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Abstract

The construction of hedging strategies that best replicate the outcomes from options (and other contingent claims) in the presence of transactions costs is an important problem. Leland [1985] presents and describes properties of a method for hedging call options when, in addition to the usual Black and Scholes assumptions, there is a proportional transactions cost. However, this method is in no sense an optimal one. Other important papers, by Davis [1988], Davis and Norman [1988], and Taksar, Klass, and Assaf [1988], describe optimal portfolio policies to maximize expected utility over an infinite horizon. While these papers are concerned with optimal policies, they are not concerned with the "financial engineering" problem of replicating a given contingent claim.

This paper describes the general problem of best replication of a contingent claim under proportional transactions costs, and other cost structures. The context in which the contingent claim is to be hedged is as follows. It is assumed that a financial intermediary or individual already has selected an optimal portfolio of assets and liabilities. An opportunity occurs to issue a contingent claim (presumably at a favorable price) and to hedge the risk involved by means of transactions in either a futures contract or the underlying asset itself, and risk-free bonds. Exact replication at finite cost is generally either impossible or too expensive to be desirable. The replication problem must therefore be defined relative to a loss function. The precise formulation of the problem is conditioned by the optimality of the original portfolio.

The problem is one of stochastic optimal control and can be characterized by a dynamic programming (Bellman-Hamilton-Jacobi) equation. In general, it is necessary to solve this equation numerically. Numerical results are provided for a realistic situation of replicating a conventional call option. The paper also derives some further insights into optimal replicating strategies by considering an alternative and simpler contigent claim. The optimal strategies are shown to be considerably better than the alternatives given by the strategies described by Leland. Finally, some general properties are described, and some further extensions are suggested.

1. Introduction

The construction of hedging strategies that best replicate the outcomes from options (and other contingent claims) in the presence of transactions costs is an important problem. Hedging is central to the theory of option pricing. Arbitrage valuation models, such as that of Black and Scholes [1973], depend on the idea that an option can be perfectly hedged using the underlying asset, so making it possible to create a portfolio that replicates the option exactly. Hedging is also widely used to reduce risk, and the kinds of delta hedging strategies implicit in Black and Scholes are commonly applied, at least approximately, by participants in options markets. Optimal hedging strategies are therefore of direct practical interest. Much of the theory of options assumes that markets are frictionless. This paper considers the impact of transaction costs on pricing and hedging.

A number of recent papers consider various aspects of the transactions cost problem. Leland [1985] presents and describes properties of a method for hedging call options when, in addition to the usual Black and Scholes assumptions, there is a proportional transactions cost. Neuhaus [1989] contributes some further theoretical insights to this approach. However, this method is in no sense an optimal one. Figlewski [1987] gives some interesting simulation results. Other important papers, by Davis [1988], Davis and Norman [1988], and Taksar, Klass, and Assaf [1988], describe optimal portfolio policies to maximize expected utility over an infinite horizon. They extend earlier work by Merton [1971] and Constantinides [1986]. However, while these papers are concerned with optimal policies, they are not directly concerned with the problems of replicating (or, similarly, hedging) contingent claims by means of the underlying asset.

This paper describes the general problem of best replication of a contingent claim under transactions costs. It describes a normative model from the perspective of a single agent, and not a model of market equilibrium. Exact replication at finite cost is generally either impossible or too expensive to be desirable. The replication problem must therefore be formulated relative to some loss function (or utility function for marginal wealth changes). We show that this problem is one of stochastic optimal control and can be characterized by a dynamic programming (Bellman-Hamilton-Jacobi) equation. In general, it would be necessary to use numerical methods of solution, and the problem involves three state variables. By using a suitable utility function (exponential), we reduce the number of state variables to two and are able to obtain numerical solutions to realistic problems. By further specializing to a particularly simple specific contingent claim, we obtain a problem with just a single state variable and we are also able to derive some analytic results. The optimal strategies are shown to be considerably better than the alternative strategies described by Leland. The paper also discusses a number of general insights concerning the nature of solutions to optimal hedging problems under transactions costs.

2. The General Framework

We consider an asset whose price, S_t, at time t evolves under the diffusion process described by

$$dS = \mu(S)dt + \sigma(S)dz \tag{1}$$

The problem is to replicate the outcomes from a contingent claim whose payoffs at a single future date, T, are given by $C(S_T)$. The replication is to be accomplished by holding x_t units of the asset plus either borrowing or lending at a constant interest rate, r. The holdings in this replicating portfolio are to be actively managed through time, but transactions in the underlying asset involve a transaction cost amounting to $k(v,S)$, where v is the volume of shares transacted (either positive or negative) and S is the (mid) share price. For most purposes, we shall specialize this to the case of costs that are a constant proportion of the value transacted,

$$k(v, S) = k\,|v|\,S, \tag{2a}$$

or a constant amount times the number of shares traded,

$$k(v, S) = k\,|v| \tag{2b}$$

We can also consider costs with a fixed and variable component, such as

$$\begin{aligned} k(v, S) &= k_1 + k_2\,|v|\,S \text{ for } v \neq 0 \\ &= 0 \text{ for } v = 0 \end{aligned} \tag{2c}$$

In general, it is either impossible or at least undesirable to replicate the contingent claim exactly. For many problems, exact replication at finite cost is impossible. For others, while exact replication at finite cost may be possible, it will be too expensive to represent an attractive policy. The replication problem is therefore ill-defined until we have specified a criterion for choosing between alternative replicating strategies. We will assume initially a fairly general state-dependent utility criterion that later will be specialized to a particular function. Thus, we assume that an initial amount of money is invested through time (managed between the asset and the risk-free rate). By the terminal date, T, after liquidating the asset holding, an amount of cash, y_T, is available to set against the contingent liability, $C(S_T)$. At this date, we have an accumulated surplus of

$$w_T = y_T - C(S_T) \tag{3}$$

net of the option value to be replicated. We define a utility function $U(w_T, S_T)$ and seek to characterize and calculate replication strategies that maximize the expected value of this utility function. We shall focus later on a utility function $U(w_T)$ that is only state dependent in the sense that the definition of w_T itself involves $C(S_T)$. We may also allow the horizon date to be later than the expiry date of the contingent claim. Also, we wish to allow w_T to be negative in some states, which precludes the use of some commonly employed utility functions, such as power or logarithmic utility functions for $U(w)$. It is worth noting that our utility function should be interpreted as the utility of wealth at the margin, after some other choices have been made, rather than as the utility of total wealth. We do this to avoid having to solve an impossibly large portfolio problem. We shall assume that $U(w,S)$ is defined for all real numbers w,S, and that its first two derivatives exist, are continuous, and satisfy the usual properties for a risk-averse utility function, i.e., that $U_w > 0$ and $U_{ww} < 0$.

We now describe the structure of the general problem. Using the notation already introduced, we define the value function

$$J(t, S, x, y) = \text{Maximum } E[U(w_T, S_T)] \tag{4}$$

as the maximum expected utility possible starting at time t when the asset price is S, with initial holding of x shares, and an amount y in cash. $E[\,.\,]$ is the expectation operator. The maximum is taken over all feasible transactions policies. At the last date T, it is clear that, by definition, $J(\,.\,)$ is obtained trivially as

$$J(T, S, x, y) = U(w_T, S_T) \tag{5}$$

where

$$w_T = x\, S_T + y_T - C(S_T) \tag{6a}$$

corresponding to no costs at termination, or

$$w_T = x\, S_T - k(x, S_T) + y_T - C(S_T) \tag{6b}$$

corresponding to cash settlement after transactions costs have been paid.

The value function is solved recursively backward through time using the dynamic programming approach of stochastic optimization. $J(\,.\,)$ evolves backward as given by

$$J(t,S,x,y) = \text{Maximum } E_{dS}[J(t+dt,\, S+dS,\, x^*,\, y(x^*))] \tag{7}$$

where the maximum is taken over the choice of the quantity of shares x^* to hold.

This optimal control problem is characterized by the second-order partial differential equation

$$J_t + \mu(s)\, J_S + \tfrac{1}{2}\, \sigma^2(S)\, J_{SS} + y\ r\ J_y = 0 \tag{8a}$$

for interior values of $x \epsilon X$, subject to the boundary conditions

$$J(T,S,x,y) = U(w_T, S_T) \tag{8b}$$

defining J at the terminal date T, and

$$J(t,S,x+u,\, y-uS-k(u,S)) \leq J(t,S,x,y) \tag{8c}$$

which defines the boundary of the region X on which transactions occur. For the special case of constant proportional transactions costs, equation (8c) simplifies to an inequality relationship between J_x and J_y.

The solution to this problem provides the optimal strategy for hedging the contingent claim, and also a valuation function for it. We believe that for any "well-behaved" cost function $k(v,S)$, the solution provides functions

$$x_-(t,S,y),\, x_+(t,S,y),\, x^*_-(t,S,y),\, x^*_+(t,S,y)$$

Once x is established within the interval $[x_-, x_+]$, the discovery at a later time that $x < x$ leads to transactions to reestablish x at the value x^*_-. Similarly, if $x > x_+$, it is reestablished to x_+. If transactions costs are simply proportional to volume, then $x^*_- = x_-$ and $x^*_+ = x_+$. If transactions costs have only a fixed component, then $x^*_- = x^*_+$.

We may define valuations of the contingent claim after adding some further simple notation. We define $J^C(t,S,x,y)$ as the expected utility (under an optimal hedging strategy) of assuming the state-contingent liability C, (e.g., $C(S_T)$ as before). The individual's selling valuation of C, $V_S(C)$ is defined as the price required to provide the same expected utility as not selling the contingent claim. Thus, V_S is defined by the equation

$$J^C(0,S,0,V_S) = J^o(0,S,0,0) \tag{9}$$

where J^o is defined as J^C, but with no state-contingent liability assumed.

Similarly, we can define the buying price, V_B, as the maximum price worth paying to buy the contingent claim, defined by the equation

$$J^{-C}(0,S,0,-V_B) = J^o(0,S,0,0). \tag{10}$$

In addition to calculating optimal hedging strategies and valuations, we can also calculate recursively as many moments of the distribution of w_T as may be of interest. The moments of w_T about zero simply accumulate as expectations conditional on the state variables involved.

Note that under this general formulation, at each date in the calculation, the value function (and also any derived moment functions) depend on the three state variables of S, x, and y. The computational effort required may be considerable unless simplifications are found. A general numerical solution would be daunting. We shall argue in the remainder of the paper that there are good reasons, both theoretical and pragmatic, to specialize the utility function to the negative exponential

$$U(w_T) = -\exp(-\lambda w_T). \tag{11}$$

This reduces the state variables by one and makes the computations relatively straightforward. It also enables us to produce strategies that have the attractively simple properties of not being wealth dependent and not creating risky positions in the absence of any contingent claim to be hedged.

3. The Formulation with Exponential Utility

We first consider the justification for assuming a utility function of the form of equation (11). We will also justify transforming the diffusion process to its risk-neutral equivalent. The simplifications that result will then be demonstrated.

The Rationale

The context in which the contingent claim is to be hedged is as follows. It is assumed that the intermediary or individual already has selected an optimal portfolio of assets and liabilities. An opportunity occurs to issue (or buy) a contingent claim at a (possibly) favorable price and to hedge the risk involved in the fashion that has been described. To obtain a more tractable problem, we wish to ignore the interactions that would normally exist between the new opportunity and the rest of the portfolio, and between the success of the replicating strategy and the agent's risk aversion. Given this background, it seems appropriate to impose restrictions that will give our optimal hedging strategies the following properties:

Property 1

The number of shares to be held in the underlying asset is to be independent of the amount of cash carried forward. We have in mind that a single hedging transaction will represent a small part of the total operations of our intermediary. The size of the cash balance partly reflects whether the hedging strategy has fared well or badly to date. We would not want the hedging strategy to depend on the cash gained or lost in other operations; it would then be inconsistent to let it be affected by the cash generated in this operation.

Property 2

If no claim is issued (or purchased), we would like the optimal incremental transactions to be nil, irrespective of how small transactions costs are. We have assumed that prior to undertaking the transaction to be hedged, the individual or institution has already optimized its portfolio. This may include plans to manage a holding of our underlying asset within a finite range. We are interested in what hedging actions should result from the sale (or purchase) of the contingent claim, and we want to make neutral assumptions regarding possible interactions with existing portfolio holdings or operations.

Property 1 immediately rationalizes our choice of negative exponential utility as defined in equation (8), since this is the only utility function that has constant absolute risk aversion. Property 2 enables us to replace the general stochastic process of equation (1) with its risk-neutral equivalent

$$dS = rSdt + \sigma(S)dz. \tag{12}$$

It is convenient (as ever) that we do not need to have information about the drift of the underlying asset. The proof of this is as follows. It is well known (for example, see Breeden and Litzenberger [1978] or Dybvig [1988]) that in complete markets, the state prices are proportional to the risk-neutral (or martingale) probabilities, π_S^*. Supposing that the original portfolio was such that there would have been no incentive to trade in a complete market, the solution to

$$\text{Maximize } \Sigma\, \pi_S\, U(w_S, S)$$

$$\text{subject to } \Sigma\, \pi_S^*\, w_S = 0$$

must be $w_S = 0$ for all states, s.

This gives the conditions,

$$\pi_S\, U'(0,S) = \gamma \pi_S^*. \tag{13}$$

We need marginal utility to be proportional to π_S^*/π_S. The simplest way to accomplish this is to retain a non-state-dependent utility function, but then adjust the probabilities to π_S^* instead of π_S. Note how the transformation to the martingale probabilities ensures very nicely that no holdings in the risky asset (either positive or negative) are generated at any date in the absence of taking a position in the contingent claim.

The Resulting Simplifications

We now examine the effects of our simplifications on the partial differential equation, its boundary conditions, and on how we obtain valuations. Since, from equations (4) and (11),

$$J(t,S,x,y) = \text{Maximum}\, E[-\exp\{-\lambda w_T\}], \tag{14}$$

and the management of x through time is independent of y,

$$J(t,S,x,y) = J(t,S,x,0)\exp\{-\lambda y e^{r(T-t)}\}. \tag{15}$$

If we define

$$H(t,S,x) = J(t,S,x,0), \tag{16}$$

then we may derive the following new equations and boundary conditions for H, which correspond to our previous equation set (8):

$$H_t + rSH_S + \frac{1}{2}\sigma^2(S)H_{SS} = 0, \quad (17a)$$

$$H(T,S,x) = -\exp\{-\lambda w_T\}, \quad (17b)$$

$$H(t,S,x+u) \geq H(t,S,x)\exp\{-\lambda(uS - k(u,S))\,e^{r(T-t)}\}. \quad (17c)$$

For the special case of a constant proportional transaction cost, $k(v,S) = k|v|S$, this last equation translates to

$$H_x = -\lambda S(1+k)\,e^{r(T-t)}H \quad (17d)$$

for $x \leq x_-$, and

$$H_x = -\lambda S(1-k)\,e^{r(T-t)}H \quad (17e)$$

for $x \geq x_+$.

Our valuation formulae for selling and buying values, V_S and V_B, simplify as follows. As before, V_S is defined by the equation

$$J^C(0,S,0,V_S) = J^o(0,S,0,0) \quad (9)$$

which now can be expressed as

$$H^C(0,S,0)\exp\{-\lambda V_S e^{rT}\} = H^o(0,S,0) = -1, \text{ so}$$

$$V_S = \frac{1}{\lambda}\,e^{-rT}\ln(-H^{-C}). \quad (18)$$

Similarly, for the buying price, V_B, we have

$$V_B = -\frac{1}{\lambda}\,e^{-rT}\ln(-H^{-C}). \quad (19)$$

4. A Numerical Example

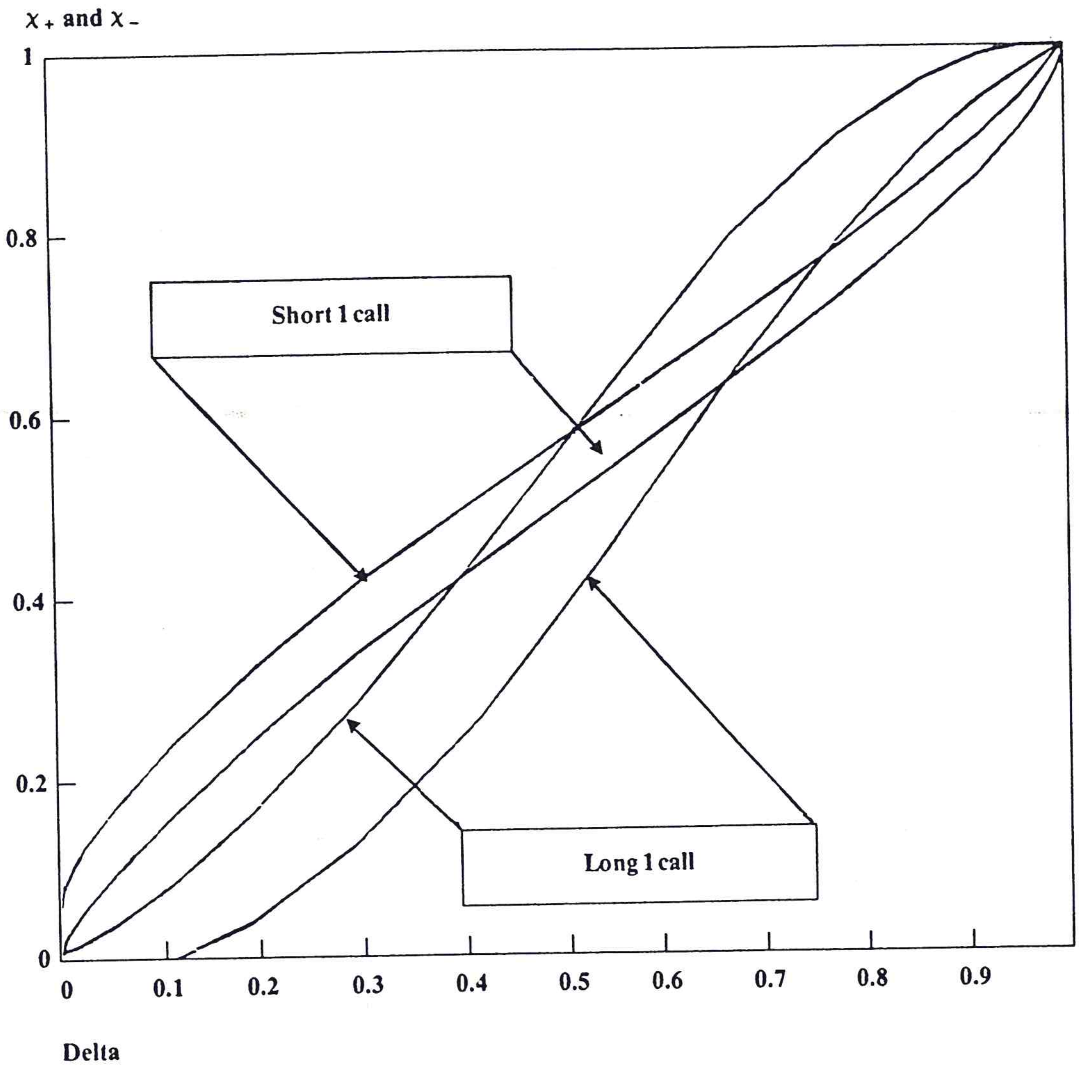
We now apply the approach we have described to a familiar Black-Scholes option example under proportional transactions costs. For our example, we have taken the case of a one-year call option with an exercise price of 100. The asset volatility is 30 percent and the riskless rate of interest is zero. Transactions costs are 2 percent of the value transacted and the risk aversion parameter, λ, is taken to be unity. This degree of risk aversion means that the hedger would be indifferent between two normally distributed outcomes where one has a mean that is x higher and a variance that is $2x$ higher than the other. The calculations were performed using a binomial lattice with a vector tabulating expected utility starting from alternative x values at each node.

In Graph A, the maximum and minimum hedge ratios (deltas), x_+ and x_-, are given both for the case where the hedger is short an option and where he is long an option. Remember that for this problem, $x^*_- = x_-$ etc., and the transactions are undertaken to just constrain x to remain between the limits of x_- and x_+.

These hedge ratios are plotted as functions of the Black-Scholes hedge ratios. Two points emerge clearly:

1. The optimal hedging strategy varies according to whether the investor is hedging a long or a short position.
2. It is quite possible for a portfolio that is perfectly hedged according to the perfect markets theory of Black-Scholes to require rebalancing when transactions costs are taken into account. In other words, $x_-(t,S)$ and $x_+(t,S)$ do not necessarily span the Black-Scholes hedge ratio of $N(d_1)$.

Graph A. Optimal hedging strategy with transaction costs (sigma = 30%, T = 1 year, transaction costs = 2%, lambda = 1, 40 time steps, 400 hedge ratios, interest rate = expected return = 0)

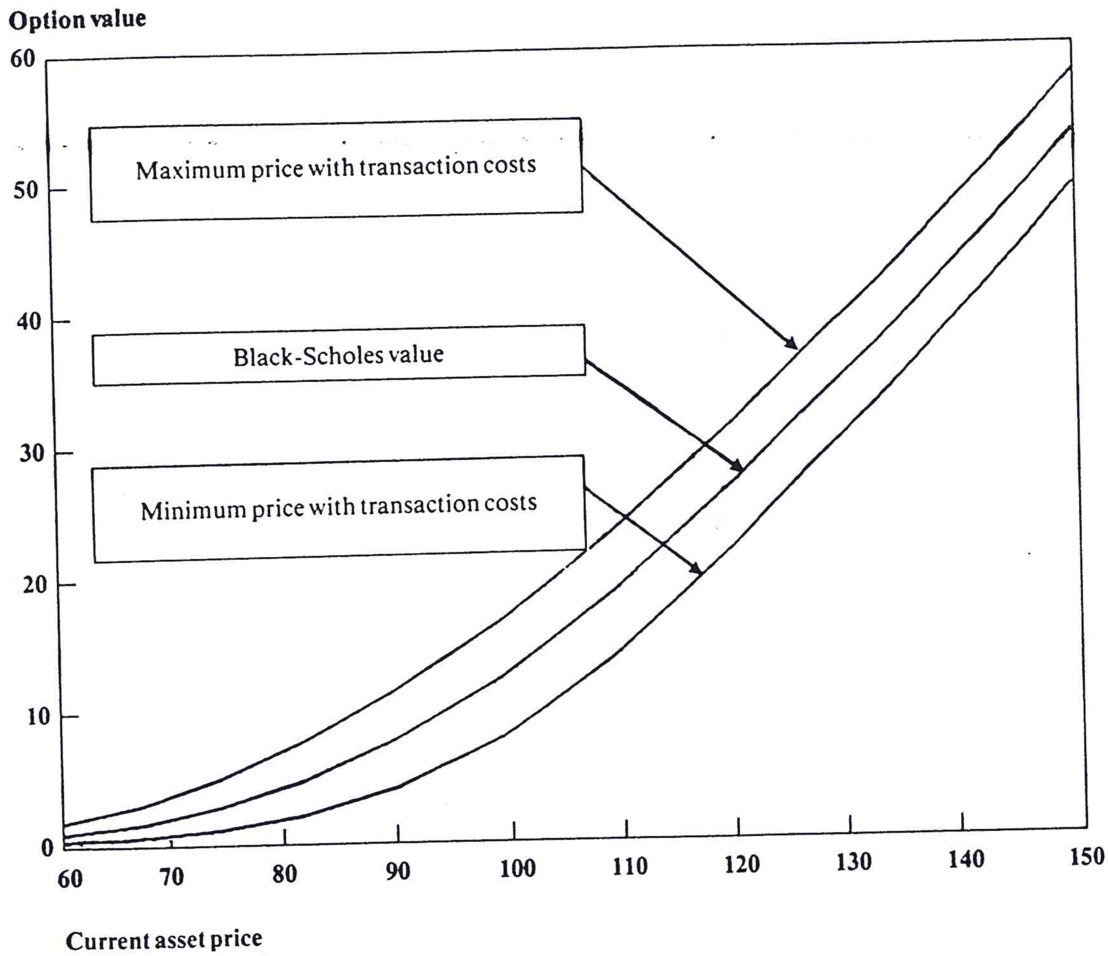


These points become less surprising when our results are compared against the analysis of Leland [1985]. Leland shows how an investor could write a European call option at a price C^+ and hedge it. By rebalancing the portfolio at regular intervals of length Δt in a prescribed way, he ensures that his terminal wealth, although uncertain, has a mean and standard deviation that both tend to zero as Δt goes to zero. The price C^+ differs from the Black-Scholes price C (and the Leland hedge ratio differs from the Black-Scholes hedge ratio) by using a modified estimate for the volatility. For a short position in the option, Leland's modified volatility is higher than the true volatility. This gives an increased call value and a hedge ratio that is a flatter function of S than normal. Conversely, if we are long the option, the volatility adjustment is downward, resulting in a lower option value and a hedge ratio that is a steeper-than-usual function of S.

The intuition for this variance adjustment is as follows. When covering a short call option position, a rise in the share price requires further shares to be purchased, and the transaction cost makes it as if an even larger rise had occurred. The same problem arises if the share price falls. It is as if the volatility had increased. Conversely, when covering a long call option position with a short asset one, a rise in the asset price implies more shares must be sold. The spread on the share price makes it as if the upward movement had been less, and a reduced volatility is called for.

V_S and V_B are plotted in Graph B for a variety of asset price levels. The figure here is very similar to what Leland would obtain, but our hedging error will be smaller than his for a given sum committed.

Graph B. Option value (1 year, 30% vol., $K = 100$, $R = 0$, $k = 2\%$, lambda = 1)



5. A Single State Variable Example

Some further insights can be obtained by considering the following even simpler hedging problem.

The Y&Z share index stands at 100. The index S follows the additive process

$$dS = \sigma\, dz \tag{20}$$

with the standard deviation of price changes, σ, being constant. Your employer at the Very Important Bank is keen to offer a new security that will pay the investor half the square of the future value of the index in 900 days time (i.e., £$1/2S^2$). It is your job to decide how much to charge for this new product, and how the funds received should be managed to meet the obligation. The interest rate is zero. You can invest in the index itself, but all intermediate portfolio revisions involve a transaction cost of k per unit transacted in the index (i.e., currently k percent, but more or less than that if the index falls or rises).

In terms of our general model, we are assuming that

1. the riskless interest rate $r = 0$
2. the asset follows an arithmetic Brownian process, so $\sigma(S) = \sigma$
3. transaction costs are a fixed amount per share, so $k(\nu,S) = k|\nu|$
4. the claim C is the entitlement to receive at time T, S_T units of the risky asset for ½ S_T per unit

No Transactions Costs Analysis

In the absence of transactions costs, the value of the claim $C(t,S)$ satisfies the partial differential equation

$$C_t = -\tfrac{1}{2}\sigma^2 C_{SS} \tag{21a}$$

subject to the terminal boundary condition

$$C(T,S) = \tfrac{1}{2}S^2. \tag{21b}$$

The solution to this problem is

$$C(t,S) = \tfrac{1}{2}S^2 + \sigma^2(T-t)/2. \tag{22}$$

The replicating portfolio involves holding S units of the index, with the remaining value $\sigma^2(T-t)/2 - \tfrac{1}{2}S^2$ in cash.

Transactions Costs Analysis

This problem has a particularly simple structure, which we will now exploit for the case where we take account of transactions costs. Since this contingent claim has C_{SS} constant, it requires only a little cunning to eliminate the index value as a state variable. The only state variable becomes the difference between the number of units held and the "ideal" number, S_t. Using the assumptions just described, and applying the methodology of Section 3, we can rewrite equation set (17) as:

$$H_t + \tfrac{1}{2}\sigma^2 H_{SS} = 0, (x_- \leq x \leq x_+) \tag{23a}$$

$$H(T,S,x) = -\exp\{-\lambda w_T\} \tag{23b}$$

$$w_T = Sx - \tfrac{1}{2}S^2 \tag{23c}$$

$$H_x = -\lambda(S+k)\,H, \ (x \leq x_-), \text{ and} \tag{23d}$$

$$H_x = -\lambda(S-k)\,H, \ (x \geq x_+), \tag{23e}$$

where $H(t,S,x)$ is the expected utility if the intermediary is holding a portfolio of x units of the risky asset and no cash.

We can simplify equation set (23) and remove one of the state variables in the following way. Recall that in the absence of transactions costs, the replicating portfolio would be long S units of the risky asset and $s^2(T-t)/2 - \tfrac{1}{2}S^2$ in cash. So we define the function:

$$G(\tau,S,z) = H(t,S,x)\exp\{-\lambda[\sigma^2(T-t)/2+S^2/2-Sx]\} \tag{24a}$$

where

$$\tau = \lambda\sigma^2(T-t), \text{ and} \tag{24b}$$

$$z = \sqrt{\lambda}(x-S). \tag{24c}$$

G is the expected utility starting with a replicating portfolio plus $z/\sqrt{\lambda}$ additional shares, offset by $zS/\sqrt{\lambda}$ of borrowing. Using (24), we can derive an appropriate partial differential equation and boundary conditions from (23). The boundary conditions become:

$$G(0,S,z) = -1 \tag{25a}$$

$$G_z = -k\sqrt{\lambda}G, \ (z \leq z_-), \tag{25b}$$

$$G_z = k\sqrt{\lambda}G, \ (z \geq z_+). \tag{25c}$$

Note that the boundary conditions do not depend on S, so the solution to the partial differential equation does not depend on S either, and $G_S = G_{SS} = G_{zS} = 0$. So (23a) becomes

$$G_t = \tfrac{1}{2}[z^2G + 2zG_z + G_{zz}], \ (z_- \leq z \leq z_+) \tag{25d}$$

and the specification of G is complete.

The solution to (25) is

$$G(\tau,z) = -\exp\{-[z^2/(1-\tau)-1]\tau/2\}\,/\sqrt{1-\tau}, \tag{26}$$

corresponding to the problem we have described, and

$$G(\tau,z) = -\exp\{[-z^2/(1+\tau)-1]\tau/2\}\,/\sqrt{1+\tau}, \tag{27}$$

corresponding to the complementary problem of hedging a long position in the contingent claim.

By considering the free boundary condition, we can obtain an inequality for the position of $z_+(\tau)$, as

$$z_+(\tau) \geq k\sqrt{\lambda}(1-\tau)/\tau, \tag{28}$$

and similarly for $z_-(\tau)$.

For small τ, this curve should be a good approximation of the barrier on which transactions occur, and the equation clearly establishes its hyperbolic behavior for small τ.

For large τ, we may hypothesize that the free boundary converges asymptotically to a constant. In this case, the intermediary will incur a constant cost per unit time in terms of risk and transactions costs in maintaining the hedge, and $G(\tau, z)$ will converge to some function $e^{bt}F(z)$, where b is a constant. The only solution to (25) of this form that is symmetrical about $z = 0$ is:

$$G(\tau, z) = A \exp\{b\tau - z^2/2\} \cosh\{\sqrt{1+2b}z\}, (z_- \leq z \leq z_+) \tag{29}$$

where A is a constant. Since G must satisfy (25c) and (25d), and also have continuous first and second derivatives at $z = z_-$ and $z = z_+$, we can solve simultaneously for b and the free boundary distances to get:

$$b = \tfrac{1}{2}[z_+ + k\sqrt{\lambda}]^2 \tag{30a}$$

$$\tanh[z_+\sqrt{1+2b}] = \sqrt{2b/(1+2b)} \tag{30b}$$

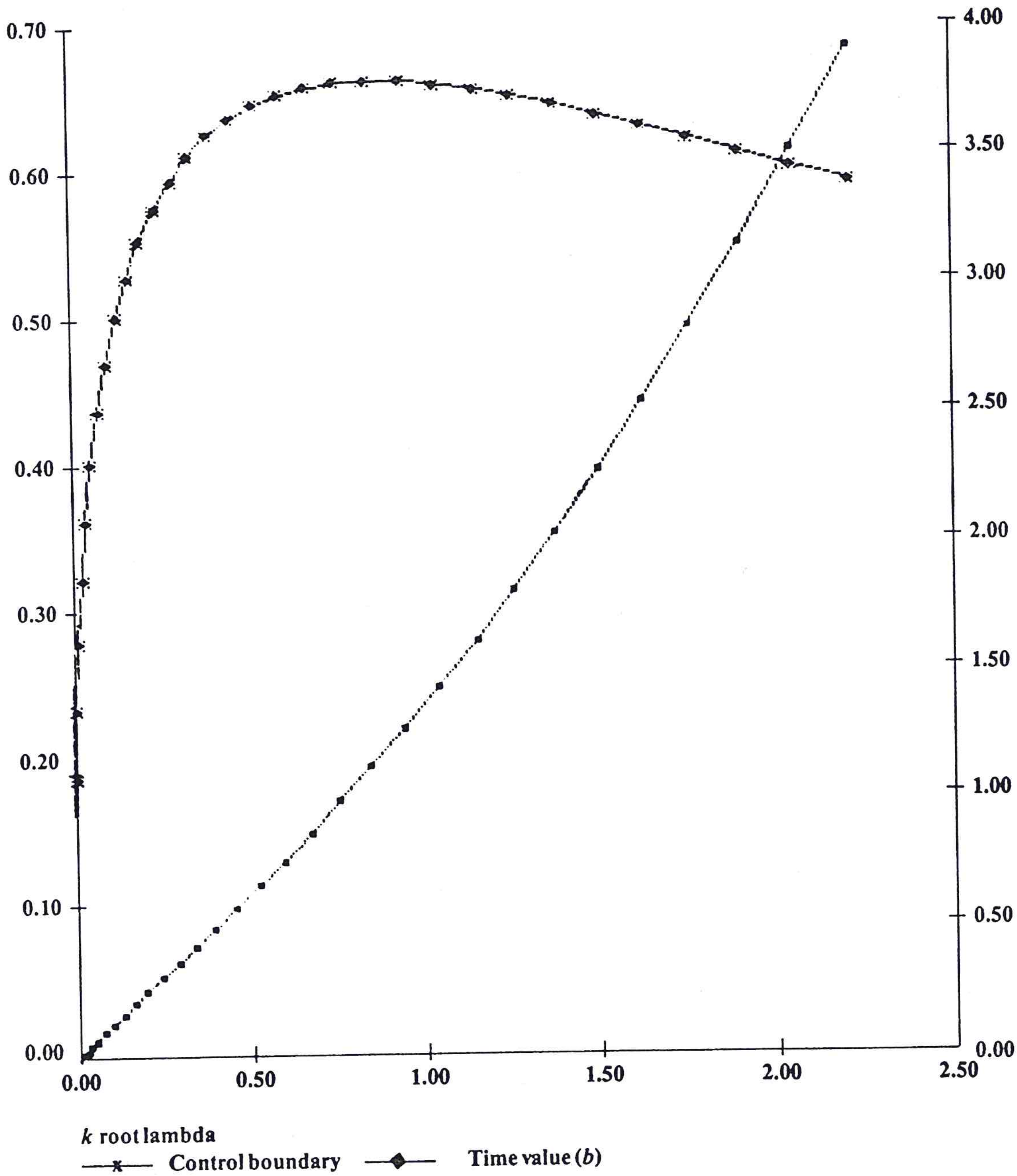
A graph of z_+ and b for different values of $k\sqrt{\lambda}$ is shown in Graph C. Rather surprisingly, z_+ is not monotonically increasing in k. This result is probably caused by the effect of the variation of transactions costs incurred. For high levels of costs or high degrees of risk aversion, it is worth having a tighter control region to reduce the variability of the costs involved.

Finally, we can compare the performance of the optimal hedging strategy against Leland's method. Graph D provides plots of the mean cost in excess of the "fair" value against the standard deviation for the two methods. A variety of rebalancing intervals, Δt, are employed for Leland's method, and a variety of risk aversion parameters, λ, for our own. Mean variance efficiency would not be an appropriate criterion for constructing hedge positions, and it is not used by either method. Nevertheless, the plot provides a useful way of describing the performance of the two methods. As expected, the optimal control approach does dominate Leland's, and particularly where high levels of turnover are involved.

6. Conclusions and Extensions

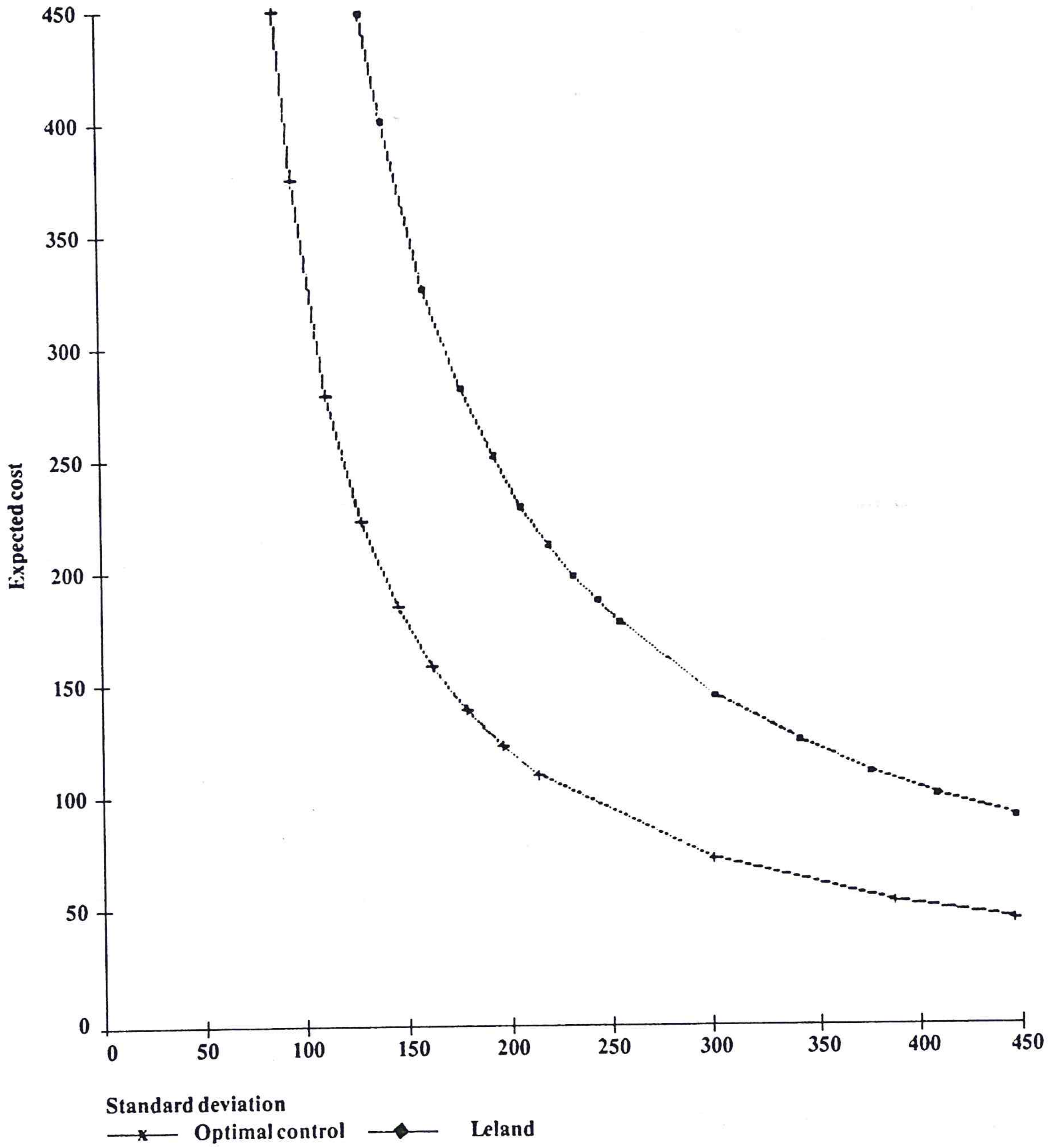
The paper has contributed some new theoretical perspectives on the problems of hedging contingent claims under transactions costs. It also describes practical procedures for computing solutions to realistic problems. The spirit that has motivated this work is a desire to relax the unrealistic assumptions that the pure Black-Scholes model requires. The imposition of transactions costs immediately constrains turnover to be bounded and implies that replication will not be exact. The problem therefore has much in common with the issues of valuation and hedging where the nature of the market dictates that trading is discontinuous, or that the asset process is such that the market is incomplete and contingent claims are not spanned by existing securities.

Graph C. Properties of simple example



We have shown that the problem of finding the optimal replication policy is one of solving the partial differential equation corresponding to (8), subject to a terminal condition and to a free boundary condition that determines the position of the control boundary. The combination of the assumption of a risk-averse utility function, $U(.)$, with the use of the equivalent risk-neutral asset process, (12), would seem to be sufficient to ensure that $J(.)$ is concave in x, and that solutions exist and are "well behaved." The optimal control x is constrained to evolve between control limits that depend on time, and on the asset price. Under the negative exponential utility assumption, the amount of cash accumulated in the replicating portfolio is irrelevant. No controlling action is taken until the control parameter x attains one of its limits.

Graph D. Comparison of two hedging methods



The optimal strategies involve a trade-off between the expected level of transactions costs and the exactness of the replication. Both the strategy and the reserve price for the contingent claim depend on the investor's preferences.

Some further, more direct, extensions of our work may also be of interest. Within the framework of our analysis and computations, it is entirely straightforward to optimize the hedging of a portfolio of contingent claims on the same asset, but with a single expiry date. It is also obviously cheaper to hedge in this way than it would be to hedge each component of the portfolio individually. Financial intermediaries clearly enjoy some benefits of scale in this kind of activity.

Our analysis has been confined to contingent claims that are European in nature. The liability crystallizes at a single known date in the future. However, the extension to American options is not too difficult. If we sell an American call option, C, then our partial differential equation, (8), is constrained by the additional free boundary condition that

$$J^{C}(t,S,x,y) \geq J^{o}(t,S,x,y-C(t,S)) \tag{31}$$

where $C(t,S)$ is the option's cash in value at date t as a function of the asset price S. Conversely, if we buy the call option C, then (8) is constrained by the condition

$$J^{-C}(t,S,x,y) \geq J^{o}(t,S,x,y+C(t,S)). \tag{32}$$

Finally, one could envisage combining these two extensions to hedge portfolios of long and short positions in American options with a range of different expiry dates and exercise prices.

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Commentary

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Introduction

The objective of Hodges and Neuberger (NH) is to find "... the best replication of a contingent claim under transactions costs." The emphasis in their paper is on *optimal* replication. The question is why should we be interested in the optimal replication of an option? One reason is to derive a pricing model for options by comparing an option to the price of its replicating portfolio. The second rationale is to control risk by forming a trading strategy in order to exploit mispricing of options relative to the underlying stock.

Replication in a World with No Transactions Costs

Black and Scholes (BS) addressed the issue of replicating an option in continuous time in their seminal paper [1973] under the condition of zero transactions costs. They use continuous-replication strategy to derive the equilibrium price of an option, *given* the price of the underlying stock. They also show how continuous hedging can be used, theoretically, to exploit mispricing of options and apply it in their empirical study [1972].

With zero transactions costs, optimal replication of a contingent claim leads simultaneously to both pricing model and trading stretegy. This characteristic is also maintained if a discrete-time framework is adopted. Rubinstein [1976] derives such a model, and by introducing compensating assumptions about the shape of the utility function of the average investor, he is able to derive a model that is identical in structure to BS.

The Problems of Introducing Transactions Costs to the Pricing of Options

In general, the introduction of transaction costs poses special problems for valuation of securities. These problems are even more significant for the valuation of contingent claims as derivative assets. The issue of the investment horizon and the time preferences of consumers becomes very important. For short investment horizons, the individual will select an asset with reduced transaction costs even if it yields lower rates of return. The rate of return on an asset will reflect its risk and also the transaction costs involved in trading it. These problems may lead to effective market segmentation, which is dependent on the preferred holding periods of different investors. By introducing transaction costs, the separation of investment decisions from the consumption decision may not be valid, and pricing of assets is no longer utility-independent.

The valuation problems are even more complex when transaction costs are incorporated in a continuous-time model. With proportional transaction costs and a diffusion process, assets cannot be traded continuously since total transactions costs are unbounded with probability one (see Merton [1989]). Hence, investors cannot perfectly hedge options and other derivative assets. With transactions costs, trading will take place at discrete points only.

Leland [1985] assumes that trading takes place at discrete, fixed intervals only. The discrete intervals lead to finite transactions costs but also to hedging errors. The trade-off is that with more frequent revisions, transactions costs tend to increase while hedging errors tend to decrease and be less correlated with the market portfolio.

As can be seen, by introducing transactions costs, continuous-time models eventually become discrete-time models. Moreover, the pricing model is not necessarily fully compatible with optimal trading strategies. In other words, the best trading rule for exploiting options mispricing does not necessarily yield the options model prices.

Trading strategies define boundary conditions for pricing options. Such is the model of Leland, which is derived under the assumption that the underlying stock follow a diffusion process and that options can be replicated at fixed points only. This approach yields boundaries for options prices that are very narrow for low, proportional transactions costs and/or short revision-intervals. Leland acknowledges that his bounds are not necessarily the narrowest due to the assumption of fixed time intervals for revising the position.

The HN Model

HN use a stochastic optional control approach to find optimal replication strategy. The trading intervals are endogenized in their model. They impose several restrictive assumptions in order to optimize the strategy subject to a well-specified loss function. While they also assume that the underlying stock follows a diffusion process, they impose two additional strong assumptions:

1. Investors possess a negative exponential utility function, which implies a constant absolute risk aversion.
2. Investors' attitude to pricing options is characterized by risk neutrality.

Though HN try to justify the two assumptions on economic grounds, it should be clear that they were imposed in order to simplify the optimization problem. These assumptions are not necessarily the most favorable ones in finance, and, expecially in a world with transactions costs, the "shadow prices" of these restrictions should be discussed.

By maximizing the utility function recursively, subject to several boundary conditions, HN find the optimal policy in terms of the number of shares of the underlying security. Two values are determined; the number of shares of the underlying security in the replicating portfolio must be kept between the two values, which may be considered as the hedge ratios. These hedge ratios are not necessarily equal to the hedge ratios of BS. The hedge ratios will also be different if the option is held long or short.

HN derive bounds on the pricing of options. The bounds may be narrower than those derived for the Leland model; but this is achieved at the cost of imposing additional restrictive assumptions.

Since, even in a world with transactions costs, there is one clearing price for an asset in an auction marketplace (and also in a dealer market, if the bid-ask spread is ignored), we know there is a "true" model that should yield the equilibrium price. HN do not achieve this goal. In this context, the approach suggested by Dumas and Luciano [1988] should be noted. They claim that when transactions costs are present, an exact replication is not generally the most efficient method of manufacturing an option. The replication technique can only yield the upper and lower bounds on the price. Therefore they employ an optimal portfolio-investment framework to derive the equilibrium option price.

HN tackle a major problem and yield interesting results. However, it is hard to fully appreciate their achievement; more accurate and detailed simulations of options strategies should be provided. Leland's paper provides an excellent illustration of the detailed examples that allow the reader to appreciate the results. HN should show the pattern of hedging over time, and the resulting cumulative transactions costs. It is important to indicate how often the portfolio is revised and by how much.

It should be noted that in order to exploit profit opportunity from mispriced options, one need not necessarily wait until the option's maturity date. This is the limiting case. It is sufficient that sometime during the life of the option, the price moves to its fair value.[1] This should be integrated into the model as a stopping rule.

[1]See Galai [1983].

When constructing a pricing model for options in a world with transactions costs, and especially when alternative models are compared, careful attention should be given to the following questions:

1. What is the structure of costs? Are they proportional, fixed, or a combination?
2. Should transactions costs be applied to the option only, or should they also be imposed on trading the underlying stock and bonds?
3. Should transactions costs be considered for opening a position, and/or for any change in a position, and/or for closing a position?
4. Who is the dominant or marginal trader that determines market prices? Should the analysis be carried out from the point of view of an average individual investor, or should it be the dealer whom the analysis should consider?

The various studies that dealt with option pricing in a world with transactions costs used different assumptions concerning the above issues. The different assumptions may lead to different results and undoubtedly complicate any comparison between alternative models. I think that replication to exploit profit opportunities (and, hence, determine boundaries for options prices) should be analyzed from the point of view of the more efficient dealers. In addition, transactions costs should be imposed on any change in a position, not only of the option but also of the stock and bonds (maybe with a reduced rate on bonds). And, it is important to compare the replicating portfolio to the option when costs are imposed on trading both the portfolio *and* the option.

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