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Two-factor models in option pricing

I Introduction

Practitioners and others are developing and using increasingly complex models for option valuation. It is now commonplace to encounter models involving two or more factors in operational use. Though we might prefer to employ simple single-factor models (such as Black–Scholes or numerical methods which take account of early exercise, e.g. the binomial method), they may become unsatisfactorily crude when faced with situations where the usual Black and Scholes assumptions are violated. Such situations are all too common, for example, in valuing long-term warrants or convertibles where the interest rate plays a role of similar importance to that of the asset value. Stochastic volatility models provide another important example: to model asset distributions with fat tails we allow the volatility of the asset to follow a stochastic process itself.

These extensions are not without their cost. Care is needed at all stages of the modelling: setting up an appropriate model, estimating its parameters, calculating model values, and finally interpreting the outputs. The purpose of the current paper is to review the issues involved in working with two-factor models. We don't claim any originality for the results presented here, but nevertheless we hope it may play a useful role in reviewing an important literature and set of problems.

The paper consists of three main sections. In Section II we discuss the choice and formulation of models. We provide a unified framework within which a variety of models are discussed, and we also consider the issue of under what circumstances the added complexity of a two-factor model is worthwhile. In this section we also discuss some issues concerning the estimation of model parameters: an area where further research is needed. Section III describes the main numerical techniques available. In general they are not difficult to program, but consume much more computing power than, say, the simple binomial method. Finally, in Section IV we use the modelling of stochastic volatility as a case study on two-factor models.

II Modelling choices

General framework

The basic framework for developing a continuous-time model involving any number of state variables is now well established. Key papers are Garman (1976) and Cox *et al.* (1985b). Hull (1989) provides an excellent introduction to this material.

Consider a world where the n state variables $S_1, \ldots, S_n$ follow the random processes

$$dS_i = \mu_i \, dt + \sigma_i \, dz_i, \tag{1}$$

where μ_i is a drift term, σ_i is the volatility of S_i, and dz_i is a Wiener process. We will write ρ_{ij} for the correlation between dz_i and dz_j, and σ_{ij} for the instantaneous covariance, $\rho_{ij}\sigma_i\sigma_j$, between S_i and S_j.

Suppose we have a contingent claim C which depends on the state variables $S_1, \ldots, S_n$ and on time, t. Then by Ito's lemma $C(S, t)$ follows the process

$$dC = \left(C_t + \sum \mu_i \, C_i + \frac{1}{2} \sum\sum \sigma_{ij} \, C_{ij} \right) dt + \sum \sigma_i \, C_i \, dz_i, \tag{2}$$

where C_t denotes $\partial C/\partial t$, C_i denotes $\partial C/\partial S_i$, and C_{ij} denotes $\partial^2 C/\partial S_i \partial S_j$. (However, note that S_i does not denote a derivative.)

We now give a simplified derivation of the partial differential equation which such a claim must satisfy. Imagine that the position in the claim is financed through borrowing its value C at the instantaneous interest rate r. The dynamic of the value associated with this leveraged position, C^L, is given by

$$dC^L = \left(C_t + \sum \mu_i \, C_i - rC + \frac{1}{2} \sum\sum \sigma_{ij} \, C_{ij} \right) dt + \sum \sigma_i \, C_i \, dz_i. \tag{3}$$

In the special case where the S_i are traded assets which pay a continuous dividend at rate α_i, the dynamic of the value associated with similar, zero-net-wealth, leveraged positions is given by

$$dS_i^L = (\mu_i - r \, S_i + \alpha_i \, S_i) \, dt + \sigma_i \, dz_i. \tag{4}$$

Finally if we form

$$dP = dC^L - \sum C_i \, dS_i^L$$

we obtain the equation of motion for a portfolio with zero investment and zero risk. It is

$$dP = \left(C_t - r\,C + \sum (r - \alpha_i) \, S_i \, C_i + \frac{1}{2} \sum\sum \sigma_{ij} \, C_{ij} \right) dt. \tag{5}$$

By economic considerations this portfolio must have a zero rate of return. Thus we obtain the general pricing equation for contingent claims:

$$\partial C/\partial t = rC - \sum (r - \alpha_i)\, S_i\, \partial C/\partial S_i - \frac{1}{2} \sum\sum \sigma_{ij}\, \partial^2 C/\partial S_i \partial S_j. \qquad (6)$$

Note that the Black–Scholes equation is just a special case of equation (6), with $n = 1$. If some of the variables S_i are not traded securities, then equation (6) can still be applied, but with two reservations. First, it is necessary to establish a value for the dividend yield α_i at which the state variable would be accepted in the market as a traded security. This is called its 'convenience yield'. As we shall see below, some researchers have equivalently worked with the 'price of risk' associated with a given untraded state variable. Second, we note that where some state variables are not traded, we can only derive equilibrium prices for contingent claims, and riskless arbitrage is not available if mispricing is observed.

We have just seen that a state variable S_i which is untraded may be included in a model provided we can estimate its convenience yield, α_i. The general pricing equation (6) involves a term $(r - \alpha_i)$ for the drift of this variable. The equation may be interpreted as a valuation equation in a risk-neutral setting. If investors were risk-neutral then the drift on S_i would need to be $(r - \alpha_i)$ for S_i to trade in equilibrium. We can interpret the difference between S_i's objective rate of drift μ_i, and this 'risk-neutral' rate of drift as a risk premium. The 'price of risk', μ_i, for the risk presented by $\sigma_i\, \mathrm{d}z_i$ may be defined by the equation

$$\lambda_i\, \sigma_i = \mu_i - (r - \alpha_i). \qquad (7)$$

In some situations it is preferable to work with the convenience yield, α_i. In others it is more satisfactory to use a price-of-risk variable, λ_i. For example, if we are building a model which involves S_i as the price of a commodity, then the natural formulation would involve its convenience yield. If a forward contract exists for t periods hence at a price F_t then the convenience yield α_i can be measured from the relationship

$$F_t = S_i \exp\{t(r - \alpha_i)\} - c, \qquad (8)$$

where r is the interest rate as before, and c is the storage cost per unit (assumed to be incurred at the end of the period). Note that we do not need to know the objective rate of drift μ_i of S_i, or the price of risk, λ_i.

Conversely, however, consider a situation involving a state variable v (perhaps an interest rate or the volatility of an asset) which follows a mean-reverting process such as

$$\mathrm{d}v = k(a - v)\, \mathrm{d}t + \sigma\, \mathrm{d}z. \qquad (9)$$

Its objective rate of drift $\mu = k(a - v)$ depends on its value v. In this situation it is natural to assume a constant price of risk associated with the shock dv. This implies a varying convenience yield. The alternative of a fixed 'convenience yield' and a varying price of risk would be a very odd assumption. Finally, just as μ may be a function of S so too it may be appropriate to model either the convenience yield or the price of risk as such a function.

Some specific models

We will now look at a number of specific two-state-variable models drawn from the literature. The objective is not to provide a complete literature review, but rather to draw out a number of salient features associated with these models. We begin by considering what types of variables are used as state variables. The three types most frequently encountered are

1. an asset price,
2. an interest rate, and
3. a volatility measure.

We shall see later that other kinds of state variables are also sometimes encountered. Table 8.1 shows the possible pairings among these three categories to form two-factor models. It also lists some examples for some of these pairings.

Two asset prices . Models involving two asset prices have been developed by Margrabe (1978) and by Stultz (1982). Margrabe's analysis shows how to value an option to exchange one asset for another. Stultz provides a more general treatment of options on the minimum or the maximum of two risky assets. Since these papers use only state variables that are traded assets, no risk premia arise in the equations. Margrabe's analysis simplifies to a single-state-variable world by treating one of the assets as numeraire. Relative to this asset the pricing formula is essentially the familiar Black–Scholes (1973) one, with a zero interest rate. Merton's (1973a) treatment of a stochastic interest rate also falls into this category because Merton actually worked with a bond price as state variable rather than with an interest rate, and again may be regarded as a special case of Margrabe's analysis. Even the more general analysis of Stultz only requires a single covariance parameter in addition to the parameters which would be needed for two Black–Scholes models. The eight parameters are S_1, S_2, X, r, T, σ_1, σ_2 and σ_{12}.

Models involving an asset price and an interest rate Models of this kind are relevant to valuing instruments such as long term warrants. A paper by Subrahmanyam (1987) has explicit processes for both the short-term interest rate and the asset value. His equations take the form:

$$dS = (\mu_1 S - \alpha)dt + \sigma_S S dz_s, \quad (10a)$$

$$dr = k(\mu_r - r)dt + \sigma_r r dz_r. \quad (10b)$$

Table 8.1 Types of two-factor models

	Asset price	*Interest rate*	*Volatility*
Asset price	1. Margrabe (1978) Stultz (1982)	2. Subrahmanyam (1987)	3. Hull & White (1987) Wiggins (1987)
Interest rate	—	4. Brennan & Schwartz (1979)	5. ?
Volatility	—	—	6. None

One disadvantage of such a formulation is that since there is no traded asset which earns the drift and shock of the interest-rate process, the model involves a risk premium parameter, λ. A similar risk premium occurs in Merton's (1973b) intertemporal capital asset pricing model scenario. However, in Subrahmanyam's model we also have two additional parameters for the process followed by r and also a dividend term for S itself. The full set of 11 parameters is S, X, T, α, r, k, u_r, λ, σ_S, σ_r, σ_{Sr}.

Models involving stochastic volatility A number of models have now been developed which have an asset price as one state variable and the volatility of the asset as the other. As mentioned earlier, this is an appropriate way to characterise both the problem of fat-tailed distributions, and the difficulties of precise hedging under movements in implied volatility. Hull and White (1987) assume away the problem of a risk premium related to the volatility process. Wiggins (1987) provides a more comprehensive treatment. We will examine this problem in more detail later in the paper.

It is logically possible to develop a model where the state variables are an interest rate and its associated volatility, in order to model bond prices. We cannot, however, recall having seen such a model. Generally authors prefer to introduce a second interest rate as a state variable before taking account of the non-constant variance of the first one. Note, in this context, that the paper of Schaefer and Schwartz (1987), which uses duration to give bonds a variance which is time-dependent, has the variance (of bonds) deterministic, and is just a single-variable model.

Models involving two interest rates Brennan and Schwartz have produced a number of papers modelling the term structure of interest rates that have used a long and a short rate as state variables. Brennan and Schwartz (1979) is a good example of their approach. Normally, with two variables involved, neither of which represents an asset price, we would need to estimate two risk premia parameters. However, Brennan and Schwartz choose the consol rate l as their definition of the long rate. The value of a perpetual government bond is always $1/l$, and it must follow a random process with risk-adjusted drift equal to the instantaneous short term interest rate. By applying Itô's lemma, Brennan and Schwartz are able to solve for the risk premium on the long-rate process. Thus only one risk premium parameter remains to be estimated, besides the parameters of the rather complicated joint stochastic process for r and l.

Other models Some other variables also make interesting state variables. One nice case study is provided by a recent paper of Gibson and Schwartz (1989) on pricing oil-contingent claims. Oil is a commodity and not a financial asset, so a convenience yield is involved from the oil price process. Comparisons between spot prices, futures prices and interest rates, enable the convenience yield to be estimated. The empirical data suggest that the convenience yield is not constant but changes randomly and with mean-reversion. Gibson and Schwartz therefore postulate the following model:

$$dS/S = \mu \, dt + \sigma_1 \, dz_1, \tag{11a}$$

$$dy = k(a - y) \, dt + \sigma_2 \, dz_2, \tag{11b}$$

where S is the spot oil price, y is its convenience yield, and dz_1 and dz_2 are correlated standard Brownian processes. What is nice about this formulation is that y already provides the risk premium for the S process. Only the risk premium for y itself remains. To fit the model it is necessary to know 10 parameters: S, y, X, r, T, k, a, σ_1, σ_2, ρ and λ, but it could easily have been worse!

Control parameters can also appear as state variables. For example in the Brennan and Schwartz (1985) paper on evaluating and controlling the exploitation of a gold mine the inventory of gold remaining in the mine is a state variable which appears in the pricing equation. In the Hodges and Neuberger paper on replicating a contingent claim under transactions costs (Chapter 3 of this volume), the number of shares held in the underlying asset is again a state variable. The mathematics by which such variables appear in the formulation is a bit different from that of our usual treatment under the heading 'General Framework', but its implications for numerical evaluation are very similar. Note that these sorts of inventory or control variables are relatively kind in the informational requirements they impose: they do not involve additional stochastic process or risk premium parameters.

Observations on modelling and parameter estimation

Once a model has been specified, two further problems remain: its parameters must be estimated, and model values must be calculated numerically. We will look at the numerical aspects of using two-factor models in the next section. It turns out that the numerical headaches are the least of our problems. Much more problematic is the increased number of parameters which need to be estimated, and the more than corresponding reduction in intuition about their role. From the examples we have described it is clear that differently specified two-factor models may involve quite different numbers of parameters. Unfortunately, the number needed will usually be dictated mainly by the nature of the application and there is not too much scope to reduce it by being clever. The familiar Black–Scholes model has, of course, five parameters (S, X, r, T and σ), but generally all except σ are observable with very little error. This makes it natural to back out implied volatilities from observed option prices and work with them, and such implied volatilities can often be estimated with a great deal of confidence. Estimating the parameters of more complex two-or-more-factor models poses problems that are much more difficult.

The data available will consist of time-series information on asset prices (and possibly other state variables). Some data on the prices of contingent claims may also be available, but cross-sections may have only a few data points and time series may be limited. One common approach has been to estimate parameters of the stochastic processes from time-series data and to infer the values of the risk premia from cross-sectional data on contingent claims. It is fair to conclude that the estimation of risk premia from time-series data would be likely to result in unacceptably large standard errors. However, for many applications, provided sufficient recent data on contingent claim prices are at hand, it could be desirable to estimate some (but not all) of the process parameters from contingent claims prices. This kind of estimation is not easy, for some claims are more affected by certain parameters than others. The literature on implied volatility estimation is testimony to the wide range of approaches and weighting schemes available. Some researchers have developed model specifications that can be estimated by maximum-likelihood methods but we have seen little evidence concerning the comparative performance of alternative procedures. This is clearly an area where more research is needed.

Given these difficulties it clearly pays to be as sparing as possible with the introduction of additional parameters. One perspective on this is provided by the view that for a European option everything really depends on the probability distributions at the expiry date. This idea is formalised in the work of Jarrow and Rudd (1982). Focusing on a single expiry date potentially provides a lot of scope for simplification, but does not help us to deal with early exercise situations. However, it is perhaps interesting to contemplate that the need for complex option models is primarily driven by the need to understand complex scenarios for early exercise, and their implications for both valuation and hedging.

III Numerical methods

Monte Carlo simulation

One of the simplest numerical approaches for valuing complex derivative securities is the use of Monte Carlo simulation. Boyle (1977) was the first (post-Black–Scholes) to describe its use for option-valuation problems. The method is as follows. Starting from the initial values of the state variables, their evolution is simulated through time using the risk-neutral form of the diffusion equations. The simulation is continued until the payoff date of the contingent claim is reached, and the final payoff is discounted back at the interest rate (or time series of short rates) to obtain a present value. The expected value obtained from repeated simulations provides the value of the contingent claim.

This approach has some obvious advantages as well as some disadvantages. Unlike most other numerical procedures, the time involved increases only linearly in the number of state variables, rather than exponentially. The method is well suited to complex kinds of lookback or down-and-out options, and it is able to deal with payoffs that are path-dependent and claims with early exercise boundaries that are known in advance. The repeated simulations also enable a standard error of the estimate to be obtained as a confidence interval on the valuation. There are two principal disadvantages. First, the approach does not provide a proper method of estimating the position of any free boundaries: for example, it would be unsuitable for valuing American put options, except relative to some guessed exercise strategy which is likely to be more or less seriously sub-optimal. Second, the approach does not automatically provide information on delta, gamma, theta and kappa, although it can be implemented to do so without too much difficulty.

Two refinements are available which can significantly improve the precision of the estimates. The first of these is the use of 'antithetic' variables in the simulation, and the second is the 'control-variate' technique. In its simplest form the antithetic-variable technique consists of running pairs of simulations which are mirror images of each other, i.e. a positive shock in one simulation is replaced by an equal and opposite one in the other. This ensures that the sample mean of the simulated variables is (across simulations) equal to their assumed population mean. However, in the context of option valuation, the variances are likely to be as important as the means. The antithetic approach can be extended to scaling sampled variables so that variances and even covariances are controlled to behave 'nicely'.

The control-variate technique (also described in Boyle, 1977) can be applied when there is a similar model to the one being solved that has an exact solution (e.g. Black–Scholes or similar analytic solution). Suppose we are trying to calculate the value of a particular contingent claim, C, and its simulated value is C^S. A related simpler derivative security (or model), D, has a value of D^S from the same simulation, but also has an exact analytic valuation of D^A. Since we can expect the errors in the two simulated valuations to be similar an improved estimate of the value of C is given by

$$C = C^S + D^A - D^S. \tag{12}$$

Lattice approaches and finite-difference methods

For many problems, particularly where free boundaries are involved, it seems more natural to employ either a lattice approach analogue of the binomial method, or a finite-difference method on a rectangular grid.

The binomial method of option valuation is too well known to be described here. (See, for example, Cox *et al.* (1979), or recent textbooks on option valuation.) With more than one state variable the usual triangular evaluation region (expanding outwards through time) may be generalised to a similar conical shape. The probability at each point on the lattice has to be chosen so that the standard deviations and correlations among the state variables are correct, and their rates of drift are consistent with the risk-neutral form of the diffusions.

The lattice approach is equivalent to a finite-difference method of solving the partial differential equation. Since each lattice point is computed only by reference to points at the next time interval, lattice approaches are examples of explicit finite-difference methods. Although the binomial method is always stable this is not true of explicit methods generally, and care is needed to guarantee convergence. For one-dimensional problems the semi-implicit Crank–Nicholson method (Smith, 1978) has much to recommend it. The method generalises to some high-dimensional problems, but more powerful techniques are to be preferred for these. Alternating directing implicit (ADI) methods were first introduced by Peaceman and Rachford (1955). Their scheme was a two-stage process, stepping first in one grid direction and then in the other. Each step individually is unstable but, provided the steps alternate, stability can be otained. This approach has been extended and refined into the hopscotch method of Gourlay and McKee (1977), which provides a reasonably efficient procedure for solving two-dimensional problems including a cross-derivative.

In the so-called 'line' method of the hopscotch algorithm, alternate lines of the space grid are solved explicitly and implicitly. First, every other line of the grid is updated using an explicit difference formula. Next, the remaining lines are filled in using a simple implicit approach which just involves inverting a tridiagonal matrix (as in Crank–Nicholson). The procedure is then reversed and repeated, with the implicit line updated by an explicit one, and vice versa. State variables following a geometric process are transformed to their logarithms before defining the grid in order to achieve constant coefficients. If possible, the first-order derivatives should also be eliminated by a suitable transformation. While the procedure is unconditionally stable with no first-order terms, this is no longer true if first-order terms remain. For the differential operator

$$L = a\,\partial^2./\partial x^2 + 2b\,\partial^2./\partial x\,\partial y + c\,\partial^2./\partial y^2 + d\,\partial./\partial x + e\,\partial./\partial y \quad (13)$$

stability requires that the space step h satisfies the inequalities:

$$h \leq 2a/d, \text{ and} \quad (14a)$$

$$h \leq 2c/e, \quad (14b)$$

in addition to the usual condition on space–time grid spacing that the time step be related to the square of the space step.

Finally, Hull and White (1988) have pointed out that there is no reason for the application of the control-variate technique to be confined to simulation methods. We may just as well apply it to solutions computed by lattice or finite-difference methods.

Other methods

The finite-difference methods we have just described work fairly well, but still suffer from a number of disadvantages. One of these is simply the nature of the grid used. With a geometric process we will have a geometric grid, while for output purposes an arithmetic one would be much more convenient. We therefore end up having to interpolate to get values in between grid points. If we are going to have to do this, it is much better to estimate the coefficients of interpolation formulae directly. In other words we should be using finite-element techniques rather than finite-difference ones.

Finally, we have rather ignored the possibility of analytical solutions or approximations to these kinds of problems. Generally, analytic solutions are not available, but the literature does contain just a few exceptions to this rule. Cox *et al.* (1985a), for example, provide closed-form expressions for bond prices, where their single-factor model for real rates is embedded in a further process for inflation. Schaefer and Schwartz (1984) provide an approximate analytic solution to another two-factor term structure model where $(l - s)$ and s have orthogonal shocks: a feature well supported by empirical work.

IV Stochastic volatility

We mentioned earlier two models which use a stochastic process for volatility to characterise the fat-tailed distributions of an underlying asset. Early work on this problem includes that of Johnson (1981), Johnson and Shanno (1985) and Scott (1986). Hull and White (1987) provide the sim-

plest formulation in which the risk premium related to the volatility process is assumed to be zero. They show that the value of a European option is an expectation of Black–Scholes values over the distribution of realisations of the variance. Hull and White calculate the moments of this distribution and use these to provide a Taylor series expansion for the option value. They suggest a Monte Carlo simulation approach to a slightly more general formulation and apply this in the context of currency options. Wiggins (1987) provides a more general treatment. His formulation includes a risk parameter on the volatility shock and he advocates Gourlay's hopscotch method for numberical solution. While all of these authors mention fat-tailed distributions, these papers contain surprisingly little information about what level of kurtosis is implied by their choices of parameters.
Wiggins formulates the problem as

$$dS = \mu_S S\, dt + v\, S\, dz_s, \tag{15a}$$

$$dv = f(v)\, dt + \theta\, v\, dz_v, \tag{15b}$$

and estimates a mean-reverting process for the drift term $f(v)$. A special method is used to obtain an unbiased estimator of $\rho\theta$, where ρ is the correlation between dz_s and dz_v. Wiggins estimates θ, ρ and the speed of mean-reversion from time-series behaviour of stock price data.

This risk premium term was re-expressed in terms of other correlations and obtained from their estimates. Wiggins thereby obtains a richer but in many ways conventional-looking model that enables him to examine how Black–Scholes-measured implied volatilities depend on the stock-to-exercise-price ratio. No serious estimations are provided for either the initial variances or for their long-term means.

Although the examples give a good illustration of the ability of this kind of model to capture a variety of exercise price effects, they also confirm the difficulties of statistical estimation.

We have computed some model values using a model of the same kind as Wiggins. Figure 8.1 shows numerically calculated call-option values as a function of the share price (S) and the initial volatility (sigma). Its behaviour is very much as we would expect. Figure 8.2 shows the early exercise boundary for an American put. If the volatility becomes high it pays to hold on and only exercise at a substantially lower stock price than usual. The only problem with this is that we have assumed that at any date the instantaneous volatility is exactly known. This brings us back again to the issue of estimation!

V **Conclusions**

The paper has provided an introduction to models containing more than one factor. We have seen that such models are relatively easily formulated

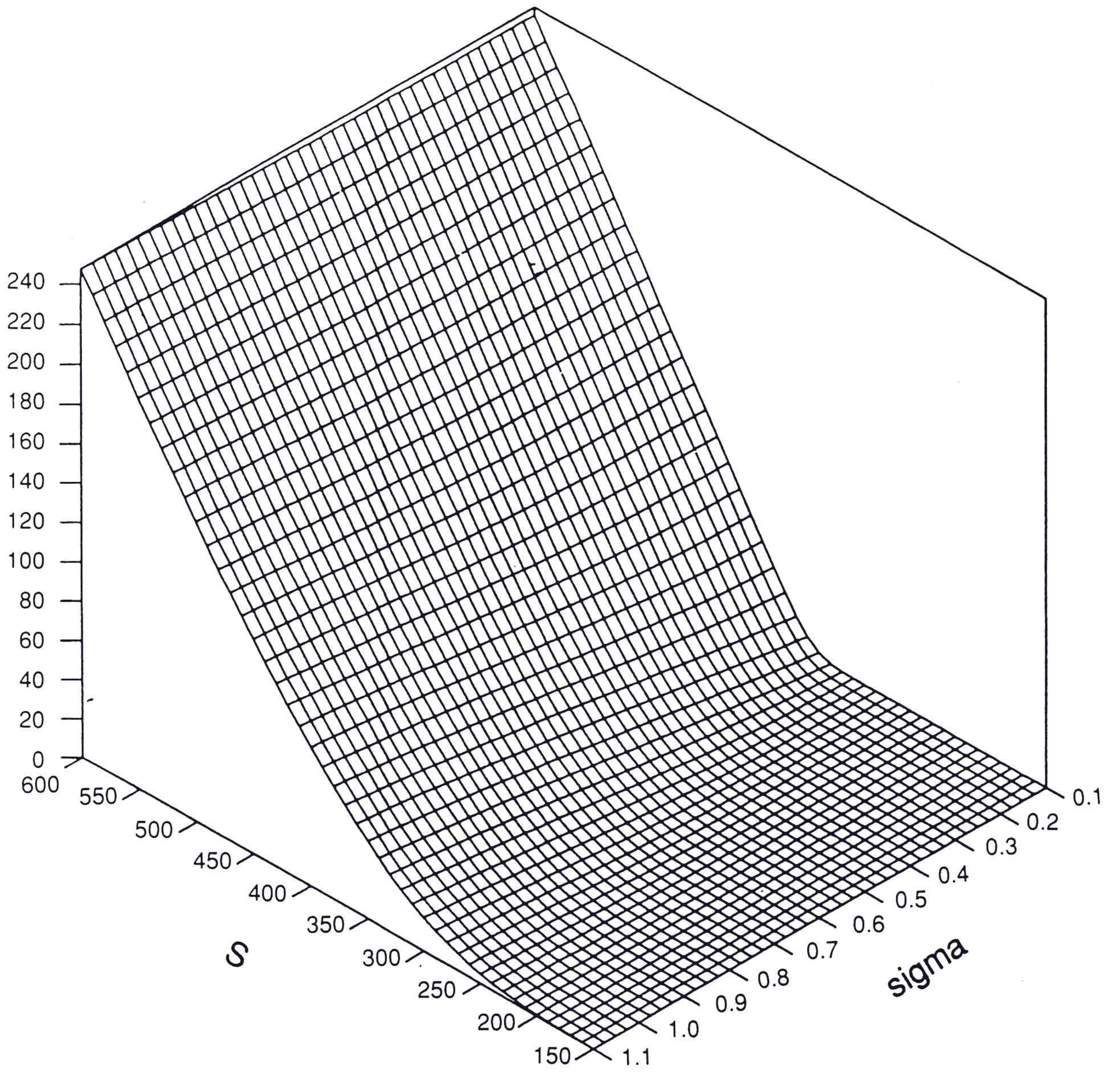

Figure 8.1 Call option values under stochastic volatility: SVOL ($X = 360$), $T = 0.08$

and that most of the numerical problems can be overcome without too much difficulty. Simulation has advantages provided a free boundary is not involved, where lattice, finite-difference, or finite-element methods apply. The most serious problems concern the estimation of parameters and also their reinterpretation in terms of implications for horizon distributions.

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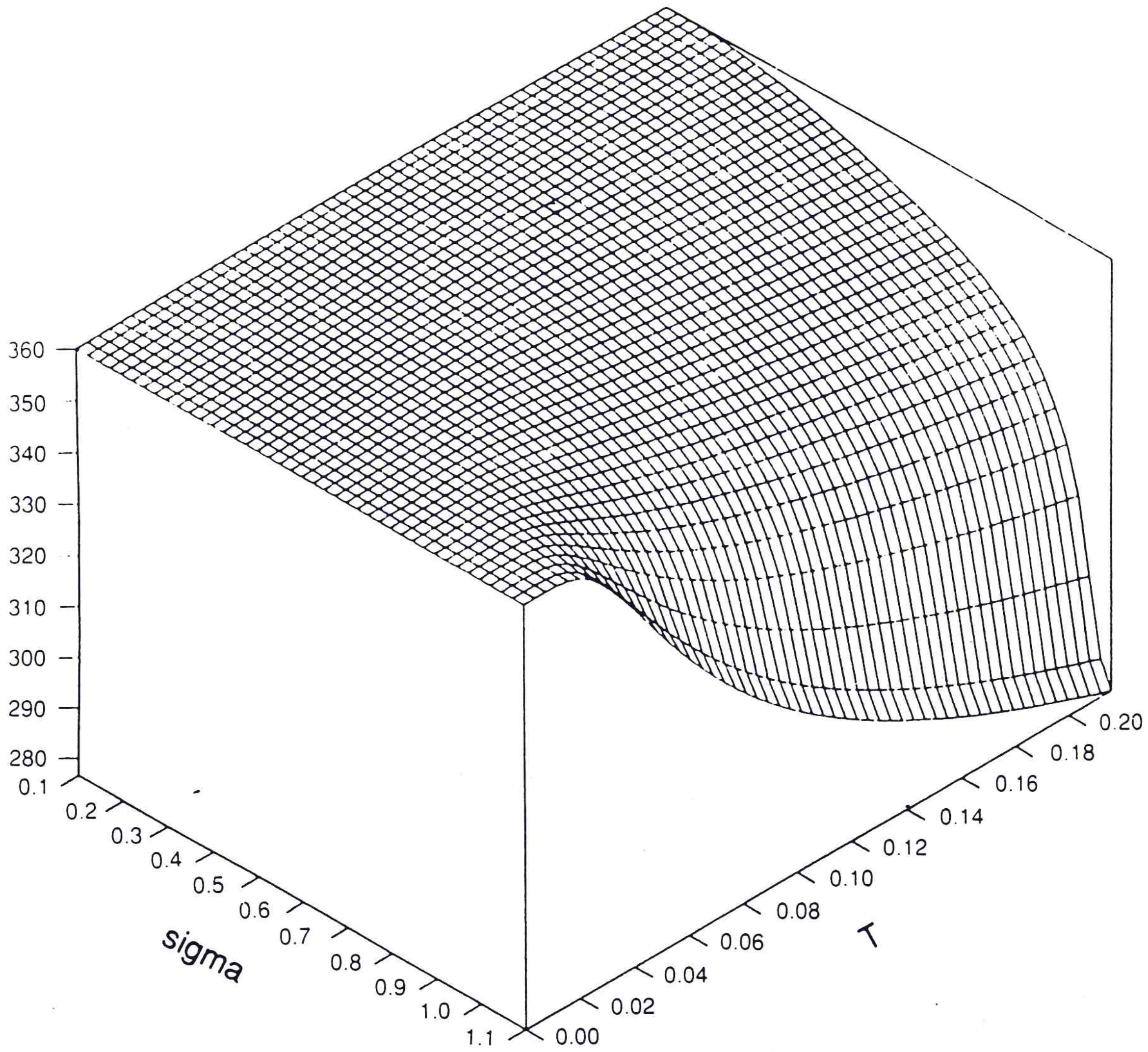

Figure 8.2 Early exercise boundary for American put option: SVOL critical S ($X = 360$)

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