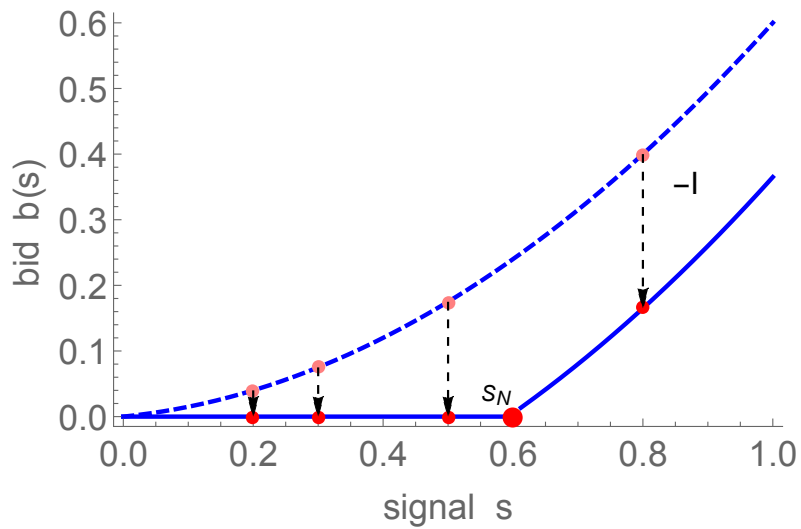
Figure 2. Bids in the setting with investments. Figure 2 shows the equilibrium bidding function in the setting with investments for four bidders.
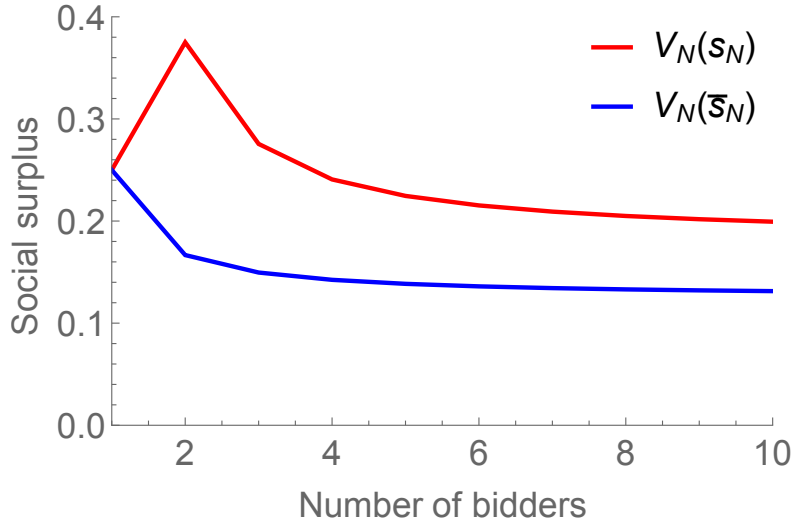
Figure 3. Market size and social surplus. Figure 3 plots social surplus as a function of number of bidders in the setting with binary signals: $f_B(s) = 1$ for all $s \in [0,1]$, $f_G(s) = 0$ for $s \in [0,1/2]$ and $f_G(s) = 2$ for $s > 1/2$. The red (blue) line corresponds to the most (least) efficient robust equilibrium.

Panel A

Panel B

Figure 4. Equilibrium market size. Panel A of Figure 4 shows social surplus gross of investor costs and the expected revenues to the entrepreneur as a function of the size of the market. Panel B shows expected gross profits to investors from participating in the auction as a function of market size, as well as a particular specification for the cost c_i of information gathering for each investor. The parameters are as follows: The project is good or bad with equal probabilities. A good project has net present value of 0.75 and a bad project has net present value of -1; $f_B(s) \equiv 1$; $f_G(s)$ is the normal distribution with mean 1 and standard deviation 0.75 truncated to the interval $[0, 1]$.

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Appendix. Proofs

Proof of Proposition 3: For convenience, the proof assumes that $f_G(s)$ and $f_B(s)$ are continuous. The proof extends to the case with jump points.

Part 1. First, we show that there is no δ-bid robust symmetric monotone equilibrium in the second-price auction with black-out level below s_N. Suppose to the contrary that there exists such an equilibrium. It must then be the case that for any $d > 0$, we can find a $\delta \in (0, d]$ such that if bids have to be made in increments of δ, there is an equilibrium black-out level below s_N. Let $\hat{s} < s_N$ be a candidate black-out level such that it is the largest signal at which the zero bid is submitted. Let Δ_1 be such that investors with signals in the interval $(\hat{s}, \hat{s} + \Delta_1]$ bid δ, and Δ_2 be such that investors with signals in the interval $(\hat{s} + \Delta_1, \hat{s} + \Delta_1 + \Delta_2]$ bid 2δ.[15] We assume that $\hat{s}$ is such that

$$E\left(V - I | Y_{1,N} = Y_{2,N} = \hat{s} + \Delta_1\right) < 0, \tag{A1}$$

$$E\left(V - I | Y_{1,N} = Y_{2,N} = Y_{3,N} = \hat{s}\right) > 0, \tag{A2}$$

which means that if there are only two bidders who bid δ the project is negative NPV. However, the project is positive NPV if there are at least three bidders who bid δ. The proof easily extends to lower values of $\hat{s}$. Consider a bidder with signal $S = \hat{s}$. For each $i \in \mathbb{N}$ define

$$\Pr_i(\hat{s}, \Delta_1) = \Pr\left(Y_{1,N-1}, \ldots, Y_{i,N-1} \in (\hat{s}, \hat{s} + \Delta_1], Y_{i+1,N-1} \leq \hat{s} | S = \hat{s}\right),$$

$$U_i(\hat{s}, \Delta_1) = E\left(V - I | Y_{1,N-1}, \ldots, Y_{i,N-1} \in (\hat{s}, \hat{s} + \Delta_1], Y_{i+1,N-1} \leq \hat{s}, S = \hat{s}\right).$$

$\Pr_i(\hat{s}, \Delta_1)$ is the conditional probability that there are exactly i bidders with signals in the range $(\hat{s}, \hat{s} + \Delta_1]$ (where bids are δ), and that the rest of the bidders get signals below or equal to $\hat{s}$. $U_i(\hat{s}, \Delta_1)$ is the corresponding expected value of the project.

The condition for the bidder with signal $\hat{s}$ to be indifferent between bidding zero or δ is

$$\sum_{i=1}^{N} \frac{\Pr_i(\hat{s}, \Delta_1)}{i+1} \times \left(\max\left[U_i(\hat{s}, \Delta_1), 0\right] - \delta\right) = 0. \tag{A3}$$

Conditions (A1) and (A2) imply that $U_1(\hat{s}, \Delta_1) < 0$ and $U_i(\hat{s}, \Delta_1) > 0$ for $i > 1$. In what follows, we let δ go to zero and show (equations (A5) and (A7)) that $\Delta_1 \sim \delta$ and $\Pr_i(\hat{s}, \Delta_1) = o(\delta^2)$ for $i > 2$. Therefore, the indifference condition (A3) takes the form:

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[15] The proof follows similar steps if the lowest bid is not δ but $k\delta$ for some $k \in \mathbb{N}$.

$$-\frac{1}{2}\delta \times \mathrm{Pr}_1(\hat{s}, \Delta_1) + \frac{1}{3}\mathrm{Pr}_2(\hat{s}, \Delta_1) \times U_2(\hat{s}, \Delta_1) + o(\delta^2) = 0. \tag{A4}$$

Let π be the ex-ante probability of the project being good. Define $z = \pi/(1 - \pi)$ and

$$z(\hat{s}) = \frac{f_G(\hat{s})}{f_B(\hat{s})}z, \quad \pi(\hat{s}) = \frac{z(\hat{s})}{1 + z(\hat{s})}.$$

Because signals are conditionally independent, and using the mean value theorem, we have

$$\mathrm{Pr}_i(\hat{s}, \Delta_1) = C^i_{N-1}\Delta^i_1\left(\pi(\hat{s})f^i_G(\bar{s}_g)F^{N-i-1}_G(\hat{s}) + (1 - \pi(\hat{s}))f^i_B(\bar{s}_b)F^{N-i-1}_B(\hat{s})\right). \tag{A5}$$

where C^i_{N-1} denotes the combinations of i elements in a set of $N - 1$, and $\bar{s}_g$ and $\bar{s}_b$ are in $(\hat{s}, \hat{s} + \Delta_1)$ and are such that

$$f_G(\bar{s}_g)\Delta_1 = \int_{\hat{s}}^{\hat{s}+\Delta_1} f_G(s)ds, \quad f_B(\bar{s}_b)\Delta_1 = \int_{\hat{s}}^{\hat{s}+\Delta_1} f_B(s)ds.$$

Let

$$z_i(\hat{s}, \Delta_1) = z(\hat{s})\frac{f^i_G(\bar{s}_g)F^{N-i-1}_G(\hat{s})}{f^i_B(\bar{s}_b)F^{N-i-1}_B(\hat{s})}, \quad \pi_i(\hat{s}, \Delta_1) = \frac{z_i(\hat{s}, \Delta_1)}{1 + z_i(\hat{s}, \Delta_1)},$$

and let

$$V_G = E[V - I|G],$$
$$V_B = E[V - I|B].$$

We have

$$U_i(\hat{s}, \Delta_1) = \pi_i(\hat{s}, \Delta_1)V_G + (1 - \pi_i(\hat{s}, \Delta_1))V_B = \frac{z_i(\hat{s}, \Delta_1)V_G + V_B}{1 + z_i(\hat{s}, \Delta_1)}.$$

Substituting expressions for $\mathrm{Pr}_1(\hat{s}, \Delta_1)$, $\mathrm{Pr}_2(\hat{s}, \Delta_1)$, and $U_2(\hat{s}, \Delta_1)$ into (A4) we have

$$\pi(\hat{s})\xi_g\Delta_1\left(C^2_{N-1}\frac{(V_G + z_2^{-1}(\hat{s}, \Delta_1)V_B)\Delta_1}{3} - C^1_{N-1}\frac{\delta}{2}\left(\frac{F_G(\hat{s})}{f_G(\bar{s}_g)} + \frac{1}{z_2(\hat{s}, \Delta_1)}\frac{F_B(\hat{s})}{f_B(\bar{s}_b)}\right)\right) + o(\delta^2) = 0, \tag{A6}$$

where

$$\xi_g = f^2_G(\bar{s}_g)F^{N-3}_G(\hat{s}).$$

Solving (A6) for Δ_1 we have

$$\Delta_1 = \delta \times \frac{3C^1_{N-1}}{2C^2_{N-1}}\frac{\left(\frac{F_G(\hat{s})}{f_G(\bar{s}_g)} + \frac{1}{z_2(\hat{s}, \Delta_1)}\frac{F_B(\hat{s})}{f_B(\bar{s}_b)}\right)}{(V_G + z_2^{-1}(\hat{s}, \Delta_1)V_B)} + o(\delta). \tag{A7}$$

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Equation A7 verifies our conjecture that Δ_1 is of the same order as δ. The bidder with signal $\hat{s}$ should be better off if she bids zero or δ rather than 2δ. Consider a deviation to a bid of 2δ. We only need to consider auction outcomes in which the bidder wins with a nonzero price, which can be either δ or 2δ. Consider first the case when the final price is δ. In this case, when the bidder bids δ she wins the auction with probability $1/2$ when there is only one more bidder with signal $s \in (\hat{s}, \hat{s} + \Delta_1]$ and with probability $1/3$ when there are two or more bidders with signals $s \in (\hat{s}, \hat{s} + \Delta_1]$. When the bidder bids 2δ she wins the auction with probability one in both cases. Using (A6), the expected gain from bidding 2δ rather than δ in the case when final price is δ can then be calculated as:

$$\Delta S = \pi(\hat{s})\xi_g\Delta_1 C^1_{N-1}\frac{\delta}{2}\left(\frac{F_G(\hat{s})}{f_G(\bar{s}_g)} + \frac{1}{z_2(\hat{s}, \Delta_1)}\frac{F_B(\hat{s})}{f_B(\bar{s}_b)}\right) + o(\delta^2). \tag{A8}$$

Consider now the case when the final price is 2δ. In this case, there is at least one other bidder with signal $s \in (\hat{s} + \Delta_1, \hat{s} + \Delta_1 + \Delta_2)$. The bidder incurs a loss if all other bidders' signals are less than $\hat{s}$. The expected loss from this event is

$$\Delta L = \frac{1}{2} \times 2\delta C^1_{N-1}\pi(\hat{s})\xi_g\Delta_2\left(\frac{f_G(\tilde{s}_g)F_G(\hat{s})}{f^2_G(\bar{s}_g)} + \frac{1}{z_2(\hat{s}, \Delta_1)}\frac{f_B(\tilde{s}_b)F_B(\hat{s})}{f^2_B(\bar{s}_g)}\right), \tag{A9}$$

where $\tilde{s}_g$ and $\tilde{s}_b$ are in $[\hat{s} + \Delta_1, \hat{s} + \Delta_1 + \Delta_2)$ and are such that

$$f_G(\tilde{s}_g)\Delta_2 = \int_{\hat{s}+\Delta_1}^{\hat{s}+\Delta_1+\Delta_2} f_G(s)ds, \quad f_B(\tilde{s}_b)\Delta_2 = \int_{\hat{s}+\Delta_1}^{\hat{s}+\Delta_1+\Delta_2} f_B(s)ds.$$

The bidder realizes a gain if there is at least one more bidder with signal $s \in (\hat{s}, \hat{s} + \Delta_1 + \Delta_2)$. The gain is at least as large as

$$\Delta G = \frac{\pi(\hat{s})\tilde{\xi}_g C^2_{N-1}(V_G + \tilde{z}_2^{-1}(\hat{s}, \Delta_1)V_B)\Delta_2^2}{3} + o(\delta^2), \tag{A10}$$

which is the gain if there is at least one more bidder with signal $s \in (\hat{s}+\Delta_1, \hat{s}+\Delta_1+\Delta_2)$, where

$$\tilde{\xi}_g = f^2_G(\tilde{s}_g)F^{N-3}_G(\hat{s} + \Delta_1),$$

and

$$\tilde{z}_2(\hat{s}, \Delta_1) = z(\hat{s})\frac{f^2_G(\tilde{s}_g)F^{N-3}_G(\hat{s} + \Delta_1)}{f^2_B(\tilde{s}_b)F^{N-3}_B(\hat{s} + \Delta_2)}.$$

Because the bidder with signal $\hat{s}$ should be better off if she bids zero or δ rather than 2δ it must be that

$$\Delta G - \Delta L + \Delta S \leq 0. \tag{A11}$$

Equation ([A11]) defines a quadratic equation for Δ_2:

$$\alpha \Delta_2^2 + \beta \Delta_2 + \gamma \leq 0, \tag{A12}$$

where

$$\alpha = \frac{\pi(\hat{s})\tilde{\xi}_g C_{N-1}^2 (V_G + \tilde{z}_2^{-1}(\hat{s}, \Delta_1)V_B)\Delta_2^2}{3}$$

$$\beta = -\delta C_{N-1}^1 \pi(\hat{s})\xi_g \left(\frac{f_G(\tilde{s}_g)F_G(\hat{s})}{f_G^2(\bar{s}_g)} + \frac{1}{z_2(\hat{s}, \Delta_1)} \frac{f_B(\tilde{s}_b)F_B(\hat{s})}{f_B^2(\bar{s}_g)} \right)$$

$$\gamma = \frac{3(C_{N-1}^1)^2 \pi(\hat{s})\xi_g \delta^2}{4C_{N-1}^2(V_G + z_2^{-1}(\hat{s}, \Delta_1)V_B)} \left(\frac{F_G(\hat{s})}{f_G(\bar{s}_g)} + \frac{1}{z_2(\hat{s}, \Delta_1)} \frac{F_B(\hat{s})}{f_B(\bar{s}_b)} \right)^2.$$

Equation ([A12]) has a solution if and only if

$$\beta^2 - 4\alpha\gamma \geq 0. \tag{A13}$$

Remark: *In fact, coefficients α and β depend on Δ_2. Below we show that $\beta^2 - 4\alpha\gamma < 0$ for any Δ_2.*

Notice that

$$\beta^2 = \left(\delta C_{N-1}^1 \pi(\hat{s}) \right)^2 \xi_g F_G^{N-1}(\hat{s}) \left(\frac{f_G(\tilde{s}_g)}{f_G(\bar{s}_g)} + \frac{1}{z(\hat{s})} \frac{F_B^{N-2}(\hat{s})}{F_G^{N-2}(\hat{s})} \frac{f_B(\tilde{s}_b)}{f_G(\bar{s}_g)} \right)^2$$

and

$$4\alpha\gamma = \left(\delta C_{N-1}^1 \pi(\hat{s}) \right)^2 \xi_g F_G^{N-3}(\hat{s} + \Delta_1)F_G^2(\hat{s}) \left(\frac{f_G(\tilde{s}_g)}{f_G(\bar{s}_g)} + \frac{1}{z(\hat{s})} \frac{F_B^{N-2}(\hat{s})}{F_G^{N-2}(\hat{s})} \frac{f_B(\bar{s}_b)f_G(\tilde{s}_g)}{f_G^2(\bar{s}_g)} \right)^2 \times$$

$$\times \frac{(V_G + \tilde{z}_2^{-1}(\hat{s}, \Delta_1)V_B)}{(V_G + z_2^{-1}(\hat{s})V_B)}.$$

Notice that $F_G(\hat{s} + \Delta_1) > F_G(\hat{s})$. The MLRP implies that

$$\frac{(V_G + \tilde{z}_2^{-1}(\hat{s}, \Delta_1)V_B)}{(V_G + z_2^{-1}(\hat{s})V_B)} > 1.$$

Observe that

$$\frac{f_B(\bar{s}_b)f_G(\tilde{s}_g)}{f_G^2(\bar{s}_g)} > \frac{f_B(\tilde{s}_b)}{f_G(\bar{s}_g)} \Leftrightarrow \frac{f_G(\tilde{s}_g)}{f_B(\tilde{s}_b)} > \frac{f_G(\bar{s}_g)}{f_B(\bar{s}_b)} \Leftrightarrow \frac{\int_{\hat{s}+\Delta_1}^{\hat{s}+\Delta_1+\Delta_2} f_G(s)ds}{\int_{\hat{s}+\Delta_1}^{\hat{s}+\Delta_1+\Delta_2} f_B(s)ds} > \frac{\int_{\hat{s}}^{\hat{s}+\Delta_1} f_G(s)ds}{\int_{\hat{s}}^{\hat{s}+\Delta_1} f_B(s)ds}.$$

By Cauchy's mean value theorem there exist $s' \in [\hat{s}, \hat{s}+\Delta_1]$ and $s'' \in [\hat{s}+\Delta_1, \hat{s}+\Delta_1+\Delta_2]$ such that

$$
\begin{aligned}
\frac{f_G(s'')}{f_B(s'')} &= \frac{\int_{\hat{s}+\Delta_1}^{\hat{s}+\Delta_1+\Delta_2} f_G(s)ds}{\int_{\hat{s}+\Delta_1}^{\hat{s}+\Delta_1+\Delta_2} f_B(s)ds}, \\
\frac{f_G(s')}{f_B(s')} &= \frac{\int_{\hat{s}}^{\hat{s}+\Delta_1} f_G(s)ds}{\int_{\hat{s}}^{\hat{s}+\Delta_1} f_B(s)ds}.
\end{aligned}
$$

The MLRP implies that

$$
\frac{f_G(s'')}{f_B(s'')} \geq \frac{f_G(s')}{f_B(s')}.
$$

Thus,

$$
\beta^2 - 4\alpha\gamma < 0.
$$

Hence, for any Δ_2 the bidder with signal $\hat{s}$ prefers bidding 2δ rather than zero or δ, a contradiction. Thus, there can be no δ-bid robust equilibrium with black-out level lower than s_N.

Part 2. We now show that there is no ε-cost robust symmetric monotone equilibrium in the second-price auction with black-out level below $\overline{s}_N$. Suppose to the contrary that the black-out level is $\hat{s} < \overline{s}_N$. First, note that in equilibrium it must be that bidding schedules are strictly monotone in some neighborhood of $\hat{s}$ for $s > \hat{s}$. If this is not the case then there is an $\hat{s}' > \hat{s}$ such that all players with a signal $S \in (\hat{s}, \hat{s}']$ submit the same bid b. Consider a deviation of the player who receives signal $\hat{s}'$ to a bid of $b + \epsilon$, where ϵ is very small. It is clear that this deviation increases the probability of winning by some $\delta > 0$, no matter how small the ϵ is. Also, conditional on winning the probability of the project being good is no less than it was before. As a result, the deviation delivers strictly higher utility to the player, which is inconsistent with equilibrium.

If a bidder with signal s wins the auction then the maximum surplus she can expect to receive is

$$
b(s; \hat{s}) = E\left[\max\left(E[V - I|\mathbf{S}_{>\hat{s}}], 0\right)|Y_{1,N} = Y_{2,N} = s\right].
$$

In the proof of Proposition 2 we showed that $b(s; \hat{s})$ is continuous in s and $b(\hat{s}; \hat{s}) = 0$ for any $\hat{s} \leq \overline{s}_N$. The gain from participating in the auction must cover the cost ε of participation. As $s \to \hat{s}$ the gain decreases to zero. Thus, there is a signal $s > \hat{s}$ such that an agent with signal s cannot recover her participation costs, which contradicts that $\hat{s}$ is a black-out level. *Q.E.D.*

Proof of Proposition 4:

Part 1. First, we show that there is no δ-bid robust symmetric monotone equilibrium in the ascending-price auction with black-out level below s_N. Suppose to the contrary that there is an equilibrium with black-out level $\hat{s} < s_N$ so that any bidder with a signal $s' > \hat{s}$ stays in the auction until the price reaches δ. For a given realization of signals, let n be the number of bidders who stay in the auction. Condition (2) implies that if $n = 2$ then in any monotone equilibrium any bidder s' with $s' \in (\hat{s}, s_N]$ should drop out at price δ. If the other bidder also has a signal in the interval $(\hat{s}, s_N]$ then each wins the auction with probability $1/2$ and realizes a loss δ. Therefore, the expected loss for a bidder with signal $s' \in (\hat{s}, s_N]$ is at least $L = \delta \times Pr(\hat{s} < Y_{1,N-1} \leq s_N, Y_{2,N-1} \leq \hat{s})/2$.

As in the proof of Proposition 3 we assume that

$$E\left(V - I | Y_{1,N} = Y_{2,N} = Y_{3,N} = \hat{s}\right) > 0, \tag{A14}$$

which implies that if $n \geq 3$ the project is positive NPV. The bidder with signal $s > \hat{s}$ can win the auction in two cases. First, she wins if all other bidders have a lower signal than s. As $s \to \hat{s}$ the probability of this event goes to zero. Since the surplus is bounded there exists $\epsilon > 0$ such that for any $s \in (\hat{s}, \hat{s} + \epsilon)$ the expected gain in this scenario is less than $L/2$.

Second, because price increases are discrete, a bidder with signal s can win if bidders with higher signals will drop at the same price as she does. Notice that as δ goes to zero the probability of this event goes to zero while the maximum gain for a bidder with signal s is no more than the price increment δ. Therefore, there exists $\delta > 0$ such that the expected gain is less than $L/2$. Thus, we have showed that for any $s \in (\hat{s}, \hat{s} + \epsilon)$ the expected loss is larger than the expected gain. Therefore, $\hat{s}$ cannot be the equilibrium black-out level.

Part 2. We now show that there is no ε-cost robust symmetric monotone equilibrium in the ascending-price auction with black-out level below $\bar{s}_N$. Suppose now that the black-out level is $\hat{s} < \bar{s}_N$. Because we restrict our attention to monotone bidding strategies, an agent with the signal just above $\hat{s}$ can win in the auction only if either all other players get a lower signal or if some players with a higher signal decide to leave the auction at the same time. In the former case, condition (4) implies that the expected benefits are lower than the cost of participation in the auction. Therefore, the player would be better off not participating in auction. In the latter case, similar to case of second-price auction, the higher type would be better off to deviate by staying a second longer. *Q.E.D.*

Proof of Proposition 6: As before, π is the ex-ante probability of the project being good, $z = \pi/(1 - \pi)$, $V_G = E[V - I | G]$, and $V_B = E[V - I | B]$. Equations (4) and (2)

imply that $\bar{s}_N$ and s_N solve the following equations

$$\frac{F_G^{N-1}(\bar{s}_N)f_G(\bar{s}_N)}{F_B^{N-1}(\bar{s}_N)f_B(\bar{s}_N)} = -\frac{V_B}{zV_G}, \tag{A15}$$

$$\frac{F_G^{N-2}(s_N)\,f_G^2(s_N)}{F_B^{N-2}(s_N)\,f_B^2(s_N)} = -\frac{V_B}{zV_G}. \tag{A16}$$

Taking the logarithm of both parts of the above equations we have

$$(N-1)\ln\left(\frac{F_G(\bar{s}_N)}{F_B(\bar{s}_N)}\right) + \ln\left(\frac{f_G(\bar{s}_N)}{f_B(\bar{s}_N)}\right) = -\ln(-zV_G/V_B), \tag{A17}$$

$$(N-2)\ln\left(\frac{F_G(s_N)}{F_B(s_N)}\right) + 2\ln\left(\frac{f_G(s_N)}{f_B(s_N)}\right) = -\ln(-zV_G/V_B). \tag{A18}$$

Equations (A17) and (A18) imply that both $\bar{s}_N$ and s_N go to one as N goes to infinity. Taking Taylor series expansions of (A17) and (A18) and using that

$$\lim_{s\to 1} F_G(s) = 1 - f_G(1)(1-s),$$
$$\lim_{s\to 1} F_B(s) = 1 - f_B(1)(1-s),$$
$$\lim_{s\to 1} \frac{f_G(s)}{f_B(s)} = \lambda,$$

we obtain that

$$1 - \bar{s}_N = \frac{a_1}{f_B(1)}\frac{1}{N} + o(1/N), \quad a_1 = \frac{\ln(-\lambda zV_G/V_B)}{\lambda - 1}, \tag{A19}$$

$$1 - s_N = \frac{a_2}{f_B(1)}\frac{1}{N} + o(1/N), \quad a_2 = \frac{\ln(-\lambda^2 zV_G/V_B)}{\lambda - 1}. \tag{A20}$$

The proposition's statements then follow from Theorem 4.2.3 of Embrechts, Klüppelberg and Mikosch (2012). *Q.E.D.*

Proof of Proposition 7: To prove the proposition we consider the comparative statics results with respect to N. To simplify the derivations we renormalize the densities f_B and f_G so that $f_B(1) \equiv 1$ and $f_G(1) = \lambda$. As before, π is the ex-ante probability of the project being good, $z = \pi/(1-\pi)$, $V_G = E[V - I|G]$, and $V_B = E[V - I|B]$. Taking Taylor series expansions of (A17) and (A18) we obtain the following results:

$$1 - \bar{s}_N = \frac{a_1}{N} + \frac{b_1}{N^2} + o(1/N^2), \tag{A21}$$

$$1 - s_N = \frac{a_2}{N} + \frac{b_2}{N^2} + o(1/N^2), \tag{A22}$$

where a_1 and a_2 are given by (A19) and (A20) respectively, and

$$b_i = \frac{\lambda a_i^2(f - \lambda(\lambda - 1)) - 4a_i f}{2\lambda(\lambda - 1)}, \quad f = f'_G(1), \quad i = 1, 2. \tag{A23}$$

In the least efficient equilibrium social surplus is

$$U_N(\bar{s}_N) = \pi V_G \Pr(Y_{1,N} > \bar{s}_N | G) + (1 - \pi) V_B \Pr(Y_{1,N} > \bar{s}_N | B) =$$

$$= \pi V_G (1 - F_G^N(\bar{s}_N)) + (1 - \pi) V_B (1 - F_B^N(\bar{s}_N)). \tag{A24}$$

Substituting (A21) into (A24) we obtain the following expression for the surplus

$$U_N(\bar{s}_N) = \pi V_G + (1 - \pi) V_B - (1 - \pi) V_B \left(-\lambda z V_G / V_B\right)^{-\frac{1}{\lambda - 1}} \left(1 - \frac{1}{\lambda}\right) \tag{A25}$$

$$- (1 - \pi) V_B \left(-\lambda z V_G / V_B\right)^{\frac{1}{\lambda - 1}} \frac{a_1^2(\lambda(\lambda - 1) - f)}{2\lambda N} + o(1/N).$$

In the equilibrium with threshold s_N the bidder who wins the auction with zero price invests if and only if his signal is higher than $\varphi(s_N)$, where $\varphi(s_N)$ is the largest solution of the following equation

$$E(V - I | Y_{1,N} = \varphi(s_N), Y_{2,N} \leq s_N) \leq 0. \tag{A26}$$

Equation (A26) implies that $\varphi(s_N)$ is defined by

$$\frac{F_G^{N-1}(s_N)}{F_B^{N-1}(s_N)} \frac{f_G(\varphi(s_N))}{f_B(\varphi(s_N))} = -\frac{V_B}{z V_G}$$

if

$$\frac{F_G^{N-1}(s_N)}{F_B^{N-1}(s_N)} \geq -\frac{V_B}{\lambda z V_G}, \tag{A27}$$

and is equal to one otherwise. Using (A16) we can write condition (A27) as

$$\frac{F_G(s_N)}{F_B(s_N)} \geq \frac{1}{\lambda} \frac{f_G^2(s_N)}{f_B^2(s_N)}. \tag{A28}$$

As N goes to infinity, the LHS of (A28) is bounded by one, while the RHS of (A28) goes to $\lambda > 1$. Thus, inequality (A28) does not hold. Hence, for N sufficiently large $\varphi(s_N) = 1$. Therefore, social surplus is given by

$$U_N(s_N) = \pi V_G \Pr(Y_{2,N} > s_N | G) + (1 - \pi) V_B \Pr(Y_{2,N} > s_N | B).$$

Notice that

$$\Pr(Y_{2,N} > s) = 1 - NF^{N-1}(s) + (N-1)F^N(s). \tag{A29}$$

Substituting (A22) into (A29) we obtain the following expression for the surplus

$$U_N(s_N) = \pi V_G + (1-\pi)V_B - (1-\pi)V_B \left(-\lambda^2 z V_G/V_B\right)^{-\frac{1}{\lambda-1}} \left(1 - \frac{1}{\lambda^2} + \frac{a_2(\lambda-1)}{\lambda}\right) \tag{A30}$$

$$- (1-\pi)V_B \left(-\lambda^2 z V_G/V_B\right)^{\frac{1}{\lambda-1}} \frac{a_2^3(\lambda(\lambda-1) - f)}{2\lambda N} + o(1/N).$$

Expressions (A25) and (A30) imply that both $U_N(\bar{s}_N)$ and $U_N(s_N)$ decrease with N if $f < \lambda(\lambda-1)$. Notice that if $f_B(s) \equiv 1$, then $\frac{F_G(s)}{F_B(s)} \frac{f_B(s)}{f_G(s)} = \frac{F_G(s)}{sf_G(s)}$. Taking the derivative of $\frac{F_G(s)}{sf_G(s)}$ at $s = 1$ we can see that it is positive if $f < \lambda(\lambda-1)$ and is negative if $f > \lambda(\lambda-1)$. Q.E.D.

Proof of Proposition 9: Suppose all costs are zero. Then by Proposition 7 there is an $\underline{N}$ such that surplus decreases with $N > \underline{N}$. Thus, the optimal size of the market cannot be larger than $\underline{N}$. Because the MLRP holds strictly all bidders earn strictly expected profit. Fix any $N > \underline{N}$. Let p_N be the expected profit of an individual investor. It is clear then that if $c < p_N$ than the size of the market will be larger than socially optimal size $\underline{N}$. To show that lowering information gathering costs can lead to a decrease in social surplus consider the following situation. Suppose that gathering costs are such that $c_N < p_{N+1}$ and $c_{N+1} > p_{N+1}$. In this case, the market size is N. Suppose that the cost c_{N+1} is reduced so that the new cost $\hat{c}_{N+1} < p_{N+1}$. As a result, the new market size is $N + 1$ and social surplus is reduced. Q.E.D.

Proof of Proposition 10: As before, π is the ex-ante probability of the project being good, $z = \pi/(1-\pi)$, $V_G = E[V - I|G]$, and $V_B = E[V - I|B]$. We first prove that the expected surplus in the K-unit auction if K is finite is strictly lower than πV_G, even if winning investors share their signals before the decision to invest is made. To prove this, we show that as N gets large the black-out level $s_{K,N}$ is

$$1 - s_{K,N} = \frac{1}{f_B(1)} \frac{a_K}{N} + o(1/N). \tag{A31}$$

Theorem 4.2.3 of Embrechts, Klüppelberg and Mikosch (2012) then implies that

$$\lim_{N \to \infty} \Pr(Y_{k,N} > s_{K,N}|G) = 1 - e^{-\lambda a_K} \sum_{r=0}^{K-1} \frac{(\lambda a_K)^r}{r!} < 1,$$

which proves that the expected surplus is less than πV_G since the project is financed only if $Y_{k,N} > s_{K,N}$.

Suppose an investor who bids just above $s_{K,N}$ is among winning bidders. The most positive signal realization possible is that $K - 1$ investors get the top signal and the $K + 1^{th}$ investor receive $s_{K,N}$ signal. In this case, the likelihood $z = \pi/(1 - \pi)$ is updated as

$$z\lambda^{K-1}\frac{F_G^{N-K-1}(s_{K,N})}{F_B^{N-K-1}(s_{K,N})}\frac{f_G^2(s_{K,N})}{f_B^2(s_{K,N})}.$$

Hence, the level of $s_{K,N}$ that makes the project break-even is

$$z\lambda^{K-1}V_G\frac{F_G^{N-K-1}(s_{K,N})}{F_B^{N-K-1}(s_{K,N})}\frac{f_G^2(s_{K,N})}{f_B^2(s_{K,N})} = -V_B. \tag{A32}$$

Condition (A32) is similar to condition (A20). Following similar steps as in the proof of Proposition 7 we obtain that

$$1 - s_{K,N} = \frac{1}{f_B(1)}\frac{a_K}{N} + o(1/N), \quad a_K = \frac{\ln(-\lambda^{K+1}zV_G/V_B)}{\lambda - 1}. \tag{A33}$$

Next, we prove that if $K/N \to (1 - \alpha)$, $\alpha \in (0, 1)$ as $N \to \infty$ then the expected surplus in the least efficient equilibrium converges to πV_G, even if bids are not revealed after the auction. We assume that the decision to start the project lies with the Kth highest bidder.

The highest black-out level possible is such that

$$\Pr(G|Y_{K,N} = s_{K,N})V_G + (1 - \Pr(G|Y_{K,N} = s_{K,N}))V_B = 0. \tag{A34}$$

If the black-out level $\hat{s}$ is higher than $s_{K,N}$ defined by (A34) then a bidder with signal $s \in (s_{K,N}, \hat{s}]$ will be better-off by deviating and bidding a strictly positive amount: If the auction results in zero price then the bidder does not loose anything. At the same time if the auction results in a positive price then there are at least K bidders with signal above the $\hat{s}$, which makes the project positive NPV.

Equation (A34) implies that

$$\frac{\pi V_G}{1 - \pi}\frac{F_G^{N-K}(s_{K,N})(1 - F_G(s_{K,N}))^{K-1}f_G(s_{K,N})}{F_B^{N-K}(s_{K,N})(1 - F_B(s_{K,N}))^{K-1}f_B(s_{K,N})} = -V_B. \tag{A35}$$

The project is started whenever $Y_{K,N} > s_{K,N}$. If $K/N = 1 - \alpha$ then we can write equation (A35) as

$$\frac{\pi V_G}{1 - \pi} \left(\frac{F_G(s_{K,N})^\alpha (1 - F_G(s_{K,N}))^{1-\alpha}}{F_B(s_{K,N})^\alpha (1 - F_B(s_{K,N}))^{1-\alpha}} \right)^N \frac{(1 - F_B(s_{K,N})) f_G(s_{K,N})}{(1 - F_G(s_{K,N})) f_B(s_{K,N})} = -V_B.$$

As N goes to infinity $s_{K,N}$ converges to the value s_α, which solves

$$F_G(s_\alpha)^\alpha (1 - F_G(s_\alpha))^{1-\alpha} = F_B(s_\alpha)^\alpha (1 - F_B(s_\alpha))^{1-\alpha}. \tag{A36}$$

Let $s_{\alpha,G}$ and $s_{\alpha,B}$ be such that $F_G(s_{\alpha,G}) = \alpha$ and $F_B(s_{\alpha,B}) = \alpha$. Because of the MLRP $s_{\alpha,B} < s_{\alpha,G}$. Notice that $x^\alpha (1 - x)^{1-\alpha}$ is a single-peaked function that reaches its maximum at $x = \alpha$. Therefore, $s_{\alpha,B} < s_\alpha < s_{\alpha,G}$.

As $N \to \infty$ and $k/N \to 1 - \alpha$, $Y_{k,N}$ becomes an αth sample quantile. It is well-known that

$$\sqrt{N}(Y_{k,N} - s_\alpha) \xrightarrow{d} N(0, \alpha(1 - \alpha)/f(s_\alpha)^2),$$

where $f(x)$ and $F(x)$ are pdf and cdf of observations and $F(s_\alpha) = \alpha$. Hence, as $N \to \infty$ the probability of undertaking the project goes to one if the project is good and goes to zero if the project is bad. *Q.E.D.*

Proof of Proposition 11:

Step 1. We first prove that $t_i(R, A(R)) = 0$ if $A(R) \neq i$ or if $E(V - I|R) < 0$, which implies bidders who expect never to receive any allocation when the project is positive NPV will have zero expected profits when revealing their signal.

If $A(R) \neq i$ or $E(V - I|R) < 0$ bidder i will walk away from the mechanism if faced with a payment $t_i(R, A(R)) > 0$ as an outcome of the mechanism. Hence, we have to have $t_i(R, A(R)) \leq 0$ whenever $A(R) \neq i$ or $E(V - I|R) < 0$. Next, suppose that $t_i(R, A(R)) < 0$ when $A(R) \neq i$ for some R, $r_i \in R$ so that a losing bidder gets a strictly positive payment. This violates the fly-by-night condition, because a fly-by-night operator reporting r_i can guarantee himself strictly positive expected profits by walking away from the mechanism for every outcome except when the vector of reports is R. Similar arguments apply if $t_i(R, A(R)) < 0$ when $E(V - I|R) < 0$ because by renegotiation proofness condition the project is not started if it is negative NPV.

Step 2. Suppose bidders with signal below $\bar{s}_N$ do not reveal their signal. We prove next that all bidders with signal $S_i > \bar{s}_N$ always reveal their signal. To see this take any $\varepsilon > 0$, and suppose bidder i with signal $s_i = \bar{s}_N + \varepsilon$ reveals his signal. In a truth-telling winner-take-all mechanism, bidder i then expects to always win when his signal is the highest, a positive probability event, plus potentially when his signal is not the

highest but bidders with higher signals do not reveal their signal. From the definition of $\bar{s}_N$, the project is therefore strictly positive NPV conditional on the information that bidder i wins the allocation. This implies that there must exist a set of reports R_{-i} by bidders other than bidder i that happen with positive probability such that $E(V - I|R) > 0$ and such that $A(R) = i$ (i.e., bidder i wins the allocation when the project is positive NPV conditional on the observed reports). The regret free condition implies that $E(V - I|R) - t_i(R, i) \geq 0$. Now take some signal $s_i' > s_i$. When bidder i observes $S_i = s_i'$ but gives the false report s_i, he will have strictly positive expected profits by following the strategy of walking away except when the the vector of reports is R, since

$$E(V - I|R_{-i}, S_i = s_i') - t_i(R, i) > E(V - I|R) - t_i(R, i) \geq 0, \qquad (A37)$$

where the first inequality follows from MLRP. Incentive compatibility requires that bidder i is at least as well off when reporting s_i' as when reporting s_i, which in turn implies that this bidder must strictly prefer to reveal his signal rather than not revealing it and getting zero expected profits. Since $\varepsilon > 0$ was picked arbitrarily, this proves that all bidders with signals above $\bar{s}_N$ strictly prefer to reveal their signal.

Step 3. Suppose bidders below $\bar{s}_N$ do not reveal their signal. Suppose that a bidder i with signal $s_i < \bar{s}_N$ reveals his signal and wins an allocation. From Step 2 and the definition of $\bar{s}_N$, and under the postulated expectations over the strategies of other bidders, this can only happen if the project is negative NPV. Hence, from Step 1, the bidder gets zero expected profits when revealing his signal. Thus, it is incentive compatible for him not to reveal his signal, which proves the first part of the proposition.

Step 4. Next, we prove the second part of the proposition. We start by showing that any participation-cost robust equilibrium must be in cut-off strategies such that bidder i reveals his signal if $S_i > \hat{s}$ and does not reveal his signal if $S_i < \hat{s}$.

First, note that any equilibrium must be such that if bidder i reveals his signal at s_i, and if there is some equilibrium R with $s_i = r_i \in R$ at which the project is positive NPV and bidder i wins an allocation with positive probability, then it must be strictly optimal to reveal the signal when $S_i > s_i$ in the equilibrium. This follows from the same steps as in the proof of Step 2 above. In order for a player not to use a cut-off strategy in equilibrium on a non-zero measure set of signals, it must then be that there is a non-zero measure set of signals at which bidder i reveals his signal and at which the project is strictly negative NPV whenever he wins. Suppose such an equilibrium is participation-cost robust, contrary to the statement in the claim. Then,

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there exists some participation cost $c > 0$ such that bidder i reveals his signal on a non-zero measure set at which the project is negative NPV whenever he wins. But then, bidder i makes strictly negative expected profits, and is better off not revealing his signal.

Restricting attention to cut-off strategies, suppose contrary to the claim in the proposition that the lowest cut-off level amongst bidders in a participation-cost robust equilibrium is $\hat{s}_N < \overline{s}_N$. By the supposition that this is a participation-cost robust equilibrium, there is an equilibrium with a nonzero cost c and reporting strategies that are arbitrary close to the cut-off equilibrium with $\hat{s}_N$. In this equilibrium, the most optimistic scenario when the bidder with signal $\hat{s}_N$ (or bidders with signals arbitrary close to $\hat{s}_N$) wins the auction is that bidders with the highest signals do not reveal their signals. However, because this set of bidders with highest signals can be made arbitrary small and by definition of $\overline{s}_N$, conditional on winning with signal $\hat{s}_N$ the NPV of the project is negative. Hence, the bidder with signal $\hat{s}_N$ strictly prefers not to reveal her signal, which contradicts that such an equilibrium exist. *Q.E.D.*

Proof that in the first-price auction only the equilibrium with the highest black-out level $\overline{s}_N$ is δ-bid and ε-cost robust.

The proof for ε-cost robustness follows exactly the same steps as the one for the second-price auction given in the proof of Proposition 3. Therefore, here we provide details for δ-bid robustness.

Let $\hat{s}$ be the largest signal at which the zero bid is submitted, Δ_1 be such that signals $(\hat{s}, \hat{s} + \Delta_1]$ induce submission of δ, and Δ_2 be such that signals $(\hat{s} + \Delta_1, \hat{s} + \Delta_1 + \Delta_2]$ induce submission of 2δ.[16] We assume that $\hat{s}$ is such that

$$E\left(V - I \mid Y_{1,N-1} = \hat{s} + \Delta_1\right) < 0, \tag{A38}$$

$$E\left(V - I \mid Y_{1,N-1} = Y_{2,N-1} = \hat{s}\right) > 0, \tag{A39}$$

which means that if there is only one bidder who bids δ the project is negative NPV. However, the project is positive NPV if there are at least two bidders who bid δ. The proof easily extends to lower values of $\hat{s}$. Consider a bidder with signal $S = \hat{s}$. Let

$$
\begin{aligned}
\mathrm{Pr}_0(\hat{s}, \Delta_1) &= \mathrm{Pr}\left(Y_{1,N-1} < \hat{s} \mid S = \hat{s}\right), \\
U_0(\hat{s}, \Delta_1) &= E\left(V - I \mid Y_{1,N-1} < \hat{s} \mid S = \hat{s}\right),
\end{aligned}
$$

---

[16]The proof follows similar steps if the lowest bid is not δ but $k\delta$ for some $k \in \mathbb{N}$.

For each $i \in \mathbb{N}$ define

$$
\begin{aligned}
\mathrm{Pr}_i(\hat{s}, \Delta_1) &= \mathrm{Pr}\left(Y_{1,N-1}, \ldots, Y_{i,N-1} \in (\hat{s}, \hat{s} + \Delta_1], Y_{i+1,N-1} \leq \hat{s} | S = \hat{s}\right), \\
U_i(\hat{s}, \Delta_1) &= E\left(V - I | Y_{1,N-1}, \ldots, Y_{i,N-1} \in (\hat{s}, \hat{s} + \Delta_1], Y_{i+1,N-1} \leq \hat{s}, S = \hat{s}\right).
\end{aligned}
$$

$\mathrm{Pr}_i(\hat{s}, \Delta_1)$ is the conditional probability that there are exactly i bidders with signal in the range $(\hat{s}, \hat{s} + \Delta_1]$, and who therefore bid δ, and that the rest of the bidders get signals below or equal to $\hat{s}$. $U_i(\hat{s}, \Delta_1)$ is the corresponding expected value of the project.

The indifference condition for the bidder with signal $\hat{s}$ to bid 0 or δ is

$$
\sum_{i=0}^{N} \frac{\mathrm{Pr}_i(\hat{s}, \Delta_1)}{i+1} \times (\max\left[U_i(\hat{s}, \Delta_1), 0\right] - \delta) = 0. \tag{A40}
$$

Conditions (A38) and (A39) imply that $U_0(\hat{s}, \Delta_1) < 0$ and $U_i(\hat{s}, \Delta_1) > 0$ for $i > 0$. In what follows, we let δ go to zero and show (equations (A42) and (A44)) that $\Delta_1 \sim \delta$ and $\mathrm{Pr}_i(\hat{s}, \Delta_1) = o(\delta)$ for $i > 1$. Therefore, the indifference condition (A40) takes the form:

$$
-\delta \times \mathrm{Pr}_0(\hat{s}, \Delta_1) + \frac{1}{2}\mathrm{Pr}_1(\hat{s}, \Delta_1) \times U_1(\hat{s}, \Delta_1) + o(\delta) = 0. \tag{A41}
$$

Let π be the ex-ante probability of the project being good. Define $z = \pi/(1 - \pi)$ and

$$
z(\hat{s}) = \frac{f_G(\hat{s})}{f_B(\hat{s})}z, \quad \pi(\hat{s}) = \frac{z(\hat{s})}{1 + z(\hat{s})}.
$$

Because signals are conditionally independent using the mean value theorem we have

$$
\mathrm{Pr}_i(\hat{s}, \Delta_1) = C_{N-1}^i \Delta_1^i \left(\pi(\hat{s})f_G^i(\bar{s}_g)F_G^{N-i-1}(\hat{s}) + (1 - \pi(\hat{s}))f_B^i(\bar{s}_b)F_B^{N-i-1}(\hat{s})\right). \tag{A42}
$$

where $\bar{s}_g$ and $\bar{s}_b$ are in $(\hat{s}, \hat{s} + \Delta_1)$ and are such that

$$
f_G(\bar{s}_g)\Delta_1 = \int_{\hat{s}}^{\hat{s}+\Delta_1} f_G(s)ds, \quad f_B(\bar{s}_b)\Delta_1 = \int_{\hat{s}}^{\hat{s}+\Delta_1} f_B(s)ds.
$$

Let

$$
z_i(\hat{s}, \Delta_1) = z(\hat{s})\frac{f_G^i(\bar{s}_g)F_G^{N-i-1}(\hat{s})}{f_B^i(\bar{s}_b)F_B^{N-i-1}(\hat{s})}, \quad \pi_i(\hat{s}, \Delta_1) = \frac{z_i(\hat{s})}{1 + z_i(\hat{s})},
$$

and let

$$
\begin{aligned}
V_G &= E[V - I|G], \\
V_B &= E[V - I|B].
\end{aligned}
$$

We have

$$U_i(\hat{s}, \Delta_1) = \pi_i(\hat{s}, \Delta_1)V_G + (1 - \pi_i(\hat{s}, \Delta_1))V_B = \frac{z_i(\hat{s}, \Delta_1)V_G + V_B}{1 + z_i(\hat{s}, \Delta_1)}.$$

Substituting expressions for $\Pr_0(\hat{s}, \Delta_1)$, $\Pr_1(\hat{s}, \Delta_1)$, and $U_1(\hat{s}, \Delta_1)$ into (A41) we have

$$\pi(\hat{s})\zeta_g\left(-\delta\left(\frac{F_G(\hat{s})}{f_G(\bar{s}_g)} + \frac{1}{z_1(\hat{s}, \Delta_1)}\frac{F_B(\hat{s})}{f_B(\bar{s}_b)}\right) + C_{N-1}^1\frac{(X - z_1^{-1}(\hat{s}, \Delta_1))\Delta_1}{2}\right) + o(\Delta_1) = 0, \quad (A43)$$

where

$$\xi_g = f_G(\bar{s}_g)F_G^{N-2}(\hat{s}).$$

Solving (A43) for Δ_1 we have

$$\Delta_1 = \delta \times \frac{2}{C_{N-1}^1}\frac{\left(\frac{F_G(\hat{s})}{f_G(\bar{s}_g)} + \frac{1}{z_1(\hat{s}, \Delta_1)}\frac{F_B(\hat{s})}{f_B(\bar{s}_b)}\right)}{(X - z_1^{-1}(\hat{s}, \Delta_1))} + o(\delta). \quad (A44)$$

The bidder with signal $\hat{s}$ should be better off if she bids δ rather than 2δ. However, it is clear that it is not true in our case. If she bids 2δ she realizes a loss only if all other bidders' signals are below $\hat{s}$. This loss is compensated by the increased probability of winning the auction for sure when there are bidders with signals in the interval $\hat{s} + \Delta_1$. Q.E.D.

Full calculations for the binary example in the paper.

We assume that signals are binary: $f_B(s) = 1$ and $f_G(s) = 0$ for $s \in [0, 1/2)$ and $f_G(s) = 2$ for $s \in [1/2, 1]$. Also, assume that $\Pr(G) = 1/2$, and $E(V - I|G) = -E(V - I|B) = 1$ so that the project is zero NPV ex ante. Define $q = 1/2$.

If there is only one bidder then the auction can stipulate any reserve price between zero and $E(V - I|s \geq 1/2) > 0$. The bidder bids the reserve price if and only if he receives a high signal. Hence, social surplus is $U_1 = \Pr(s \geq 1/2) \times E(V - I|s \geq 1/2)$. Note that this is equivalent to the first-best surplus with one signal.

When there are two bidders then in the most efficient equilibrium each bidder submits a nonzero bid only if he receives a high signal. The project is started only if the auction price is greater than zero. Hence, social surplus is $U_2 = \pi - (1 - \pi)q^2 = 3/8$, where $(1 - \pi)q^2$ is the probability that the project is bad and both bidders get a high signal. This is equivalent to the first-best surplus with two signals. In the least efficient equilibrium each bidder submits a nonzero bid only if he receives a signal $s \in [\bar{s}_2, 1]$ where $\bar{s}_2$ solves $E(V - I|Y_{1,2} = \bar{s}_2) = 0$, which using Bayes' theorem can be calculated as

$$\bar{s}_2 = \frac{1 - q}{1 - q^2\frac{1-\pi}{\pi}} = \frac{2}{3}.$$

If $N > 2$, then the blackout level s_N in the most efficient equilibrium can be calculated as:

$$s_N = \frac{1-q}{1 - q\left(q^2 \frac{1-\pi}{\pi}\right)^{\frac{1}{N-2}}}.$$

From Equation A26 one can calculate that $\varphi(s_N) = 1$, so a winner never invests unless the second-highest bidder puts in a strictly positive bid, which happens when $Y_{2,N} > s_N$. We can then calculate the surplus as:

$$U_N(s_N) = \pi \Pr\left(Y_{2,N} > s_N | G\right) - (1-\pi)\Pr\left(Y_{2,N} > s_N | B\right).$$

Similarly, if $N > 2$, the blackout level $\bar{s}_N$ in the least efficient equilibrium can be calculated from (4) as:

$$\bar{s}_N = \frac{1-q}{1 - q\left(q\frac{1-\pi}{\pi}\right)^{\frac{1}{N-1}}}.$$

Therefore,

$$U_N(\bar{s}_N) = \pi \Pr\left(\{Y_{1,N} > \bar{s}_N\} | G\right) - (1-\pi)\Pr\left(\{Y_{1,N} > \bar{s}_N\} | B\right).$$