Optimal R&D investment for a risk-averse entrepreneur *

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Abstract

The technical uncertainty associated with the cost to completion of an R&D project, whilst idiosyncratic, is also inherently unhedgeable. We extend existing real options models of R&D investment to incorporate the cost of bearing this unhedgeable risk and find it decreases risk-averse entrepreneurs' valuations of R&D projects and increases the minimum NPVs required for continued investment in R&D (threshold NPVs) relative to 'unpriced risk' values and threshold NPVs, by more for larger projects and more risk-averse entrepreneurs. Overall, threshold NPVs for risk-averse entrepreneurs can be positive or negative and can increase or decrease with technical uncertainty depending on risk aversion and project size.

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1 Introduction

The valuation of and decisions about investments in research and development (R&D) are perhaps even more important for founders and potential investors in research-based start-up firms, than for large companies in research intensive industries. Estimated figures for 2005 (Wolfe (2007)) show total spending by U.S. small firms on R&D was \$22*bn*. Moreover small firms had above average R&D expenditure as a percentage of sales revenue (9.4% *vs* 3.7% over all firms) and employed disporpotionate numbers of scientists or engineers working on R&D relative to their size[1].

In this paper we model the investment choices of a risk-averse entrepreneur[2] in an R&D project. Risk aversion is more likely to be an important factor in investors' decision-making in smaller, privately-owned firms than in large public corporations. The model is thus particularly relevant for *e.g.* research-based start-up firms. We focus on a risk that is peculiar to R&D: uncertainty over the rate of progress in R&D. This risk is specific to the individual R&D project. It thus cannot be hedged and must be borne by the entrepreneur whilst he is investing in the R&D project. We show that a risk-averse entrepreneur's investment choices and valuation of R&D projects is affected by his risk aversion and can differ considerably from those of an investor who does not price this unhedgeable risk. The model thus also suggests a potential source of agency conflicts relating to R&D investments: different investors

[1]Small firms employed just over 14% of all R&D scientists, whereas firms with more than $25,000$ employees employed 29% of R&D scientists.

[2]In this paper we will use the term 'entrepreneur' for a risk-averse investor to distinguish them from a risk-neutral investor. Neither the entrepreneur nor the risk-neutral investor is capital constrained in our model.

in an R&D project may disagree about the optimal investment strategy depending on their level of risk-aversion. We leave detailed modeling of such implications for further work.

In general the evaluation of investments in R&D is complicated by features specific to these investments, many of them relating to the various uncertainties associated with the eventual profitability of the R&D project. Not only is the value of the output of the R&D project generally uncertain, but there is often even greater uncertainty over the expected cost to completion of the R&D project.

Research and development projects typically take time to complete. Moreover, the time to completion or difficulty of the project is initially uncertain, and this uncertainty often forms the primary component of the uncertainty associated with the overall R&D project value. Such *technical uncertainty*, as defined by Pindyck (1993), can only be resolved by investing[3]; indeed changes in the expected time to completion occur *only* whilst R&D is being undertaken. Working on the research project generates additional information about the project and its expected cost to completion; if R&D stops, there is no new information to update expectations, so the expected cost to completion does not change. Furthermore, whilst continuing investment in R&D is expected to decrease the remaining expected time to completion, this is not certain: due to *e.g.* unexpected setbacks or unanticipated difficulties, which can arise because of the exploratory nature of the research and development process, the expected remaining time to completion can *increase* whilst investment in R&D continues. The actual time to completion is known for certain

[3]Prices, on the other hand, fluctuate whether or not investment is made. Thus technical uncertainty differs from the standard *price uncertainty* more usually considered in real-option investment models.

only once the R&D project is complete.

This technical uncertainty about the eventual cost to completion is specific to the individual R&D project and is thus completely idiosyncratic, uncorrelated with economy-wide or market uncertainty, and so cannot be hedged using tradeable assets.[4] Existing models of optimal dynamic R&D investment which allow for uncertain evolution of the expected cost to completion of the R&D project (*e.g.* Pindyck (1993)) typically assume that, since the technical uncertainty associated with cost to completion is diversifiable, no additional return is required for this by unconstrained and hence well-diversified investors. Whilst this assumption may be reasonable for large corporations or their shareholders, it is less so for an entrepreneur, for whom the R&D project may represent a sizable proportion of his overall wealth[5] and who is unable to hedge these risks by trading in the market because of their very nature.

In this paper we incorporate an entrepreneur's risk-aversion into the valuation of an R&D project subject to unhedgeable technical uncertainty about its cost to completion. We use utility-based valuation methods to obtain the certainty equivalent value of the R&D project to the entrepreneur and his optimal investment strategy in R&D. In order to isolate the effects of risk aversion and incomplete markets on technical uncertainty, we initially assume that the value of a completed

[4]The more innovative it is, the more likely that the value of the completed R&D is also partially unhedgeable.

[5]Moskowitz & Vissing-Jorgensen (2002) find substantial concentrations of private equity holdings by entrepreneurial investors: "about 75 percent of all private equity is owned by households for whom it constitutes at least half of their total net worth. Furthermore, households with entrepreneurial equity invest on average more than 70 percent of their private holdings in a single private company in which they have an active management interest.".

R&D project is deterministic, and use CARA utility, which enables us to obtain closed form solutions. A more general framework for evaluating R&D projects in incomplete markets incorporating partially unhedgeable price uncertainties is discussed in section 4.

We find that there are two opposing influences on both the value of the R&D project and the optimal investment/abandonment threshold (threshold between regions where it is worthwhile continuing to invest in the R&D project and where it is not, so investment stops[6]). The first influence is the information generation value of investing in R&D. Only by investing is additional information about the eventual cost to complete the R&D project generated. Absent risk-aversion and price uncertainty, this implies it is worth continuing to invest whilst the expected cost to completion is less than a maximum value which is *strictly greater* than the value of the completed R&D project, so the expected NPV at the threshold where R&D ceases is *negative*. Increases in technical uncertainty further reduce the NPV at the investment/abandonment threshold. The countervailing effect arises from risk aversion and incomplete markets. Risk aversion reduces the entrepreneur's certainty equivalent value of continuing to invest in R&D, because of the unhedgeable risk the continued investment forces him to bear. Since the value of continuing to invest is lower, investment stops for a lower maximum expected cost to completion, *i.e.* the expected NPV at the threshold increases. We show that the magnitude of this effect is positively related to both the investor's absolute risk aversion and the value of the completed project, and also increases with the level of technical

[6] If technical uncertainty is the only uncertainty associated with the R&D project the project is effectively abandoned.

uncertainty.

Either effect (information generation or risk aversion) can dominate for different ranges of parameter values, so the expected cost to completion at the investment/abandonment threshold can be either greater than or less than the value of the completed project. In the limit as the investor's risk-aversion tends to zero we recover the equations and results given in Pindyck (1993). For low levels of risk aversion and small projects, the information generation effect dominates, so that, as in the unpriced risk case, investment in R&D can continue even whilst the expected NPV is negative, and the expected NPV at the threshold becomes more negative as technical uncertainty increases. However, for high levels of risk aversion and large projects, the effects of risk aversion dominate, so investment in R&D ceases when the expected NPV is still positive. In these cases the reduction in R&D project value due to the increased cost of unhedgeable risk outweighs the increase in the option value of generating information under technical uncertainty, so the expected NPV at the investment/abandonment threshold becomes more positive as technical uncertainty increases. Thus sufficiently high levels of risk aversion or large projects can outweigh the information generation value of continuing to invest in R&D, reversing both the sign of the NPV at the investment/abandonment threshold and the sensitivity of the threshold to changes in technical uncertainty.

Existing real options/decision-theoretic models of R&D investment which have incorporated technical uncertainty in the expected R&D cost[7] all assume that, since

[7]Other models which use similar formulations for technical uncertainty include Kort (1998), who investigates the effects of both allowing greater resolution of uncertainty in the early stages of an R&D project and of R&D subsidies on investment/abandonment threshold and project values, Schwartz & Moon (2000), who, in addition to technical uncertainty, consider price uncertainty of the payoff to R&D

the risk associated with the cost to completion is diversifiable, no risk premium is required. All find the effects of the option value of information generation from investing in R&D described above[8]. We show that incorporating the 'risk-aversion' effect due to the unhedgeability of technical uncertainty for risk-averse entrepreneurs can lead to distinctly different conclusions: not only that sufficiently risk-averse entrepreneurs can stop investing in R&D whilst the expected NPV is strictly positive (whereas risk-neutral investors would continue to invest until the expected NPV was strictly negative), but also, for sufficiently risk-averse entrepreneurs, that increased levels of technical uncertainty can *increase* the minimum expected NPV required for continued investment, in direct contrast to the effect of increased technical uncertainty for risk-neutral investors, which is to decrease the minimum expected NPV required for continued investment.

The paper is also related to the recent strand of the real options literature which considers the effects of partial unhedgeablilty of price uncertainty on investment timing decisions *e.g.* Henderson (2007) and Miao & Wang (2007)[9]. However,

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and the possibility of catastrophic complete loss of value in an application to multiple-staged drug development, Schwartz (2004), who also incorporates technical uncertainty, price uncertainty and catastrophic loss of value in a model of patent valuation and Kyle & Meng (2004), who also consider both technical uncertainty in costs and price uncertainty of the payoff but focus on competitive interactions and takeover incentives when firms of different sizes undertake R&D in a patent race.

[8] *I.e.* investment in R&D continues whilst the expected cost to completion is below a threshold which corresponds to a strictly negative expected NPV for the research project. Furthermore increased technical uncertainty increases the value of the option to generate information about the cost to completion by continuing to invest, and is thus positively related to investment in R&D (since investment continues for a wider range of expected costs to completion).

[9] Henderson (2007) considers a risk-averse entrepreneur with exponential utility who can decide when

these papers considered only the case of price or market-based uncertainty. Under complete markets or unpriced idiosyncratic risk, increases in price uncertainty and technical uncertainty have opposite effects on the optimal investment threshold. Price uncertainty increases the NPV threshold required for optimal investment above zero, whereas technical uncertainty decreases the NPV threshold, so investment can be undertaken even when the expected cost to completion exceeds the value of the completed R&D project. It is thus of interest to compare the effects of risk aversion on the two types of investment-related uncertainty. We discuss the relationship between our results and other models of real options under market incompleteness in section 4.

In section 2 we set up a utility based model for the value of an R&D project to a risk-averse entrepreneur who is unable to hedge the technical uncertainty associated with the expected cost to completion of the R&D and find closed form solutions for the project value and investment/abandonment threshold. Proofs of all results are given in the Appendix. Section 3 analyses the results of the model, section 4 discusses these results in the wider context of real options models in complete and incomplete markets and section 5 concludes.

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to invest to obtain a lump-sum payoff. The payoff's uncertainty is only partially hedgeable by other traded assets. In a similar setting Miao & Wang (2007) incorporate optimal consumption and consider both lump-sum and flow payoffs but present results primarily for the completely unhedgeable case. Van den Goorbergh, Huisman & Kort (2003) consider a slightly different utility function with payoff a flow of completely unhedgeable cashflows. Hugonnier & Morellec (2005) also consider the flow payoff case. Andrikopoulos (2006) argues for the application of such models to R&D because of the innovative nature of the output of R&D output and applies a binomial model based on Smith & McCardle (1998) to study the effects of agency conflicts between debt and equity holders.

2 Model

We consider an infinite-horizon continuous-time model, in which a risk-averse entrepreneur decides whether and how much to invest in research and development of a new product at each point in time, *i.e.* he optimises over the amount he invests $I \geq 0$. No cashflows arise from the R&D project before completion.[10] The value on successful completion of the R&D project, V, is deterministic[11], however the remaining cost to completion of the R&D, $\tilde{K}$, is unknown.

We follow Pindyck (1993) and model technical uncertainty in the evolution of the entrepreneur's expected cost to completion, $K = E[\tilde{K}]$, over time using

$$dK = -Idt + \phi I^{\frac{1}{2}} K^{\frac{1}{2}} dZ_K. \tag{1}$$

where dZ_K is Brownian motion. Here I is the rate of investment by the entrepreneur in R&D per period and is constrained to be less than some maximum rate ($0 \leq I \leq i_{\max}(t)$).[12] [13] The expected decrease in K equals the rate of investment in R&D, $E[dK] = -Idt$. However, investment generates additional information about

[10] In order to concentrate on the effects of unhedgeable technical uncertainty and risk aversion we abstract from price uncertainty of either the value of completed R&D or input costs as well as issues relating to agency problems, financing considerations and competitive interactions.

[11] We take the risk-free bond as numeraire and so express all amounts in discounted units. A constant V in the model is thus equivalent to a monetary value on completion of R&D of Ve^{rt}.

[12] This constraint can represent physical constraints (or a simple representation of diseconomies of scale) on the feasible rate of investment. Kort (1998) justifies a similar assumption by arguing that R&D projects in practice need to be self-financed, so $i_{\max}(t)$ represents the maximum cash available from the firm's or entrepreneur's other activities. We use exponential utility and thus do not consider wealth effects.

[13] Pindyck's formulation also has the useful features that if investment occurs at the maximum rate until the project is complete, then K is the actual cost to completion: $E_0 \int_0^{\tilde{T}} i_{\max}(t)dt = K$. Furthermore,

future costs, so the expected cost to completion can change by more or less than this and can even increase if the new information is sufficiently bad. Additional information about the eventual cost to complete the R&D project is generated only by investing (the variance of the change in the cost to completion, $Var[dK] = \phi^2 IKdt$, is non-zero only whilst investment in R&D continues). Hence the expected cost to completion changes only whilst investment occurs ($I \neq 0$).

All risk associated with the Brownian motion dZ_K is diversifiable, *i.e.* dZ_K is uncorrelated with the traded risky asset, $dZ_K dZ_M = 0$ (see (2)). However, although technical uncertainty is completely idiosyncratic, it cannot be hedged using traded assets. Thus the entrepreneur is forced to bear this risk as long as he continues to invest in the R&D project but exposure to this risk ceases as soon as investment stops.

In addition to the R&D project, the entrepreneur also possesses outside liquid wealth, which he can invest in risk-free bonds or a traded risky asset, M, which evolves as

$$dM = \lambda\sigma M dt + \sigma dZ_M \tag{2}$$

where σ and λ are the asset's volatility and Sharpe ratio respectively. At each instant, the entrepreneur chooses the amount he holds in the traded risky asset, $\theta(t)$, and the rate of investment in the R&D project, $0 \leq I \leq i_{max}(t)$. His wealth,

the variance of the cost to completion, $Var[K] = \frac{\phi^2}{2-\phi^2}K^2$, is finite for all finite K and tends to zero as the expected cost to completion tends to zero (see Pindyck (1993)).

W, expressed in discounted units,[14] evolves as

$$dW = (\theta\lambda\sigma - I)dt + \theta\sigma dZ_M \tag{3}$$

where $\theta(t)$ is the amount invested in the traded risky asset. The entrepreneur finances the investment in R&D from his outside liquid wealth. We assume there are no frictions in transfers between his holdings in risk-free bonds and either the traded risky asset or the R&D project.

We reflect the entrepreneur's choices over θ and I in his objective function, which is given by

$$J = \sup_{I_s,\theta_s,t\le s\le \tilde{T}} E_t\left[U_{\tilde{T}}(W_{\tilde{T}} + V_{\tilde{T}}))\right] \tag{4}$$

subject to (1) and (3) where $\tilde{T}$ is defined by $K_{\tilde{T}} = 0$ (so $\tilde{T}$ is the stochastic time at which R&D is successfully completed), and $U_{\tilde{T}}(x)$ is the horizon-unbiased exponential utility function[15]

$$U_{\tilde{T}}(x) = -\frac{1}{\gamma}e^{\frac{\lambda^2}{2}(\tilde{T}-t)}e^{-\gamma x} \tag{5}$$

Before completion of the R&D project the entrepreneur chooses the investment strategy for his liquid wealth and the amount of ongoing investment in R&D in order to maximise his expected horizon-unbiased utility on completion of the R&D project. After completion he continues to invest his liquid wealth, which now includes the

[14] Recall we take the risk-free bond as numeraire so the entrepreneur's wealth is expressed in units of the risk-free bond.

[15] Henderson & Hobson (2007) show that requiring an investor to be unbiased or have no preference with respect to horizon, $\tilde{T}$, restricts the choice of discount rate in an intertemporal utility function. Effectively, horizon-unbiased utility functions incorporate optimal investment by the entrepreneur after the horizon date, $\tilde{T}$, *i.e.* after investment or abandonment or, in this case, completion of the R&D project.

value obtained on successful completion of the R&D project, optimally; this is reflected in the particular form (horizon-unbiasedness) of the utility function in his objective. The current value of his objective function depends on his current liquid wealth, W, and on his current valuation of the R&D project via the expected cost to completion, K, (and more generally on the value on completion, V) but is independent of the current time: $J \equiv J(W, K; V)$.

Standard arguments give the HJB equation for J:

$$\frac{\lambda^2}{2}J + \theta^* \lambda \sigma J_W + \frac{\sigma^2}{2}\theta^{*2} J_{WW} + I^* \left(-J_W - J_K + \frac{\phi^2}{2} K J_{KK} \right) = 0 \tag{6}$$

where θ^*, I^* represent the optimal amount held in the traded risky asset and optimal investment in the R&D project respectively. Since the technical uncertainty is uncorrelated with the traded risky asset, the entrepreneur cannot hedge the R&D risk, so his liquid wealth investment strategy is unaffected by the R&D project: $\theta^* = \frac{-\lambda J_W}{\sigma J_{WW}}$. The optimal strategy for investing in the R&D project is:

$$I^* = \begin{cases} i_{max} & \text{if } \ \frac{1}{2}\phi^2 K J_{KK} - J_K \geq J_W \\ 0 & \text{if } \ \frac{1}{2}\phi^2 K J_{KK} - J_K < J_W \end{cases} \tag{7}$$

This is equivalent to investing maximally if the expected cost to completion, K, is less than a threshold K_*; if $K > K_*$, investment in R&D ceases.

In the Appendix we show that we can write

$$J(W, K) = -\frac{1}{\gamma} e^{-\gamma(W + F(K))} \tag{8}$$

where $F(K)$ is the certainty equivalent value of the R&D project (the amount of incremental wealth, invested optimally, which makes the entrepreneur indifferent between that amount of liquid wealth and the R&D project). As usual under exponential utility, the certainty equivalent value is independent of the entrepreneur's liquid wealth and, since the value on completion of R&D is deterministic, depends only on the expected cost to completion of the project. Proposition 1 characterises the equations satisfied by the certainty equivalent value of R&D.

Proposition 1 *The certainty equivalent value of the R&D project to the entrepreneur, $F(K)$, satisfies:*

$$\frac{1}{2}\phi^2 K(F_{KK} - \gamma F_K^2) - F_K = 1 \qquad 0 \leq K \leq K_* \tag{9}$$

subject to the following boundary conditions

$$F(0) = V \tag{10}$$

$$F(K_*) = 0 \tag{11}$$

$$F_K(K_*) = 0 \tag{12}$$

The entrepreneur's optimal investment strategy for R&D is

$$I^* = \begin{cases} i_{max} & \text{if} \quad K \leq K_* \\ 0 & \text{if} \quad K > K_* \end{cases} \tag{13}$$

He invests an amount $\theta_0^* = \frac{\lambda}{\gamma\sigma}$ of his liquid wealth in the risky asset and the remainder in the risk-free asset.

The optimal investment strategy in R&D is thus either to invest at the maximum rate, $I = i_{\max}$, if $K \leq K_*$, or not to invest, $I = 0$, if the expected cost to completion is too high, $K > K_*$. The threshold K_* must be found as part of the solution. Since the expected cost to completion only changes whilst investment continues and K_* is constant, once K exceeds K_* and investment stops, K remains above K_* thereafter so the R&D project is effectively abandoned at K_*[16]and its value to the

[16]In contrast if there were additional input cost uncertainty or if the value on completion, V, were also

entrepreneur is zero. This is reflected in the value matching boundary condition (11) for F at K_*. Equation (12) is the corresponding smooth-pasting condition. Investment only starts if initially $K \leq K_*$; it continues at the maximum rate, i_{max}, until either K increases to K_* or the project is complete ($K = 0$). On completion, the value of the R&D project equals V, so $F(0) = V$.

F satisfies a nonlinear ordinary differential equation due to the term $-\frac{1}{2}\gamma\phi^2 K F_K^2 < 0$, which represents the cost of the unhedgeable technical uncertainty the entrepreneur is forced to bear whilst investment in R&D continues. The magnitude of this cost increases with the entrepreneur's risk aversion, γ, and with his exposure to the unhedgeable risk via the instantaneous variance of the technical uncertainty ($\phi^2 K$) and the exposure of the R&D project to this (F_K^2).

We are able to derive a solution to (9) - (12) in terms of convergent infinite series (details of the solution procedure are given in the Appendix). For ease of interpretation, we scale by the deterministic value on completion, V, and consider both the expected cost to completion and also the certainty equivalent value of R&D as proportions of the value obtained on completion, $\xi = \frac{K}{V}$ and $f = \frac{F}{V}$ respectively.

Proposition 2 *The solution to (9) - (12) is given by* $F = Vf(\xi)$ *with* $\xi = \frac{K}{V}$, $\xi_* = \frac{K_*}{V}$ *and*

$$f(\xi) = -\frac{1}{\gamma V} \ln \left[\sum_{n=0}^{\infty} a_n \xi^n - \beta(\xi_*) \left(\frac{\xi}{\xi^*} \right)^{1+\frac{2}{\phi^2}} \sum_{n=0}^{\infty} b_n \xi^n \right] \tag{14}$$

where $a_0 = b_0 = 1$ *and*

$$a_n = \Pi_{m=1}^{n} \left(\frac{(-2\gamma V/\phi^2)}{m(m-1-\frac{2}{\phi^2})} \right), \qquad b_n = \Pi_{m=1}^{n} \left(\frac{(-2\gamma V/\phi^2)}{m(m+1+\frac{2}{\phi^2})} \right), \tag{15}$$

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stochastic, investment could restart so K_* would be a suspension threshold and could vary with V. We discuss this further in section 4.

$$\beta(\xi_*) = \frac{\sum_{n=1}^{\infty} n a_n \xi_*^n}{\sum_{n=0}^{\infty}(n+1+\frac{2}{\phi^2}) b_n \xi_*^n} \tag{16}$$

and ξ_* is the solution to

$$\gamma V = \ln\left[\sum_{n=0}^{\infty} a_n \xi_*^n - \beta(\xi_*) \sum_{n=0}^{\infty} b_n \xi_*^n\right] \tag{17}$$

So f is the certainty equivalent value of R&D and ξ_* the expected cost to completion at the investment/abandonment threshold, *i.e.* the maximum expected cost to completion for which investment in R&D continues. Both f and ξ_* are normalised by the value obtained on completion of the R&D project. The conventional NPV at the investment/abandonment threshold relative to the value on completion is $\xi_* - 1$. Note f and ξ_* depend only on two parameter combinations, ϕ^2 and $\alpha \equiv \gamma V$, representing technical uncertainty and effective risk aversion. In the next section we analyse the effects of technical uncertainty, risk aversion and project size on the entrepreneur's valuation of the project and the investment/abandonment threshold.

3 Results

If there is no technical uncertainty ($\phi = 0$), both risk-neutral and risk-averse entrepreneurs invest at the conventional NPV threshold ($\xi_* = 1$ or $K_* = V$) and, if they decide to invest, value the project at the conventional NPV: $F = V - K$ ($f = 1 - \xi$). Risk-aversion thus has no effect in the absence of unhedgeable risk (technical uncertainty). Technical uncertainty does however impact both the entrepreneur's valuation of the project and the optimal investment/abandonment threshold in the absence of risk-aversion. We thus first consider the effect of technical uncertainty in the absence of risk aversion and then identify the additional effects in the full problem due to risk-aversion.

Proposition 3 *As $\gamma \to 0$, the model reduces to the unpriced idiosyncratic technical uncertainty model of Pindyck (1993): $F^{(0)}$ satisfies*

$$\frac{1}{2}\phi^2 K F^{(0)}_{KK} - F^{(0)}_{K} = 1, \qquad 0 \leq K \leq K_{*(0)} \tag{18}$$

subject to $F^{(0)}(0) = V$, $F^{(0)}(K_{*(0)}) = 0$, and $F^{(0)}_{K}(K_{*(0)}) = 0$. The optimal investment strategy is

$$I^* = \begin{cases} i_{max} & \text{if} \quad K \leq K_{*(0)} \\ 0 & \text{if} \quad K > K_{*(0)} \end{cases} \tag{19}$$

This has solution $F^{(0)} = Vf^{(0)}(\xi)$ with $\xi = \frac{K}{V}$, $\xi_{*(0)} = \frac{K_{*(0)}}{V}$ and

$$f^{(0)}(\xi) = 1 - \xi + \frac{\phi^2}{(2+\phi^2)}\xi_{*(0)}\left(\frac{\xi}{\xi_{*(0)}}\right)^{1+\frac{2}{\phi^2}} \tag{20}$$

$$\xi_{*(0)} = 1 + \frac{\phi^2}{2} \tag{21}$$

If $\gamma = 0$ and idiosyncratic technical uncertainty is unpriced, investment continues until the project's conventional NPV is strictly negative ($\xi_{*(0)} > 1$ or $V - K_{*(0)} < 0$). Furthermore both the value of the R&D project, $F^{(0)}$, and the investment/abandonment threshold, $K_{*(0)}$, increase with technical uncertainty, ϕ. So projects with greater risk surrounding their time or cost to completion are both more highly valued and are continued for a wider range of expected costs to completion: the expected NPV can become negative before R&D is halted.

These findings are due to the information generation value of investing in R&D. Continuing to invest in R&D brings benefits to the entrepreneur, not only reducing the expected cost to completion of the project by the amount invested (and thus increasing the project's conventional NPV) but also generating additional information about future costs of completing the R&D, which can *only* be achieved by investing.

Once investment ceases, since there is no price uncertainty in this model, there is no new information to update the expectation of future costs, so the expected cost to completion, K, does not change. Since no changes in the project's NPV occur, there is no option value to waiting and so it is never optimal to restart investment. Whilst he continues to invest, the entrepreneur retains the option to stop investing at any time. This can be viewed as an option to stop (in this case abandon) R&D, which can, crucially, only be retained by continuing to invest in R&D.

In the unpriced technical uncertainty model the value of the R&D project thus consists of the project's conventional NPV[17], ($V - K = V(1-\xi)$), given by the first two terms in (20), plus the information generation value of investing in R&D including the value of the option to stop researching, given by the final term. The latter increases the value of the project to the entrepreneur above the conventional NPV, and increases in value with technical uncertainty. Increased technical uncertainty increases the value of continuing to invest to generate additional information about costs to completion of the R&D when there is an option to stop investing. Since the conventional NPV and the value on stopping R&D are both unaffected by technical uncertainty, the threshold for stopping investment also increases with increases in technical uncertainty: the entrepreneur is willing to continue investing for a more negative conventional NPV because he also takes into account the greater option value of information generation so his overall value of continuing to invest remains positive for higher expected costs to completion.

[17]Recall we work in discounted values, so V represents the discounted value of the completed project and K the discounted expected cost to completion.

Note $f^{(0)}$ and $\xi_{*(0)}$ are both independent of the value on completion, V, so R&D project values and investment/abandonment thresholds both scale linearly with the size of the project, *e.g.* investment is stopped in both large and small projects at the same proportion of the respective values on completion, providing they have the same level of technical uncertainty, ϕ.

We now turn to the full model with $\gamma > 0$. The enforced exposure to risk whilst continuing to invest in the R&D project (due to the unhedgeability of the technical uncertainty associated with the project's expected cost to completion) decreases the value of continuing to invest to a risk-averse entrepreneur relative to the value when this idiosyncratic technical uncertainty is unpriced (unpriced risk value)[18]. Since the value on stopping R&D is the same for both the risk-averse and unpriced risk values, the cost threshold for stopping investment is also lower under risk-aversion. A risk-averse entrepreneur thus places a lower value on continuing to invest in R&D and so will wish to stop investing at a lower expected cost to completion. Thus the minimum NPV for continued investment in R&D is higher for a risk-averse entrepreneur than in the unpriced risk case.

Insert Figure 1 here

Both these effects are shown in Figure 1, which plots the value of the R&D project as a proportion of the value on completion, ($\frac{F}{V} = f$) as a function of $\xi = \frac{K}{V}$, the expected cost to completion as a proportion of the value on completion for

[18]This can be seen by comparing the equations satisfied by the project values in the risk-averse (9) and $\gamma = 0$ cases (18). The risk-averse equation, (9), contains an additional term, $-\frac{1}{2}(\gamma V)\phi^2\xi f_\xi^2 < 0$, which, as discussed in section 2, represents the cost of unhedgeable technical uncertainty. Since the term is always negative, by a comparison argument it decreases the project value in the risk-averse case.

different values of effective risk aversion $\alpha = \gamma V$[19]. Higher α corresponds to a more risk-averse entrepreneur (higher γ) or a larger project (higher V). We discuss the interpretation of results for projects of different size (V) later; for now we concentrate on the effects of changing risk aversion, γ.

The highest, solid curve in Figure 1 represents the risk-neutral value, $\gamma = \alpha = 0$, as in Pindyck (1993). In this case the value of the project is strictly greater than the project's conventional NPV (represented by the dotted straight line $1 - \xi$) and the NPV when investment ceases is strictly negative ($\xi_{*(0)} > 1$). Curves for successively higher values of α show that as risk aversion increases, project values decrease monotonically for all costs to completion, often dropping below the conventional NPV. So risk averse entrepreneurs can value risky R&D projects at less than the conventional NPV, even including the associated option. Figure 1 also shows that successive thresholds, where the project value $f = F/V$ reaches 0, also decrease monotonically with effective risk aversion, α, and for high levels of effective risk aversion the NPV when investment ceases can be strictly positive (*i.e.* $\xi_* < 1$).

The reversal of the sign of the NPV at the investment/abandonment threshold when the effects of risk aversion are high is a striking result, so we focus on the investment/abandonment threshold in more detail. We investigate how the threshold varies with effective risk aversion, α. Expanding the threshold ξ_* for $\alpha = \gamma V \ll 1$ shows the behaviour of the threshold for low levels of effective risk aversion:

$$\xi_* \approx \xi_{*(0)} - \gamma V \frac{\phi^2 \left(1 + \frac{\phi^2}{2}\right)}{4(1+\phi^2)} + O\left((\gamma V)^2\right) \tag{22}$$

[19]Recall f depends only on two parameter combinations, ϕ^2 and γV. In Figure 1 $\phi = 0.6$. This corresponds to a standard deviation of the project's cost of just under 50% of the expected cost, which is not unusual (Pindyck (1993)). Previous studies of R&D investment use values of ϕ in the range $0.34 - 1$.

So to leading order $\xi < \xi_{*(0)}$ for $\gamma V > 0$. A risk-averse entrepreneur ceases investment at a lower expected cost to completion than the risk-neutral/unpriced idiosyncratic technical uncertainty threshold, even for low levels of effective risk-aversion. Moreover, for low levels of effective risk aversion the difference between risk averse and unpriced risk thresholds increases with technical uncertainty.

Insert Figure 2 here

For higher levels of effective risk aversion we must use numerical methods to implement the full model. Figure 2 plots the normalised optimal investment/abandonment threshold $\xi_* = \frac{K_*}{V}$ against effective risk aversion $\alpha = \gamma V$ for different levels of technical uncertainty, ϕ. It shows the expected cost to completion at the investment/abandonment threshold decreases as the effective risk aversion increases for all levels of technical uncertainty, and, as suggested by the asymptotic expansion for small α, is more pronounced for higher levels of technical uncertainty. Normalised risk-neutral thresholds ($\alpha = 0$) are all greater than 1: $\xi_* > 1$. This corresponds to a negative conventional NPV at the investment/abandonment threshold, so the entrepreneur continues to invest in projects which have a conventional NPV which is negative (providing it is not too negative). For low levels of effective risk aversion the expected cost to completion at investment/abandonment threshold, whilst lower than in the unpriced risk case, remains above the value of completed R&D ($\xi_* > 1$). However, for higher levels of effective risk aversion the threshold $\xi_* < 1$, so the entrepreneur stops investing whilst the project's conventional NPV is still positive. For these cases the entrepreneur's cost of continuing to bear the unhedgeable technical uncertainty associated with the R&D project's cost exceeds the information generating value, even including the option to stop investing in the future.

In this model, the entrepreneur's risk aversion, γ and the size of the R&D project (value on completion, V) affect the normalised project value and investment/abandonment thresholds only via the combination $\alpha = \gamma V$, the effective risk aversion. We can thus interpret results relating to effective risk aversion in two ways. Firstly, a particular entrepreneur stops investing in an R&D project earlier (for lower normalised expected cost to completion), the larger the project. The intuition is that the larger the project the larger the expected outlay or cost to completion and so the greater the exposure to unhedgeable risk, which reduces the value of continuing investment. Secondly, for research projects of the same size, more risk-averse entrepreneurs stop investing in R&D earlier, since the cost to them of the unhedgeable risk exposure is greater. Hence, under risk aversion the normalised value of the project and the optimal investment strategy are no longer independent of the scale of the project. The normalised expected cost to completion at the investment/abandonment threshold is lower (and can be less than one) for larger projects as well as for more risk-averse entrepreneurs.

Another distinction between high and low levels of effective risk aversion, α, shown in Figure 2 is the sensitivity of the investment/abandonment threshold ξ_* to changes in levels of technical uncertainty. Comparing the plots for $\phi = 0.4$, 0.6 and 0.8, we see that for small α, increased technical uncertainty increases ξ_*, whereas for large α, increased technical uncertainty decreases ξ_*. Thus for less risk averse entrepreneurs investing in smaller projects, for whom α is small, we continue to obtain the same result as in the unpriced risk case: the threshold expected cost to completion increases with technical uncertainty, further decreasing the expected NPV at the threshold below zero. The information generating option effect, which increases with uncertainty, thus dominates. The risk aversion effect also increases with uncertainty, however, and for entrepreneurs who are sufficiently risk-averse or considering sufficiently large R&D projects (large α), increased (unhedgeable) technical uncertainty *decreases* the expected cost to completion at the investment/abandonment threshold, further *increasing* the expected positive NPV at the threshold. Expanding the equation for the optimal threshold, ξ_* for small ϕ^2 shows

$$\xi_* \approx 1 + \phi^2 \left(\frac{1}{2} - \frac{\gamma V}{4} \right) + O(\phi^4), \tag{23}$$

so, to leading order, this distinction persists even for low levels of technical uncertainty.

Insert Figure 3 here

Figure 3 illustrates the two possible effects of increases in technical uncertainty on the value placed by the entrepreneur on the R&D project. The outer lines represent the cases for $\phi = 0.8$ and the inner lines for $\phi = 0.4$. The middle straight dotted line represents the conventional NPV, which corresponds to the project value if $\phi = 0$. The top two lines represent R&D project values for low levels of effective risk aversion ($\alpha = 0.1$), where the information generation option effect dominates. This option increases the value of the R&D project to the entrepreneur above the project's conventional NPV, so investment continues whilst the conventional NPV is negative. The project value and optimal cost threshold, above which investment ceases, are both greater for $\phi = 0.8$ than $\phi = 0.4$, showing both increase with technical uncertainty. The bottom two lines show the case for higher levels of effective risk aversion ($\alpha = 4$). In this case the risk-aversion effect dominates, so the

R&D project value to the risk-averse entrepreneur is lower than the conventional NPV, and decreases as (the cost of unhedgeable) technical uncertainty increases (the solid line representing $\phi = 0.8$ is lower than the dash-dotted line representing $\phi = 0.4$). Investment in R&D thus ceases whilst the conventional NPV is still positive, and the minimum NPV required for continued investment increases with uncertainty (more than doubling from around $0.1 = 1 - 0.9$ for $\phi = 0.4$ to 0.24 for $\phi = 0.8$).

4 Discussion

The results in this paper differ markedly from those from real options models which consider the more usual *price uncertainty.* We summarise the implications of different types of uncertainty on the minimum conventional NPV required for investment in each case in table 1. For price uncertainty under risk aversion, the comparative static results differ depending on whether the payoff on investment involves a lump-sum amount which depends on the value of the unhedgeable asset but which can then be traded, or instead gives the right to a stream of cashflows (flow payoff), each of which depend on the contemporaneous value of the unhedgeable asset.

Insert Table 1 here

For the moment ignore the flow payoff column: we will come back to this later. From the table we see that different types of uncertainty not only have opposite effects on the conventional threshold NPV in the risk-neutral case but that the effect on the threshold of increasing risk-aversion in incomplete markets also has the opposite sign for the two types of uncertainty: the NPV at the investment threshold decreases with risk-aversion for price uncertainty with lump-sum payoffs, but increases with risk-aversion for technical uncertainty. Moreover, whilst the sensitivity of the threshold NPV to uncertainty can be either positive or negative for both types of uncertainty depending on the entrepreneur's level of effective risk-aversion,[20] again the signs of the sensitivities for technical uncertainty and price uncertainty with lump-sum payoffs are opposite. These differences complicate empirical testing significantly. Thus empirically it is important to control for the type of uncertainty (price or technical) and additionally for entrepreneurial risk aversion and the size of the project when testing relationships between uncertainty, risk aversion, project size and investment. Moreover the signs of sensitivities for price uncertainty with flow payoffs have a different profile to those for either lump-sum payoffs or technical uncertainty, which complicates matters still further.

However, the seemingly disparate results in Table 1 can be viewed in a common framework. Consider first the top line about the sign of the conventional NPVs in 'standard' real options models. Here the distinction between lump sum and flow payoffs is irrelevant. Both types of uncertainty can be thought of as giving rise to additional option effects (either of waiting for new information or of generating information). In both cases the option has value to its holder, who therefore takes this value into account when making optimal decisions. For investment under price uncertainty, investing involves giving up the option, so, absent risk aversion, the investment threshold occurs when the payoff, or conventional NPV, is positive and equals the value of the option exercised. In contrast, the 'investment' decision

[20]Under price uncertainty the effect of risk aversion is also scaled by a measure of the project's size, generally the investment cost.

for R&D is actually when to *stop* investing, which involves giving up the option to continue to invest. Again, absent risk aversion the threshold occurs when the payoff to stopping equals the positive option value. This is equivalent to investing until the conventional NPV of continuing to invest is strictly negative. Moreover, option values under both price and technical uncertainty increase with the level of uncertainty, so the magnitude of the conventional NPVs at the thresholds also increase with uncertainty in both cases.

Risk aversion and market incompleteness (partial or complete inability to hedge) create an additional effect due to the cost of unhedgeable risk, which increases with the level of unhedgeable or idiosyncratic risk and the entrepreneur's effective risk aversion. This cost of unhedgeable risk reduces values of investments or options to entrepreneurs exposed to market incompleteness and hence affects investment thresholds and sensitivities to changes in uncertainty levels. Under price uncertainty with lump sum payoffs, incompleteness is eliminated at the moment of investment. Risk aversion decreases the value of the option to invest but has no effect on the value of the payoff and thus also reduces the conventional NPV at the investment threshold, by more the greater the risk aversion, the larger the project or the greater the level of unhedgeable risk. Under technical uncertainty, incompleteness is present only whilst investing but is, in expectation, resolved gradually with continuing investment. The cost of unhedgeable risk decreases the overall value of continuing to invest, including the option to abandon, and hence induces earlier abandonment, which is equivalent to increasing the minimum conventional NPV required for investment. Under price uncertainty with flow payoffs, incompleteness is present both before and after investment, but is 'locked in' by investing. Risk aversion decreases the value of the investment itself as well as the option to invest, but the absence of flexibility means its effect is greater on the investment than the option. Hence investment is optimally delayed further than in the complete markets case, further increasing the conventional NPV at the investment threshold.

Thus when incompleteness is resolved by investing (for lump sum payoffs under price uncertainty and under technical uncertainty), both the cost of unhedgeable risk and option effects are present only before investment is complete. Since the option effect increases values, whilst the cost of unhedgeable risk reduces values, these act in opposition to each other, and the final effect on values and thresholds (relative to the conventional NPV value and threshold) depends on the relative strengths of each effect[21]. However, when incompleteness continues after investment (the flow payoff case), the cost of unhedgeable risk is at least as great after investment as before. Thus the two effects reinforce each other: investment both eliminates flexibility and increases the cost of unhedgeable risk and is thus delayed until the conventional NPV on investment is greater.

Turning to the sensitivity of thresholds to changes in unhedgeable risk, note in incomplete markets idiosyncratic or unhedgeable risk increases both option values and the cost of unhedgeable risk. For both types of uncertainty, for low levels of risk

[21]Miao & Wang (2006) consider the optimal abandonment of an entrepreneurial venture where an undiversified risk-averse entrepreneur learns about the venture's profitability over time. They find two opposing effects: an option value of learning, which increases the value of continuing to invest and induces delay in abandoning, and a cost of unhedged risk effect, which reduces the value of continued investment and induces earlier abandonment. Both effects are only present before abandonment. They also obtain that either can dominate, so the NPV of the decision can be either positive or negative, depending on the relative strengths of the effects, in particular on the entrepreneur's risk-aversion.

aversion, the option value effect is more important, so the difference between the optimal and zero conventional NPV threshold increases[22]. However the importance of the cost of unhedgeable risk increases with effective risk aversion and eventually dominates, causing a change in the sign of the sensitivity of the thresholds to changes in unhedgeable risk in the cases when option and cost of unhedgeable risk effects act in opposite directions.

Two distinctions between price and technical uncertainty are firstly that technical uncertainty is completely idiosyncratic. Hence the cost of unhedgeable risk effect will generally be greater than under price uncertainty, where risk is partly systematic and hence hedgeable. Secondly, whilst for price uncertainty optimal threshold NPVs under market incompleteness are closer to the conventional NPV threshold but remain positive (the option effect always remains greater than the associated cost of unhedgeable risk), for technical uncertainty the cost of bearing the unhedgeable risk associated with continuing with the R&D project can be greater than the project's information generation option value, so the net effect of uncertainty can be to reduce project values for the entrepreneur below the project's conventional NPV, reversing the sign of the NPV at the investment/abandonment threshold.

In general, R&D projects are likely to be subject to both types of uncertainty due to price uncertainty about input costs and uncertainty about the market price or demand for the R&D output, as well as technical uncertainty about the expected costs to completion. Moreover, to the extent R&D produces new products these

[22]For price uncertainty, the conventional NPV at the threshold becomes more positive as idiosyncratic risk increases, whereas for technical uncertainty, the conventional NPV of investment at the threshold becomes more negative.

will also be at least partially unhedgeable.

In the Appendix we relax the assumption that V is deterministic and allow the resulting price risk to be partially hedgeable using traded assets. We derive equations satisfied by the project's certainty equivalent value under exponential utility, which is now a function of both the expected cost to completion and the value on completion of the R&D, $F(K, V)$. Detailed investigation of this more general problem is beyond the scope of this paper; however we can make the following observations. Firstly, as in the unpriced risk case, the threshold K_* is now an investment/*suspension* threshold and varies with V, and the R&D project has value even whilst investment in R&D is suspended. Secondly, even in the unpriced risk case with both technical and price uncertainties in K and V respectively it has been shown (Schwartz & Moon (2000)) that the investment/suspension threshold can correspond to conventional NPVs which are either positive or negative, depending on the expected cost to completion and the relative sizes of technical and price uncertainties, which produce two opposing effects as outlined above. Finally, risk aversion in incomplete markets will decrease the values of the options to both suspend and restart investment, but by different amounts depending on the project's size (because of the differences in the form of the stochastic processes) and the degree of partial hedgeability of the completed value of R&D, as well as the relative sizes of the technical and price uncertainties as before. If the entrepreneur retains the project and receives a stream of partially unhedgeable cashflows on completion of the R&D, the value on completion will also be reduced because of the cost of unhedgeable risk. These will all affect the threshold's location and comparative statics.

Thus with multiple sources of uncertainty and market incompleteness the location of the investment/suspension threshold for R&D (positive or negative NPV) and its sensitivity to project scale, risk aversion and different forms of uncertainty will depend on the interaction of a number of effects which have been considered individually above. When effects are reinforcing or independent, comparative static results can be inferred[23]; more generally they will depend on the relative sizes of each effect. However, the effects of technical uncertainty in this incomplete markets setting should not be ignored, especially if the normalised conventional NPV is small: when the project is 'close' to the investment/abandonment threshold the location of the threshold is particularly important.

5 Conclusion

R&D involves the creation of something new or different and is an inherently uncertain process. A key source of uncertainty is the technical uncertainty associated with the difficulty of completing R&D, which is unaffected by economy-wide uncertainty, and so cannot be hedged. Undertaking R&D exposes entrepreneurs to this unhedgeable risk but is the only way of generating information about and eventually resolving the uncertainty about costs and completing the R&D project.

In this paper we have used utility based methods to derive the value of an R&D

[23] *E.g.* if the entrepreneur retains the project on completion (the flow payoff case for the price uncertainty), then the conventional NPV at the investment/suspension threshold should increase with the entrepreneur's risk aversion, both hedgeable and unhedgeable price uncertainty about the cashflows on completion of the R&D project and, for sufficiently high risk aversion, also with technical uncertainty about the expected cost to completion of the R&D project.

project with unhedgeable technical uncertainty and have characterised the optimal investment/abandonment threshold as the root of a single non-linear equation. We find two opposing effects of technical uncertainty in incomplete markets, one due to the generation of information and the second to the unhedgeable risk the entrepreneur is forced to bear. The option effect of continuing to invest and generate information increases the value of the project to the entrepreneur, delaying abandonment or suspension of R&D until the conventional NPV of continuing investment is strictly negative. The cost of unhedgeable risk reduces the project value, inducing abandonment at a lower expected cost to completion, when the conventional NPV of continuing investment is still strictly positive. Either effect can dominate depending on the entrepreneur's risk aversion and the size of the project, so the NPV at the investment/abandonment threshold can be either positive or negative. Moreover, both effects increase with technical uncertainty, so the NPV at the threshold can either increase or decrease as technical uncertainty increases.

In practice an R&D project may be subject to both price and technical uncertainty. Evaluating such a project for a risk-averse entrepreneur incorporating his likely inability to hedge some, if not all, of the risks he is forced to bear in order to invest in the project is complex in general as it involves the interaction of a number of option and cost of unhedgeable risk effects. The effects of price uncertainty on values and optimal investment thresholds have received considerable attention in both complete and, more recently, incomplete market settings; those of technical uncertainty much less so, even in the existing literature which, whilst recognising this risk is idiosyncratic, assumes that it is unpriced. Nevertheless, we have shown in this paper that the effects of technical uncertainty under market incompleteness are quite distinct from those of price uncertainty. It is thus important to incorporate both technical uncertainty and its inherent market incompleteness when analysing investments in R&D, particularly for founders and potential investors in small private research-based start-up firms.

6 Appendix

Proof of Proposition 1

The entrepreneur solves

$$J(W,K) = \sup_{I_s,\theta_s,t\le s\le \tilde{T}} E_t\left[-\frac{1}{\gamma}e^{\frac{\lambda^2}{2}(\tilde{T}-t)}e^{-\gamma(W_{\tilde{T}}+V)}\right] \tag{24}$$

subject to (1) and (3) where $\tilde{T}$ is defined by $K_{\tilde{T}}=0$.

Itô's lemma gives the HJB equation for J

$$\frac{\lambda^2 J}{2} + \sup_{\theta}\left(\theta\lambda\sigma J_W + \frac{\sigma^2\theta^2}{2}J_{WW}\right) + \sup_I\left(I\left(-J_W - J_K + \frac{\phi^2 K}{2}J_{KK}\right)\right) = 0 \tag{25}$$

Optimising with respect to θ and I gives the optimal amount held in the traded risky asset and optimal investment in the R&D project respectively: $\theta^* = \frac{-\lambda J_W}{\sigma J_{WW}}$ and

$$I^* = \begin{cases} i_{max} & \text{if } \frac{1}{2}\phi^2 K J_{KK} - J_K \ge J_W \\ 0 & \text{if } \frac{1}{2}\phi^2 K J_{KK} - J_K < J_W \end{cases} \tag{26}$$

We look for a solution of the form $J(W,K) = -\frac{1}{\gamma}e^{-\gamma(W+F(K))}$. Substituting into (25) gives an equation for F:

$$\sup_I\left(I\left(-1 - F_K + \frac{\phi^2}{2}K(F_{KK} - \gamma F_K^2)\right)\right) = 0 \tag{27}$$

s.t. $F(0)=V$. So $\theta_0^* = \frac{\lambda}{\gamma\sigma}$ and F is indeed independent of outside wealth, W. Optimising with respect to I gives rise to two regions separated by a free boundary

K_*: a continuation region in which investment proceeds at the maximum rate ($I = i_{\max}$ if $K \leq K_*$) and in which F satisfies the second order o.d.e. (9) with one free boundary, and a stopping region in which investment ceases ($I = 0$ if $K > K_*$) and the project value is identically zero. Three boundary conditions are thus required for (9): value matching and smooth pasting conditions at the free boundary K_* given by (11) and (12), since $F = F_K = 0$ in the stopping region, and a boundary condition at zero (10).

Proof of Proposition 2

Substituting $F = Vf(\xi)$ with $\xi = \frac{K}{V}$ gives

$$\frac{1}{2}\phi^2\xi(f_{\xi\xi} - \alpha f_\xi^2) - f_\xi = 1 \qquad\qquad 0 \leq \xi \leq \xi_* \tag{28}$$

s.t. $f(0) = 1, f(\xi_*) = 0$ and $f_\xi(\xi_*) = 0$.

Substituting $g(\xi) = f_\xi(\xi)$ shows g satisfies the Ricatti equation

$$\frac{1}{2}\phi^2\xi(g_\xi - \alpha g^2) - g - 1 = 0 \tag{29}$$

which has general solution

$$g(\xi) = -\frac{1}{\alpha}\frac{A\sum_{n=1}^{\infty} n a_n \xi^{n-1} + B\sum_{n=0}^{\infty}(n+1+\frac{2}{\phi^2})b_n\xi^{n+\frac{2}{\phi^2}}}{A\sum_{n=0}^{\infty} a_n\xi^n + B\sum_{n=0}^{\infty} b_n\xi^{n+1+\frac{2}{\phi^2}}} \tag{30}$$

where A, B are arbitrary constants, $a_0 = b_0 = 1$ and $a_n = \Pi_{m=1}^{n}\left(\frac{(-2\alpha/\phi^2)}{m(m-1-\frac{2}{\phi^2})}\right)$, $b_n = \Pi_{m=1}^{n}\left(\frac{(-2\alpha/\phi^2)}{m(m+1+\frac{2}{\phi^2})}\right)$. The infinite series $\sum_{n=0}^{\infty} a_n\xi^n$ and $\sum_{n=0}^{\infty} b_n\xi^n$ are convergent for all ξ by the ratio test.

Integrating this gives

$$f(\xi) \quad = \quad -\frac{1}{\alpha}\ln\left[A\sum_{n=0}^{\infty} a_n\xi^n + B\xi^{1+\frac{2}{\phi^2}}\sum_{n=0}^{\infty} b_n\xi^n\right] + C \tag{31}$$

Applying $f(\xi_*) = 0$ and $f_\xi(\xi_*) = 0$ gives $B = -A\xi_*^{-\left(1+\frac{2}{\phi^2}\right)}\beta(\xi_*)$ and $C - \frac{1}{\alpha}\ln A = \frac{1}{\alpha}\ln\left[\sum_{n=0}^{\infty} a_n\xi_*^n + \beta(\xi_*)\sum_{n=0}^{\infty} b_n\xi_*^n\right]$ where $\beta(\xi_*) = \frac{\sum_{n=1}^{\infty} n a_n\xi_*^n}{\sum_{n=0}^{\infty}(n+1+\frac{2}{\phi^2})b_n\xi_*^n}$. Substituting into (31) shows terms in A cancel, leaving (14). Finally $f(0) = 1$ gives the nonlinear equation which needs to be solved for ξ_*:

$$1 \quad = \quad \frac{1}{\alpha} \ln \left[\sum_{n=0}^{\infty} a_n \xi_*^n - \beta(\xi_*) \sum_{n=0}^{\infty} b_n \xi_*^n \right] \tag{32}$$

Proof of Proposition 3

Taking the limit in equation (9) as $\gamma \to 0$ we obtain equation (18); the boundary conditions are unchanged. That the solution is (20) - (21) can be verified by direct substitution.

Approximation for small α

We rewrite equation (17) which determines ξ_* as

$$\begin{aligned} & e^{\alpha} \left(\sum_{n=0}^{\infty} (n+1+\frac{2}{\phi^2}) b_n \xi_*^n \right) \\ &= \left(\sum_{n=0}^{\infty} a_n \xi_*^n \right) \left(\sum_{n=0}^{\infty} (n+1+\frac{2}{\phi^2}) b_n \xi_*^n \right) - \left(\sum_{n=0}^{\infty} b_n \xi_*^n \right) \left(\sum_{n=0}^{\infty} n a_n \xi_*^n \right) \end{aligned} \tag{33}$$

and write $a_n = c_n \alpha^n$, $b_n = d_n \alpha^n$ where $c_n = \Pi_{m=1}^{n} \left(\frac{(-2/\phi^2)}{m(m-1-\frac{2}{\phi^2})} \right)$ and $d_n = \Pi_{m=1}^{n} \left(\frac{(-2/\phi^2)}{m(m+1+\frac{2}{\phi^2})} \right)$ are independent of α. We then substitute $\xi_* \approx \xi_{*0} + \alpha \xi_{*1} + O(\alpha^2)$, expand in powers of α and consider terms of successively higher order powers in α.

The $O(1)$ terms cancel out, so the leading order terms are $O(\alpha)$:

$$d_0 \left(1 + \frac{2}{\phi^2} \right) + d_1 \xi_{*0} \left(2 + \frac{2}{\phi^2} \right) \quad = \quad c_1 d_0 \xi_{*0} \left(1 + \frac{2}{\phi^2} \right) - c_1 d_0 \xi_{*0} + c_0 d_1 \xi_{*0} \left(2 + \frac{2}{\phi^2} \right) \tag{34}$$

which after rearranging shows the leading order threshold equals the in the unpriced risk case given in (21): $\xi_{*0} = 1 + \frac{\phi^2}{2} = \xi_{*(0)}$.

The $O(\alpha^2)$ equation is:

$$d_1 \xi_{*1} \left(2 + \frac{2}{\phi^2} \right) + d_2 \xi_{*0}^2 \left(3 + \frac{2}{\phi^2} \right) + d_1 \xi_{*0} \left(2 + \frac{2}{\phi^2} \right) + \frac{d_0}{2} \left(1 + \frac{2}{\phi^2} \right)$$

$$
\begin{aligned}
= \quad & c_0 d_1 \xi_{*1}\left(2+\frac{2}{\phi^2}\right) + c_0 d_2 \xi_{*0}^2\left(3+\frac{2}{\phi^2}\right) + c_1 d_1 \xi_{*0}^2\left(2+\frac{2}{\phi^2}\right) \\
& + c_1 d_0 \xi_{*1}\left(1+\frac{2}{\phi^2}\right) + c_2 d_0 \xi_{*0}^2\left(1+\frac{2}{\phi^2}\right) \\
& - \left(c_1 d_0 \xi_{*1} + 2 c_2 d_0 \xi_{*0}^2 + c_1 d_1 \xi_{*0}^2\right)
\end{aligned}
\tag{35}
$$

Rearranging gives the leading order effect of risk aversion, $\xi_{*1} = -\frac{\phi^2\left(1+\frac{\phi^2}{2}\right)}{4(1+\phi^2)}$, which is less than zero for all $\phi > 0$.

Numerical solution

We solve (28) and associated boundary conditions directly by finding the long horizon limit of

$$
f_t + \frac{1}{2}\phi^2\xi(f_{\xi\xi} - \alpha f_\xi^2) - f_\xi = 1 \qquad\qquad 0 \le \xi \le \xi_*(t) \tag{36}
$$

s.t. $f(0,T) = 1, f(\xi_*(t),t) = 0$ and $f_\xi(\xi_*(t),t) = 0$ using standard explicit finite difference methods. We take T sufficiently large that $f_t \to 0$ (typically we take $T \ge 50$ and check that $f(\xi,t)$ no longer changes with t).

Approximation for small ϕ^2

We write $\xi_* \approx \xi_{*0}\left(1+\phi^2\frac{\xi_{*1}}{\xi_{*0}} + O(\phi^4)\right)$ and substitute into (33), expanding in powers of ϕ^2. Note the coefficients in the various power series also need to be expanded in powers of ϕ^2, *i.e.* we write $a_n = a_{n,0} + \phi^2 a_{n,1} + O(\phi^4)$, $b_n = b_{n,0} + \phi^2 b_{n,1} + O(\phi^4)$ and $(n+1+\frac{2}{\phi^2})b_n = \phi^{-2}e_{n,-1} + e_{n,0} + \phi^2 e_{n,1} + O(\phi^4)$ with $a_{n,0} = \frac{\alpha^n}{n!}$, $a_{n,1} = \frac{\alpha^n}{n!}\frac{n(n-1)}{4}$, $b_{n,0} = \frac{(-\alpha)^n}{n!}$, $b_{n,1} = \frac{(-\alpha)^n}{n!}\frac{n(n+3)}{4}$, $e_{n,-1} = 2\frac{-\alpha^n}{n!}$ and $e_{n,0} = -\frac{(-\alpha)^n}{n!}\frac{(n-1)(n+2)}{2}$. Collecting terms of successively higher powers of ϕ^2, we find the $O(\phi^{-2})$ terms give

$$
2e^\alpha\left(\sum_{n=0}^\infty b_{n,0}\xi_{*0}^n\right) = 2\left(\sum_{n=0}^\infty a_{n,0}\xi_{*0}^n\right)\left(\sum_{n=0}^\infty b_{n,0}\xi_{*0}^n\right) \tag{37}
$$

or $\sum_{n=0}^\infty a_{n,0}\xi_{*0}^n = e^\alpha$, which gives $\xi_{*0} = 1$: in the absence of uncertainty, the threshold equals the conventional NPV threshold.

After some cancellations, the $O(1)$ terms give

$$\left(\sum_{n=0}^{\infty} b_{n,0}\xi_{*0}^{n}\right)\left(2\xi_{*1}\sum_{n=0}^{\infty}(na_{n,0}) + 2\sum_{n=0}^{\infty} a_{n,1} - \sum_{n=0}^{\infty}(na_{n,0})\right) = 0 \tag{38}$$

which gives $\xi_{*1} = \frac{1}{2} - \frac{\frac{\alpha^2}{4}e^{\alpha}}{\alpha e^{\alpha}} = \frac{1}{2} - \frac{\alpha}{4}$.

Equations incorporating partially unhedgeable price risk in V

If the value of completed R&D, V, is non-traded and evolves as[24]:

$$dV = (\mu - \delta)V dt + \eta V dZ_V \tag{39}$$

with $dZ_V dZ_M = \rho dt$, then the entrepreneur's objective function becomes:

$$J(W, K, V) \quad = \quad \sup_{I_s, \theta_s, t \leq s \leq \tilde{T}} E_t \left[-\frac{1}{\gamma} e^{\frac{\lambda^2}{2}(\tilde{T}-t)} e^{-\gamma(W_{\tilde{T}}^{\theta} + V_{\tilde{T}})} \right] \tag{40}$$

subject to (39), (1) and (3) where $\tilde{T}$ is defined by $K_{\tilde{T}} = 0$.

The HJB equation for J is now

$$\frac{\lambda^2}{2}J + \frac{\eta^2 V^2}{2}J_{VV} + (\xi\eta - \delta)VJ_V + \sup_I \left(I \left(-J_W - J_K + \frac{\phi^2 K}{2}J_{KK} \right) \right) \tag{41}$$
$$+ \sup_\theta \left(\theta\lambda\sigma J_W + \theta\rho\sigma\eta V J_{VW} + \frac{\sigma^2\theta^2}{2}J_{WW} \right) = 0$$

The optimal amount held in the traded risky asset now includes an additional hedging term due to the partial correlation between V and W: $\theta^* = \frac{-(\lambda J_W + \rho\eta V J_{VW})}{\sigma J_{WW}}$, but the form of the condition for investment in the R&D project is unchanged:

$$I^* \quad = \quad \begin{cases} i_{max} & \text{if} \quad \frac{1}{2}\phi^2 K J_{KK} - J_K \geq J_W \\ 0 & \text{if} \quad \frac{1}{2}\phi^2 K J_{KK} - J_K < J_W \end{cases} \tag{42}$$

We look for a solution of the form $J(W, K) = -\frac{1}{\gamma}e^{-\gamma(W+F(K,V))}$. Substituting into (41) gives an equation for F:

$$\frac{\eta^2 V^2}{2}(F_{VV} - \gamma(1-\rho^2)F_V^2) - \delta V F_V$$

[24]Recall we work with discounted values, equivalently using risk-free bond as numeraire, so $\mu = \xi\eta$ with ξ the Sharpe ratio for V.

$$+ \sup_I \left(I \left(-1 - F_K + \frac{\phi^2}{2} K (F_{KK} - \gamma F_K^2) \right) \right) = 0 \tag{43}$$

where we have used $\xi\eta = \mu_V - r = \frac{\rho\eta}{\sigma}(\mu_M - r) = \rho\eta\lambda$. There are still two regions separated by a free boundary which now varies with V, $K_*(V)$. In the continuation region investment proceeds at the maximum rate ($I = i_{\max}$ if $K \leq K_*(V)$); in the stopping region investment ceases ($I = 0$ if $K > K_*(V)$) but the project still has some value because of the option to restart investment. Let $F^{(inv)}$ denote the value of the project whist investment is ongoing, and $F^{(susp)}$ the value when investment is suspended. We rewrite (43) as

$$\begin{aligned} \frac{\eta^2 V^2}{2} (F_{VV}^{(inv)} - \gamma(1-\rho^2) F_V^{(inv)2}) - \delta V F_V^{(inv)} & \\ + i_{max} \left(-1 - F_K^{(inv)} + \frac{\phi^2}{2} K (F_{KK}^{(inv)} - \gamma F_K^{(inv)2}) \right) &= 0 \qquad (44) \\ \frac{\eta^2 V^2}{2} (F_{VV}^{(susp)} - \gamma(1-\rho^2) F_V^{(susp)2}) - \delta V F_V^{(susp)} &= 0 \qquad (45) \end{aligned}$$

subject to the following seven boundary conditions:

$$\begin{aligned} F^{(inv)}(0, V) &= G(V) && (46) \\ F^{(inv)}(K_*(V), V) &= F^{(susp)}(K_*(V), V) && (47) \\ F_K^{(inv)}(K_*(V), V) &= F_K^{(susp)}(K_*(V), V) = 0 && (48) \\ F_V^{(inv)}(K_*(V), V) &= F_V^{(susp)}(K_*(V), V) && (49) \\ F_{KK}^{(inv)}(K_*(V), V) &= \frac{2}{\phi^2 K} && (50) \\ F^{(susp)}(K, 0) &= 0 && (51) \\ lim_{V \to \infty} \left(F_{VV}^{(inv)} - \gamma(1-\rho^2) F_V^2 \right) &= 0 && (52) \end{aligned}$$

The first condition fixes the value on completion of the R&D. If the incompleteness is resolved on completion of the R&D project, *e.g.* because the right to produce the new product is sold at the market price, then $G(V) = V$. If the entrepreneur retains the project and receives a stream of partially unhedgeable cashflows, $G(V) < V$ represents the certainty equivalent of this cashflow stream. Conditions (47) - (49) represent the value matching and smooth pasting conditions with respect to each variable at the investment/suspension threshold, $K_*(V)$, whilst condition (50) follows from the optimality condition and (48). Equations (51) and (52) represent the behaviour of the project value at extreme values of the market value of completed R&D. If the market value of the completed project tends to infinity (for finite K), investment will be ongoing in the project, whereas if the market value of the completed project reaches zero, the project will be suspended, and the value of the option to recommence R&D will also be zero.

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Captions for figures

1. Value of R&D project as a proportion of value on completion, $f = \frac{F}{V}$ *vs* expected time to completion as a proportion of value on completion $\xi = \frac{K}{V}$ for unpriced risk and utility-based models with different levels of effective risk aversion $\alpha = \gamma V$. Parameter values where not stated: $\phi = 0.6$.

2. R&D investment/abandonment threshold as a proportion of value on completion, $\xi_* = \frac{K_*}{V}$ *vs* effective risk aversion $\alpha = \gamma V$ for different levels of technical uncertainty, ϕ.

3. Value of R&D project as a proportion of value on completion, $f = \frac{F}{V}$ *vs* expected time to completion as a proportion of value on completion $\xi = \frac{K}{V}$ for different levels of technical uncertainty, ϕ and effective risk aversion $\alpha = \gamma V$.

Uncertainty type:		Technical	Price	
Payoff type:			Lump-sum	Flow
$NPV^*\|_{\gamma=0}$		-ve	+ve	+ve
Sensitivity of NPV^* w.r.t. γ		+ve	-ve	+ve
Sensitivity of NPV^* w.r.t.	γ low	-ve	+ve	+ve
unhedgeable uncertainty	γ high	+ve	-ve	+ve

Table 1: Characterisation of effects of uncertainty type and payoff type on NPV at optimal threshold and its sensitivities.

*

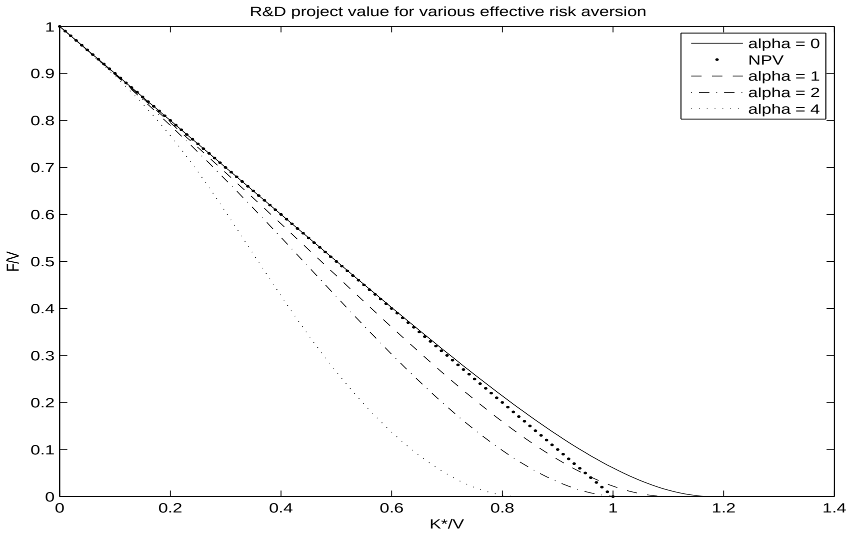
R&D project value for various effective risk aversion
alpha = 0
NPV
alpha = 1
alpha = 2
alpha = 4
F/V
K*/V
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.2
1.4

Figure 1:

*

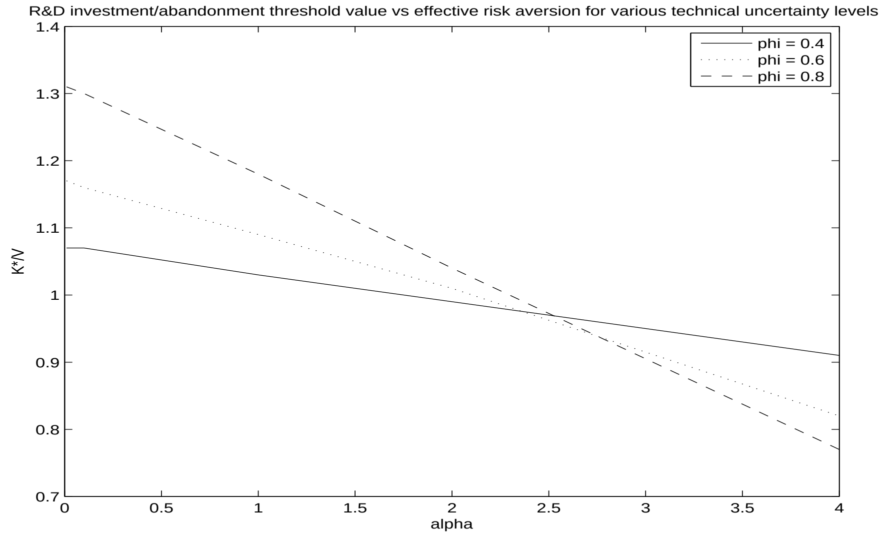
R&D investment/abandonment threshold value vs effective risk aversion for various technical uncertainty levels
phi = 0.4
phi = 0.6
phi = 0.8
K*/V
alpha
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0
0.5
1
1.5
2
2.5
3
3.5
4

Figure 2:

*

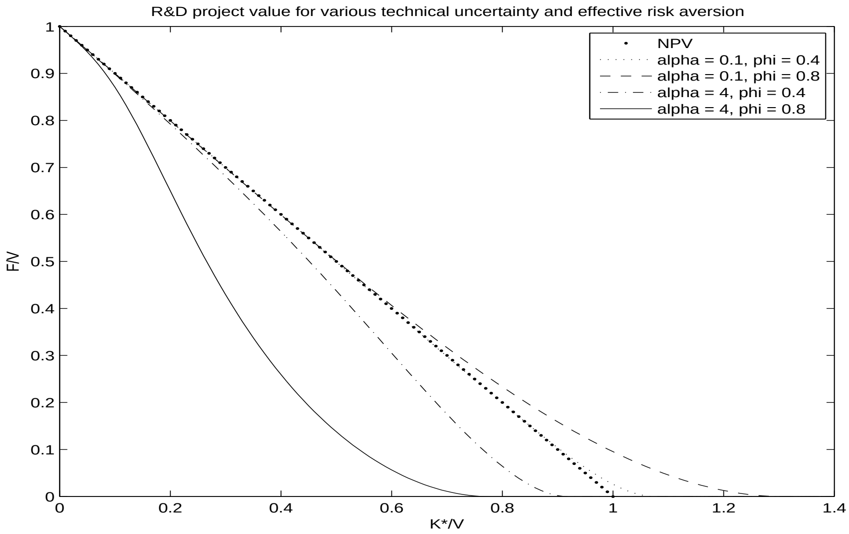
R&D project value for various technical uncertainty and effective risk aversion
NPV
alpha = 0.1, phi = 0.4
alpha = 0.1, phi = 0.8
alpha = 4, phi = 0.4
alpha = 4, phi = 0.8
F/V
K*/V
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.2
1.4

Figure 3:

*